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QUANTITATIVE UNIQUENESS FOR TIME-PERIODIC HEAT EQUATION WITH POTENTIAL AND ITS APPLICATIONS

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Abstract. In this paper, we establish a quantitative unique continuation property for some time-periodic linear parabolic equations in a bounded domain Ω. We prove that for a time-periodic heat equation with particular time-periodic potential, its solution $u(x,t)$ satisfies $\|u(\cdot,0)\|_{L^2(\Omega)} \leq C\,\|u(\cdot,0)\|_{L^2(\omega)}$ where $\omega \subset \Omega$. Also we deduce the asymptotic controllability for the heat equation with an even, time-periodic potential. Moreover, the controller belongs to a finite dimensional subspace and is explicitly computed.

1. Introduction and main results

Throughout this paper, Ω is a connected bounded domain in $\mathbb{R}^d$, $d \geq 1$, with a boundary $\partial\Omega$ of class C^2, ω is a non-empty open subset of Ω. Let $\ell_1 > 0$ be the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary condition (i.e., the smallest strictly positive eigenvalue of $-\Delta$ in $H_0^1(\Omega)$). Let $T > 0$ be a real number; we denote by $\|.\|_\infty$ the usual norm in $L^\infty(\Omega\times(0,T))$.

In this paper, we study an infinite-dimensional system generated by the following parabolic equation

$$\begin{cases} \partial_t y - \Delta y + ay = f & \text{in } \Omega\times(0,+\infty), \\ y = 0 & \text{on } \partial\Omega\times(0,+\infty), \end{cases} \tag{1.1}$$

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with an even, T-periodic, bounded potential $a = a(x,t) \in L^\infty(\Omega \times \mathbb{R})$ satisfying

$$\ell_1 \le \|a\|_\infty \quad \text{and} \quad a(x,t+T) = a(x,t) = a(x,-t) \quad \text{in } \Omega \times \mathbb{R}. \tag{1.2}$$

We assume that $y(\cdot,0) = y^o \in L^2(\Omega)$ and $f = f(x,t) \in L^1_{loc}(0,+\infty; L^2(\Omega))$, so that (1.1) admits a unique solution $y = y(x,t) \in C([0,+\infty); L^2(\Omega))$. Let $\{G(t,s)\}_{0\le s\le t<+\infty}$ be the T-periodic evolutionary process on $L^2(\Omega)$, such that

$$y(\cdot,t) = G(t,s)y(\cdot,s) + \int_s^t G(t,r)f(\cdot,r)dr, \tag{1.3}$$

for all $0 \le s \le t$, where the T-periodicity of $G(t,s)$ means $G(t+T,s+T) = G(t,s)$.

Notice that any bounded function in $L^\infty(\Omega \times (0,T/2))$ can be extended to be an even, T-periodic potential $a \in L^\infty(\Omega \times \mathbb{R})$. Here, we are only interested in the case where $(\|a\|_\infty - \ell_1) \ge 0$ and a is not positive, because if $(\|a\|_\infty - \ell_1) < 0$ or $a \ge 0$, then the system (1.1) is clearly stable when $f = 0$.

The Poincaré map is usually defined by $G(T+t,t)$. We restrict our attention on the operator $G(T,0)$

$$\begin{array}{rccc} G(T,0): & L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & y(\cdot,0) & \longmapsto & y(\cdot,T). \end{array} \tag{1.4}$$

It is well known that $G(T,0)$ is compact by the smoothing action of the diffusion. Since a is even and T-periodic, we see that $G(T,0)$ is self-adjoint. Consequently, there is a complete orthonormal set in $L^2(\Omega)$ formed of eigenfunctions $(z_j^o)_{j\ge1}$ of $G(T,0)$ corresponding to eigenvalues $(\lambda_j)_{j\ge1}$, where $\lambda_j = \lambda_j(T)$ depends on T, $\lambda_j \in \mathbb{R}$, $\lambda_j \to 0$ as $j \to +\infty$. Thus, we may arrange the eigenfunctions so that the sequence $\{|\lambda_j|\}_{j\ge1}$ is non-increasing, $\cdots \le |\lambda_{m+1}| \le |\lambda_m| \le \cdots \le |\lambda_1|$. Clearly, $\lambda_j \ne 0$ from the backward uniqueness property for the linear parabolic equation. Furthermore, we will see that for all $T > 0$ and for all even, T-periodic, bounded potentials a, any eigenvalue $\lambda_j(T)$ of $G(T,0)$ satisfies

$$\ell_1 + \frac{\ln|\lambda_j(T)|}{T} \le \|a\|_\infty . \tag{1.5}$$

The first result of the paper is as follows.

Theorem 1. *There exist two constants $c_o \in (0,1)$ and $C > 0$, both only depending on ω and Ω, such that if we choose the time-periodicity $T \in (0, c_o]$*

and an even, T-periodic potential $a \in L^\infty(\Omega \times \mathbb{R})$ such that

$$\ell_1 \leq \|a\|_\infty \leq \ell_1 \big(\frac{c_o}{T}\big)^{1/4},$$

then any eigenfunction z_j^o of $G(T,0)$ in $L^2(\Omega)$, corresponding to the eigenvalue $\lambda_j(T)$ with $|\lambda_j(T)| \geq 1$, satisfies

$$\int_\Omega \big|G(t,0)z_j^o(x)\big|^2 \, dx \leq e^{C\|a\|_\infty^{4/3}} \int_\omega \big|G(t,0)z_j^o(x)\big|^2 \, dx$$

for all $t \geq 0$.

The knowledge of the eigencouple of the Poincaré map plays a key role in the study of periodic parabolic systems (see [14], [15]). Theorem 1 implies clearly that the eigenfunctions z_j^o of the Poincaré map corresponding to the eigenvalues λ_j with $|\lambda_j| \geq 1$ have the unique continuation property.

It is classical in control theory for linear partial differential equations that the unique continuation property is linked with the approximate controllability and, more precisely, a quantitative uniqueness result yields an estimate of the cost of the approximate control (see e.g. [12]). Here, our second theorem establishes an asymptotic controllability (or open loop stabilizability) for the heat equation with time-periodic potential.

Theorem 2. *There exist two constants $c_o \in (0,1)$ and $C > 0$, both only depending on Ω and ω, such that if we choose the time-periodicity $T \in (0, c_o]$ and an even, T-periodic potential $a \in L^\infty(\Omega \times \mathbb{R})$ such that*

$$\ell_1 \leq \|a\|_\infty \leq \ell_1 \big(\frac{c_o}{T}\big)^{1/4},$$

and if there exists $m > 0$ such that

$$\cdots \leq |\lambda_{m+1}(T)| < 1 \leq |\lambda_m(T)| = \cdots = |\lambda_2(T)| = |\lambda_1(T)|\,,$$

then for each initial data $y^o \in L^2(\Omega)$, *the control function* $f_c \in C([0,T];L^2(\Omega))$ *satisfying*

$$\begin{cases} (i) \quad f_c(x,T-t) = \sum_{j=1,\cdot\cdot,m} \sigma_j(t) G(t,0) z_j^o(x), \\ (ii) \quad \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{pmatrix} = -\frac{1}{T}\left[\left(\int_\omega G(t,0) z_\kappa^o(x) G(t,0) z_j^o(x) dx \right)_{1\leq\kappa,j\leq m} \right]^{-1} \\ \qquad\qquad \times \begin{pmatrix} \int_\Omega y^o(x) G(T,0) z_1^o(x) dx \\ \vdots \\ \int_\Omega y^o(x) G(T,0) z_m^o(x) dx \end{pmatrix}, \\ (iii) \quad \int_0^T \left\| f_c(\cdot,s) \right\|_{L^2(\omega)} ds \leq e^{C\|a\|_\infty^{4/3}} e^{\left(\|a\|_\infty - \ell_1\right)T} \left\| y^o \right\|_{L^2(\Omega)}, \end{cases}$$

implies that the solution $y = y(x,t) \in C([0,+\infty);L^2(\Omega))$ *of the following heat equation with potential* a *and control function* f_c,

$$\begin{cases} \partial_t y - \Delta y + ay = f_c \cdot 1_{|\omega\times(0,T)} \quad \text{in } \Omega\times(0,+\infty), \\ y = 0 \quad \text{on } \partial\Omega\times(0,+\infty), \\ y(\cdot,0) = y^o \quad \text{in } \Omega, \end{cases}$$

satisfies, for all $t \geq T$,

$$\left\| y(\cdot,t) \right\|_{L^2(\Omega)} \leq C_m \ e^{-\gamma t} \left\| y^o \right\|_{L^2(\Omega)},$$

where

$$\gamma = \frac{-\ln|\lambda_{m+1}(T)|}{T} > 0 \text{ and } C_m = \frac{2e^{C\left(\|a\|_\infty^{4/3} + \|a\|_\infty T\right)}}{|\lambda_{m+1}(T)|^2}.$$

The notion of asymptotic controllability is standard in the nonlinear control theory of finite-dimensional systems (see e.g. [7]). Here, (i) says that our control has a finite-dimensional structure (see also [4], [5]). (ii) implies that the operator associating the initial data with the control function is linear which gives a kind of robustness property of our control. (iii) gives us an explicit expression of the cost of the control, from which we see easily that we can act a control on the equation in a very short time T, but as a payment, evenness and T-periodicity for the potential a are required.

From the following property of the Poincaré map,

$$G(nT,0)z_j^o = (\lambda_j)^n z_j^o \tag{1.6}$$

for all $n \in \mathbb{N}$, it is clear that if all eigenvalues λ_j satisfy $|\lambda_j| < 1$, then the system (1.1) is stable without control. On the other hand, if the modulus of some eigenvalues of $G(T,0)$ are bigger than one, then the equation

(1.1) with $f = 0$ is unstable. We call such eigenvalues unstable eigenvalues. Theorem 2 amounts to saying that if all unstable eigenvalues have the same absolute value then the equation (1.1) with $f = f_c \cdot 1_{|\omega\times(0,T)}$ can be stabilized by a control f_c in a finite-dimensional subspace of $L^2(\Omega)$ spanned by $(z_j^o)_{j=1,\cdots,m}$ which are the eigenfunctions of $G(T,0)$ corresponding to the unstable eigenvalues. So it is important to study the unique continuation of the eigenfunctions of $G(T,0)$ corresponding to the unstable eigenvalues. This is why we restrict our attention in Theorem 1 to the case where $|\lambda_j| \geq 1$.

The unique continuation property or more precisely the number of zeros of a one-dimensional parabolic equation is well described in [1] which leads to the study of the asymptotic behaviour of the periodic $1d$ quasilinear parabolic equation [3], or to the consideration of the structural stability of the periodic $1d$ semilinear heat equation [6].

In order to get Theorem 1 and Theorem 2, we need a more general unique continuation result in bounded domains $\Omega \subset \mathbb{R}^d$, $d \geq 1$, described as follows.

Theorem 3. *There exist two constants $c_o \in (0,1)$ and $C > 0$, both only depending on Ω and ω, such that for all $T \in (0, c_o]$, $\beta \geq 0$ and $q \in L^\infty(\Omega \times (0,T))$ satisfying*

$$\ell_1 + \beta \leq \|q\|_\infty \leq \ell_1 \left(\frac{c_o}{T}\right)^{1/4},$$

we have

$$\int_\Omega |u(x,0)|^2 \, dx \leq e^{C\|q\|_\infty^{4/3}} \int_\omega |u(x,0)|^2 \, dx,$$

where $u \in C([0,T]; L^2(\Omega))$ solves the following T-periodic linear heat equation

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u(\cdot,T) \quad \text{in } \Omega. \end{cases}$$

Strong unique continuation for the heat equation with potential in bounded domains $\Omega \subset \mathbb{R}^d$, $d \geq 1$, has been widely studied, (see for instance [10], [11] and references therein). In [17], one obtains that any solution v of the linear heat equation with time-independent Lipschitz continuous potential such that $v(\cdot, t_o) = 0$ in ω, for some $t_o \in (0,T)$, satisfies $v = 0$ in $\Omega \times (0,T)$. The idea of the proof in [17] is to transform the solution v to a solution of an elliptic equation where the unique continuation property for elliptic operators can be used (see also [8]). Here we are interested in the unique continuation and its quantification of the linear time-periodic heat equation with time-dependent potential. The basic technique used here is to combine some properties of time-periodic solutions of the heat equation with a kind of global Carleman estimate for an elliptic operator.

This paper is organized as follows. In Section 2, we give an introduction of the Poincaré map and some related preliminary results, then we prove Theorem 1. In Section 3 and Section 4, we give the proofs of Theorem 2 and Theorem 3 respectively. In Section 5, we point out some related results for a particular case where the potential depends only on time. Finally, in the Appendix, we give the proof of the Global Carleman inequality we need.

2. The Poincaré map

Let a be an even, T-periodic, bounded function in $L^\infty(\Omega\times\mathbb{R})$; more precisely $a(x,t+T)=a(x,t)=a(x,-t)$ in $\Omega\times\mathbb{R}$. We consider the following heat equation with the potential $a=a(x,t)$,

$$\begin{cases} \partial_t y-\Delta y+ay=f \quad \text{in } \Omega\times(s,+\infty),\\ y=0 \quad \text{on } \partial\Omega\times(s,+\infty),\end{cases} \tag{2.1}$$

where $s\geq 0$ and $f\in L^1_{loc}(0,+\infty;L^2(\Omega))$. Let $y\in C([s,+\infty);L^2(\Omega))$ be the mild solution of (2.1) and $G(t,s)$ be the T-periodic evolutionary process generated by $-\Delta+aI$ in $L^2(\Omega)$. More precisely, $\{G(t,s)\}_{0\leq s\leq t<+\infty}$ is a family of bounded linear operators from $L^2(\Omega)$ to itself, such that

$$\begin{cases} (i) & G(t,t)=I \quad \text{for all } t\geq 0,\\ (ii) & G(t,r)G(r,s)=G(t,s) \quad \text{for all } 0\leq s\leq r\leq t,\\ (iii) & \text{The map } (t,s)\mapsto G(t,s)g \text{ is continuous for every fixed } g\in L^2(\Omega),\\ (iv) & G(t+T,s+T)=G(t,s) \quad \text{for all } 0\leq s\leq t,\\ (v) & \|G(t,s)\|\leq e^{(\|a\|_\infty-\ell_1)(t-s)} \quad \text{for all } 0\leq s\leq t.\end{cases} \tag{2.2}$$

Thus, the mild solution y of equation (2.1) can be written in the following form

$$y(\cdot,t)=G(t,s)y(\cdot,s)+\int_s^t G(t,r)f(\cdot,r)dr \tag{2.3}$$

for all $0\leq s\leq t$.

Now, we are going to focus our attention on the Poincaré map $G(T,0)$ and obtain some properties in our case. Since a is even and T-periodic, we get that $G(T,0)$ is self-adjoint. Indeed, let $p^T\in L^2(\Omega)$ and $p\in C([0,T];L^2(\Omega))$ be the solution of

$$\begin{cases} -\partial_t p-\Delta p+ap=0 \quad \text{in } \Omega\times(0,T),\\ p=0 \quad \text{on } \partial\Omega\times(0,T),\\ p(\cdot,T)=p^T \quad \text{in } \Omega,\end{cases} \tag{2.4}$$

then for all $y^o \in L^2(\Omega)$,

$$\int_\Omega y^o(x)p(x,0)dx = \int_\Omega \left[G(T,0)y^o(x)\right] p^T(x)dx; \tag{2.5}$$

that is,

$$G^*(T,0)p^T(x) = p(x,0). \tag{2.6}$$

On the other hand, because $a(x,T-t) = a(x,t)$, the solution $w(x,t) = p(x,T-t)$ solves

$$\begin{cases} \partial_t w - \Delta w + aw = 0 \quad \text{in } \Omega \times (0,T), \\ w = 0 \quad \text{on } \partial\Omega \times (0,T), \\ w(\cdot,0) = p^T \quad \text{in } \Omega, \\ w(\cdot,T) = p(\cdot,0) \quad \text{in } \Omega. \end{cases} \tag{2.7}$$

Finally, for all $p^T \in L^2(\Omega)$,

$$G^*(T,0)p^T(x) = G(T,0)p^T(x). \tag{2.8}$$

As $G(T,0)$ is a compact self-adjoint operator on $L^2(\Omega)$, we deduce that there is a complete orthonormal set in $L^2(\Omega)$ formed by the eigenfunctions $(z_j^o)_{j\geq 1}$ of $G(T,0)$ with corresponding eigenvalues $\lambda_j = \lambda_j(T) \in \mathbb{R}$, which converge to 0 as $j \to +\infty$. Thus, we can arrange the eigenfunctions so that the sequence $\{|\lambda_j|\}_{j\geq 1}$ is non-increasing.

Now we are able to give some properties of the eigencouple $(z_j^o, \lambda_j(T))$ associated to the compact self-adjoint operator $G(T,0)$.

Lemma 2-1. *For all $s \in [0,T]$ and $\phi \in L^2(\Omega)$,*

$$\int_\Omega \phi(x)G(T-s,0)z_j^o(x)dx = \int_\Omega G(T,s)\phi(x)z_j^o(x)dx.$$

It follows immediately from Lemma 2-1 that for all $s \in [0,T]$,

$$\int_\Omega G(s,0)z_i^o(x)\frac{G(T-s,0)z_j^o(x)}{\lambda_j}dx = \delta_{ij}. \tag{2.8}$$

Lemma 2-2. *As soon as the time-periodicity $T > 0$ and an even, T-periodic potential $a \in L^\infty(\Omega \times \mathbb{R})$ are chosen, then any eigenvalue $\lambda_j(T)$ of $G(T,0)$ satisfies*

$$\ell_1 + \frac{\ln|\lambda_j(T)|}{T} \leq \|a\|_\infty .$$

Note that in particular, under the hypothesis of Theorem 1, the eigenvalues $\lambda_j(T)$ have to satisfy

$$\ell_1 + \frac{\ln|\lambda_j(T)|}{T} \leq \|a\|_\infty \leq \ell_1\big(\frac{c_o}{T}\big)^{1/4}. \tag{2.9}$$

2.1. **Proof of Lemma 2-1.** Let $g = g(x,t) \in C([s,T]; L^2(\Omega))$ be the solution of

$$\begin{cases} \partial_t g - \Delta g + ag = 0 \quad \text{in } \Omega \times (s,T), \\ g = 0 \quad \text{on } \partial\Omega \times (s,T), \\ g(\cdot, s) = \phi \quad \text{in } \Omega, \end{cases} \tag{2.10}$$

and $z_j = z_j(x,t) \in C([0,T]; L^2(\Omega))$ be the solution of

$$\begin{cases} \partial_t z_j - \Delta z_j + az_j = 0 \quad \text{in } \Omega \times (0,T), \\ z_j = 0 \quad \text{on } \partial\Omega \times (0,T), \\ z_j(\cdot, 0) = z_j^o \quad \text{in } \Omega. \end{cases} \tag{2.11}$$

Now, from the relation $a(\cdot, T-t) = a(\cdot, t)$, we obtain that the function $t \mapsto \int_\Omega g(x,t) z_j(x, T-t) dx$ is constant. Indeed,

$$\begin{aligned} &\frac{d}{dt} \int_\Omega g(x,t) z_j(x, T-t) dx \\ &= \int_\Omega \partial_t g(x,t) z_j(x,T-t) dx + \int_\Omega g(x,t) \big(-(\partial_t z_j)(x, T-t) \big) dx \\ &= \int_\Omega -(-\Delta + a(x,t)) g(x,t) z_j(x, T-t) dx \\ &\quad + \int_\Omega g(x,t)(-\Delta + a(x,t)) z_j(x,T-t) dx = 0. \end{aligned} \tag{2.12}$$

Consequently, for all $t \in [s,T]$,

$$\int_\Omega g(x,s) z_j(x,T-s) dx = \int_\Omega g(x,t) z_j(x,T-t) dx = \int_\Omega g(x,T) z_j^o(x) dx. \tag{2.13}$$

That completes the proof of Lemma 2-1.

2.2. **Proof of Lemma 2-2.** Let

$$\widetilde{\beta} = \frac{\ln |\lambda_j(T)|}{T} \in \mathbb{R} \tag{2.14}$$

and $u = u(x,t) \in C([0,2T]; L^2(\Omega))$ be the solution of

$$\begin{cases} \partial_t u - \Delta u + \widetilde{\beta} u + au = 0 \quad \text{in } \Omega \times (0, 2T), \\ u = 0 \quad \text{on } \partial\Omega \times (0, 2T), \\ u(\cdot, 0) = z_j^o \quad \text{in } \Omega. \end{cases} \tag{2.15}$$

By an energy method, we have the following equality

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u(x,t)|^2 \, dx + \int_\Omega |\nabla u(x,t)|^2 \, dx + \widetilde{\beta} \int_\Omega |u(x,t)|^2 \, dx$$

$$+\int_\Omega a(x,t)\,|u(x,t)|^2\,dx = 0. \tag{2.16}$$

Also, notice that $u(x,t) = e^{-\widetilde{\beta}t} z_j(x,t) = e^{-\widetilde{\beta}t} G(t,0) z_j^o(x) \neq 0$ and

$$u(\cdot,2T) = u(\cdot,0) \quad \text{in } \Omega. \tag{2.17}$$

From (2.16) and (2.17), we get

$$\Big(\ell_1 + \widetilde{\beta} - \|a\|_\infty\Big) \int_0^{2T} \int_\Omega |u(x,t)|^2\,dx \le 0. \tag{2.18}$$

Consequently,

$$\ell_1 + \widetilde{\beta} = \ell_1 + \frac{\ln|\lambda_j(T)|}{T} \le \|a\|_\infty\,. \tag{2.19}$$

This is the desired inequality.

2.3. **Proof of Theorem 1.** For $|\lambda_j(T)| \ge 1$, we have that

$$\beta = \frac{\ln|\lambda_j(T)|}{T} \ge 0 \tag{2.20}$$

and $u(x,t) = e^{-\beta t} z_j(x,t) \in C([0,2T]; L^2(\Omega))$ satisfies

$$\begin{cases} \partial_t u - \Delta u + \beta u + au = 0 \quad \text{in } \Omega \times (0,2T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,2T), \\ u(\cdot,2T) = u(\cdot,0) \quad \text{in } \Omega. \end{cases} \tag{2.21}$$

Now, we apply Theorem 3 for the $2T$-periodic solution u of (2.21) with $q = a \in L^\infty(\Omega \times (0,2T))$ satisfying

$$\ell_1 + \frac{\ln|\lambda_j(T)|}{T} \le \|a\|_\infty \le \ell_1 \big(\frac{c_o}{2T}\big)^{1/4} \tag{2.22}$$

to obtain

$$\int_\Omega |u(x,0)|^2\,dx \le e^{C\|a\|_\infty^{4/3}} \int_\omega |u(x,0)|^2\,dx. \tag{2.23}$$

It follows by translation in time and by the $2T$-periodicity of u that for all $t \in [0,2T]$,

$$\int_\Omega |u(x,t)|^2\,dx \le e^{C\|a\|_\infty^{4/3}} \int_\omega |u(x,t)|^2\,dx. \tag{2.24}$$

Consequently, we obtain that for all $t \in [0,2T]$,

$$\int_\Omega \left|G(t,0) z_j^o(x)\right|^2 dx \le e^{C\|a\|_\infty^{4/3}} \int_\omega \left|G(t,0) z_j^o(x)\right|^2 dx. \tag{2.25}$$

Hence, if we choose c_o in Theorem 1 to be $\frac{c_o}{2}$ in Theorem 3 and the same C, we complete the proof of Theorem 1.

3. The asymptotic controllability problem

In this section, we prove Theorem 2. Recall that if $(\|a\|_\infty - \ell_1) < 0$, then the system (1.1) is already stable with $f = 0$. We consider the solution of the following heat equation with a potential $a = a(x,t) \in L^\infty(\Omega \times \mathbb{R})$ satisfying $a(x, t+T) = a(x,t) = a(x,-t)$ in $\Omega \times \mathbb{R}$ and $(\|a\|_\infty - \ell_1) \geq 0$,

$$\left\{ \begin{array}{c} \partial_t y - \Delta y + ay = f_c \cdot 1_{|\omega\times(0,T)} \quad \text{in } \Omega \times (0,+\infty), \\ y = 0 \quad \text{on } \partial\Omega \times (0,+\infty), \\ y(\cdot,0) = y^o \quad \text{in } \Omega, \end{array} \right. \tag{3.1}$$

where $y^o \in L^2(\Omega)$ and $f_c \in C([0,T]; L^2(\Omega))$. It is clear that for all integers $n \geq 1$, we have

$$y(\cdot, nT) = G(nT,0)y^o + \int_0^{nT} G(nT,s)\big(f_c(\cdot,s)\cdot 1_{|\omega\times(0,T)}\big)ds. \tag{3.2}$$

Let $(z_j^o, \lambda_j(T))$ be the eigencouple of the compact self-adjoint operator $G(T,0)$, then

$$y^o = \sum_{j\geq 1} \Big(\int_\Omega y^o(x) z_j^o(x) dx \Big) z_j^o. \tag{3.3}$$

It follows from Lemma 2-1 that for any $s \in [0,T]$,

$$\begin{aligned} & G(T,s)\big(f_c(x,s)\cdot 1_{|\omega\times(0,T)}\big) \\ & = \sum_{j\geq 1} \Big(\int_\Omega G(T,s)(f_c(x,s)\cdot 1_{|\omega\times(0,T)}) z_j^o(x) dx \Big) z_j^o(x) \\ & = \sum_{j\geq 1} \Big(\int_\Omega \big(f_c(x,s)\cdot 1_{|\omega\times(0,T)}\big) G(T-s,0) z_j^o(x) dx \Big) z_j^o(x). \end{aligned} \tag{3.4}$$

By (3.3), (3.4) and by the periodicity of $G(t,s)$, we infer

$$\begin{aligned} G(nT,0)y^o & = \sum_{j\geq 1} \Big(\int_\Omega y^o(x) z_j^o(x) dx \Big) G(nT,0) z_j^o \\ & = \sum_{j\geq 1} \lambda_j^n \Big(\int_\Omega y^o(x) z_j^o(x) dx \Big) z_j^o \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & G(nT,s)\big(f_c(x,s)\cdot 1_{|\omega\times(0,T)}\big) = G(nT,T)G(T,s)\big(f_c(x,s)\cdot 1_{|\omega\times(0,T)}\big) \\ & = \sum_{j\geq 1} \Big(\int_\Omega \big(f_c(x,s)\cdot 1_{|\omega\times(0,T)}\big) G(T-s,0) z_j^o(x) dx \Big) G(nT,T) z_j^o(x) \end{aligned}$$

$$= \sum_{j\geq 1} \Big(1_{|(0,T)} \cdot \int_\omega f_c(x,s) G(T-s,0) z_j^o(x) dx \Big) G((n-1)T, 0) z_j^o(x)$$

$$= \sum_{j\geq 1} \lambda_j^{n-1} \Big(1_{|(0,T)} \cdot \int_\omega f_c(x,s) G(T-s,0) z_j^o(x) dx \Big) z_j^o(x). \tag{3.6}$$

We conclude from (3.5) and (3.6) that $y(x,nT) = \sum_{j\geq 1} \lambda_j^n Y_j(T) z_j^o(x)$ for all integers $n \geq 1$, where

$$Y_j(T) = \int_\Omega y^o(x) z_j^o(x) dx + \frac{1}{\lambda_j} \int_0^T \int_\omega f_c(x,s) G(T-s,0) z_j^o(x) dx ds. \tag{3.7}$$

Recall that we have arranged the eigenfunctions so that $\{|\lambda_j|\}_{j\geq 1}$ is a non-decreasing sequence. Let $m > 0$ be such that $|\lambda_j| \leq |\lambda_{m+1}| < 1$ for all $j \geq m+1$ and $1 \leq |\lambda_m| \leq |\lambda_j|$ for all $j = 1, \cdots, m$. Now we decompose $y(x,nT)$ as follows

$$y(x,nT) = y_{1,m}(x,nT) + y_{2,m}(x,nT), \tag{3.8}$$

where

$$y_{1,m}(x,nT) = \sum_{j=1,\cdots,m} \lambda_j^n Y_j(T) z_j^o(x) \tag{3.9}$$

and

$$y_{2,m}(x,nT) = \sum_{j\geq m+1} \lambda_j^n Y_j(T) z_j^o(x). \tag{3.10}$$

Note that $y_{1,m}$ is the projection of y in the subspace of $L^2(\Omega)$ spanned by the eigenfunctions $(z_j^o)_{j=1,\cdots,m}$ of $G(T,0)$ corresponding to the unstable eigenvalues $(\lambda_j)_{j=1,\cdots,m}$ while $y_{2,m}$ is the projection of y in the subspace of $L^2(\Omega)$ spanned by the eigenfunctions $(z_j^o)_{j\geq m+1}$ of $G(T,0)$ corresponding to the stable eigenvalues $(\lambda_j)_{j\geq m+1}$. Next, we estimate respectively $\int_\Omega |y_{2,m}(x,nT)|^2 dx$ and $\int_\Omega |y_{1,m}(x,nT)|^2 dx$.

It follows first from (3.10) that

$$\int_\Omega |y_{2,m}(x,nT)|^2 dx = \sum_{j\geq m+1} |\lambda_j|^{2n} |Y_j(T)|^2$$

$$= \sum_{j\geq m+1} |\lambda_j|^{2n} \Big| \int_\Omega y(x,T) \frac{z_j^o(x)}{\lambda_j} dx \Big|^2. \tag{3.11}$$

Indeed, because of the evenness and T-periodicity of a, we have

$$\int_0^T \int_\omega f_c(x,s) \frac{G(T-s,0) z_j^o(x)}{\lambda_j} dx ds$$

$$= \int_0^T \int_\Omega \big(\partial_t y(x,s) - \Delta y(x,s) + a(x,T-s)y(x,s)\big) \frac{G(T-s,0)z_j^o(x)}{\lambda_j} dxds$$
$$= \int_\Omega y(x,T)\frac{z_j^o(x)}{\lambda_j}dx - \int_\Omega y^o(x)z_j^o(x)dx. \tag{3.12}$$

As $|\lambda_j| \leq |\lambda_{m+1}| < 1$ for all $j \geq m+1$, we deduce from (3.11) that

$$\int_\Omega |y_{2,m}(x,nT)|^2 \, dx \leq |\lambda_{m+1}|^{2(n-1)} \sum_{j\geq m+1} \Big| \int_\Omega y(x,T)z_j^o(x)dx \Big|^2$$
$$\leq |\lambda_{m+1}|^{2(n-1)} \int_\Omega |y(x,T)|^2 \, dx. \tag{3.13}$$

On the other hand, we have

$$\int_\Omega |y_{1,m}(x,nT)|^2 \, dx = \sum_{j=1,\cdots,m} |\lambda_j|^{2n} |Y_j(T)|^2$$

by (3.9) where

$$Y_j(T) = \int_0^T \Big[\frac{1}{T}\int_\Omega y^o(x)z_j^o(x)dx + \frac{1}{\lambda_j}\int_\omega f_c(x,s)G(T-s,0)z_j^o(x)dx\Big]ds. \tag{3.14}$$

Now we are in position to construct a control function $f_c = f_c(x,s) \in C([0,T];L^2(\Omega))$ such that for all $x \in \Omega$, for all $s \in [0,T]$,

$$f_c(x,T-s) = \sum_{i=1,\cdots,m} \sigma_i(s)G(s,0)z_i^o(x), \tag{3.15}$$

where $\sigma_i = \sigma_i(s) \in C([0,T])$ and

$$\frac{1}{T}\int_\Omega y^o(x)z_j^o(x)dx + \frac{1}{\lambda_j}\int_\omega f_c(x,s)G(T-s,0)z_j^o(x)dx = 0 \tag{3.16}$$

for all $j = 1,\cdots,m$, which implies $\int_\Omega |y_{1,m}(x,nT)|^2 \, dx = 0$.

By (3.15) and (3.16), it suffices to obtain $\{\sigma_i(s)\}_{i=1,\cdots,m}$ by solving the following linear system

$$\sum_{i=1,\cdots,m} \sigma_i(s)\int_\omega z_i(x,s)z_j(x,s)dx = -\frac{\lambda_j}{T}\int_\Omega y^o(x)z_j^o(x)dx; \tag{3.17}$$

that is,

$$\begin{bmatrix} \int_\omega z_1(x,s)z_1(x,s)dx & \cdots & \int_\omega z_m(x,s)z_1(x,s)dx \\ \vdots & \ddots & \vdots \\ \int_\omega z_1(x,s)z_m(x,s)dx & \cdots & \int_\omega z_m(x,s)z_m(x,s)dx \end{bmatrix} \begin{pmatrix} \sigma_1(s) \\ \vdots \\ \sigma_m(s) \end{pmatrix}$$

$$= -\frac{1}{T}\begin{pmatrix} \int_\Omega y^o(x) z_1(x,T)dx \\ \vdots \\ \int_\Omega y^o(x) z_m(x,T)dx \end{pmatrix} \tag{3.18}$$

where $z_j = z_j(x,t) \in C([0,+\infty); L^2(\Omega))$ is the solution of

$$\begin{cases} \partial_t z_j - \Delta z_j + a z_j = 0 \quad \text{in } \Omega \times (0,+\infty), \\ z_j = 0 \quad \text{on } \partial\Omega \times (0,+\infty), \\ z_j(\cdot,0) = z_j^o \quad \text{in } \Omega. \end{cases} \tag{3.19}$$

The existence of such $\sigma_i(s) \in C([0,T])$ can be proved if and only if for all $s \in [0,T]$, the matrix $\Big(\int_\omega G(s,0) z_i^o(x) G(s,0) z_j^o(x) dx \Big)_{1 \le i,j \le m}$ is invertible.

3.1. **Invertibility of** $\Big(\int_\omega G(s,0) z_i^o(x) G(s,0) z_j^o(x) dx \Big)_{1 \le i,j \le m}$**.** By contradiction, we assume that $M(s) = \big[\int_\omega z_i(x,s) z_j(x,s) dx \big]_{1 \le i,j \le m}$ is a not invertible matrix for some $s \in [0,T]$ fixed. Then the lines of $M(s)$ are linearly dependent which implies that $\{z_i(x,s)\}_{i=1,\cdots,m}$ are linearly dependent in $L^2(\omega)$. Hence there exists $\big(\mu_1(s), \cdots, \mu_{m-1}(s)\big)$ a non-zero vector in $\mathbb{R}^{m-1}$ such that

$$\mu_1(s) z_1(\cdot,s) + \cdots + \mu_{m-1}(s) z_{m-1}(\cdot,s) - z_m(\cdot,s) = 0 \quad \text{in } \omega. \tag{3.20}$$

However, by the assumption that $|\lambda_{m+1}| < 1 \le |\lambda_m| = \cdots = |\lambda_2| = |\lambda_1|$, it follows that the function

$$\varkappa_s(x,t) = \mu_1(s) z_1(x,t) + \cdots + \mu_{m-1}(s) z_{m-1}(x,t) - z_m(x,t) \tag{3.21}$$

satisfies

$$\begin{cases} \partial_t \varkappa_s - \Delta \varkappa_s + a \varkappa_s = 0 \quad \text{in } \Omega \times (0,2T), \\ \varkappa_s = 0 \quad \text{on } \partial\Omega \times (0,2T), \\ \varkappa_s(\cdot,2T) = |\lambda_1|^2 \varkappa_s(\cdot,0) \quad \text{in } \Omega. \end{cases} \tag{3.22}$$

Let $u(x,t) = e^{-\frac{\ln|\lambda_1|}{T} t} \varkappa_s(x,t)$, then $u \in C([0,2T]; L^2(\Omega))$ solves

$$\begin{cases} \partial_t u - \Delta u + \frac{\ln|\lambda_1|}{T} u + a u = 0 \quad \text{in } \Omega \times (0,2T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,2T), \\ u(\cdot,2T) = u(\cdot,0) \quad \text{in } \Omega. \end{cases} \tag{3.23}$$

By Theorem 3, there exists a constant $c_1 > 0$ such that if we choose $a \in L^\infty(\Omega \times \mathbb{R})$ such that

$$\ell_1 \le \|a\|_\infty \le \ell_1 \big(\frac{c_1}{2T}\big)^{1/4}, \tag{3.24}$$

and noting that

$$\ell_1 + \frac{\ln|\lambda_1|}{T} \le \|a\|_\infty \quad \text{and} \quad \frac{\ln|\lambda_1|}{T} \ge 0, \tag{3.25}$$

which follows from Lemma 2-2 and the fact that $|\lambda_1| \ge 1$, then the solution $u = u(x,t)$ of (3.23) satisfies

$$u(\cdot,0) = 0 \text{ in } \omega \Longrightarrow u(\cdot,0) = 0 \text{ in } \Omega. \tag{3.26}$$

It follows by translation in time and by the $2T$-periodicity of u that

$$u(\cdot,s) = 0 \text{ in } \omega \Longrightarrow u(\cdot,s) = 0 \text{ in } \Omega. \tag{3.27}$$

So $\varkappa_s(\cdot,s) = 0$ in Ω and

$$\mu_1(s)z_1^o + \cdots + \mu_{m-1}(s)z_{m-1}^o - z_m^o = 0 \quad \text{in } \Omega. \tag{3.28}$$

This contradiction shows that the matrix $\left[\int_\omega z_i(x,s)z_j(x,s)dx\right]_{1\le i,j\le m}$ is invertible for all $s \in [0,T]$ and

$$\begin{pmatrix} \sigma_1(s) \\ \vdots \\ \sigma_m(s) \end{pmatrix} = \begin{bmatrix} \int_\omega z_1(x,s)z_1(x,s)dx & \cdots & \int_\omega z_m(x,s)z_1(x,s)dx \\ \vdots & \ddots & \vdots \\ \int_\omega z_1(x,s)z_N(x,s)dx & \cdots & \int_\omega z_m(x,s)z_m(x,s)dx \end{bmatrix}^{-1} \times \begin{pmatrix} -\frac{1}{T}\int_\Omega y^o(x)z_1(x,T)dx \\ \vdots \\ -\frac{1}{T}\int_\Omega y^o(x)z_m(x,T)dx \end{pmatrix}. \tag{3.29}$$

Consequently, if we take the control function $f_c \in C([0,T];L^2(\Omega))$ in equation (3.1) as

$$f_c(x,t) = \sum_{i=1,\cdots,m} \sigma_i(T-t)z_i(x,T-t), \tag{3.30}$$

then the corresponding solution $y \in C([0,+\infty);L^2(\Omega))$ satisfies

$$\begin{aligned} \|y(\cdot,nT)\|_{L^2(\Omega)} = \|y_{2,m}(\cdot,nT)\|_{L^2(\Omega)} &\le |\lambda_{m+1}|^{(n-1)} \|y(\cdot,T)\|_{L^2(\Omega)} \\ &\le |\lambda_{m+1}|^{(n-1)} \|y^o\|_{L^2(\Omega)} e^{(\|a\|_\infty - \ell_1)T} \\ &\quad + |\lambda_{m+1}|^{(n-1)} \int_0^T \|f_c(\cdot,s)\|_{L^2(\omega)} e^{(\|a\|_\infty - \ell_1)(T-s)} ds, \end{aligned} \tag{3.31}$$

for all integers $n \ge 1$, where λ_{m+1} is the first eigenvalue such that $|\lambda_{m+1}| < 1$.

3.2. **Estimate of $\|f_c(\cdot,s)\|_{L^2(\omega)}$.** It follows from (3.31) that we would like to compute $\int_0^T \|f_c(\cdot,s)\|_{L^2(\omega)} e^{(\|a\|_\infty-\ell_1)(T-s)}ds$. We have

$$\begin{aligned}&\begin{pmatrix} \sigma_1(s) & \cdots & \sigma_m(s) \end{pmatrix} \cdot \Big[\Big(\int_\omega z_i(x,s)z_j(x,s)dx\Big)_{1\le i,j\le m}\Big] \begin{pmatrix} \sigma_1(s) \\ \vdots \\ \sigma_m(s) \end{pmatrix}\\ &= -\frac{1}{T}\begin{pmatrix} \sigma_1(s) & \cdots & \sigma_m(s) \end{pmatrix} \cdot \begin{pmatrix} \int_\Omega y^o(x)z_1(x,T)dx \\ \vdots \\ \int_\Omega y^o(x)z_m(x,T)dx \end{pmatrix}, \end{aligned} \tag{3.32}$$

or equivalently

$$\int_\omega \Big|\sum_{i=1,\cdots,m} \sigma_i(s)z_i(x,s)\Big|^2 dx = -\frac{1}{T}\int_\Omega y^o(x)\sum_{i=1,\cdots,m}\sigma_i(s)z_i(x,T)dx, \tag{3.33}$$

which implies

$$\begin{aligned}\Big\|\sum_{i=1,\cdots,m}\sigma_i(s)z_i(\cdot,s)\Big\|^2_{L^2(\omega)} &\le \frac{1}{T}\|y^o\|_{L^2(\Omega)}\Big\|\sum_{i=1,\cdots,m}\sigma_i(s)z_i(\cdot,T)\Big\|_{L^2(\Omega)}\\ &\le \frac{1}{T}\|y^o\|_{L^2(\Omega)}\Big\|\sum_{i=1,\cdots,m}\sigma_i(s)G(T,s)z_i(\cdot,s)\Big\|_{L^2(\Omega)}\\ &\le \frac{1}{T}\|y^o\|_{L^2(\Omega)}\Big\|G(T,s)\sum_{i=1,\cdots,m}\sigma_i(s)z_i(\cdot,s)\Big\|_{L^2(\Omega)}\\ &\le \frac{1}{T}\|y^o\|_{L^2(\Omega)}e^{(\|a\|_\infty-\ell_1)(T-s)}\Big\|\sum_{i=1,\cdots,m}\sigma_i(s)z_i(\cdot,s)\Big\|_{L^2(\Omega)}. \end{aligned}\tag{3.34}$$

Now we claim that there exists a constant $c>0$ such that

$$\Big\|\sum_{i=1,\cdots,m}\sigma_i(s)z_i(\cdot,s)\Big\|_{L^2(\Omega)} \le c\Big\|\sum_{i=1,\cdots,m}\sigma_i(s)z_i(\cdot,s)\Big\|_{L^2(\omega)}. \tag{3.35}$$

Indeed, since $|\lambda_{m+1}|<1\le|\lambda_m|=\cdots=|\lambda_2|=|\lambda_1|$, the function

$$\varkappa_s(x,t)=\sum_{i=1,\cdots,m}\sigma_i(s)z_i(x,t) \tag{3.36}$$

satisfies

$$\begin{cases} \partial_t\varkappa_s-\Delta\varkappa_s+a\varkappa_s=0 \quad \text{in } \Omega\times(0,2T),\\ \varkappa_s=0 \quad \text{on } \partial\Omega\times(0,2T),\\ \varkappa_s(\cdot,2T)=|\lambda_1|^2\varkappa_s(\cdot,0) \quad \text{in } \Omega. \end{cases}\tag{3.37}$$

Let $u(x,t)=e^{-\frac{\ln|\lambda_1|}{T}t}\varkappa_s(x,t)$, then $u\in C([0,2T];L^2(\Omega))$ satisfies

$$\left\{\begin{array}{c}\partial_t u-\Delta u+\frac{\ln|\lambda_1|}{T}u+au=0 \quad \text{in } \Omega\times(0,2T),\\ u=0 \quad \text{on } \partial\Omega\times(0,2T),\\ u(\cdot,2T)=u(\cdot,0) \quad \text{in } \Omega.\end{array}\right. \tag{3.38}$$

By Theorem 3, there exists a constant $c_1>0$, which is the same as that in (3.24), such that if we choose $a\in L^\infty(\Omega\times\mathbb{R})$ satisfying (3.24) and note (3.25) then the following estimate holds

$$\int_\Omega |u(x,0)|^2\,dx\le e^{C\|a\|_\infty^{4/3}}\int_\omega |u(x,0)|^2\,dx, \tag{3.39}$$

where $C>0$ is a constant which depends only on ω and Ω. It follows by translation in time and by the $2T$-periodicity of u solution of (3.38) that for all $s\in[0,T]$,

$$\int_\Omega |u(x,s)|^2\,dx\le e^{C\|a\|_\infty^{4/3}}\int_\omega |u(x,s)|^2\,dx, \tag{3.40}$$

which implies

$$\int_\Omega |\varkappa_s(x,s)|^2\,dx\le e^{C\|a\|_\infty^{4/3}}\int_\omega |\varkappa_s(x,s)|^2\,dx \tag{3.41}$$

or equivalently

$$\Big\|\sum_{i=1,\cdots,m}\sigma_i(s)z_i(\cdot,s)\Big\|_{L^2(\Omega)}\le e^{C\|a\|_\infty^{4/3}}\Big\|\sum_{i=1,\cdots,m}\sigma_i(s)z_i(\cdot,s)\Big\|_{L^2(\omega)}, \tag{3.42}$$

as desired. By (3.42) and (3.34) we get

$$\Big\|\sum_{i=1,\cdots,m}\sigma_i(s)z_i(\cdot,s)\Big\|_{L^2(\omega)}\le\frac{1}{T}\|y^o\|_{L^2(\Omega)}e^{\left(\|a\|_\infty-\ell_1\right)(T-s)}e^{C\|a\|_\infty^{4/3}}. \tag{3.43}$$

Now we conclude that if we take $c_o=\frac{c_1}{2}$ where c_1 is given in (3.39) then for all $a\in L^\infty(\Omega\times\mathbb{R})$ satisfying

$$\ell_1\le\|a\|_\infty\le\ell_1\big(\frac{c_o}{T}\big)^{1/4}, \tag{3.44}$$

we may take the control function $f_c\in C([0,T];L^2(\Omega))$ of the form (3.15) with estimates

$$\begin{aligned}\int_0^T\|f_c(\cdot,s)\|_{L^2(\omega)}\,ds&\le\frac{1}{T}e^{C\|a\|_\infty^{4/3}}\|y^o\|_{L^2(\Omega)}\int_0^T e^{\left(\|a\|_\infty-\ell_1\right)(T-s)}ds\\ &\le e^{C\|a\|_\infty^{4/3}}e^{\left(\|a\|_\infty-\ell_1\right)T}\|y^o\|_{L^2(\Omega)}\end{aligned} \tag{3.45}$$

and

$$\int_0^T \|f_c(\cdot,s)\|_{L^2(\omega)}\, e^{\left(\|a\|_\infty-\ell_1\right)(T-s)}ds \le e^{C\|a\|_\infty^{4/3}} e^{2\left(\|a\|_\infty-\ell_1\right)T}\, \|y^o\|_{L^2(\Omega)}\,, \tag{3.46}$$

so that the corresponding solution $y \in C([0,+\infty);L^2(\Omega))$ of (3.1) satisfies (3.31).

3.3. **Exponential decay of the solution.** Now we turn to obtaining the exponential decay of the solution. Let $C_{T,a} = 1 + e^{C\|a\|_\infty^{4/3}} e^{(\|a\|_\infty-\ell_1)T}$. By (3.31) and (3.46), it follows that

$$\|y(\cdot,nT)\|_{L^2(\Omega)} \le |\lambda_{m+1}|^{(n-1)}\, C_{T,a}\ e^{(\|a\|_\infty-\ell_1)T}\, \|y^o\|_{L^2(\Omega)} \tag{3.47}$$

for all integers $n \ge 1$.

For any $t \ge T$, we may write $t = nT + \sigma$ where $n \ge 1$ and $0 \le \sigma < T$. Since $y(\cdot,t) = G(t,nT)y(\cdot,nT)$, it follows from (3.47) that

$$\begin{aligned}\|y(\cdot,t)\|_{L^2(\Omega)} &= \|y(\cdot,nT)\|_{L^2(\Omega)} e^{(\|a\|_\infty-\ell_1)(t-nT)} \le \|y(\cdot,nT)\|_{L^2(\Omega)} e^{(\|a\|_\infty-\ell_1)T}\\ &\le \exp\Big(-(n+1)\ln\big(\frac{1}{|\lambda_{m+1}(T)|}\big)\Big)\frac{1}{|\lambda_{m+1}(T)|^2} C_{T,a}\ e^{2(\|a\|_\infty-\ell_1)T}\, \|y^o\|_{L^2(\Omega)}\,,\end{aligned} \tag{3.48}$$

which implies

$$\|y(\cdot,t)\|_{L^2(\Omega)} \le \exp\Big(-\frac{t}{T}\ln\big(\frac{1}{|\lambda_{m+1}(T)|}\big)\Big)\frac{C_{T,a}\ e^{2(\|a\|_\infty-\ell_1)T}}{|\lambda_{m+1}(T)|^2}\, \|y^o\|_{L^2(\Omega)} \tag{3.49}$$

for all $t \ge T$, and completes the proof of Theorem 2.

4. THE QUANTITATIVE UNIQUE CONTINUATION FOR TIME-PERIODIC HEAT EQUATION WITH POTENTIAL

In this section, we shall show Theorem 3 by dividing the proof into three steps. In the first step, we study the solutions u locally in $\omega\times\{t=0\}$ then, in the second step, we apply a global Carleman inequality for elliptic operators where time $t \in (0,T)$ is seen as a parameter. In the last step, we conclude the proof of Theorem 3 by choosing an adequate time T.

4.1. **Step 1: Local property of u near $t = 0$.** The goal of this subsection is to prove the following result:

Lemma 4-1. *There exist a non-empty open subset $\widetilde{\omega}$ of ω and a constant $C_\omega > 1$ depending only on ω and Ω such that for all $T > 0$ and integers $M \geq 2$, we have*

$$\int_{\widetilde{\omega}} |u(x,t)|^2 \, dx \leq e^{2\|q\|_\infty t} \left(C_\omega (M+2)^{2+\frac{d}{M}} \; t \right)^M \int_\Omega |u(x,0)|^2 \, dx$$
$$+ e^{2\|q\|_\infty t} (M+2)^{\frac{d}{M}} \left(\frac{1 - \left(C_\omega (M+2)^2 t \right)^M}{1 - C_\omega (M+2)^2 t} \right) \int_\omega |u(x,0)|^2 dx$$

for all $t \in (0,T)$, where the solution u solves the following linear heat equation with potential

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) \in L^2(\Omega). \end{cases}$$

Proof of Lemma 4-1. Let $x^o \in \mathbb{R}^d$ and $r_o > 0$; we denote

$$\begin{aligned} B(x^o, r) &= \left\{ x \in \mathbb{R}^d : |x - x^o| \leq r_o \right\}, \\ D(x^o, r) &= \left\{ x \in \mathbb{R}^d : |x_i - x_i^o| \leq r_o \quad , \text{for all } i = 1, \cdots, d \right\}. \end{aligned} \tag{4.1}$$

Without loss of generality, we suppose that there exists $r > 0$ such that $D(0, 2r) \subset \omega$. Then $B(x^o, r) \subset \omega$ for all $x^o \in [-r, r]^d$. Let $N \geq 4$ be an arbitrary but fixed integer; we have

$$B(x^o, r/N) \subset B(x^o, 2r/N) \subset \cdots \subset B(x^o, r) \subset \omega. \tag{4.2}$$

Let us define a sequence of smooth functions $\Phi_n \in C_0^\infty(B(x^o, nr/N))$ for $n = 2, \cdots, N$ such that

$$\begin{cases} 0 \leq \Phi_n \leq 1, \\ \Phi_n = 1 \text{ in } B(x^o, (n-1)r/N), \\ |\nabla \Phi_n| \leq 2N/r. \end{cases} \tag{4.3}$$

The construction of such a sequence Φ_n is standard.

Let $\Phi \in C_0^\infty(B(x^o, r_o))$; multiplying the equation in Lemma 4-1 by $\Phi^2 u$, we get the usual energy equality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_\Omega |\Phi u(x,t)|^2 \, dx + \int_\Omega |\Phi \nabla u(x,t)|^2 \, dx + \beta \int_\Omega |\Phi u(x,t)|^2 \, dx \\ &= - \int_\Omega q(x,t) \, |\Phi u(x,t)|^2 \, dx - 2 \int_\Omega \Phi(x) \nabla u(x,t) \cdot \nabla \Phi(x) u(x,t) dx, \end{aligned} \tag{4.4}$$

from which, after some calculation involving the Cauchy-Schwarz inequality, we infer

$$\frac{1}{2}\frac{d}{dt}\int_{B(x^o,r_o)}|\Phi u(x,t)|^2\,dx \tag{4.5}$$
$$\le \|q\|_\infty\int_{B(x^o,r_o)}|\Phi u(x,t)|^2\,dx+\|\nabla\Phi\|^2_{L^\infty(B(x^o,r_o))}\int_{B(x^o,r_o)}|u(x,t)|^2\,dx.$$

Multiplying (4.5) by $e^{-2\|q\|_\infty t}$, we obtain

$$\frac{d}{dt}\Big(\int_{B(x^o,r_o)}|\Phi u(x,t)|^2\,dx\ e^{-2\|q\|_\infty t}\Big)$$
$$\le 2\,\|\nabla\Phi\|^2_{L^\infty(B(x^o,r_o))}\Big(\int_{B(x^o,r_o)}|u(x,t)|^2\,dx\ e^{-2\|q\|_\infty t}\Big). \tag{4.6}$$

Taking $\Phi=\Phi_n$ and $r_o=nr/N$ in (4.6), using (4.3), it follows that

$$\frac{d}{dt}g_n(t)\le 8(N/r)^2\ g_{n+1}(t) \tag{4.7}$$

for all $n=2,\cdots,N-1$, where

$$g_n(t)=\int_{B(x^o,nr/N)}|\Phi_n u(x,t)|^2\,dx\ e^{-2\|q\|_\infty t}. \tag{4.8}$$

Integrating (4.7) over $(0,t)$, we obtain that for all $n=2,\cdots,N-1$,

$$g_n(t)\le 8(N/r)^2\int_0^t g_{n+1}(\tau)d\tau+g_n(0). \tag{4.9}$$

By induction, we deduce that for $n=0,\cdots,N-3$,

$$g_2(t)\le\big(8(N/r)^2\big)^{n+1}t^n\int_0^t g_{n+3}(\tau)d\tau+\sum_{j=0,\cdots,n}\big(8(N/r)^2t\big)^j g_{j+2}(0). \tag{4.10}$$

By (4.8), (4.3) and (4.10), after some computation, we have

$$\int_{D(x^o,r/(N\sqrt{d}))}|u(x,t)|^2\,dx\ e^{-2\|q\|_\infty t}\le\int_{B(x^o,r/N)}|u(x,t)|^2\,dx\ e^{-2\|q\|_\infty t}$$
$$\le\int_{B(x^o,2r/N)}|\Phi_2u(x,t)|^2\,dx\ e^{-2\|q\|_\infty t}\le g_2(t)$$
$$\le(8(N/r)^2)^{N-2}t^{N-3}\int_0^t g_N(\tau)d\tau+\sum_{j=0,\cdots,N-3}(8(N/r)^2t)^j g_{j+2}(0)$$
$$\le(8(N/r)^2)^{N-2}t^{N-3}\int_0^t\int_{B(x^o,r)}|\Phi_N u(x,\tau)|^2\,dx e^{-2\|q\|_\infty\tau}d\tau$$

$$+\sum_{j=0,\cdots,N-3}(8(N/r)^2t)^j\int_{B(x^o,(j+2)r/N)}\left|\Phi_{j+2}u(x,0)\right|^2dx$$

$$\leq(8(N/r)^2)^{N-2}t^{N-3}\int_0^t\int_\Omega\left|u(x,\tau)\right|^2dxe^{-2\|q\|_\infty\tau}d\tau$$

$$+\Big(\frac{1-(8(N/r)^2t)^{N-2}}{1-8(N/r)^2t}\Big)\int_\omega\left|u(x,0)\right|^2dx. \tag{4.11}$$

On the other hand, it is easy to check that for all $\tau\in(0,T)$,

$$\int_\Omega\left|u(x,\tau)\right|^2dx\ e^{-2\|q\|_\infty\tau}\leq\int_\Omega\left|u(x,0)\right|^2dx, \tag{4.12}$$

which together with (4.11) implies

$$\int_{D(x^o,r/(N\sqrt{d}))}\left|u(x,t)\right|^2dx\leq e^{2\|q\|_\infty t}(8(N/r)^2)^{N-2}t^{N-2}\int_\Omega\left|u(x,0)\right|^2dx$$

$$+e^{2\|q\|_\infty t}\Big(\frac{1-(8(N/r)^2t)^{N-2}}{1-8(N/r)^2t}\Big)\int_\omega\left|u(x,0)\right|^2dx. \tag{4.13}$$

Now, we break the rectangle $[0,\frac{r}{\sqrt{d}}]^d\subset\mathbb{R}^d$ into N^d small rectangles D_j, $j=1,\cdots,N^d$, having the same shape, by dividing each interval $[0,\frac{r}{\sqrt{d}}]$ into N small pieces (i.e., $[0,\frac{r}{\sqrt{d}}]=\bigcup_{k=0,\cdot\cdot,N-1}[k\frac{r}{N\sqrt{d}},(k+1)\frac{r}{N\sqrt{d}}]$ and $[0,\frac{r}{\sqrt{d}}]^2=\Big(\bigcup_{k=0,\cdot\cdot,N-1}\Big[k\frac{r}{N\sqrt{d}},(k+1)\frac{r}{N\sqrt{d}}\Big]\Big)\times\Big(\bigcup_{k=0,\cdots,N-1}\Big[k\frac{r}{N\sqrt{d}},(k+1)\frac{r}{N\sqrt{d}}\Big]\Big)$, ...). It is clear that $meas\big(D_{j_1}\cap D_{j_2}\big)=0$ for $j_1\neq j_2$ and also

$$\int_{\left[0,\frac{r}{\sqrt{d}}\right]^d}\left|u\left(x,t\right)\right|^2dx=\sum_{j=1,\cdot\cdot,N^d}\int_{D_j}\left|u\left(x,t\right)\right|^2dx. \tag{4.14}$$

One can check that for each D_j, $j=1,\cdots,N^d$,

$$D_j\subset D\Big(x^o,k\frac{r}{N\sqrt{d}}\Big), \tag{4.15}$$

if $x^o=(x_1^o,\cdots,x_d^o)\in[-r,r]^d$ with x_i^o, $j=1,\cdots,d$, equals $k(r/N\sqrt{d})$ for some $k=0,\cdots,N-1$, which implies by (4.14) that

$$\int_{\left[0,\frac{r}{\sqrt{d}}\right]^d}\left|u(x,t)\right|^2dx\leq N^de^{2\|q\|_\infty t}(8(N/r)^2t)^{N-2}\int_\Omega\left|u(x,0)\right|^2dx$$

$$+N^de^{2\|q\|_\infty t}\Big(\frac{1-(8(N/r)^2t)^{N-2}}{1-8(N/r)^2t}\Big)\int_\omega\left|u(x,0)\right|^2dx. \tag{4.16}$$

Let $M = N - 2 \geq 2$; we have

$$\int_{\frac{r}{\sqrt{d}}[0,1]^d} |u(x,t)|^2\,dx \leq e^{2\|q\|_\infty t}\Big(\frac{8}{r^2}(M+2)^{2+\frac{d}{M}}\ t\Big)^M \int_\Omega |u(x,0)|^2\,dx$$
$$+ e^{2\|q\|_\infty t}(M+2)^{\frac{d}{M}}\Big(\frac{1-\big(\frac{8}{r^2}(M+2)^2 t\big)^M}{1-\frac{8}{r^2}(M+2)^2 t}\Big)\int_\omega |u(x,0)|^2\,dx. \qquad (4.17)$$

Lemma 4-1 follows.

4.2. **Step 2: Observability estimate for periodic-parabolic solutions.** The goal of this section is to prove the following observability inequality:

Theorem 4-1. *Let $\widetilde{\omega}$ be a non-empty open subset of ω. There exists a constant $c > 0$ such that for all $T > 0$, $\beta \geq 0$ and $q \in L^\infty(\Omega \times (0,T))$, we have*

$$\int_\Omega |u(x,0)|^2\,dx \leq \frac{e^{2\|q\|_\infty T}}{T}\ \exp\big(c(1+\beta^{2/3}+\|q\|_\infty^{4/3})\big)\int_0^T\int_{\widetilde{\omega}} |u(x,t)|^2\,dxdt$$

where $u \in C([0,T];L^2(\Omega))$ solves the following T-periodic in time linear heat equation with potential

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega. \end{cases}$$

As we know, for initial condition problems of parabolic equations, the constant of observability grows in an exponentially fast way of order $e^{1/T}$ (see [12]). However, Theorem 4-1 shows that the constant of observability explodes for small time only in a polynomially fast way of order $1/\sqrt{T}$ for parabolic equations with time-periodic conditions.

In order to prove Theorem 4-1, we need the following propositions.

Proposition 4-1. *Let ω_o be a non-empty open subset of Ω. There exist a smooth function $\varphi = \varphi(x) \geq 1$ and a constant $c > 0$, such that for all $T > 0$, $\beta \geq 0$, $q \in L^\infty(\Omega \times (0,T))$ and $\theta \geq c\big(1+\beta^{2/3}+\|q\|_\infty^{4/3}\big)$, we have*

$$\frac{d}{dt}\Big[\frac{1}{2}\int_\Omega e^{2\theta\varphi(x)}|\nabla u(x,t)|^2\,dx + \frac{1}{2}\beta\int_\Omega e^{2\theta\varphi(x)}|u(x,t)|^2\,dx$$
$$- \theta^2\int_\Omega e^{2\theta\varphi(x)}|\nabla\varphi(x)|^2|u(x,t)|^2\,dx\Big]$$
$$+ \frac{1}{c}\Big(\int_\Omega e^{2\theta\varphi(x)}|\nabla u(x,t)|^2\,dx + \theta^2\int_\Omega e^{2\theta\varphi(x)}|u(x,t)|^2\,dx\Big)$$

$$\leq c\Big(\theta \int_{\omega_o} e^{2\theta\varphi(x)} \left|\nabla u(x,t)\right|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta\varphi(x)} \left|u(x,t)\right|^2 dx\Big)$$

for almost every $t \in (0,T)$, where $u = u(x,t)$ solves the following linear heat equation with potential

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) \in H^2(\Omega) \cap H_0^1(\Omega). \end{cases}$$

Proposition 4-2. (Global Carleman inequality for the operator $\Delta - \beta$) *Let ω_o be a non-empty open subset of Ω. There exist a smooth function $\varphi = \varphi(x) \geq 1$ and a constant $c > 0$, such that for all $\beta \geq 0$, $\widetilde{q} \in L^\infty(\Omega)$, $\theta > c\big(1 + \beta^{2/3} + \|\widetilde{q}\|_{L^\infty(\Omega)}^{4/3}\big)$ and $w \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\begin{aligned} &\left\|2\theta\nabla\varphi \cdot \nabla(e^{\theta\varphi}w) - \widetilde{q}e^{\theta\varphi}w\right\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{c}\Big(\theta \int_\Omega e^{2\theta\varphi(x)}|\nabla w(x)|^2 dx + \theta^3 \int_\Omega e^{2\theta\varphi(x)}|w(x)|^2 dx\Big) \\ &\leq \left\|e^{\theta\varphi}(\Delta - \beta)w\right\|_{L^2(\Omega)}^2 c\Big(\theta \int_{\omega_o} e^{2\theta\varphi(x)}|\nabla w(x)|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta\varphi(x)}|w(x)|^2 dx\Big). \end{aligned}$$

The proof of Proposition 4-2 is given in the Appendix.

Proof of Proposition 4-1. We apply Proposition 4-2 to the solution u of the linear heat equation with potential, then we deduce that there exist a smooth function $\varphi = \varphi(x) \geq 1$ and a constant $c > 0$, such that for all $\beta \geq 0$, $q \in L^\infty(\Omega \times (0,T))$, and $\theta > c\big(1 + \beta^{2/3} + \|q\|_\infty^{4/3}\big)$, we have

$$\begin{aligned} &\left\|2\theta\nabla\varphi \cdot \nabla\big(e^{\theta\varphi}u(\cdot,t)\big) - q(\cdot,t)e^{\theta\varphi}u(\cdot,t)\right\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{c}\Big(\theta \int_\Omega e^{2\theta\varphi(x)}|\nabla u(x,t)|^2 dx + \theta^3 \int_\Omega e^{2\theta\varphi(x)}|u(x,t)|^2 dx\Big) \\ &\leq \left\|e^{\theta\varphi}(\Delta - \beta)u(\cdot,t)\right\|_{L^2(\Omega)}^2 \\ &\quad + c\Big(\theta \int_{\omega_o} e^{2\theta\varphi(x)}|\nabla u(x,t)|^2 dx + \theta^3 \int_{\omega_o} e^{2\theta\varphi(x)}|u(x,t)|^2 dx\Big) \qquad (4.18) \end{aligned}$$

for almost every $t \in (0,T)$.

On the other hand, we multiply equation $-\partial_t u + (\Delta - \beta)u - qu = 0$ by $e^{2\theta\varphi}(\Delta - \beta)u$ and integrate it over Ω to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega e^{2\theta\varphi(x)} \left|\nabla u(x,t)\right|^2 dx + \frac{1}{2}\beta\frac{d}{dt}\int_\Omega e^{2\theta\varphi(x)} \left|u(x,t)\right|^2 dx$$

$$
\begin{aligned}
&+ \left\| e^{\theta\varphi}(\Delta-\beta)u(\cdot,t)\right\|^2_{L^2(\Omega)} \\
&= \int_\Omega q(x,t)u(x,t)(\Delta-\beta)u(x,t)e^{2\theta\varphi(x)}dx \\
&\quad - \int_\Omega \partial_t u(x,t) 2\theta\nabla\varphi(x)\cdot\nabla u(x,t)e^{2\theta\varphi(x)}dx \\
&= \int_\Omega q(x,t)u(x,t)(\Delta-\beta)u(x,t)e^{2\theta\varphi(x)}dx \\
&\quad - \int_\Omega \partial_t u(x,t)e^{\theta\varphi(x)}2\theta\nabla\varphi(x)\cdot\nabla\big(e^{\theta\varphi(x)}u(x,t)\big)dx \\
&\quad + \int_\Omega \partial_t u(x,t)e^{\theta\varphi(x)}2\theta\nabla\varphi(x)\cdot(\theta\nabla\varphi(x)e^{\theta\varphi(x)})u(x,t)dx
\end{aligned}
\tag{4.19}
$$

which implies, with $\Xi(t) = \int_\Omega e^{2\theta\varphi(x)}\left|\nabla u(x,t)\right|^2 dx + \beta\int_\Omega e^{2\theta\varphi(x)}\left|u(x,t)\right|^2 dx - 2\theta^2\int_\Omega \left|\nabla\varphi(x)\right|^2 e^{2\theta\varphi(x)}\left|u(x,t)\right|^2 dx$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Xi(t) + \left\| e^{\theta\varphi}(\Delta-\beta)u(\cdot,t)\right\|^2_{L^2(\Omega)} \\
&\qquad - \int_\Omega q(x,t)u(x,t)(\Delta-\beta)u(x,t)e^{2\theta\varphi(x)}dx \\
&\quad = - \int_\Omega \partial_t u(x,t)e^{\theta\varphi(x)}2\theta\nabla\varphi(x)\cdot\nabla\big(e^{\theta\varphi(x)}u(x,t)\big)dx.
\end{aligned}
\tag{4.20}
$$

Replacing $\partial_t u$ by $(\Delta-\beta)\,u - qu$ in the right-hand side of (4.20), we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Xi(t) + \left\| e^{\theta\varphi}(\Delta-\beta)u(\cdot,t)\right\|^2_{L^2(\Omega)} \\
&\quad + \int_\Omega e^{\theta\varphi(x)}(\Delta-\beta)u(x,t)\left[2\theta\nabla\varphi(x)\cdot\nabla\big(e^{\theta\varphi(x)}u(x,t)\big)\right]dx \\
&\quad + \int_\Omega e^{\theta\varphi(x)}(\Delta-\beta)u(x,t)\left[-q(x,t)e^{\theta\varphi(x)}u(x,t)\right]dx \\
&= \int_\Omega q(x,t)u(x,t)e^{\theta\varphi(x)}2\theta\nabla\varphi(x)\cdot\nabla\big(e^{\theta\varphi(x)}u(x,t)\big)dx.
\end{aligned}
\tag{4.21}
$$

By the Cauchy-Schwarz inequality, the right-hand side of (4.21) is bounded by

$$
\int_\Omega q(x,t)u(x,t)e^{\theta\varphi(x)}2\theta\nabla\varphi(x)\cdot\theta\nabla\varphi(x)e^{\theta\varphi(x)}u(x,t)dx
$$

$$
\begin{aligned}
&+\int_\Omega q(x,t)u(x,t)e^{\theta\varphi(x)}2\theta\nabla\varphi(x)\cdot e^{\theta\varphi(x)}\nabla u(x,t)dx\\
&\le 2\theta^2\left\|q\right\|_\infty\left\|\nabla\varphi\right\|^2_{L^\infty(\Omega)}\left\|e^{\theta\varphi}u\right\|^2_{L^2(\Omega)}\\
&\quad+2\theta\left\|q\right\|_\infty\left\|\nabla\varphi\right\|_{L^\infty(\Omega)}\left\|e^{\theta\varphi}u\right\|_{L^2(\Omega)}\left\|e^{\theta\varphi}\nabla u\right\|_{L^2(\Omega)}.
\end{aligned}\tag{4.22}
$$

Then, we conclude by (4.21) and (4.22) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Xi(t)+\left\|e^{\theta\varphi}(\Delta-\beta)u(\cdot,t)\right\|^2_{L^2(\Omega)}\\
&+\int_\Omega e^{\theta\varphi(x)}(\Delta-\beta)u(x,t)\left[2\theta\nabla\varphi(x)\cdot\nabla\left(e^{\theta\varphi(x)}u(x,t)\right)\right]dx\\
&+\int_\Omega e^{\theta\varphi(x)}(\Delta-\beta)u(x,t)\left[-q(x,t)e^{\theta\varphi(x)}u(x,t)\right]dx\\
&\le 2\theta^2\left\|q\right\|_\infty\left\|\nabla\varphi\right\|^2_{L^\infty(\Omega)}\left\|e^{\theta\varphi}u\right\|^2_{L^2(\Omega)}\\
&\quad+2\theta\left\|q\right\|_\infty\left\|\nabla\varphi\right\|_{L^\infty(\Omega)}\left\|e^{\theta\varphi}u\right\|_{L^2(\Omega)}\left\|e^{\theta\varphi}\nabla u\right\|_{L^2(\Omega)}.
\end{aligned}\tag{4.23}
$$

Consequently, there exists a constant $\widetilde{c}>0$ such for all $\beta\ge 0$ and $q\in L^\infty(\Omega\times(0,T))$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Xi(t)+\left\|e^{\theta\varphi}(\Delta-\beta)u(\cdot,t)\right\|^2_{L^2(\Omega)}\\
&+\int_\Omega e^{\theta\varphi(x)}(\Delta-\beta)u(x,t)\left[2\theta\nabla\varphi(x)\cdot\nabla\left(e^{\theta\varphi(x)}u(x,t)\right)\right]dx\\
&+\int_\Omega e^{\theta\varphi(x)}(\Delta-\beta)u(x,t)\left[-q(x,t)e^{\theta\varphi(x)}u(x,t)\right]dx\\
&\le\widetilde{c}\left\|q\right\|_\infty\Big(\int_\Omega e^{2\theta\varphi(x)}\left|\nabla u(x,t)\right|^2dx+\theta^2\int_\Omega e^{2\theta\varphi(x)}\left|u(x,t)\right|^2dx\Big)
\end{aligned}\tag{4.24}
$$

for almost every $t\in(0,T)$. By (4.18) and (4.24), we get

$$
\begin{aligned}
&\frac{1}{2}\left\|2\theta\nabla\varphi\cdot\nabla\left(e^{\theta\varphi}u(\cdot,t)\right)-q(\cdot,t)e^{\theta\varphi}u(\cdot,t)\right\|^2_{L^2(\Omega)}\\
&+\frac{1}{2c}\Big(\theta\int_\Omega e^{2\theta\varphi(x)}\left|\nabla u(x,t)\right|^2dx+\theta^3\int_\Omega e^{2\theta\varphi(x)}\left|u(x,t)\right|^2dx\Big)\\
&-\frac{c}{2}\Big(\theta\int_{\omega_o} e^{2\theta\varphi(x)}\left|\nabla u(x,t)\right|^2dx+\theta^3\int_{\omega_o} e^{2\theta\varphi(x)}\left|u(x,t)\right|^2dx\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left\| e^{\theta\varphi}(\Delta-\beta)u(\cdot,t)\right\|^2_{L^2(\Omega)} \\
&\leq -\frac{1}{2}\frac{d}{dt}\Xi(t) - \frac{1}{2}\left\| e^{\theta\varphi}(\Delta-\beta)u(\cdot,t)\right\|^2_{L^2(\Omega)} \\
&\quad - \int_\Omega e^{\theta\varphi(x)}(\Delta-\beta)u(x,t)\left[2\theta\nabla\varphi(x)\cdot\nabla\big(e^{\theta\varphi(x)}u(x,t)\big)\right]dx \\
&\quad - \int_\Omega e^{\theta\varphi(x)}(\Delta-\beta)u(x,t)\left[-q(x,t)e^{\theta\varphi(x)}u(x,t)\right]dx \\
&\quad + \widetilde{c}\,\|q\|_\infty\Big(\int_\Omega e^{2\theta\varphi(x)}\left|\nabla u(x,t)\right|^2 dx + \theta^2\int_\Omega e^{2\theta\varphi(x)}\left|u(x,t)\right|^2 dx\Big) \qquad (4.25)
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\frac{1}{2}\left\| e^{\theta\varphi}(\Delta-\beta)u(\cdot,t) + 2\theta\nabla\varphi\cdot\nabla\big(e^{\theta\varphi}u(\cdot,t)\big) - q(\cdot,t)e^{\theta\varphi}u(\cdot,t)\right\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2}\frac{d}{dt}\Xi(t) + \frac{1}{2c}\theta\Big(\int_\Omega e^{2\theta\varphi}\left|\nabla u(x,t)\right|^2 dx + \theta^2\int_\Omega e^{2\theta\varphi}\left|u(x,t)\right|^2 dx\Big) \\
&\leq \widetilde{c}\,\|q\|_\infty\Big(\int_\Omega e^{2\theta\varphi}\left|\nabla u(x,t)\right|^2 dx + \theta^2\int_\Omega e^{2\theta\varphi}\left|u(x,t)\right|^2 dx\Big) \\
&\quad + \frac{c}{2}\Big(\theta\int_{\omega_o} e^{2\theta\varphi}\left|\nabla u(x,t)\right|^2 dx + \theta^3\int_{\omega_o} e^{2\theta\varphi}\left|u(x,t)\right|^2 dx\Big). \qquad (4.26)
\end{aligned}
$$

By taking $\theta \geq c\big(1+\beta^{2/3}+\|q\|_\infty^{4/3}\big) + 4c\widetilde{c}\,\|q\|_\infty$ in (4.26), we deduce that the inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Xi(t) + c\Big(\int_\Omega e^{2\theta\varphi(x)}\left|\nabla u(x,t)\right|^2 dx + \theta^2\int_\Omega e^{2\theta\varphi(x)}\left|u(x,t)\right|^2 dx\Big) \\
&\leq \frac{c}{2}\Big(\theta\int_{\omega_o} e^{2\theta\varphi(x)}\left|\nabla u(x,t)\right|^2 dx + \theta^3\int_{\omega_o} e^{2\theta\varphi(x)}\left|u(x,t)\right|^2 dx\Big) \qquad (4.27)
\end{aligned}
$$

holds for all $\theta \geq c_1\big(1+\beta^{2/3}+\|q\|_\infty^{4/3}\big)$ where $c_1 > 0$ is large enough. This completes the proof of Proposition 4-1.

Proof of Theorem 4-1. Throughout the proof of Theorem 4-1, c denotes several positive constants only depending on the geometry. By integrating the estimate in Proposition 4-1 over $(0,T)$, we get

$$
\begin{aligned}
&\int_0^T\int_\Omega e^{2\theta\varphi(x)}\left|\nabla u(x,t)\right|^2 dxdt + \theta^2\int_0^T\int_\Omega e^{2\theta\varphi(x)}\left|u(x,t)\right|^2 dxdt \qquad (4.28) \\
&\leq c\Big(\theta\int_0^T\int_{\omega_o} e^{2\theta\varphi(x)}\left|\nabla u(x,t)\right|^2 dxdt + \theta^3\int_0^T\int_{\omega_o} e^{2\theta\varphi(x)}\left|u(x,t)\right|^2 dxdt\Big),
\end{aligned}
$$

thanks to the T-periodicity of u. With an adequate choice of $\theta > c\big(1+\beta^{2/3}+\|q\|_\infty^{4/3}\big)$ in (4.28), it follows that, denoting $C_{\beta,q} = \exp\left(c(1+\beta^{2/3}+\|q\|_\infty^{4/3})\right)$,

$$\int_0^T\int_\Omega |u(x,t)|^2\,dxdt \le C_{\beta,q}\Big(\int_0^T\int_{\omega_o}|u(x,t)|^2\,dx + \int_0^T\int_{\omega_o}|\nabla u(x,t)|^2\,dx\Big). \tag{4.29}$$

By the usual energy method (i.e., an equality like (4.4)), we also have

$$\int_0^T\int_{\omega_o}|\nabla u(x,t)|^2\,dxdt \le (c+\|q\|_\infty)\int_0^T\int_{\widetilde{\omega}}|u(x,t)|^2\,dx \tag{4.30}$$

for some $\omega_o \subset \widetilde{\omega} \subset \omega$, using the T-periodicity of u.

Consequently, for all $T>0$, $\beta \ge 0$ and $q \in L^\infty(\Omega\times(0,T))$,

$$\int_0^T\int_\Omega |u(x,t)|^2\,dxdt \le \exp\left(c(1+\beta^{2/3}+\|q\|_\infty^{4/3})\right)\int_0^T\int_{\widetilde{\omega}}|u(x,t)|^2\,dx. \tag{4.31}$$

where u solves the following T-periodic in time linear heat equation with potential

$$\left\{\begin{array}{c} \partial_t u - \Delta u + \beta u + qu = 0 \quad \text{in } \Omega\times(0,T), \\ u = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = u(\cdot,T) \quad \text{in } \Omega, \\ u(\cdot,0) \in H^2(\Omega)\cap H_0^1(\Omega). \end{array}\right. \tag{4.32}$$

By density of $H^2(\Omega)\cap H_0^1(\Omega)$ in $L^2(\Omega)$, this last estimate is also true for solutions u with initial data in $L^2(\Omega)$.

On the other hand, we recall that

$$e^{2\|q\|_\infty t}\int_\Omega |u(x,T)|^2\,dx \le e^{2\|q\|_\infty T}\int_\Omega |u(x,t)|^2\,dx \tag{4.33}$$

and

$$T \le \frac{1}{2\|q\|_\infty}\big(e^{2\|q\|_\infty T}-1\big) = \int_0^T e^{2\|q\|_\infty t}dt. \tag{4.34}$$

Finally, it follows from (4.33), (4.34) and (4.31) that

$$\begin{aligned}\int_\Omega |u(x,0)|^2\,dx = \int_\Omega |u(x,T)|^2\,dx \le \frac{e^{2\|q\|_\infty T}}{T}\int_0^T\int_\Omega |u(x,t)|^2\,dxdt \\ \le \frac{e^{2\|q\|_\infty T}}{T}\ \exp\left(c(1+\beta^{2/3}+\|q\|_\infty^{4/3})\right)\int_0^T\int_{\widetilde{\omega}}|u(x,t)|^2\,dxdt.\end{aligned} \tag{4.35}$$

This completes the proof of Theorem 4-1.

4.3. **Step 3: Choice of T.** Now we are able to conclude the proof of Theorem 3 as follows. By integrating over $(0,T)$ the estimate in Lemma 4-1, we have

$$\frac{1}{T}\int_0^T\int_{\widetilde{\omega}}|u(x,t)|^2\,dx \leq e^{2\|q\|_\infty T}\left(C_\omega(M+2)^{2+\frac{d}{M}}\ T\right)^M\int_\Omega |u(x,0)|^2\,dx$$
$$+ e^{2\|q\|_\infty T}(M+2)^{\frac{d}{M}}\Big(\frac{1-\left(C_\omega(M+2)^2T\right)^M}{1-C_\omega(M+2)^2T}\Big)\int_\omega |u(x,0)|^2\,dx. \quad (4.36)$$

Combining the latter with Theorem 4-1, we deduce that there exists a constant $c>1$ such that for all $T>0$, $q\in L^\infty(\Omega\times(0,T))$ and integers $M\geq 2$, we have, denoting $C_{\beta,q}=\exp\left(c(1+\beta^{2/3}+\|q\|_\infty^{4/3})\right)$,

$$\int_\Omega |u(x,0)|^2\,dx \leq C_{\beta,q}e^{4\|q\|_\infty T}\left(C_\omega\,(M+2)^{2+d/M}\ T\right)^M\int_\Omega |u\,(x,0)|^2\,dx$$
$$+ C_{\beta,q}e^{4\|q\|_\infty T}\,(M+2)^{d/M}\Big(\frac{1-\left(C_\omega\,(M+2)^2\,T\right)^M}{1-C_\omega\,(M+2)^2\,T}\Big)\int_\omega |u\,(x,0)|^2\,dx. \quad (4.37)$$

From this we infer that there exists a constant $c>1$ such that for all $T>0$, $\beta\geq 0$ and $q\in L^\infty(\Omega\times(0,T))$ satisfying $\ell_1+\beta\leq\|q\|_\infty$, the inequality

$$\int_\Omega |u(x,0)|^2\,dx \leq e^{c(1+T)\|q\|_\infty^{4/3}}\left(C_\omega(M+2)^{2+d/M}\ T\right)^M\int_\Omega |u(x,0)|^2\,dx$$
$$+ e^{c(1+T)\|q\|_\infty^{4/3}}(M+2)^{d/M}\Big(\frac{1-\left(C_\omega(M+2)^2T\right)^M}{1-C_\omega(M+2)^2T}\Big)\int_\omega |u(x,0)|^2\,dx \quad (4.38)$$

holds for all integers $M\geq 2$.

First we choose an integer $M\geq 2$ so that

$$M < 2\,\|q\|_\infty^{4/3}\Big(\frac{1+d}{\ell_1^{4/3}}+2c\Big)\leq M+1, \quad (4.39)$$

which can be done because $\|q\|_\infty\geq\ell_1$. By (4.39), it follows that

$$1+d\leq M, \qquad 1+2c\,\|q\|_\infty^{4/3}\leq M. \quad (4.40)$$

By (4.40) and (4.38), we infer

$$\int_\Omega |u\,(x,0)|^2\,dx \leq e^{M-1+c(T-1)\|q\|_\infty^{4/3}}\left(C_\omega\,(2M)^3\ T\right)^M\int_\Omega |u\,(x,0)|^2\,dx$$
$$+ e^{c(1+T)\|q\|_\infty^{4/3}}\,(2M)\Big(\frac{1-\left(C_\omega\,(2M)^2\,T\right)^M}{1-C_\omega\,(2M)^2\,T}\Big)\int_\omega |u\,(x,0)|^2\,dx. \quad (4.41)$$

Next, we choose $T>0$, so that $T \le \frac{1}{eC_\omega\left(4\|q\|_\infty^{4/3}\left(\frac{1+d}{\ell_1^{4/3}}+2c\right)\right)^3} < \frac{1}{eC_\omega(2M)^3} \le 1$, which can be done because of (4.39). Then by (4.41), we get

$$\int_\Omega |u(x,0)|^2 dx \le e^{-1}\int_\Omega |u(x,0)|^2 dx + e^{2c\|q\|_\infty^{4/3}}(2M)^2\int_\omega |u(x,0)|^2 dx, \quad (4.42)$$

which gives

$$\int_\Omega |u(x,0)|^2\,dx \le \frac{e^{2c\|q\|_\infty^{4/3}}}{1-e^{-1}}\left[4\,\|q\|_\infty^{4/3}\left(\frac{1+d}{\ell_1^{4/3}}+2c\right)\right]^2\int_\omega |u(x,0)|^2\,dx. \quad (4.43)$$

Now let $c_o = \frac{1}{eC_\omega\left(4\left(\frac{1+d}{\ell_1^{4/3}}+2c\right)\right)^3\ell_1^4}$, then by (4.43), we conclude that if $\ell_1+\beta \le \|q\|_\infty$ and $\|q\|_\infty \le \ell_1\left(\frac{c_o}{T}\right)^{1/4}$, then there exists a constant $C>0$ such that

$$\int_\Omega |u(x,0)|^2\,dx \le e^{C\|q\|_\infty^{4/3}}\int_\omega |u(x,0)|^2\,dx, \quad (4.44)$$

where the solution $u \in C([0,T];L^2(\Omega))$ satisfies the following T-periodic in time linear heat equation with potential

$$\begin{cases} \partial_t u - \Delta u + \beta u + qu = 0 \quad \text{in } \Omega\times(0,T), \\ u = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = u(\cdot,T) \quad \text{in } \Omega. \end{cases} \quad (4.45)$$

The proof of Theorem 3 is now completed.

5. The autonomous case

Let us denote by ℓ_i, with $0<\ell_1<\ell_2\le\ell_3\le\cdots$, and by e_i for $i\ge 1$, the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$. Recall here that $(e_i)_{i\ge1}$ forms an orthonormal basis of the Hilbert space $L^2(\Omega)$.

Let $a=a(t)\in L^\infty(\mathbb{R})$ be an even, T-periodic potential. From a Galerkin method, $G(t,s)$ can be explicitly expressed by $G(t,s) = e^{-\int_s^t a(\tau)dt}e^{\Delta(t-s)}$ and one can check easily that the eigencouple (z_j^o,λ_j) of the compact self-adjoint operator $G(T,0)$ satisfies

$$\lambda_j = e^{-\ell_j T - \int_0^T a(\tau)d\tau} \qquad z_j^o = e_j.$$

Furthermore, we obtain the following asymptotic controllability result.

Theorem 5-1. *Suppose that $a = a(t) \in L^\infty(\mathbb{R})$ is a space-independent potential satisfying*

$$\begin{cases} (i) & a(t+T) = a(t) = a(-t) \quad in\ \mathbb{R}, \\ (ii) & \ell_1 \leq -\frac{1}{T}\int_0^T a(t)dt < \ell_2; \end{cases}$$

then for all initial data $y^o \in L^2(\Omega)$,the control function $f \in L^1(0,T;L^2(\Omega))$ given by

$$f(x,t) = -\frac{1}{T} e^{-\ell_1 t - \int_{T-t}^T a(\tau)d\tau} \Big(\frac{\int_\Omega y^o(x)e_1(x)dx}{\int_\omega |e_1(x)|^2 \, dx} \Big) e_1(x)$$

implies that the solution $y = y(x,t) \in C([0,+\infty); L^2(\Omega))$ of the following heat equation with potential $a = a(t)$,

$$\begin{cases} \partial_t y - \Delta y + ay = f \cdot 1_{|\omega \times (0,T)} \quad in\ \Omega \times (0,+\infty), \\ y = 0 \quad on\ \partial\Omega \times (0,+\infty), \\ y(\cdot,0) = y^o \quad in\ \Omega, \end{cases}$$

satisfies for all $t \geq T$,

$$\|y(t)\|_{L^2(\Omega)} \leq e^{C\left(\|a\|_\infty^{\alpha/2} + \|a\|_\infty T\right)} \exp\Big(- \Big(\ell_2 + \frac{1}{T}\int_0^T a(\tau)d\tau \Big) t \Big) \|y^o\|_{L^2(\Omega)}$$

and

$$\int_0^T \|f(s)\|_{L^2(\omega)} \, ds \leq e^{C\|a\|_\infty^{\alpha/2}} e^{\left(\|a\|_\infty - \ell_1\right)T} \|y^o\|_{L^2(\Omega)} \,,$$

where $C > 0$ is a constant which only depends on the geometry and $\alpha \geq 0$. More precisely, $\alpha = \frac{2}{3}$ for general non-empty subdomains $\omega \subset \Omega$ and $\alpha = 0$ under the geometrical control condition of the work of Bardos-Lebeau-Rauch (i.e., $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is of class C^∞, there is no infinite order of contact between the boundary $\partial\Omega$ and the bicharacteristics of $\partial_t^2 - \Delta$, and all generalized bicharacteristic ray of $\partial_t^2 - \Delta$ meets $\omega \times (0,T_c)$ for some $0 < T_c < +\infty$).

Note that no hypotheses on $T > 0$ are required with a space- independent potential which satisfies $\left(\|a\|_\infty - \ell_1 \right) \geq 0$ thanks to (ii). We omit details of the proof here. For general non-empty subdomains $\omega \subset \Omega$, we apply Theorem A (see Appendix) to get a quantification of the unique continuation property for the eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ and deduce $\alpha = \frac{2}{3}$. When the geometrical control condition of the work of Bardos-Lebeau-Rauch [2] is satisfied, then we apply the observability estimate for the wave equation for a fixed frequency and deduce $\alpha = 0$.

If the even, T-periodic bounded potential $a = a(t)$ depends only on the time t, then $G(T, 0)$ is a positive operator and the first eigenvalue is of multiplicity one (see the Krein-Rutman theorem in e.g. [15]). We do not know if such a result is still true for a potential $a = a(x, t)$ depending on space and time. Observe that here no positivity hypotheses on the potential or on the initial data are imposed as in [15] or as in [9].

6. Appendix: Global Carleman inequality for the operator $(\Delta - \beta)$

Carleman estimates are widely used in applied problems (see [16] for inverse problems or e.g. [13], [12], [18] for controllability problems). In what follows, we present the proof of the global Carleman inequality for the operator $(\Delta - \beta)$ that we have used (see Proposition 4-2 or Theorem A below). In our inequality in Theorem A, the presence of the first term in the left-hand side plays a key role.

Theorem A. *Let Ω be a bounded connected domain in $\mathbb{R}^d$, $d \geq 1$, with a boundary $\partial\Omega$ of class C^2, and ω_o be a non-empty open subset of Ω. Then there exist a smooth function $\varphi = \varphi(x) \geq 1$ and a constant $C > 0$, such that for all $\widetilde{\beta} \in \mathbb{R}$, for all $\widetilde{q} \in L^\infty(\Omega)$, for all $\theta > C\big(1 + |\widetilde{\beta}|^{2/3} + \|\widetilde{q}\|^{4/3}_{L^\infty(\Omega)}\big)$, for all $w \in H^2(\Omega) \cap H^1_0(\Omega)$, we have*

$$\begin{aligned} &\left\|2\theta\nabla\varphi \cdot \nabla(e^{\theta\varphi}w) - \widetilde{q}e^{\theta\varphi}w\right\|^2_{L^2(\Omega)} \\ &\quad + \frac{1}{C}\Big(\theta^3 \int_\Omega e^{2\theta\varphi(x)}|w(x)|^2 dx + \theta \int_\Omega e^{2\theta\varphi(x)}|\nabla w(x)|^2 dx\Big) \\ \leq &\left\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\right\|^2_{L^2(\Omega)} \\ &\quad + C\Big(\theta^3 \int_{\omega_o} e^{2\theta\varphi(x)}|w(x)|^2 dx + \theta \int_{\omega_o} e^{2\theta\varphi(x)}|\nabla w(x)|^2 dx\Big). \end{aligned}$$

6.1. **An equality with weight.** We denote $\|.\| = \big(\int (\cdot)^2 dx\big)^{1/2}$ the usual norm in $L^2(\Omega)$ associated to the scalar product $(\cdot,\cdot) = \int (\cdot,\cdot)dx$. From now on, we shall omit all xs in the functions of x.

Let $\theta > 0$, $\varphi = \varphi(x) \in C^2(\mathbb{R}^d)$ be arbitrary but given. We shall prove first that the function $v(x) = e^{\theta\varphi(x)}w(x)$ where $w \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfies the following

$$\left\{\begin{array}{l} \left\|\Delta v-\theta\Delta\varphi v+\theta^2(\nabla\varphi)^2v-\widetilde{\beta}v-\widetilde{q}v\right\|^2+\|2\theta\nabla\varphi\nabla v+\widetilde{q}v\|^2 \\ +2(boundary\ terms) \\ +4\theta\int\sum_i\partial_i^2\varphi(\partial_iv)^2+2\theta^3\int\nabla\varphi\nabla\big((\nabla\varphi)^2\big)v^2 \\ +4\theta\int\sum_i\sum_{j\neq i}\partial_{ij}\varphi\partial_iv\partial_jv \\ =\left\|e^{\theta\varphi}(\Delta-\widetilde{\beta})w\right\|^2+2\theta^2\int div(\Delta\varphi\nabla\varphi)v^2 \\ -4\theta\int\nabla\varphi\widetilde{q}v\nabla v-2\int\left[\Delta v-\theta\Delta\varphi v+\theta^2(\nabla\varphi)^2v-\widetilde{\beta}v-\widetilde{q}v\right]\widetilde{q}v \\ +2\theta\widetilde{\beta}\int\Delta\varphi v^2+2\theta\int\Delta\varphi(\nabla v)^2-2\theta^3\int\Delta\varphi(\nabla\varphi)^2v^2, \end{array}\right. \tag{A1}$$

$$\left\{\begin{array}{l} 2\theta\widetilde{\beta}\int\Delta\varphi v^2+2\theta\int\Delta\varphi(\nabla v)^2-2\theta^3\int\Delta\varphi(\nabla\varphi)^2v^2 \\ =-2\theta\int\Delta\varphi v\left[\Delta v-\theta\Delta\varphi v+\theta^2(\nabla\varphi)^2v-\widetilde{\beta}v-\widetilde{q}v\right] \\ -2\theta^2\int(\Delta\varphi)^2v^2-2\theta\int\Delta\varphi\widetilde{q}v^2+\theta\int\Delta^2\varphi v^2 \end{array}\right. \tag{A2}$$

and

$$boundary\ terms=-\theta\int_{\partial\Omega}(\partial_nv)^2\partial_n\varphi. \tag{A3}$$

Proof of (A1). We shall start with computation of all derivatives of $v=e^{\theta\varphi}w$ and then write the term $e^{\theta\varphi}(\Delta-\widetilde{\beta})w$ as the sum of P_1v and P_2v where P_1 is a differential operator of order 1 and P_2 is a differential operator of order 2 given below; after that we calculate $\int_\Omega(P_2v+P_1v)^2dx$ where the term $\int_\Omega(P_2v,P_1v)dx$ plays a key role.

One can check easily that

$$\begin{aligned}\Delta v&=\Delta e^{\theta\varphi}w+2\nabla e^{\theta\varphi}\nabla w+e^{\theta\varphi}\Delta w\\ &=\theta\Delta\varphi v-\theta^2(\nabla\varphi)^2v+2\theta\nabla\varphi\nabla v+e^{\theta\varphi}\Delta w\end{aligned} \tag{A4}$$

and

$$P_2v+P_1v=e^{\theta\varphi}(\Delta-\widetilde{\beta})w, \tag{A5}$$

where

$$P_1v=-2\theta\nabla\varphi\nabla v+\widetilde{q}v \tag{6.1}$$

$$P_2v=\Delta v-\theta\Delta\varphi v+\theta^2(\nabla\varphi)^2v-\widetilde{\beta}v-\widetilde{q}v. \tag{A6}$$

Then we have

$$\begin{array}{l}\left\|e^{\theta\varphi}(\Delta-\widetilde{\beta})w\right\|^2=\left\|\Delta v-\theta\Delta\varphi v+\theta^2(\nabla\varphi)^2v-\widetilde{\beta}v-\widetilde{q}v\right\|^2\\ +\left\|-2\theta\nabla\varphi\nabla v+\widetilde{q}v\right\|^2+2(P_2v,P_1v).\end{array} \tag{A7}$$

However, a direct calculation leads us to

$$
\begin{aligned}
&(P_2v, P_1v) = \big(\Delta v - \theta\Delta\varphi v + \theta^2(\nabla\varphi)^2 v - \widetilde{\beta} v - \widetilde{q}v, -2\theta\nabla\varphi\nabla v + \widetilde{q}v\big) \\
&= -2\theta\int \nabla\varphi\nabla v\Delta v + 2\theta^2\int \Delta\varphi\nabla\varphi v\nabla v - 2\theta^3\int\nabla\varphi(\nabla\varphi)^2 v\nabla v \qquad \text{(A8)}\\
&\quad + 2\theta\widetilde{\beta}\int\nabla\varphi v\nabla v + 2\theta\int\nabla\varphi\widetilde{q}v\nabla v + (P_2 v, \widetilde{q}v) \\
&= -2\theta\int\sum_i \partial_i\varphi\partial_i v\partial_i^2 v - 2\theta\int\sum_i\partial_i\varphi\partial_i v\sum_{j\neq i}\partial_j^2 v + \theta^2\int\Delta\varphi\nabla\varphi\nabla(v^2) \\
&\quad - \theta^3\int\nabla\varphi(\nabla\varphi)^2\nabla(v^2) + \theta\widetilde{\beta}\int\nabla\varphi\nabla(v^2) + 2\theta\int\nabla\varphi\widetilde{q}v\nabla v + (P_2v, \widetilde{q}v) \\
&= -\theta\int\sum_i\partial_i\varphi\partial_i(\partial_i v)^2 - 2\theta\int\sum_i\sum_{j\neq i}\partial_j(\partial_i\varphi\partial_i v\partial_j v) \\
&\quad + 2\theta\int\sum_i\sum_{j\neq i}\partial_{ij}\varphi\partial_i v\partial_j v + \theta\int\sum_i\sum_{j\neq i}\partial_i\varphi\partial_i(\partial_j v)^2 \\
&\quad + \theta^2\int div\big((\Delta\varphi\nabla\varphi)v^2\big) - \theta^2\int div(\Delta\varphi\nabla\varphi)v^2 - \theta^3\int div\big((\nabla\varphi(\nabla\varphi)^2)v^2\big) \\
&\quad + \theta^3\int div\big(\nabla\varphi(\nabla\varphi)^2\big)v^2 - \theta\widetilde{\beta}\int\Delta\varphi v^2 + 2\theta\int\nabla\varphi\widetilde{q}v\nabla v + (P_2v, \widetilde{q}v).
\end{aligned}
$$

From this we get by integration by parts

$$
\begin{aligned}
&\big(\Delta v - \theta\Delta\varphi v + \theta^2(\nabla\varphi)^2 v - \widetilde{\beta}v - \widetilde{q}v, -2\theta\nabla\varphi\nabla v + \widetilde{q}v\big) = (\text{boundary terms}) \\
&\quad + \theta\int\sum_i\partial_i^2\varphi(\partial_i v)^2 + 2\theta\int\sum_i\sum_{j\neq i}\partial_{ij}\varphi\partial_i v\partial_j v - \theta\int\sum_i\sum_{j\neq i}\partial_i^2\varphi(\partial_j v)^2 \\
&\quad - \theta^2\int div(\Delta\varphi\nabla\varphi)v^2 + \theta^3\int div\big(\nabla\varphi(\nabla\varphi)^2\big)v^2 - \theta\widetilde{\beta}\int\Delta\varphi v^2 \\
&\quad + 2\theta\int\nabla\varphi\widetilde{q}v\nabla v + (P_2v, \widetilde{q}v). \qquad \text{(A9)}
\end{aligned}
$$

Then by (A7) and (A9), we infer

$$
\begin{aligned}
&\left\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\right\|^2 - \Big(\left\|\Delta v - \theta\Delta\varphi v + \theta^2(\nabla\varphi)^2 v - \widetilde{\beta}v - \widetilde{q}v\right\|^2 \\
&\qquad\qquad + \|-2\theta\nabla\varphi\nabla v + \widetilde{q}v\|^2\Big) \\
&= 2\big(\Delta v - \theta\Delta\varphi v + \theta^2(\nabla\varphi)^2 v - \widetilde{\beta}v - \widetilde{q}v, -2\theta\nabla\varphi\nabla v + \widetilde{q}v\big)
\end{aligned}
$$

$$= 2(\text{boundary terms}) + 2\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 \tag{A10}$$

$$+ 4\theta \int \sum_i \sum_{j\neq i} \partial_{ij}\varphi \partial_i v \partial_j v - 2\theta \int \sum_i \sum_{j\neq i} \partial_i^2 \varphi (\partial_j v)^2 - 2\theta^2 \int div(\Delta\varphi\nabla\varphi)v^2$$

$$+ 2\theta^3 \int div\big(\nabla\varphi(\nabla\varphi)^2\big)v^2 - 2\theta\widetilde{\beta} \int \Delta\varphi v^2 + 4\theta \int \nabla\varphi\widetilde{q}v\nabla v + 2(P_2 v, \widetilde{q}v)$$

or equivalently

$$\|P_2 v\|^2 + \|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 + 2(\text{boundary terms})$$

$$+ 2\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 + 2\theta^3 \int div\big(\nabla\varphi(\nabla\varphi)^2\big)v^2 + 4\theta \int \sum_i \sum_{j\neq i} \partial_{ij}\varphi \partial_i v \partial_j v$$

$$= \Big\| e^{\theta\varphi}(\Delta - \widetilde{\beta})w \Big\|^2 + 2\theta^2 \int div(\Delta\varphi\nabla\varphi)v^2 - 4\theta \int \nabla\varphi\widetilde{q}v\nabla v - 2(P_2 v, \widetilde{q}v)$$

$$+ 2\theta \int \sum_i \sum_{j\neq i} \partial_i^2 \varphi (\partial_j v)^2 + 2\theta\widetilde{\beta} \int \Delta\varphi v^2. \tag{A11}$$

By adding $2\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2$ on both sides of (A11) and noting that

$$div\big(\nabla\varphi(\nabla\varphi)^2\big) = \nabla\varphi\nabla\big((\nabla\varphi)^2\big) + \Delta\varphi(\nabla\varphi)^2,$$

we get

$$\|P_2 v\|^2 + \|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 + 2(\text{boundary terms})$$

$$+ 4\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 + 2\theta^3 \int \nabla\varphi\nabla\big((\nabla\varphi)^2\big)v^2 + 4\theta \int \sum_i \sum_{j\neq i} \partial_{ij}\varphi \partial_i v \partial_j v$$

$$= \Big\| e^{\theta\varphi}(\Delta - \widetilde{\beta})w \Big\|^2 + 2\theta^2 \int div(\Delta\varphi\nabla\varphi)v^2 - 4\theta \int \nabla\varphi\widetilde{q}v\nabla v - 2(P_2 v, \widetilde{q}v)$$

$$+ 2\theta \int \sum_i \sum_{j\neq i} \partial_i^2 \varphi (\partial_j v)^2 + 2\theta\widetilde{\beta} \int \Delta\varphi v^2$$

$$+ 2\theta \int \sum_i \partial_i^2 \varphi (\partial_i v)^2 - 2\theta^3 \int \Delta\varphi(\nabla\varphi)^2 v^2$$

$$= \Big\| e^{\theta\varphi}(\Delta - \widetilde{\beta})w \Big\|^2 + 2\theta^2 \int div(\Delta\varphi\nabla\varphi)v^2 - 4\theta \int \nabla\varphi\widetilde{q}v\nabla v - 2(P_2 v, \widetilde{q}v)$$

$$+ 2\theta\widetilde{\beta} \int \Delta\varphi v^2 + 2\theta \int \Delta\varphi(\nabla v)^2 - 2\theta^3 \int \Delta\varphi(\nabla\varphi)^2 v^2, \tag{A12}$$

as desired.

Proof of (A2). By integration by parts and the fact that $v \in H^2(\Omega) \cap H_0^1(\Omega)$, it follows that

$$\begin{aligned} &2\theta \int \Delta\varphi \nabla v \nabla v = -2\theta \int \Delta\varphi v \Delta v - 2\theta \int v \nabla\Delta\varphi \nabla v \\ &= -2\theta\widetilde{\beta} \int \Delta\varphi v^2 - 2\theta \int \Delta\varphi v (\Delta - \widetilde{\beta}) v - 2\theta \int v \nabla\Delta\varphi\nabla v \\ &= -2\theta\widetilde{\beta} \int \Delta\varphi v^2 - 2\theta \int \nabla\Delta\varphi \frac{1}{2} \nabla(v^2) \\ &\quad - 2\theta \int \Delta\varphi v \left[P_2 v + \theta\Delta\varphi v - \theta^2 (\nabla\varphi)^2 v + \widetilde{q} v\right] \end{aligned} \tag{A13}$$

or equivalently

$$\begin{aligned} &2\theta\widetilde{\beta} \int \Delta\varphi v^2 + 2\theta \int \Delta\varphi (\nabla v)^2 - 2\theta^3 \int \Delta\varphi (\nabla\varphi)^2 v^2 \\ &= -2\theta \int \Delta\varphi v P_2 v - 2\theta^2 \int (\Delta\varphi)^2 v^2 - 2\theta \int \Delta\varphi \widetilde{q} v^2 + \theta \int \Delta^2\varphi v^2, \end{aligned} \tag{A14}$$

as desired.

Proof of (A3). By (A8) and (A9) we can check that

$$\begin{aligned} &\text{boundary terms} = -\theta \int \sum_i \partial_i\big(\partial_i\varphi (\partial_i v)^2\big) \\ &- 2\theta \int \sum_i \sum_{j\neq i} \partial_j (\partial_i\varphi \partial_i v \partial_j v) + \theta \int \sum_i \sum_{j\neq i} \partial_i\big(\partial_i\varphi (\partial_j v)^2\big) \\ &+ \theta^2 \int div\big((\Delta\varphi\nabla\varphi) v^2\big) - \theta^3 \int div\big((\nabla\varphi(\nabla\varphi)^2) v^2\big) \\ &= -\theta \int_{\partial\Omega} \sum_i \partial_i\varphi (\partial_i v)^2 n_i - 2\theta \int_{\partial\Omega} \sum_i \sum_{j\neq i} \partial_i\varphi \partial_i v \partial_j v n_j \qquad \text{(A15)} \\ &+ \theta \int_{\partial\Omega} \sum_i \sum_{j\neq i} \partial_i\varphi (\partial_j v)^2 n_i + \theta^2 \int_{\partial\Omega} (\Delta\varphi\nabla\varphi) v^2 \cdot n - \theta^3 \int_{\partial\Omega} \nabla\varphi (\nabla\varphi)^2 v^2 \cdot n. \end{aligned}$$

Since $v \in H^2(\Omega) \cap H_0^1(\Omega)$, we deduce from (A15) that

$$\begin{aligned} &\text{boundary terms} = -\theta \int_{\partial\Omega} \sum_i \partial_i\varphi (\partial_i v)^2 n_i \\ &- 2\theta \int_{\partial\Omega} \sum_i \sum_{j\neq i} \partial_i\varphi \partial_i v \partial_j v n_j + \theta \int_{\partial\Omega} \sum_i \sum_{j\neq i} \partial_i\varphi (\partial_j v)^2 n_i \end{aligned}$$

$$
\begin{aligned}
&= -2\theta \int_{\partial\Omega} \sum_i \partial_i\varphi(\partial_i v)^2 n_i - 2\theta \int_{\partial\Omega} \sum_i \sum_{j\neq i} \partial_i\varphi \partial_i v \partial_j v n_j \\
&\quad + \theta \int_{\partial\Omega} \sum_i \sum_{j\neq i} \partial_i\varphi(\partial_j v)^2 n_i + \theta \int_{\partial\Omega} \sum_i \partial_i\varphi(\partial_i v)^2 n_i \\
&= -2\theta \int_{\partial\Omega} \sum_i \sum_j \partial_i\varphi \partial_i v \partial_j v n_j + \theta \int_{\partial\Omega} \sum_i \sum_j \partial_i\varphi(\partial_j v)^2 n_i \\
&= -2\theta \int_{\partial\Omega} \nabla\varphi \nabla v \partial_n v + \theta \int_{\partial\Omega} (\nabla v)^2 \partial_n \varphi. \qquad \text{(A16)}
\end{aligned}
$$

Since $v_{|\partial\Omega} = 0$, $\nabla v = (\nabla v \cdot n)n$ and then we have

$$
\text{boundary terms} = -2\theta \int_{\partial\Omega} (\partial_n v)^2 \partial_n\varphi + \theta \int_{\partial\Omega} (\nabla v)^2 \partial_n \varphi = -\theta \int_{\partial\Omega} (\partial_n v)^2 \partial_n \varphi. \qquad \text{(A17)}
$$

This completes the proof of (A3).

6.2. **Choice of the weight function.** We shall prove that with a particular choice of $\varphi = \varphi(x) \geq 1$ there exists a constant $C > 0$ such that for all $\widetilde{\beta} \in \mathbb{R}$, for all $\widetilde{q} \in L^\infty(\Omega)$, and $\theta > C\big(1 + |\widetilde{\beta}|^{2/3} + \|\widetilde{q}\|^{4/3}_{L^\infty(\Omega)}\big)$, the estimate

$$
\begin{aligned}
&\left\| 2\theta\nabla\varphi\nabla(e^{\theta\varphi}w) - \widetilde{q}e^{\theta\varphi}w \right\|^2 + \frac{1}{C}\Big(\theta^3 \int_\Omega (e^{\theta\varphi}w)^2 + \theta \int_\Omega \big(\nabla(e^{\theta\varphi}w)\big)^2\Big) \\
&\leq \left\| e^{\theta\varphi}(\Delta - \widetilde{\beta})w \right\|^2 + C\Big(\theta^3 \int_{\omega_o} (e^{\theta\varphi}w)^2 + \theta \int_{\omega_o} \big(\nabla(e^{\theta\varphi}w)\big)^2\Big) \qquad \text{(A18)}
\end{aligned}
$$

holds for all $w \in H^2(\Omega) \cap H^1_0(\Omega)$.

Proof of (A18). We study the particular case that $\varphi(x) = e^{\rho\psi(x)}$, where $\rho > 0$ and ψ is a smooth non-negative function. One can check that

$$
\Delta\varphi = \rho^2(\nabla\psi)^2\varphi + \rho\Delta\psi\varphi \qquad \text{(A19)}
$$

and

$$
\begin{aligned}
\partial_i\varphi\partial^2_{ij}\varphi &= \rho(\partial_i\varphi)^2\partial_j\psi + \partial_i\varphi\rho\partial^2_{ij}\psi\varphi \\
\partial_i\varphi\partial_j\varphi\partial^2_{ij}\varphi &= \rho^2(\partial_j\psi)^2(\partial_i\varphi)^2\varphi + \rho\partial_i\varphi\partial_j\varphi\partial^2_{ij}\psi\varphi.
\end{aligned} \qquad \text{(A20)}
$$

Next we compute $\nabla\varphi\nabla\big((\nabla\varphi)^2\big) = \sum_i \partial_i\varphi\partial_i(\nabla\varphi)^2$. Since

$$
\begin{aligned}
&\partial_i\varphi\partial_i(\nabla\varphi)^2 = 2\partial_i\varphi\nabla\varphi\nabla\partial_i\varphi = 2\partial^2_i\varphi(\partial_i\varphi)^2 + 2\partial_i\varphi \sum_{j\neq i} \partial_j\varphi\partial^2_{ij}\varphi \\
&= 2\left[\rho^2(\partial_i\psi)^2\varphi + \rho\partial^2_i\psi\varphi\right](\partial_i\varphi)^2 + 2\partial_i\varphi \sum_{j\neq i} \partial_j\varphi\left[\rho^2\partial_i\psi\varphi\partial_j\psi + \rho\partial^2_{ij}\psi\varphi\right]
\end{aligned}
$$

$$= 2\left[\rho^2(\partial_i\psi)^2\varphi + \rho\partial_i^2\psi\varphi\right](\partial_i\varphi)^2 + 2\partial_i\varphi\sum_{j\neq i}(\rho\partial_j\psi\varphi)(\rho\partial_i\psi\varphi)\rho\partial_j\psi$$
$$+ 2\partial_i\varphi\sum_{j\neq i}\partial_j\varphi\rho\partial_{ij}^2\psi\varphi = 2\left[\rho^2(\partial_i\psi)^2\varphi + \rho\partial_i^2\psi\varphi\right](\partial_i\varphi)^2$$
$$+ 2\rho^2\partial_i\varphi\sum_{j\neq i}(\partial_j\psi)^2\varphi\partial_i\varphi + 2\partial_i\varphi\sum_{j\neq i}\partial_j\varphi\rho\partial_{ij}^2\psi\varphi$$
$$= 2\rho^2\sum_j(\partial_j\psi)^2\varphi(\partial_i\varphi)^2 + 2\rho\Big[\partial_i^2\psi\varphi(\partial_i\varphi)^2 + \partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\varphi\Big], \tag{A21}$$

it follows that

$$\nabla\varphi\nabla\big((\nabla\varphi)^2\big) = 2\rho^2(\nabla\psi)^2\varphi(\nabla\varphi)^2 + 2\rho\sum_i\Big[\partial_i^2\psi\,(\partial_i\varphi)^2 + \partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\Big]\varphi. \tag{A22}$$

On the other hand, by (A1) and (A2), we infer

$$\|P_2v\|^2 + \|2\theta\nabla\varphi\nabla v + \widetilde{q}v\|^2 + 2(\text{boundary terms})$$
$$+ 4\theta\int\sum_i\partial_i^2\varphi(\partial_iv)^2 + 2\theta^3\int\nabla\varphi\nabla\big((\nabla\varphi)^2\big)v^2 + 4\theta\int\sum_i\sum_{j\neq i}\partial_{ij}\varphi\partial_iv\partial_jv$$
$$= \Big\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\Big\|^2 + 2\theta^2\int div(\Delta\varphi\nabla\varphi)v^2 - 4\theta\int\nabla\varphi\widetilde{q}v\nabla v - 2(P_2v, \widetilde{q}v)$$
$$- 2\theta\int\Delta\varphi vP_2v - 2\theta^2\int(\Delta\varphi)^2v^2 - 2\theta\int\Delta\varphi\widetilde{q}v^2 + \theta\int\Delta^2\varphi v^2 \tag{A23}$$

which together with (A22) gives

$$\|P_2v\|^2 + \|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 + 2\ (\text{boundary terms})$$
$$+ 4\theta\rho^2\int\varphi\sum_i\big((\partial_i\psi)^2(\partial_iv)^2 + \sum_{j\neq i}\partial_i\psi\partial_j\psi\partial_iv\partial_jv\big)$$
$$+ 4\theta\rho\int\varphi\sum_i\big(\partial_i^2\psi(\partial_iv)^2 + \sum_{j\neq i}\partial_{ij}^2\psi\partial_iv\partial_jv\big) + 4\theta^3\rho^2\int(\nabla\psi)^2(\nabla\varphi)^2\varphi v^2$$
$$+ 4\theta^3\rho\int\sum_i\Big[\partial_i^2\psi(\partial_i\varphi)^2 + \partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\Big]\varphi v^2$$
$$= \Big\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\Big\|^2 + 2\theta^2\int div(\Delta\varphi\nabla\varphi)v^2 - 4\theta\int\nabla\varphi\widetilde{q}v\nabla v - 2(P_2v, \widetilde{q}v)$$

$$-2\theta\int\Delta\varphi vP_2v-2\theta^2\int(\Delta\varphi)^2v^2-2\theta\int\Delta\varphi\widetilde{q}v^2+\theta\int\Delta^2\varphi v^2 \tag{A24}$$

or equivalently

$$\begin{aligned}&\|P_2v\|^2+\|2\theta\nabla\varphi\nabla v-\widetilde{q}v\|^2+2(\text{boundary terms}) \qquad (6.2)\\
&+4\theta\rho^2\int\varphi\Big(\sum_i\partial_i\psi\partial_iv\Big)^2+4\theta^3\rho^2\int(\nabla\psi)^2(\nabla\varphi)^2\varphi v^2\\
&=\Big\|e^{\theta\varphi}(\Delta-\widetilde{\beta})w\Big\|^2+2\theta^2\int div(\Delta\varphi\nabla\varphi)v^2-4\theta\int\nabla\varphi\widetilde{q}v\nabla v-2(P_2v,\widetilde{q}v)\\
&\quad-2\theta\int\Delta\varphi vP_2v-2\theta^2\int(\Delta\varphi^2v^2-2\theta\int\Delta\varphi\widetilde{q}v^2+\theta\int\Delta^2\varphi v^2\\
&\quad-4\theta\rho\int\sum_i\varphi\big(\partial_i^2\psi(\partial_iv)^2+\sum_{j\neq i}\partial_{ij}^2\psi\partial_iv\partial_jv\big)\\
&\quad-4\theta^3\rho\int\sum_i\Big[\partial_i^2\psi(\partial_i\varphi)^2+\partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\Big]\varphi v^2. \qquad (A25)\end{aligned}$$

Next we compute the term $2\theta\rho^2\int(\nabla\psi)^2\varphi(\nabla v)^2$. To this end, we observe by (A2) that

$$\begin{aligned}&2\theta\int\left[\rho^2(\nabla\psi)^2\varphi+\rho\Delta\psi\varphi\right](\nabla v)^2=2\theta\int\Delta\varphi(\nabla v)^2\\
&=-2\theta\widetilde{\beta}\int\Delta\varphi v^2+2\theta^3\int\Delta\varphi(\nabla\varphi)^2v^2\\
&\quad-2\theta\int\Delta\varphi vP_2v-2\theta^2\int(\Delta\varphi)^2v^2-2\theta\int\Delta\varphi\widetilde{q}v^2+\theta\int\Delta^2\varphi v^2\\
&=-2\theta\widetilde{\beta}\int\Delta\varphi v^2+2\theta^3\int\left[\rho^2(\nabla\psi)^2\varphi+\rho\Delta\psi\varphi\right](\nabla\varphi)^2v^2\\
&\quad-2\theta\int\Delta\varphi vP_2v-2\theta^2\int(\Delta\varphi)^2v^2-2\theta\int\Delta\varphi\widetilde{q}v^2+\theta\int\Delta^2\varphi v^2 \qquad (A26)\end{aligned}$$

which implies

$$\begin{aligned}&2\theta\rho^2\int(\nabla\psi)^2\varphi(\nabla v)^2=-2\theta\rho\int\Delta\psi\varphi(\nabla v)^2+2\theta^3\rho^2\int(\nabla\psi)^2(\nabla\varphi)^2\varphi v^2\\
&\quad+2\theta^3\rho\int\Delta\psi(\nabla\varphi)^2\varphi v^2-2\theta\int\Delta\varphi vP_2v-2\theta^2\int(\Delta\varphi)^2v^2\\
&-2\theta\int\Delta\varphi(\widetilde{q}+\widetilde{\beta})v^2+\theta\int\Delta^2\varphi v^2. \qquad (A27)\end{aligned}$$

Finally, it follows from (A25) and (A27) that

$$\begin{aligned}
&\|P_2v\|^2+\|2\theta\nabla\varphi\nabla v-\widetilde{q}v\|^2+2(\text{boundary terms})+4\theta\rho^2\int\Big(\sum_i\partial_i\psi\partial_iv\Big)^2\varphi\\
&\quad+4\theta^3\rho^2\int(\nabla\psi)^2(\nabla\varphi)^2\varphi v^2+2\theta\rho^2\int(\nabla\psi)^2\varphi(\nabla v)^2\\
&=\Big\|e^{\theta\varphi}(\Delta-\widetilde{\beta})w\Big\|^2+2\theta^2\int div(\Delta\varphi\nabla\varphi)v^2-4\theta\int\nabla\varphi\widetilde{q}v\nabla v-2(P_2v,\widetilde{q}v)\\
&\quad-2\Big(2\theta\int\Delta\varphi vP_2v+2\theta^2\int(\Delta\varphi)^2v^2+2\theta\int\Delta\varphi\Big(\widetilde{q}+\frac{\widetilde{\beta}}{2}\Big)v^2-\theta\int\Delta^2\varphi v^2\Big)\\
&\quad-4\theta\rho\int\sum_i\Big(\partial_i^2\psi\varphi(\partial_iv)^2+\sum_{j\neq i}\partial_{ij}^2\psi\varphi\partial_iv\partial_jv\Big) &\text{(A28)}\\
&\quad-4\theta^3\rho\int\sum_i\Big[\partial_i^2\psi(\partial_i\varphi)^2+\partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\Big]\varphi v^2\\
&\quad-2\theta\rho\int\Delta\psi\varphi(\nabla v)^2+2\theta^3\rho^2\int(\nabla\psi)^2(\nabla\varphi)^2\varphi v^2+2\theta^3\rho\int\Delta\psi(\nabla\varphi)^2\varphi v^2
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\|P_2v\|^2+\|2\theta\nabla\varphi\nabla v-\widetilde{q}v\|^2+2(\text{boundary terms})+4\theta\rho^2\int\Big(\sum_i\partial_i\psi\partial_iv\Big)^2\varphi\\
&\quad+2\theta^3\rho^2\int(\nabla\psi)^2(\nabla\varphi)^2\varphi v^2+2\theta\rho^2\int(\nabla\psi)^2\varphi(\nabla v)^2\\
&=\Big\|e^{\theta\varphi}(\Delta-\widetilde{\beta})w\Big\|^2+2\theta^2\int div(\Delta\varphi\nabla\varphi)v^2-4\theta\int\nabla\varphi\widetilde{q}v\nabla v-2(P_2v,\widetilde{q}v)\\
&\quad-2\Big(2\theta\int\Delta\varphi vP_2v+2\theta^2\int(\Delta\varphi)^2v^2+2\theta\int\Delta\varphi\Big(\widetilde{q}+\frac{\widetilde{\beta}}{2}\Big)v^2-\theta\int\Delta^2\varphi v^2\Big)\\
&\quad-4\theta\rho\int\sum_i\Big(\partial_i^2\psi\varphi(\partial_iv)^2+\sum_{j\neq i}\partial_{ij}^2\psi\varphi\partial_iv\partial_jv\Big)\\
&\quad-4\theta^3\rho\int\sum_i\Big[\partial_i^2\psi(\partial_i\varphi)^2+\partial_i\varphi\sum_{j\neq i}\partial_j\varphi\partial_{ij}^2\psi\Big]\varphi v^2\\
&\quad-2\theta\rho\int\Delta\psi\varphi(\nabla v)^2+2\theta^3\eta\int\Delta\psi\varphi(\nabla\varphi)^2v^2. &\text{(A29)}
\end{aligned}$$

Now, we choose ψ such that $\psi\in C^2(\overline{\Omega})$, $\psi(x)>0$ in Ω, $\psi(x)=0$ on $\partial\Omega$ and

$$|\nabla\psi(x)|>0\quad\forall x\in\overline{\Omega\setminus\omega_o}, \tag{A30}$$

where $\omega_o \subset \omega$ is an arbitrary fixed subdomain of Ω such that $\overline{\omega_o} \subset \omega$. The existence of such a function ψ was proved in [13]. Then the boundary terms (A3) are non-negative. More precisely, for all $\rho > 0$, $\theta > 0$,

$$\text{boundary terms} = -\theta \int_{\partial\Omega} (\partial_n v)^2 (\rho \partial_n \psi)\varphi \geq 0. \tag{A31}$$

By (A30), (A31) and (A29), there exists a constant $C > 0$, such that for all $\rho > 0$ there is a constant $C_\rho > 0$ such that for all $\theta > 0$, we have

$$\begin{aligned}
&\|P_2 v\|^2 + \|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 + \frac{1}{C}\Big(\theta^3\rho^2 \int_{\Omega\setminus\omega_o} (\nabla\varphi)^2\varphi v^2 + \theta\rho^2 \int_{\Omega\setminus\omega_o} \varphi(\nabla v)^2\Big) \\
&\quad \leq \|P_2 v\|^2 + \|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 \\
&\qquad + \Big(2(\mathit{boundary\ terms}) + 4\theta\rho^2 \int \Big(\sum_i \partial_i\psi\partial_i v\Big)^2 \varphi\Big) \\
&\qquad + 2\theta^3\rho^2 \int (\nabla\psi)^2(\nabla\varphi)^2\varphi v^2 + 2\theta\rho^2 \int (\nabla\psi)^2\varphi(\nabla v)^2 \\
&= \Big\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\Big\|^2 + 2\theta^2 \int div(\Delta\varphi\nabla\varphi)v^2 - 4\theta \int \nabla\varphi\widetilde{q}v\nabla v - 2(P_2 v, \widetilde{q}v) \\
&\quad - 2\theta\Big(2\int \Delta\varphi v P_2 v + 2\theta \int (\Delta\varphi)^2 v^2 + 2\int \Delta\varphi(\widetilde{q} + \frac{\widetilde{\beta}}{2})v^2 - \int \Delta^2\varphi v^2\Big) \\
&\quad - 4\theta\rho \int \sum_i \Big(\partial_i^2\psi\varphi(\partial_i v)^2 + \sum_{j\neq i} \partial_{ij}^2\psi\varphi\partial_i v\partial_j v\Big) \\
&\quad - 4\theta^3\rho \int \sum_i \Big[\partial_i^2\psi(\partial_i\varphi)^2 + \partial_i\varphi \sum_{j\neq i} \partial_j\varphi\partial_{ij}^2\psi\Big]\varphi v^2 \\
&\quad - 2\theta\rho \int \Delta\psi\varphi(\nabla v)^2 + 2\theta^3\rho \int \Delta\psi\varphi(\nabla\varphi)^2 v^2.
\end{aligned} \tag{A32}$$

Using the Cauchy-Schwarz inequality in (A32), we get

$$\begin{aligned}
&\|P_2 v\|^2 + \|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 + \frac{1}{C}\Big(\theta^3\rho^2 \int_{\Omega\setminus\omega_o} (\nabla\varphi)^2\varphi v^2 + \theta\rho^2 \int_{\Omega\setminus\omega_o} \varphi(\nabla v)^2\Big) \\
&\leq \Big\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\Big\|^2 + \theta^2 C_\rho \int v^2 + \theta^2 \|\widetilde{q}\|_{L^\infty}^2 C_\rho \int v^2 + \int (\nabla v)^2 \\
&+ 2\|\widetilde{q}\|_{L^\infty}^2 \int v^2 + \frac{1}{2}\|P_2 v\|^2 + \theta\big(\|\widetilde{q}\|_{L^\infty} + |\widetilde{\beta}|\big)C_\rho \int v^2 + \theta C_\rho \int v^2 \\
&+ C\theta\rho \int \varphi(\nabla v)^2 + C\theta^3\rho \int (\nabla\varphi)^2\varphi v^2 + C\theta\rho \int \varphi(\nabla v)^2 + C\theta^3\rho \int \varphi(\nabla\varphi)^2 v^2.
\end{aligned} \tag{A33}$$

Finally, we obtain that, (recall that $\varphi \geq 1$) for all $\rho > 1$, for all $\theta > 1$,

$$\begin{aligned}
&\frac{1}{2}\|P_2 v\|^2 + \|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 + \frac{1}{C}\rho^2\Big(\theta^3\int_{\Omega\backslash\omega_o}(\nabla\varphi)^2\varphi v^2 + \theta\int_{\Omega\backslash\omega_o}\varphi(\nabla v)^2\Big)\\
&\leq \Big\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\Big\|^2 + \theta^2\big(1 + |\widetilde{\beta}| + \|\widetilde{q}\|_{L^\infty} + \|\widetilde{q}\|^2_{L^\infty}\big)C_\rho\int v^2\\
&\quad + C\rho\Big(\theta^3\int(\nabla\varphi)^2\varphi v^2 + \theta\int\varphi(\nabla v)^2\Big)\\
&\leq \Big\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\Big\|^2 + 2\theta^2\big(1 + |\widetilde{\beta}| + \|\widetilde{q}\|^2_{L^\infty}\big)C_\rho\int v^2\\
&\quad + C\rho\Big(\theta^3\int_{\omega_o}(\nabla\varphi)^2\varphi v^2 + \theta\int_{\omega_o}\varphi(\nabla v)^2\Big)\\
&\quad + C\rho\Big(\theta^3\int_{\Omega\backslash\omega_o}(\nabla\varphi)^2\varphi v^2 + \theta\int_{\Omega\backslash\omega_o}\varphi(\nabla v)^2\Big),
\end{aligned} \tag{A34}$$

where $C > 0$ is a constant independent of ρ and θ. Now we may choose and fix $\rho > 1$ large enough in (A34) to get

$$\begin{aligned}
&\|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 + \frac{1}{C}\Big(\theta^3\int_{\Omega\backslash\omega_o}(\nabla\varphi)^2\varphi v^2 + \theta\int_{\Omega\backslash\omega_o}\varphi(\nabla v)^2\Big)\\
&\leq \Big\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\Big\|^2 + \theta^2\big(1 + |\widetilde{\beta}| + \|\widetilde{q}\|^2_{L^\infty}\big)C\Big(\int_{\Omega\backslash\omega_o}v^2 + \int_{\omega_o}v^2\Big)\\
&\quad + C\Big(\theta^3\int_{\omega_o}(\nabla\varphi)^2\varphi v^2 + \theta\int_{\omega_o}\varphi(\nabla v)^2\Big),
\end{aligned} \tag{A35}$$

where $C > 0$ is a constant independent of θ. Then by (A35), using properties of ψ, we may choose θ large enough so that $\theta^{3/2} > C_1\big(1 + |\widetilde{\beta}| + \|\widetilde{q}\|^2_{L^\infty(\Omega)}\big)$ to get

$$\begin{aligned}
&\|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 + \frac{1}{C_2}\big(\theta^3\int_{\Omega\backslash\omega_o}v^2 + \theta\int_{\Omega\backslash\omega_o}(\nabla v)^2\big)\\
&\leq \Big\|e^{\theta\varphi}(\Delta - \widetilde{\beta})w\Big\|^2 + C_2\big(\theta^3\int_{\omega_o}(\nabla\varphi)^2\varphi v^2 + \theta\int_{\omega_o}\varphi(\nabla v)^2\big),
\end{aligned} \tag{A36}$$

where C_1 and C_2 are two constants independent of θ.

Finally, we have proved that there exists a constant $C > 0$ such that for all $\widetilde{\beta} \in \mathbb{R}$, $\widetilde{q} \in L^\infty(\Omega)$ and $\theta > C\big(1 + |\widetilde{\beta}|^{2/3} + \|\widetilde{q}\|^{4/3}_{L^\infty(\Omega)}\big)$, the following estimate holds

$$\|2\theta\nabla\varphi\nabla v - \widetilde{q}v\|^2 + \frac{1}{C}\Big(\theta^3\int_\Omega v^2 + \theta\int_\Omega(\nabla v)^2\Big)$$

$$\leq \left\| e^{\theta\varphi}(\Delta - \widetilde{\beta})w \right\|^2 + C\Big(\theta^3 \int_{\omega_o} v^2 + \theta \int_{\omega_o} (\nabla v)^2\Big). \tag{A37}$$

Thus, coming back to the function w, for all $\theta > C\big(1 + |\widetilde{\beta}|^{2/3} + \|\widetilde{q}\|^{4/3}_{L^\infty(\Omega)}\big)$,

$$\begin{aligned} &\left\| 2\theta\nabla\varphi\nabla(e^{\theta\varphi}w) - \widetilde{q}e^{\theta\varphi}w \right\|^2 + \frac{1}{C}\Big(\theta^3 \int_\Omega (e^{\theta\varphi}w)^2 + \theta \int_\Omega \big(\nabla(e^{\theta\varphi}w)\big)^2\Big) \\ &\leq \left\| e^{\theta\varphi}(\Delta - \widetilde{\beta})w \right\|^2 + C\Big(\theta^3 \int_{\omega_o} (e^{\theta\varphi}w)^2 + \theta \int_{\omega_o} \big(\nabla(e^{\theta\varphi}w)\big)^2\Big), \end{aligned} \tag{A38}$$

which completes the proof of (A18).

Finally, we deduce from (A38),

$$\begin{aligned} &\left\| 2\theta\nabla\varphi\nabla(e^{\theta\varphi}w) - \widetilde{q}e^{\theta\varphi}w \right\|^2 + \frac{1}{C}\Big(\theta^3 \int_\Omega (e^{\theta\varphi}w)^2 + \theta \int_\Omega (e^{\theta\varphi}\nabla w)^2\Big) \\ &\leq \left\| e^{\theta\varphi}(\Delta - \widetilde{\beta})w \right\|^2 + C\Big(\theta^3 \int_{\omega_o} (e^{\theta\varphi}w)^2 + \theta \int_{\omega_o} (e^{\theta\varphi}\nabla w)^2\Big), \end{aligned} \tag{A39}$$

which implies Theorem A.

This closes the Appendix on the global Carleman inequality for the operator $\Delta - \widetilde{\beta}$ where $\widetilde{\beta} \in \mathbb{R}$.

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