Differential and Integral Equations Volume 19, Number 7 (2006), 799–830

BLOW–UP FOR THE SEMILINEAR WAVE EQUATION IN THE SCHWARZSCHILD METRIC

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Dedicated to Professor Yvonne Choquet–Bruhat in occasion of her 80th year.

Abstract. We study the Cauchy problem for the semilinear wave equation in the Schwarzschild metric ((3 + 1)–dimensional space–time). First, we establish that the problem is locally well posed in H^σ for any $\sigma \in [1, p+1)$; then we prove the blow–up of the solution in two cases: *a)* $p \in (1, 1+\sqrt{2})$ and small initial data supported far away from the black hole, *b)* $p \in (2, 1+\sqrt{2})$ and large data supported near the black hole. In both cases, we also give an estimate from above for the lifespan of the solution.

1. Introduction

Consider the manifold

$$\mathbf{M} = \mathbf{R} \times \mathbf{\Omega}, \qquad \mathbf{\Omega} = \{(r,\omega) : r > 2M,\, \omega \in \mathbf{S}^2\} = (2M,\infty) \times \mathbf{S}^2,$$

equipped with the Schwarzschild metric having the form (see Chapter V in [3] or Chapter 31 in [14]):

$$g = F(r)\, dt^2 - F(r)^{-1}\, dr^2 - r^2\, d\omega^2. \tag{1.1}$$

Here

$$F(r) = 1 - \frac{2M}{r}, \tag{1.2}$$

the constant $M > 0$ has the interpretation of a mass and $d\omega^2$ is the standard metric on the unit sphere $\mathbf{S}^2$.

Accepted for publication: April 2006.
AMS Subject Classifications: 35L70, 35L05, 58J45.
The authors are partially supported by Contract PRIN-2005010482 (2005-2007) with the Italian Ministry of Education, University and Research and the INDAM Contract “Mathematical modelling and numerical analysis of quantum systems with applications to nanosciences”. The second author was visiting the University of Nantes in May 2004, where part of this work was prepared. He is grateful to Professor Georgi Vodev for the hospitality during his visit.

The D'Alembert operator associated with the metric g is

$$\Box_g = \frac{1}{F}\Big(\partial_t^2 - \frac{F}{r^2}\partial_r(r^2F)\partial_r - \frac{F}{r^2}\Delta_{\mathbf{S}^2}\Big),$$

where $\Delta_{\mathbf{S}^2}$ denotes the standard Laplace–Beltrami operator on $\mathbf{S}^2$.

Our goal is to study the long–time behavior of solutions to the corresponding Cauchy problem for the semilinear wave equation

$$\Box_g u = |u|^p \qquad \text{in } [0,\infty)\times\mathbf{\Omega}, \tag{1.3}$$

where $p > 1$. This problem can be considered as a natural analogue of the classical semilinear wave equation

$$\Box_{g_0} u = |u|^p \qquad \text{in } [0,\infty)\times\mathbf{R}^n, \tag{1.4}$$

where g_0 is the flat Minkowski metric

$$g_0 = dt^2 - dr^2 - r^2\,d\omega^2. \tag{1.5}$$

It is well known (see [9], [10], [7], [18], [19], [20], [21], [22], [6], [23], [12] or the review in [5] for a more complete list of references on the subject) that, for any space dimension $n \geqslant 2$, there exists a critical value $p_0 = p_0(n) > 1$ such that the Cauchy problem for (1.4) admits a global small data solution provided $p > p_0(n)$. For subcritical values of $p \leqslant p_0(n)$, a blow–up phenomenon in the flat background is manifested. In the case of a space dimension $n = 3$, the critical exponent is $p_0(3) = 1 + \sqrt{2}$, while in the general case of a space dimension $n \geqslant 2$, the critical exponent is defined as the positive solution to

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

The blow–up results in [9], [10], [7], [18], [19] require a suitable comparison principle for the free wave equation. One further remark is connected with the fact that the critical exponent $p_0(n)$ is the same for the smaller class of radially symmetric solutions.

For the case of the Schwarzschild metric, the dispersive properties of the solution to the linear problem

$$\Box_g u = \Phi \qquad \text{in } [0,\infty)\times\mathbf{\Omega} \tag{1.6}$$

with zero initial data depend essentially on the distribution of resonances for the operator

$$P = \frac{F}{r^2}\partial_r(r^2F)\partial_r + \frac{F}{r^2}\Delta_{\mathbf{S}^2}. \tag{1.7}$$

This problem is studied in [1], [17] (see also [2]) and in [17] it is shown that the resolvent $R(z) = (z^2 - P)^{-1}$ can be extended as a meromorphic function (as an operator from $\mathcal{C}_0^\infty(\Omega)$ to $\mathcal{C}^\infty(\Omega)$) from $\{z \in \mathbf{C} : \Im z > 0\}$ to $\mathbf{C}\setminus i\mathbf{R}$.

The result in [17] shows that the resolvent can be extended further to a meromorphic function in the whole complex plane $\mathbf{C}$. The corresponding poles of the resolvent are called resonances and they are isolated and have finite rank. Moreover, there exists a strip of the type

$$\{z \in \mathbf{C} \,:\, |\Im z| < \varepsilon\} \tag{1.8}$$

free of resonances. This phenomenon is similar to the situation of an exterior domain of several convex obstacles, studied in [8], where a similar domain free of resonances is found. The approach in [8] leads to an exponential decay of the local energy with a derivative loss. Similar exponential energy decay with derivative losses is assumed in [13] for the case of the wave equation in the exterior of compact obstacles. In this work, some weighted space–time a priori L^2–estimates are obtained and further applications to a quasilinear wave equation in the exterior of compact obstacles are done.

Our main goal in this work is to study the semilinear wave equation in the presence of the Schwarzschild metric and to show a blow–up result for $1 < p < 1+\sqrt{2}$. A natural idea here is to adapt the approach of F. John from [9], [10] to the semilinear problem (1.3) in the flat metric and to show the blow–up for some subcritical values of p. This approach meets the essential difficulty that there is no simple explicit representation of the corresponding fundamental solution to the D'Alambert operator in the Schwarzschild metric.

An important tool that will reduce the case of a radially symmetric wave equation in the Schwarzschild metric to the case of a one–dimensional wave equation with suitable potential is the use of the Regge–Wheeler coordinate

$$s(r) = r + 2M \log(r - 2M). \tag{1.9}$$

We can rewrite equation (1.3) as

$$\partial_t^2 u - \partial_s^2 u - \frac{2F}{r(s)}\partial_s u - \frac{F}{r(s)^2}\Delta_{\mathbf{S}^2} u = F|u|^p, \tag{1.10}$$

where $F = F(s) = 1 - \frac{2M}{r(s)}$ and $r(s)$ is the function inverse to (1.9).

For simplicity (and with no loss of generality), we shall restrict our considerations to the case of solutions of the form $u = u(t, s)$. Then (1.10) is simplified to the following equation:

$$\partial_t^2 u - \partial_s^2 u - \frac{2F}{r(s)}\partial_s u = F|u|^p. \tag{1.11}$$

Making further the substitution

$$u(t, s) = \frac{v(t, s)}{r(s)},$$

we obtain the semilinear Cauchy problem:

$$\begin{cases} [\partial_{tt} - \partial_{ss} + W(s)]\, v = f(s)|v|^p, & (t,s) \in [0,\infty) \times \mathbf{R}, \\ v(0,s) = v_0(s), \quad \partial_t\, v(0,s) = v_1(s), & s \in \mathbf{R}, \end{cases} \tag{1.12}$$

where

$$W(s) = \frac{2MF}{r^3}, \qquad f(s) = Fr(s)^{1-p}. \tag{1.13}$$

It is easy to see that $W(s)$, $f(s) \in \mathcal{C}(\mathbf{R})$ satisfy the following estimates

$$W(s) > 0, \qquad f(s) > 0 \qquad \forall s \in \mathbf{R}, \tag{1.14}$$

$$W(s) \sim s^{-3}, \qquad f(s) \sim s^{1-p} \qquad \forall s \geqslant 1, \tag{1.15}$$

$$W(s) \sim \mathrm{e}^{s/(2M)}, \qquad f(s) \sim \mathrm{e}^{s/(2M)} \qquad \forall s \leqslant 0. \tag{1.16}$$

Here and below we shall use often the notation $f \lesssim g$, which means the existence of a positive constant C so that $f \leqslant Cg$. The standard notation $f \sim g$ is equivalent to $f \lesssim g$ and $g \lesssim f$.

For this reason, the study of the long–time behavior of solutions to the wave equation in the Schwarzschild metric is reduced to the study of the semilinear one–dimensional wave equation in (1.12). We shall study the existence of local solutions and the blow–up phenomenon for (1.12) without using the explicit representation (1.13) and assuming that $W(s)$, $f(s)$ obey the asymptotic properties (1.14), (1.15), (1.16) only.

First, we state a local existence result for the Cauchy problem (1.12).

Theorem 1.1. *Given any* $\sigma \in [1, p+1)$ *and any real number* $R > 0$*, one can find* $T = T(R)$ *so that if the initial data*

$$v_0 \in \mathrm{H}^\sigma(\mathbf{R}), \qquad v_1 \in \mathrm{H}^{\sigma-1}(\mathbf{R})$$

satisfy

$$\|v_0\|_{\mathrm{H}^\sigma(\mathbf{R})} + \|v_1\|_{\mathrm{H}^{\sigma-1}(\mathbf{R})} \leqslant R,$$

then the Cauchy problem (1.12) *has a unique solution*

$$v(t,s) \in \mathcal{C}^0([0,T); \mathrm{H}^\sigma(\mathbf{R})) \cap \mathcal{C}^1([0,T); \mathrm{H}^{\sigma-1}(\mathbf{R})).$$

To study the maximal time interval of existence of solutions to (1.12), we choose the following initial data:

$$v_0(s) = \rho(\varepsilon)\chi_0\big(s - s_0(\varepsilon)\big), \qquad v_1(s) = \rho(\varepsilon)\chi_1\big(s - s_0(\varepsilon)\big), \tag{1.17}$$

where $\chi_j \in \mathcal{C}_0^\infty(\mathbf{R})$ satisfy, for $j = 1, 2$, the conditions

$$\chi_j(s) \geqslant 0, \qquad s \in \mathbf{R}, \tag{1.18}$$

$$\chi_j(s) = 1, \qquad s \in [-R/2, R/2], \tag{1.19}$$

$$\operatorname{supp}\chi_j \subseteq [-R, R] \tag{1.20}$$

for a positive constant R. The function $\rho(\varepsilon)$ will be chosen appropriately later on.

It is not difficult to see that

$$||v_0||_{\mathrm{H}^\sigma(\mathbf{R})} + ||v_1||_{\mathrm{H}^{\sigma-1}(\mathbf{R})} \sim \rho(\varepsilon) \tag{1.21}$$

for each $\sigma \geqslant 1$, so the initial data in (1.17) have small $\mathrm{H}^\sigma \times \mathrm{H}^{\sigma-1}$ norms provided $\rho(\varepsilon)$ is small.

Recall that a big Regge–Wheeler coordinate s corresponds to the domain where one is far away from the black hole ($\mathbf{R}^3 \setminus \Omega$), i.e. the domain with almost flat metric. On the other hand, $s \to -\infty$ corresponds to the domain close to the black hole.

First, we shall choose $\varepsilon > 0$ sufficiently small and shall set

$$s_0(\varepsilon) = \varepsilon^{-\vartheta}, \qquad \rho(\varepsilon) = \varepsilon, \tag{1.22}$$

where ϑ satisfies

$$\vartheta \geqslant \frac{b(p-1)}{-p^2+2p+1}, \qquad \vartheta \geqslant 1 + \frac{b(3p-5)}{-p^2+2p+1}, \tag{1.23}$$

and $b = p$ if $p \in [2, 1+\sqrt{2})$, $b = p^2$ if $p \in (1, 2)$.

In this case, the initial data in (1.17) have support far away from the black hole and our next result asserts that small data solutions manifest a blow–up phenomenon in the subcritical case.

Theorem 1.2. *For any p, $1 < p < 1+\sqrt{2}$, there exists a positive number ε_0 so that for any $\varepsilon \in (0, \varepsilon_0)$ and any initial data of type* (1.17) *satisfying* (1.22)*, there exists a positive number $T = T(\varepsilon) < \infty$ and a solution*

$$v \in \bigcap_{k=0}^{2} \mathcal{C}^k\big([0,T); \mathrm{H}^{2-k}(\mathbf{R})\big)$$

of (1.12) *such that*

$$\lim_{t \uparrow T} ||v(t)||_{\mathrm{L}^2(\mathbf{R})} = \infty.$$

The above result means that, when the initial data are supported far away from the black hole, the wave equation in the Schwarzschild metric has a critical exponent similar to that of the free wave equation. In this region, we can estimate from above the lifespan of the solution.

The situation changes completely when one tries to approach the black hole. To have a model that simulates this phenomenon, we take initial data of the type (1.17) choosing

$$s_0(\varepsilon) = -T_2(\varepsilon), \tag{1.24}$$

where $T_2(\varepsilon) > 0$ grows very rapidly as $\varepsilon \to 0$. More precisely, we take $T_2(\varepsilon) \in \Sigma$, where Σ is the following class of functions:

$$\Sigma = \{T(\varepsilon) \in \mathcal{C}((0,1]) \,:\, \forall A \geqslant 1,\, \lim_{\varepsilon \downarrow 0} \varepsilon^A T(\varepsilon) = \infty\}. \tag{1.25}$$

A typical example is $T(\varepsilon) = \mathrm{e}^{1/\varepsilon}$.

Approaching the black hole, one meets an essential difficulty in overcoming the attraction force of the black hole. In this case, the coefficient $f(s)$ in the source term in (1.12) decays exponentially and this dissipative phenomenon is in competition with the blow–up properties of the source term. Because of this, the blow–up mechanism which we propose is based on a different choice of the quantity $\rho(\varepsilon)$ that measures the Sobolev norm of the initial data according to (1.21). We have to take $\rho(\varepsilon) \in \Sigma$; i.e. the initial data are large.

Then we have the following blow–up result.

Theorem 1.3. *For any p, $2 < p < 1 + \sqrt{2}$, there exists a positive number ε_0 so that for any $\varepsilon \in (0, \varepsilon_0)$ and any initial data of type* (1.17) *satisfying* (1.24)*, there exists a function $\rho(\varepsilon) \in \Sigma$, a positive number $T = T(\varepsilon) < \infty$ and a solution*

$$v \in \bigcap_{k=0}^{2} \mathcal{C}^k\big([0,T); \mathrm{H}^{2-k}(\mathbf{R})\big)$$

of (1.12) *such that*

$$\lim_{t \uparrow T} ||v(t)||_{\mathrm{L}^2(\mathbf{R})} = \infty.$$

However, the proof we follow here suggests that the lifespan of the solution has a completely different behavior — it might be much longer than in the corresponding flat case (see Lemma 2.3 below).

The main difficulty in establishing the blow–up of the solution is connected with the sign–changing properties of the fundamental solution of the linear wave equation in the Schwarzschild metric (or more generally in curved metrics). In the case of the flat $(1+3)$–Minkowski metric, the fundamental solution is non–negative and this property is used effectively in the study of the blow–up phenomenon for the corresponding semilinear wave equation.

Our blow–up analysis, in the case of initial data supported far away from the black hole, is based on the study of the asymptotic behavior of the following quantities:

$$U(t) = \Big(\int_{\mathbf{R}} \varphi_0(s) |v(t,s)|^p f(s)\, ds \Big)^{1/p}, \tag{1.26}$$

$$V(t) = \int_{\mathbf{R}} \varphi_0(s) v(t,s)\, ds. \tag{1.27}$$

Here φ_0 is a solution to the equation

$$\partial_{ss}\, \varphi_0(s) - W(s)\varphi_0(s) = 0. \tag{1.28}$$

Lemma 5.3 implies that there exists a solution φ_0 to (1.28) such that

$$\varphi_0(s) = \psi_0(s) + D \qquad \forall s \in \mathbf{R} \tag{1.29}$$

for some positive constant D and $\psi_0(s) \geqslant 0$ has the asymptotic expansion

$$\psi_0(s) \sim \begin{cases} s & s \to \infty, \\ \mathrm{e}^{s/(2M)} & s \to -\infty. \end{cases} \tag{1.30}$$

The key point in the proof of Theorem 1.2 is to verify the following a priori estimates for V:

$$V(0) \geqslant C_0 \varepsilon^\beta, \qquad V'(0) \geqslant C_0 \varepsilon^\beta, \tag{1.31}$$

$$V(t) \geqslant C_0 \varepsilon^b (t+R)^a \qquad \forall t \in [T_0, T_1 + T_0), \tag{1.32}$$

$$V''(t) \geqslant C_0 (t+R)^{-q} V(t)^p \qquad \forall t \in [T_0, T_1 + T_0), \tag{1.33}$$

and then to apply a suitable variant of the classical Kato lemma (see Lemma 2.2 below).

For the case of initial data supported close to the black hole we modify U, V as follows:

$$U(t) = \Big(\int_{\mathbf{R}} \psi_0(s) |v(t,s)|^p f(s)\, ds \Big)^{1/p}, \tag{1.34}$$

$$V(t) = \int_{\mathbf{R}} \psi_0(s) v(t,s)\, ds. \tag{1.35}$$

The main new phenomenon manifested in this case is the possible loss of positivity of $V(t)$. Indeed, one can show that V satisfies the differential equation (see (3.46) below)

$$V''(t) = U(t)^p + D \int vW(s)\, ds,$$

so the positivity of $V(t)$ cannot be obtained as a trivial consequence of the positivity of $V(0), V'(0)$ and the differential equation satisfied by $V(t)$.

We use in this case another variant of the classical Kato lemma stated in Lemma 2.3 and involving two functions U and V.

The plan of the work is the following:

- Section 2 contains the statement of the Kato lemma and the statements and the proofs of its needed variants;
- in Section 3 we prove the main blow–up results, that is Theorem 1.2 and Theorem 1.3;
- Section 4 is devoted to the verification of some asymptotic estimates (which imply the conditions (1.14), (1.15) and (1.16)) and the proof of the local existence theorem, i.e., Theorem 1.1;
- in Section 5 we find the asymptotic estimates for φ_0 and for φ_1, another function that we shall use in Section 2 which is a solution of an associated elliptic problem.

The authors are grateful to the referees for important remarks concerning the initial version of this work.

2. Variants of the classical Kato lemma

A key point in the blow–up argument that we follow is an appropriate modification of the following lemma due to Kato (see [11]).

Lemma 2.1 (classical Kato lemma). *Assume that* $a \geqslant 1$, $p > 1$ *and*

$$0 \leqslant q < (p-1)a + 2. \tag{2.1}$$

Given any positive constants R, C_0 *one can find a constant* $C_1 = C_1(a, p, q, R, C_0) > 0$ *so that for any* $T_0 \geqslant 0$ *the condition*

a) there exists a non–negative function $V \in \mathcal{C}^2([T_0, T_0 + T))$ *so that*

$$V(T_0) \geqslant 0, \qquad V'(T_0) \geqslant 0, \tag{2.2}$$

$$V(t) \geqslant C_0(t+R)^a \qquad \forall t \in [T_0, T_0+T), \tag{2.3}$$

$$V''(t) \geqslant C_0(t+R)^{-q} V(t)^p \qquad \forall t \in [T_0, T_0+T) \tag{2.4}$$

will imply

$$T \leqslant C_1(1+T_0). \tag{2.5}$$

Proof. For $0 \leqslant T_0 \leqslant 1$ this is the classical assertion of the Kato lemma. For $T_0 \geqslant 1$ the inequality (2.5) is equivalent to $T \leqslant C_1 T_0$ and we can use a rescaling argument. More precisely, we shall perform the following transform:

$$t = T_0 \tau, \qquad V(t) = T_0^a \tilde{V}(\tau). \tag{2.6}$$

Setting $\tau_1 = T/T_0$, we see that

$$\begin{cases} \tilde{V}(1) \geqslant 0, \qquad \tilde{V}'(1) \geqslant 0, \\ \tilde{V}(\tau) \geqslant C_0 \tau^a & \forall \tau \in [1, 1+\tau_1), \\ \tilde{V}''(\tau) \geqslant C_0 T_0^{a(p-1)-q+2} \tau^{-q} \tilde{V}(\tau)^p & \forall \tau \in [1, 1+\tau_1). \end{cases} \tag{2.7}$$

Assumption (2.1) implies that $T_0^{a(p-1)-q+2} \geqslant 1$ so applying to (2.7) the classical Kato's argument, we get $\tau_1 \leqslant C_1$. This completes the proof. □

The above lemma gives the following more precise information for the lifespan interval in t.

Lemma 2.2 (finite lifespan for the classical Kato lemma)**.** *Assume that $a \geqslant 1$, $p > 1$ satisfy*

$$0 \leqslant q < (p-1)a + 2,$$

R, C_0 are two positive constants and $b \geqslant 0$. There are positive constants

$$C_1 = C_1(a, p, q, R, C_0, b),$$

so that for any $\varepsilon \in (0,1)$ and for any couple of real numbers $T_0 \geqslant 0$, $T_1 > 0$, the condition

(*a*) *there exists a non–negative function $V \in \mathcal{C}^2([T_0, T_0 + T_1))$, such that*

$$T_0 \geqslant C_0 \varepsilon^\alpha, \qquad \alpha = -\frac{b(p-1)}{2 + (p-1)a - q} \leqslant 0, \tag{2.8}$$

$$V(T_0) \geqslant C_0 \varepsilon^\beta, \quad V'(T_0) \geqslant C_0 \varepsilon^\beta, \quad \beta = -b(q-2)/(2 + (p-1)a - q), \tag{2.9}$$

$$V(t) \geqslant C_0 \varepsilon^b (t+R)^a \qquad \forall t \in [T_0, T_0 + T_1), \tag{2.10}$$

$$V''(t) \geqslant C_0 (t+R)^{-q} V(t)^p \qquad \forall t \in [T_0, T_0 + T_1) \tag{2.11}$$

implies

(*b*) $T_1 \leqslant C_1(T_0 + \varepsilon^\alpha)$.

Proof. For $b = 0$ the assertion of the lemma follows directly from the classical Kato lemma 2.1. Take $b > 0$; making the transform

$$t = \varepsilon^\alpha \tau, \qquad V(t) = \varepsilon^\beta \tilde{V}(\tau) \tag{2.12}$$

and setting $\tau_0 = T_0 \varepsilon^{-\alpha}$, $\tau_1 = (T_0 + T_1)\varepsilon^{-\alpha}$, we see that

$$\begin{cases} \tilde{V}(\tau_0) \geqslant C_0, \qquad \tilde{V}'(\tau_0) \geqslant C_0 \varepsilon^\alpha, \\ \tilde{V}(\tau) \geqslant C_0 \varepsilon^{b-\beta+\alpha a} (\tau + R\varepsilon^{-\alpha})^a & \forall \tau \in [\tau_0, \tau_1), \\ \tilde{V}''(\tau) \geqslant C_0 \varepsilon^{\beta(p-1) - \alpha(q-2)} (\tau + R\varepsilon^{-\alpha})^{-q} \tilde{V}(\tau)^p & \forall \tau \in [\tau_0, \tau_1). \end{cases} \tag{2.13}$$

We choose α, β so that

$$b - \beta + \alpha a = \beta(p-1) - \alpha(q-2) = 0.$$

The solution to the above system is given by

$$\alpha = -\frac{b(p-1)}{(2+(p-1)a-q)}\ , \qquad \beta = -\frac{b(q-2)}{(2+(p-1)a-q)}.$$

Then (2.13) becomes

$$\begin{cases} \tilde{V}(\tau_0) \geqslant C_0, \qquad \tilde{V}'(\tau_0) \geqslant C_0\varepsilon^{\alpha}, \\ \tilde{V}(\tau) \geqslant C_0(\tau + R\varepsilon^{-\alpha})^a \geqslant C_0\tau^a \qquad \forall \tau \in [\tau_0, \tau_1), \\ \tilde{V}''(\tau) \geqslant C_0(\tau + R\varepsilon^{-\alpha})^{-q}\tilde{V}(\tau)^p \qquad \forall \tau \in [\tau_0, \tau_1). \end{cases} \tag{2.14}$$

Since $\alpha \leqslant 0$, $q \geqslant 0$ and $0 < \varepsilon < 1$, we have

$$(\tau + R\varepsilon^{-\alpha})^{-q} \geqslant (\tau + R)^{-q}.$$

Note that assumption (2.8) implies $\tau_0 = T_0\varepsilon^{-\alpha} \geqslant C_0$. We are now in the situation to apply the classical Kato lemma 2.1 and get $\tau_1 \leqslant C_1(1 + \tau_0)$, which implies

(*b*) $T_1 \leqslant C_1(T_0 + \varepsilon^{\alpha})$.

This completes the proof of the lemma. □

We shall need another variant of the Kato lemma. For the purpose, we introduce the following class of functions:

$$\Sigma = \{T(\varepsilon) \in C((0,1]) \,:\, \forall A \geqslant 1,\, \lim_{\varepsilon \downarrow 0} \varepsilon^A T(\varepsilon) = \infty\}. \tag{2.15}$$

Lemma 2.3. *Assume that $a \geqslant 1$, $p > 1$ satisfy*

$$0 \leqslant q < (p-1)a + 2, \tag{2.16}$$

while R, C_0, C_1 are positive constants. Then there exist two constants

$$D_1 = D_1(a,p,q,R,C_0,C_1) > 0, \qquad \varepsilon_0 = \varepsilon_0(a,p,q,R,C_0,C_1) > 0,$$

so that for any positive $T_0(\varepsilon) \in \Sigma$, for any $\varepsilon \in (0, \varepsilon_0)$ and for any real number $T_1 > 0$, the condition

(*a*) *there exist two functions $U \in \mathcal{C}([0, T_0(\varepsilon)+T_1))$, $V \in \mathcal{C}^2([0, T_0(\varepsilon)+T_1))$, so that*

$$V(0) \geqslant 0, \quad V'(0) \geqslant 0, \quad U(t) \geqslant 0 \qquad \forall t \in [0, T_0(\varepsilon) + T_1), \tag{2.17}$$

$$U(t)^p \geqslant C_0(t+R)^{-q}|V(t)|^p + C_0\varepsilon^p(t+R)^{a-2} \quad \forall t \in [0, T_0(\varepsilon) + T_1), \tag{2.18}$$

$$V''(t) \geqslant C_0U(t)^p - C_1U(t) \qquad \forall t \in [0, T_0(\varepsilon) + T_1), \tag{2.19}$$

$$V''(t) \geqslant C_0U(t)^p - C_1\,\mathrm{e}^{-C_0T_0(\varepsilon)}\,U(t) \qquad \forall t \in [0, T_0(\varepsilon)) \tag{2.20}$$

implies

(*b*) $T_1 \leqslant D_1T_0(\varepsilon)$.

Proof. Consider the function

$$K(x) = x^p - Cx, \tag{2.21}$$

where $C = C_1/C_0 > 0$. For

$$x > x_0 \doteq 2C^{1/(p-1)}, \tag{2.22}$$

we have

$$K(x) \gtrsim x^p. \tag{2.23}$$

In a similar way, given $T_0(\varepsilon) \in \Sigma$, we can consider the function

$$K_{T_0(\varepsilon)}(x) = x^p - C\,\mathrm{e}^{-C_0 T_0(\varepsilon)}\,x. \tag{2.24}$$

For

$$x > x_0(\varepsilon) \doteq 2C^{1/(p-1)}\,\mathrm{e}^{-C_0 T_0(\varepsilon)/(p-1)}, \tag{2.25}$$

we have

$$K_{T_0(\varepsilon)}(x) \gtrsim x^p. \tag{2.26}$$

Note that inequality (2.18) assures that

$$U(t) \gtrsim \varepsilon(t+R)^{(a-2)/p} \geqslant \varepsilon(T_0+R)^{(a-2)/p} \tag{2.27}$$

for $t \in [0, T_0]$ if $a \leqslant 2$, and

$$U(t) \gtrsim \varepsilon(t+R)^{(a-2)/p} \geqslant \varepsilon R^{(a-2)/p} \tag{2.28}$$

for $t \in [0, T_0]$ if $a > 2$. Now it is clear that choosing $T_0 = T_0(\varepsilon) \in \Sigma$, we can guarantee the following analogue of inequality (2.25):

$$U(t) > x_0(\varepsilon) \doteq 2C^{1/(p-1)}\,\mathrm{e}^{-C_0 T_0(\varepsilon)/(p-1)} \quad \text{for } t \in [0, T_0(\varepsilon)],\ \varepsilon \in (0, \varepsilon_0). \tag{2.29}$$

Indeed, the lower bound of $U(t)$ is at most polynomially decaying (in T_0) due to (2.27) and (2.28), while $x_0(\varepsilon)$ decays exponentially in $T_0(\varepsilon)$ as $\varepsilon \to 0$.

Since (2.25) implies (2.26), we conclude that

$$V''(t) \gtrsim U(t)^p \gtrsim \varepsilon^p (t+R)^{a-2} \qquad \text{for } t \in [0, T_0),$$

and integrating this inequality twice, we obtain

$$V(t) \gtrsim \varepsilon^p (t+R)^a, \quad V'(t) \gtrsim \varepsilon^p (t+R)^{a-1} \quad \text{for } t \in [T_0/2, T_0), \tag{2.30}$$

because of (2.17). Note that we have also the inequality

$$V''(t) \gtrsim (t+R)^{-q} V(t)^p \qquad \text{for } t \in [T_0/2, T_0]. \tag{2.31}$$

Now we are in the situation when estimates similar to the estimates (2.3) and (2.4) of the classical Kato lemma are fulfilled. Setting $a_0 = a$, $p_0 = p$, we define the recurrence sequence

$$a_{k+1} = pa_k - q + 2, \qquad p_{k+1} = p_k p. \tag{2.32}$$

It is clear that

$$a_{k+1} = a + \frac{p^{k+1}-1}{p-1}\left((p-1)a - q + 2\right), \qquad p_k = p^{k+1}, \tag{2.33}$$

so integrating twice the differential inequality (2.31) in combination with (2.30), we prove inductively in k the following estimates

$$V(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k}, \qquad V'(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k-1} \tag{2.34}$$

for $t \in [(1-2^{-(k+1)})T_0, T_0]$. It is not difficult to see that a_k tends to infinity, due to (2.33) and (2.16).

We can choose in particular $k \geqslant 0$ so that $a_k > 2$ and fix this k. Then estimate (2.34) combined with (2.18) implies

$$\begin{cases} V(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k}, \qquad V'(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k-1}, \\ U(t)^p \gtrsim \varepsilon^{p_{k+1}}(t+R)^{a_{k+1}-2} \end{cases} \tag{2.35}$$

for $t \in [(1-2^{-(k+1)})T_0, T_0]$. Thus, we have

$$U(t) \gtrsim \varepsilon^{p_{k+1}/p}(T_0(\varepsilon)+R)^{(a_{k+1}-2)/p} > x_0 \doteq 2C^{1/(p-1)} \tag{2.36}$$

for $t \in [(1-2^{-(k+1)})T_0, T_0]$, due to the property $T_0 \in \Sigma$ and the definition (2.15) of the class Σ (we assume that ε is sufficiently small). Once this inequality is satisfied, we can use (2.18), (2.36) and (2.19), and see that

$$U(t)^p \gtrsim (t+R)^{-q}|V(t)|^p + \varepsilon^p(t+R)^{a-2}, \tag{2.37}$$

$$V''(t) \gtrsim U(t)^p - CU(t) \tag{2.38}$$

for $t \in [(1-2^{-(k+1)})T_0, T_0+T_1)$ and

$$U(t) > x_0, \quad V'(t) \geqslant 0 \quad \text{for } t \in [(1-2^{-(k+1)})T_0, T_0]. \tag{2.39}$$

Let

$$T_2 \doteq \sup\{T \in [0,T_1] \,:\, U(t) > x_0,\ t \in [(1-2^{-(k+1)})T_0, T_0+T)\}.$$

One can show that $T_2 = T_1$. Indeed, for $t \in [(1-2^{-(k+1)})T_0, T_0+T_2)$, we have

$$V''(t) \gtrsim U(t)^p \gtrsim (t+R)^{-q}|V(t)|^p + \varepsilon^p(t+R)^{a-2}; \tag{2.40}$$

so we are in a situation to repeat the argument of the proof of (2.35) and we can prove inductively in m the following estimates:

$$V(t) \gtrsim \varepsilon^{p_m}(t+R)^{a_m}, \qquad V'(t) \gtrsim \varepsilon^{p_m}(t+R)^{a_m-1} \tag{2.41}$$

for all $t \in [(1-2^{-(k+m+1)})T_0, T_0+T_2)$. Take $m=k$ so that $a_k>2$. Then estimate (2.41) combined with (2.40) implies

$$\begin{cases} V(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k}, \qquad V'(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k-1}, \\ U(t)^p \gtrsim \varepsilon^{p_{k+1}}(t+R)^{a_{k+1}-2}, \end{cases} \tag{2.42}$$

for $t \in [(1-2^{-(2k+1)})T_0, T_0+T_2)$. Since, provided ε is small, we have

$$\varepsilon^{p_{k+1}}(T_0(\varepsilon)+R)^{a_{k+1}-2} > x_0^p$$

due to the property $T_0 \in \Sigma$, we get also $U(t) > x_0$ near $t = T_0+T_2$. This inequality, combined with the definition of T_2, guarantees that $T_2 = T_1$.

From this conclusion we see that the functions $U \in \mathcal{C}([0, T_0(\varepsilon)+T_1)), V \in \mathcal{C}^2([0, T_0(\varepsilon)+T_1))$ satisfying the condition (a), will obey also the following stronger estimates

$$V(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k}, \qquad V'(t) \gtrsim \varepsilon^{p_k}(t+R)^{a_k-1}, \tag{2.43}$$

$$V''(t) \gtrsim (t+R)^{-q} V(t)^p \qquad \forall t \in [(1-2^{-(2k+1)})T_0(\varepsilon), T_0(\varepsilon)+T_1). \tag{2.44}$$

It remains to verify condition (2.9) in order to apply the lifespan estimate of Lemma 2.2. Indeed, (2.43) implies

$$V(T_0(\varepsilon)) \gtrsim \varepsilon^{p_k}(T_0(\varepsilon)+R)^{a_k},$$

therefore the fact that $T_0 \in \Sigma$ and $a_k > 0$ imply (2.9), so applying Lemma 2.2 we complete the proof. □

3. Proofs of theorems 1.2 and 1.3

Since the proofs of Theorems 1.2 and 1.3 use similar tools, we present them in the next two subsections.

3.1. **Blow–up with small data far away from the black hole and $p \in (1, 1+\sqrt{2})$.** In this subsection, we verify the hypotheses of Lemma 2.2 to get Theorem 1.2. In particular, we take

$$p > 1, \qquad q = 3(p-1), \qquad a = 4-p, \tag{3.1}$$

and prove (2.9), (2.10) and (2.11) for

$$U(t) = \left(\int_{\mathbf{R}} \varphi_0(s) |v(t,s)|^p f(s)\, ds \right)^{1/p}, \tag{3.2}$$

$$V(t) = \int_{\mathbf{R}} \varphi_0(s) v(t,s)\, ds, \tag{3.3}$$

where v is a solution to the Cauchy problem (1.12) with

$$\rho(\varepsilon) = \varepsilon, \qquad s_0(\varepsilon) = \varepsilon^{-\vartheta} \tag{3.4}$$

and

$$\vartheta \geqslant \frac{b(p-1)}{\delta}, \qquad \vartheta \geqslant 1 + \frac{b(3p-5)}{\delta}, \tag{3.5}$$

$$b = \begin{cases} p & \text{if } 2 \leqslant p < 1+\sqrt{2}, \\ p^2 & \text{if } 1 < p < 2, \end{cases} \tag{3.6}$$

$$\delta = (p-1)a - q + 2 > 0. \tag{3.7}$$

Let us notice that (3.7) is equivalent to $p^2-2p-1 < 0$; that is, $1 < p < 1+\sqrt{2}$; in particular we have $a > 1$. Moreover, (3.4) guarantees that

$$||v_0||_{\mathrm{H}^2(\mathbf{R})} + ||v_1||_{\mathrm{H}^1(\mathbf{R})} \sim \varepsilon, \tag{3.8}$$

therefore the initial data in (1.17) have small $\mathrm{H}^2 \times \mathrm{H}^1$ norms.

First, we check (2.9). We apply Lemma 3.1 (see the end of this subsection) and get that there exists a positive constant C_0 such that

$$V(0) \geqslant C_0\varepsilon^\beta, \qquad V'(0) \geqslant C_0\varepsilon^\beta, \qquad \beta = -\frac{b(3p-5)}{\delta}$$

thanks to the second part of condition (3.5); that is, $1 - \vartheta \leqslant \beta$.

Further, we have the relation

$$\begin{aligned} V''(t) &= \int_{\mathbf{R}} v_{tt}(t,s)\varphi_0(s)\,ds \\ &= \int_{\mathbf{R}} v_{ss}(t,s)\varphi_0(s)\,ds - \int_{\mathbf{R}} W(s)v(t,s)\varphi_0(s) + \int_{\mathbf{R}} f(s)|v(t,s)|^p\varphi_0(s)\,ds, \end{aligned}$$

so (1.28) implies

$$V''(t) = \int_{\mathbf{R}} f(s)|v(t,s)|^p\varphi_0(s)\,ds = U(t)^p \tag{3.9}$$

and we conclude that $V''(t)$ is a non–negative function. This argument guarantees that setting

$$T_0 = T_0(\varepsilon) = 2C_1 s_0(\varepsilon), \tag{3.10}$$

where $C_1 > 0$ is the constant from Lemma 2.2, one has

$$V(T_0) \geqslant C_0\varepsilon^\beta, \qquad V'(T_0) \geqslant C_0\varepsilon^\beta, \tag{3.11}$$

thus (2.9) is verified.

The estimates of (2.10) and (2.11) are based on a finite dependence domain argument, that is

$$\operatorname{supp} v(t,s) \subseteq \{(t,s) \,:\, |s - s_0(\varepsilon)| \leqslant t + R\}. \tag{3.12}$$

This dependence domain property implies that the support of $v(t,s)$ is in the domain $s>1$ for $0\leqslant t\leqslant 2T_0$, provided $\varepsilon>0$ is sufficiently small. Hence we can use Hölder's inequality and get, for $0\leqslant t\leqslant 2T_0$,

$$\begin{aligned}|V(t)| &\leqslant \int_{\mathbf{R}} |v(t,s)|\varphi_0(s)\,ds\\ &= \int_{|s-s_0|\leqslant t+R} \left(f^{1/p}|v|\varphi_0^{1/p}\right)\left(f^{-1/p}\varphi_0^{(p-1)/p}\right)\,ds\\ &\leqslant \Big(\int_{\mathbf{R}} f|v|^p\varphi_0\,ds\Big)^{1/p}\Big(\int_{|s-s_0|\leqslant t+R} f^{-1/(p-1)}\varphi_0\,ds\Big)^{(p-1)/p};\end{aligned}$$

since $f(s)^{-1/(p-1)}\varphi_0(s)\lesssim(1+|s|)^2$, we conclude

$$|V(t)|\leqslant CU(t)(t+R)^{3(p-1)/p}$$

for each $t\in[T_0,2T_0]$ (i.e., $s_0(\varepsilon)\leqslant t/2C_1$), where $C>0$ is a constant independent of ε, T_0. Thus, we have

$$U(t)^p\geqslant C(t+R)^{-q}|V(t)|^p,\qquad q=3(p-1), \tag{3.13}$$

and the inequality (2.11) is verified.

It is easy to show that estimate (2.10) follows from an estimate of the type

$$U(t)^p\geqslant C\varepsilon^b(t+R)^{a-2},\qquad t\in[T_0,2T_0]. \tag{3.14}$$

Indeed, using (3.9) and integrating twice (3.14) we get (2.10), since the initial data are non–negative due to (3.11).

To verify (3.14), we use the auxiliary quantity

$$F_1(t)=\mathrm{e}^{-t/(2M)}\int_{\mathbf{R}} v(t,s)\varphi_1(s)\,ds, \tag{3.15}$$

where φ_1 is a solution to

$$-\partial_{ss}\varphi_1+W(s)\varphi_1+\frac{1}{4M^2}\varphi_1=0 \tag{3.16}$$

having asymptotic behavior

$$\varphi_1(s)\sim\mathrm{e}^{s/(2M)}\qquad\text{as } |s|\longrightarrow\infty. \tag{3.17}$$

The existence of such a function is shown in Lemma 5.4.

Proceeding as before, we obtain

$$\begin{aligned} F_1(t) &\leqslant e^{-t/2M} \int_{\mathbf{R}} |v| \varphi_1 \, ds \\ &= e^{-t/2M} \int_{|s-s_0| \leqslant t+R} \left(f^{\frac{1}{p}} |v| {\varphi_0}^{\frac{1}{p}} \right) \left(f^{-\frac{1}{p}} {\varphi_0}^{-\frac{1}{p}} \varphi_1 \right) \, ds \\ &\leqslant U(t) \Big(e^{-\frac{t}{2M}\frac{p}{p-1}} \int_{|s-s_0| \leqslant t+R} f^{-\frac{1}{p-1}} {\varphi_0}^{-\frac{1}{p-1}} {\varphi_1}^{\frac{p}{p-1}} \, ds \Big)^{\frac{p-1}{p}}; \end{aligned} \tag{3.18}$$

thanks to the hypotheses on f, φ_0, φ_1 (see (1.15), (1.29), (1.30) and (3.17)), we deduce

$$F_1(t) \leqslant CU(t) \Big(\int_{|s-s_0| \leqslant t+R} (1+s)^{\frac{p-2}{p-1}} \, e^{\frac{s-t}{2M}\frac{p}{p-1}} \, ds \Big)^{\frac{p-1}{p}}. \tag{3.19}$$

Now we distinguish two cases.

Case 1: $p \in [2, 1+\sqrt{2})$. Let us observe that, given any $\alpha \geqslant 0$, $\beta > 0$ and $R > 0$, one can find a positive constant $C = C(R, \alpha, \beta)$ so that the inequality

$$\int_{0 \leqslant s \leqslant t+R+s_0(\varepsilon)} (s+R)^\alpha \, e^{\beta(s-t)} \, ds \leqslant C \, (t+R+s_0(\varepsilon))^\alpha \, e^{\beta s_0(\varepsilon)} \tag{3.20}$$

holds for any $t \geqslant 0$ and $\varepsilon \in (0,1]$. Applying (3.20) with

$$\alpha = \frac{p-2}{p-1} \geqslant 0, \qquad \beta = \frac{p}{2M(p-1)} > 0$$

in (3.19), we derive

$$F_1(t) \leqslant CU(t)(t+R+s_0(\varepsilon))^{\frac{p-2}{p}} \, e^{\frac{s_0(\varepsilon)}{2M}}.$$

Now, Lemma 3.2 assures that $F_1(t) \geqslant C'\varepsilon \, e^{\frac{s_0(\varepsilon)}{2M}}$; so, taking $t \in [T_0, 2T_0]$ (i.e., $t \sim s_0(\varepsilon)$) and the power p in both members, we get

$$U(t)^p \geqslant C\varepsilon^p \, (t+R)^{2-p}, \tag{3.21}$$

that is the desired estimate (3.14) with $a = 4-p$, $b = p$.

Case 2: $p \in (1, 2]$. Given any $\alpha \leqslant 0$, $\beta > 0$ and $R > 0$, one can find a positive constant $C = C(R, \alpha, \beta)$ so that the inequality

$$\int_{0 \leqslant s \leqslant t+R+s_0(\varepsilon)} (s+R)^\alpha \, e^{\beta(s-t)} \, ds \leqslant C \, e^{\beta s_0(\varepsilon)} \tag{3.22}$$

holds for any $t \geqslant 0$ and $\varepsilon \in (0,1]$. Applying this estimate and (3.19), we derive

$$F_1(t) \leqslant CU(t) \, e^{\frac{s_0(\varepsilon)}{2M}}.$$

Lemma 3.2 implies that

$$C'\varepsilon\,\mathrm{e}^{\frac{s_0(\varepsilon)}{2M}} \leqslant F_1(t)$$

and hence the inequalities

$$C'\varepsilon\,\mathrm{e}^{\frac{s_0(\varepsilon)}{2M}} \leqslant F_1(t) \leqslant CU(t)\,\mathrm{e}^{\frac{s_0(\varepsilon)}{2M}}$$

imply $U(t)^p \geqslant C\varepsilon^p$, so we get

$$V''(t) = U(t)^p \geqslant C\varepsilon^p. \tag{3.23}$$

Integrating twice, we obtain

$$V(t) \geqslant C\varepsilon^p(t+R)^2 \tag{3.24}$$

which, substituted in (3.13), yields

$$V''(t) = U(t)^p \geqslant C\varepsilon^{p^2}(t+R)^{3-p} \tag{3.25}$$

and consequently

$$V(t) \geqslant C\varepsilon^{p^2}(t+R)^{5-p} \geqslant C\varepsilon^{p^2}(t+R)^a, \tag{3.26}$$

that is (2.10), with $a = 4-p$, $b = p^2$.

Now, suppose that the lifespan $T(\varepsilon)$ for the solution to (1.12) is greater than $(C_1+1)T_0(\varepsilon)$. Then we can apply Lemma 2.2 with $T_1 = (C_1+1)T_0(\varepsilon)$ and derive the inequality $T_1 \leqslant C_1(T_0 + \varepsilon^{-b(p-1)/\delta})$, so

$$T_0 \leqslant C_1\varepsilon^{-b(p-1)/\delta}$$

and this is in contradiction with the choice

$$T_0 = 2C_1\varepsilon^{-\vartheta}, \qquad \vartheta \geqslant \frac{b(p-1)}{\delta}.$$

The contradiction shows that $V(t)$ must blow up in a finite time $T(\varepsilon) \leqslant (C_1+1)T_0(\varepsilon)$.

To conclude, we recall that $|\varphi_0(s)| \lesssim 1+|s|$ (see (1.29) and (1.30)) and apply the Cauchy–Schwartz inequality, obtaining

$$V(t) \lesssim \int_{|s-s_0|\leqslant t+R} (1+|s|)|v(t,s)|\,ds \lesssim (t+R)^{3/2}||v(t)||_{\mathrm{L}^2(\mathbf{R})}. \tag{3.27}$$

Since V blows up in finite time, the same happens to v and this concludes the proof of Theorem 1.2.

Lemma 3.1. *There exists $\varepsilon_0 > 0$ and a positive constant $C_0 = C_0(R)$ independent of ε_0 such that, for each $\varepsilon \in (0, \varepsilon_0)$, one has*

$$V(0) \geqslant C_0\varepsilon^{1-\vartheta}, \qquad V'(0) \geqslant C_0\varepsilon^{1-\vartheta}. \tag{3.28}$$

Proof. First of all, let us observe that, thanks to our hypotheses on χ_j and the non–negativity of the integrand functions, we have

$$\begin{aligned} V(0) = \int_{\mathbf{R}} \varphi_0(s) v(0,s)\, ds &= \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R} \varphi_0(s)\chi_0(s - s_0(\varepsilon))\, ds \\ &\geqslant \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R/2} \varphi_0(s)\, ds \end{aligned} \tag{3.29}$$

and similarly

$$V'(0) \geqslant \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R/2} \varphi_0(s)\, ds; \tag{3.30}$$

hence it is sufficient to prove that

$$I(\varepsilon) \doteq \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R/2} \varphi_0(s)\, ds \geqslant C_0 \varepsilon^{1-\vartheta} \qquad \forall \varepsilon \in (0, \varepsilon_0) \tag{3.31}$$

for suitable constants $C_0 = C_0(R) > 0$ and $\varepsilon_0 > 0$.

Since $s_0(\varepsilon) \sim \varepsilon^{-\vartheta}$ and $\varphi_0(s) \gtrsim s$ for every $s \geqslant \bar{s}$, where $\bar{s} > 0$ is sufficiently large (independent of ε_0), we can choose $\varepsilon_0 > 0$ small enough, such that

$$s_0(\varepsilon) \geqslant R, \qquad\qquad s_0(\varepsilon) \geqslant 2\bar{s}, \tag{3.32}$$

so that

$$s_0(\varepsilon) - \frac{R}{2} \geqslant \frac{s_0(\varepsilon)}{2} \geqslant \bar{s} \qquad \forall \varepsilon \in (0, \varepsilon_0) \tag{3.33}$$

and consequently

$$I(\varepsilon) \gtrsim \varepsilon \int_{|s-s_0(\varepsilon)|\leqslant R/2} s\, ds = R\varepsilon s_0(\varepsilon) \sim \varepsilon^{1-\vartheta} \qquad \forall \varepsilon \in (0, \varepsilon_0) \tag{3.34}$$

(note that throughout the proof, the implicit constants in "$\gtrsim$" and "$\sim$" are independent of ε and ε_0). □

3.2. **Blow–up with large data close to the black hole and $p \in (2, 1 + \sqrt{2})$.** In this subsection, we verify the hypotheses of Lemma 2.3 to get Theorem 1.3. With this purpose, we set

$$U(t) = \Big(\int_{\mathbf{R}} \psi_0(s) |v(t,s)|^p f(s)\, ds \Big)^{1/p}, \tag{3.35}$$

$$V(t) = \int_{\mathbf{R}} \psi_0(s) v(t,s)\, ds, \tag{3.36}$$

where v is a solution to the Cauchy problem (1.12) with

$$T_2(\varepsilon) \in \Sigma, \qquad s_0(\varepsilon) = -T_2(\varepsilon) \qquad \rho(\varepsilon) = \varepsilon\, \mathrm{e}^{T_2(\varepsilon)/2M}\, . \tag{3.37}$$

The function $\psi_0(s)$ can be represented as $\psi_0(s) = \varphi_0(s) - D$, where D is an appropriate constant (see Lemma 5.3 below) and $\varphi_0(s)$ is the solution to

(1.28), obeying the asymptotic properties of Lemma 5.3. Equation (1.28) implies further the relation

$$\partial_s^2 \psi_0 - W(s)\psi_0 = DW(s). \tag{3.38}$$

We assume further

$$p \in (2, 1+\sqrt{2}); \tag{3.39}$$

as to the other hypotheses, we do not make any change.

We set $T_0(\varepsilon) = T_2(\varepsilon)/2$ and suppose that $T_1 > 0$ is chosen so that $T_0(\varepsilon) + T_1$ is the lifespan of the solution to (1.12); i.e., for any $T < T_1 + T_0(\varepsilon)$ there exists a solution

$$v \in \bigcap_{k=0}^{2} \mathcal{C}^k\big([0,T); \mathrm{H}^{2-k}(\mathbf{R})\big)$$

of (1.12).

Let notice that in this case, (1.21) and (3.37) imply

$$||v_0||_{\mathrm{H}^2(\mathbf{R})} + ||v_1||_{\mathrm{H}^1(\mathbf{R})} \sim \rho(\varepsilon) \sim \varepsilon\, \mathrm{e}^{T_2(\varepsilon)/2M}, \tag{3.40}$$

therefore the initial data in (1.17) have large $\mathrm{H}^2 \times \mathrm{H}^1$ norms.

First of all, we observe that the conditions in (2.17) are trivially satisfied. Moreover, to prove estimate (2.18), we proceed exactly as in the previous subsection, with ψ_0 instead of φ_0. In fact, we have the inequality

$$\begin{aligned} |V(t)| &\leqslant \int_{\mathbf{R}} |v(t,s)|\psi_0(s)\, ds \\ &\leqslant \Big(\int_{\mathbf{R}} f|v|^p \psi_0\, ds\Big)^{1/p} \Big(\int_{|s-s_0|\leqslant t+R} f^{-1/(p-1)}\psi_0\, ds\Big)^{(p-1)/p} \end{aligned}$$

due to the Hölder inequality. Further, from Lemma 5.3 and the asymptotic expansions (1.14)–(1.16), we get

$$f(s)^{-1/(p-1)}\psi_0(s) \lesssim \begin{cases} (1+s)^2 & \text{if } s \geqslant 0, \\ \mathrm{e}^{(p-2)s/2M(p-1)} & \text{if } s < 0. \end{cases} \tag{3.41}$$

Hence, we can use the assumption $p > 2$ and deduce estimate (3.13) with $q = 3(p-1)$.

To derive an estimate similar to (3.19), we use the quantity $F_1(t)$ and the estimate

$$\begin{aligned} F_1(t) &\leqslant \mathrm{e}^{-t/2M} \int_{\mathbf{R}} |v|\varphi_1\, ds \\ &\leqslant U(t)\Big(\mathrm{e}^{-\frac{t}{2M}\frac{p}{p-1}} \int_{|s-s_0|\leqslant t+R} f^{-\frac{1}{p-1}}\psi_0{}^{-\frac{1}{p-1}}\varphi_1{}^{\frac{p}{p-1}}\, ds\Big)^{\frac{p-1}{p}}; \end{aligned} \tag{3.42}$$

from the estimate

$$f^{-1/(p-1)}(s)\psi_0^{-1/(p-1)}(s)\varphi_1{}^{\frac{p}{p-1}}(s) \lesssim \begin{cases} (1+s)^{\frac{p-2}{p-1}}\,\mathrm{e}^{\frac{ps}{2M(p-1)}} & \text{if } s \geqslant 0, \\ \mathrm{e}^{\frac{(p-2)s}{2M(p-1)}} & \text{if } s < 0, \end{cases} \tag{3.43}$$

we get

$$F_1(t) \leqslant CU(t)(t+R)^{\frac{p-2}{p}} \qquad \text{for } t \geqslant 0. \tag{3.44}$$

To finish the proof of (2.18), it is sufficient to take $a = 4-p,\ q = 3(p-1)$ and to mention that, in this case, Lemma 3.2 implies

$$F_1(t) \geqslant C'\rho(\varepsilon)\,\mathrm{e}^{s_0(\varepsilon)/(2M)} = C'\varepsilon.$$

In order to prove estimate (2.19), we multiply equation (1.12) by ψ_0, integrate on $\mathbf{R}$ and then integrate by parts:

$$\begin{aligned} V''(t) &= \int_{\mathbf{R}} v_{tt}(t,s)\psi_0(s)\,ds = \int (v_{ss} - W(s)v)\,\psi_0\,ds + \int f|v|^p\psi_0\,ds \\ &= \int v\left(\psi_0'' - W(s)\psi_0\right)\,ds + \int f|v|^p\psi_0\,ds; \end{aligned} \tag{3.45}$$

using the relation (3.38), we deduce

$$V''(t) = U(t)^p + D\int vW(s)\,ds. \tag{3.46}$$

From Hölder's inequality, we get

$$\begin{aligned} \int vW &\geqslant -\int |v|W = -\int \left(f^{\frac{1}{p}}|v|\psi_0^{\frac{1}{p}}\right)\left(f^{-\frac{1}{p}}W\psi_0^{-\frac{1}{p}}\right) \\ &\geqslant -U(t)\Big(\int (f\psi_0)^{-\frac{1}{p-1}}W^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}. \end{aligned}$$

Using Lemma 5.3 and the assumptions (1.14), (1.15) and (1.16), we obtain

$$(f\psi_0)^{-\frac{1}{p-1}}W^{\frac{p}{p-1}} \in \mathrm{L}^1(\mathbf{R}) \qquad \text{for } p > 2 \tag{3.47}$$

(see also (4.4) below, where this property is verified in details). Thus, we get

$$\int vW \gtrsim -U(t),$$

and (3.46) implies

$$V''(t) \geqslant U(t)^p - CU(t) \tag{3.48}$$

for a suitable positive constant C.

Now, it remains to prove estimate (2.20) in $[0, T_0(\varepsilon))$, where we recall that $T_0(\varepsilon) = T_2(\varepsilon)/2$.

To begin, let us observe that if $t \in [0, T_0(\varepsilon))$, $\operatorname{supp} v \subset \{s < R\}$, since

$$s \leqslant s_0 + T_0(\varepsilon) + R = -T_2(\varepsilon)/2 + R = -T_0(\varepsilon) + R,$$

we can consider only (1.16) and the asymptotic behavior of ψ_0 for $s \to -\infty$, which yields

$$(f\psi_0)^{-\frac{1}{p-1}} W^{\frac{p}{p-1}} \lesssim \mathrm{e}^{\frac{s}{2M}\frac{p-2}{p-1}}. \tag{3.49}$$

Recalling the argument of the proof of (3.48), we obtain

$$\begin{aligned}\int vW &\geqslant -U(t)\Big(\int_{|s-s_0|\leqslant t+R} (f\psi_0)^{-\frac{1}{p-1}} W^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\\ &\gtrsim -U(t)\Big(\int_{|s-s_0|\leqslant t+R} \mathrm{e}^{\frac{s}{2M}\frac{p-2}{p-1}}\, ds\Big)^{\frac{p-1}{p}}\\ &\gtrsim -U(t)\,\mathrm{e}^{\frac{s_0+T_0+R}{2M}\frac{p-2}{p}} \gtrsim -U(t)\,\mathrm{e}^{-C_0T_0},\end{aligned}$$

where we have chosen $C_0 \leqslant \frac{p-2}{2Mp}$; from (3.46), we finally deduce estimate (2.20).

To finish the proof, we suppose that the lifespan $T(\varepsilon)$ of the solution to (1.12) is greater than $(D_1 + 2)T_0(\varepsilon)$, where $D_1 > 0$ is the constant from Lemma 2.3. Then we apply Lemma 2.3 with $T_1 = (D_1 + 1)T_0(\varepsilon)$ and get

$$(D_1 + 1)T_0(\varepsilon) \leqslant D_1 T_0(\varepsilon).$$

This is an obvious contradiction and shows that $T(\varepsilon) \leqslant (D_1+2)T_0(\varepsilon)$, which concludes the proof of Theorem 1.3.

3.3. Estimate of $F_1(t)$.

Lemma 3.2. *There exists a positive constant C' independent of ε such that*

$$F_1(t) \geqslant C'\rho(\varepsilon)\,\mathrm{e}^{s_0(\varepsilon)/(2M)}$$

for all $t \geqslant 0$.

Proof. We multiply equation (1.12) by $\psi(t,s) \doteq \mathrm{e}^{-t/2M}\,\varphi_1(s)$ and integrate over $\mathbf{R}$ in s and over $[0,\tau]$ in t:

$$\int_0^\tau \int_{\mathbf{R}} (v_{tt} - v_{ss} + W(s)v)\,\psi\, ds\, dt = \int_0^\tau \int_{\mathbf{R}} f|v|^p\psi\, ds\, dt.$$

Since $v \in \cap_{k=0}^2 \mathcal{C}^k([0,T[;\mathrm{H}^{2-k}(\mathbf{R}))$, we can apply an integration by parts argument and obtain

$$-\int_0^\tau \int_{\mathbf{R}} v\,(\psi_{tt} - \psi_{ss} + W(s)\psi)\; ds\, dt + \int_0^\tau \int_{\mathbf{R}} f|v|^p\psi\, ds\, dt$$

$$= \int_{\mathbf{R}} (v_t\psi - v\psi_t)\, ds\Big|_{t=\tau} - \int_{\mathbf{R}} (v_t\psi - v\psi_t)\, ds\Big|_{t=0}.$$

The right side of this equality can be rewritten as

$$\begin{aligned}
&\int_{\mathbf{R}} (v_t\psi - v\psi_t)\, ds\Big|_{t=\tau} - \int_{\mathbf{R}} (v_t\psi - v\psi_t)\, ds\Big|_{t=0} \\
&= \int_{\mathbf{R}} (v_t\psi + v\psi_t)\, ds\Big|_{t=\tau} - 2\int_{\mathbf{R}} v\psi_t\, ds\Big|_{t=\tau} - \int_{\mathbf{R}} \mathrm{e}^{-t/2M} \Big(v_t + \frac{v}{2M}\Big)\, \varphi_1\, ds\Big|_{t=0} \\
&\qquad = \frac{d}{d\tau}\int_{\mathbf{R}} v\psi\, ds + \frac{1}{M}\int_{\mathbf{R}} v\psi\, ds - \int_{\mathbf{R}} \Big(\frac{v_0}{2M} + v_1\Big)\, \varphi_1\, ds
\end{aligned}$$

due to the property $\psi_t = -\psi/2M$. The relation

$$\psi_{tt} - \psi_{ss} + W(s)\psi = \mathrm{e}^{-t/2M} \Big(-\partial_{ss}^2 + W(s) + \frac{1}{4M^2}\Big)\varphi_1 = 0$$

implies

$$-\int_0^\tau \int_{\mathbf{R}} v\, (\psi_{tt} - \psi_{ss} + W(s)\psi)\ ds\, dt = 0;$$

so, since

$$F_1(t) = \mathrm{e}^{-t/2M} \int_{\mathbf{R}} v(t,s)\varphi_1(s)\, ds = \int_{\mathbf{R}} v(t,s)\psi(s)\, ds,$$

we arrive at

$$F_1'(\tau) + \frac{1}{M}F_1(\tau) = \int_0^\tau \int_{\mathbf{R}} f|v|^p\psi\, ds\, dt + \int_{\mathbf{R}} \Big(\frac{v_0}{2M} + v_1\Big)\, \varphi_1\, ds.$$

The right side of this identity is greater than $\rho(\varepsilon)\,\mathrm{e}^{s_0(\varepsilon)/(2M)}$ multiplied by a positive constant, since

$$\int v_j(s)\varphi_1(s)\, ds = \rho(\varepsilon) \int_{|s-s_0(\varepsilon)|\leqslant R} \chi_j\big(s - s_0(\varepsilon)\big)\varphi_1(s)\, ds, \tag{3.50}$$

with $j = 0, 1$. For $\varepsilon > 0$ small enough, we are in position to apply the asymptotic expansion derived in Lemma 5.4 and we find

$$\int v_j(s)\varphi_1(s)\, ds \geqslant C\rho(\varepsilon)\,\mathrm{e}^{s_0(\varepsilon)/(2M)} \int_{\mathbf{R}} \chi_j\big(s - s_0(\varepsilon)\big)\, ds \geqslant C'\rho(\varepsilon)\,\mathrm{e}^{s_0(\varepsilon)/(2M)}, \tag{3.51}$$

so we get

$$F_1'(\tau) + \frac{1}{M}F_1(\tau) \geqslant C'\rho(\varepsilon)\,\mathrm{e}^{s_0(\varepsilon)/(2M)}.$$

Now, we multiply both sides by $\mathrm{e}^{\tau/M}$ obtaining

$$\frac{d}{d\tau}\Big(\mathrm{e}^{\tau/M}\, F_1(\tau)\Big) \geqslant C'\,\mathrm{e}^{\tau/M}\,\rho(\varepsilon)\,\mathrm{e}^{s_0(\varepsilon)/(2M)}$$

and integrating on $\tau \in [0,t]$ we deduce

$$\begin{aligned} e^{t/M} F_1(t) &\gtrsim F_1(0) + M(e^{t/M} -1)\rho(\varepsilon)\, e^{s_0(\varepsilon)/(2M)} \\ &= \int_{\mathbf{R}} v_0\varphi_1\, ds + M(e^{t/M} -1)\rho(\varepsilon)\, e^{s_0(\varepsilon)/(2M)} \\ &\gtrsim [1 + M(e^{t/M} -1)]\rho(\varepsilon)\, e^{s_0(\varepsilon)/(2M)} \gtrsim e^{t/M}\rho(\varepsilon)\, e^{s_0(\varepsilon)/(2M)}, \end{aligned} \tag{3.52}$$

namely

$$F_1(t) \geqslant C'\rho(\varepsilon)\, e^{s_0(\varepsilon)/(2M)};$$

that is the claim. □

4. Appendix I: Asymptotic estimates and the local existence theorem

The Regge–Wheeler coordinate

$$s(r) = r + 2M\log(r-2M) \tag{4.1}$$

satisfies the relation

$$\frac{r}{r-2M}\, dr = ds.$$

If $r(s)$ denotes the function inverse to (4.1), then one can find positive constants C_1, C_2 so that we have the following asymptotic behaviors:

$$\begin{cases} C_1 s \leqslant r(s) \leqslant C_2 s & \text{if } \ s \geqslant 2, \\ C_1 \leqslant r(s) \leqslant C_2 & \text{if } \ |s| \leqslant 2, \\ C_1\, e^{s/2M} \leqslant r(s) - 2M \leqslant C_2\, e^{s/2M} & \text{if } \ s \leqslant -2; \end{cases} \tag{4.2}$$

further, the coefficient $F(r(s))$ defined in (1.2) satisfies

$$\begin{cases} |F(s) - 1| \leqslant C_2/s & \text{if } \ s \geqslant 2, \\ C_1 \leqslant F(s) \leqslant C_2 & \text{if } \ |s| \leqslant 2, \\ C_1\, e^{s/2M} \leqslant F(s) \leqslant C_2\, e^{s/2M} & \text{if } \ s \leqslant -2. \end{cases} \tag{4.3}$$

Moreover, we can use the definitions (1.13) of the potential $W(s)$ and of the coefficient $f(s)$ to conclude that (1.14), (1.15) and (1.16) are satisfied.

We need also the property

$$\Psi(s) = (f\psi_0)^{-\frac{1}{p-1}} W^{\frac{p}{p-1}} \in L^1(\mathbf{R}) \qquad \text{for } p > 2. \tag{4.4}$$

Indeed, Lemma 5.3 implies that

$$\psi_0(s) = \varphi_0(s) - D \sim \begin{cases} s & \text{as } s \to +\infty, \\ e^{s/2M} & \text{as } s \to -\infty, \end{cases} \tag{4.5}$$

hence we get

$$\Psi(s) \sim \begin{cases} s^{-2(p+1)/(p-1)} & \text{as } s \to +\infty, \\ \mathrm{e}^{\frac{s}{2M}\frac{p-2}{p-1}} & \text{as } s \to -\infty, \end{cases} \tag{4.6}$$

and we conclude that (4.4) is verified.

Now we can show the local existence result given in Theorem 1.1.

Proof of Theorem 1.1. It is not difficult to see that

$$G = -\partial_s^2 + W(s)$$

is a non–negative symmetric operator in the Hilbert space $\mathrm{L}^2(\mathbf{R}, ds)$ with dense domain $\mathrm{H}^2(\mathbf{R})$. Thus the estimates (1.14), (1.15) and (1.16), together with the KLMN–theorem (see Theorem 10.17 in [15]), imply that G is a non–negative self–adjoint operator.

Let consider the Cauchy problem

$$\begin{cases} \partial_t^2 v + Gv = \Phi, \\ v(0,s) = v_0(s), \quad \partial_t v(0,s) = v_1(s); \end{cases} \tag{4.7}$$

then the solution can be represented in the form

$$v(t) = \cos(t\sqrt{G})v_0 + \frac{\sin(t\sqrt{G})}{\sqrt{G}}v_1 + \int_0^t \frac{\sin((t-\tau)\sqrt{G})}{\sqrt{G}}\Phi(\tau)d\tau.$$

From this representation, we find

$$\|v(t)\|_{\mathrm{L}^2(\mathbf{R},ds)} \leqslant \|v_0\|_{\mathrm{L}^2(\mathbf{R},ds)} + t\|v_1\|_{\mathrm{L}^2(\mathbf{R},ds)} + \int_0^t |t-\tau| \|\Phi(\tau)\|_{\mathrm{L}^2(\mathbf{R},ds)}\, d\tau \tag{4.8}$$

and for any $\sigma \geqslant 1$ we have

$$\begin{aligned} \|G^{\sigma/2}v(t)\|_{\mathrm{L}^2(\mathbf{R},ds)} \leqslant \|G^{\sigma/2}v_0\|_{\mathrm{L}^2(\mathbf{R},ds)} + \|G^{(\sigma-1)/2}v_1\|_{\mathrm{L}^2(\mathbf{R},ds)} \\ + \int_0^t \|G^{(\sigma-1)/2}\Phi(\tau)\|_{\mathrm{L}^2(\mathbf{R},ds)}\, d\tau; \end{aligned} \tag{4.9}$$

thus, using the equivalence

$$\|G^{\sigma/2}h\|_{\mathrm{L}^2(\mathbf{R},ds)} + \|h\|_{\mathrm{L}^2(\mathbf{R},ds)} \sim \|h\|_{\mathrm{H}^\sigma(\mathbf{R})},$$

we arrive at the energy estimate

$$\begin{aligned} \|v(t)\|_{\mathrm{H}^\sigma(\mathbf{R})} \leqslant C\|v_0\|_{\mathrm{H}^\sigma(\mathbf{R})} + C(1+t)\|v_1\|_{\mathrm{H}^{\sigma-1}(\mathbf{R})} \\ + C\int_0^t (1+|t-\tau|)\|\Phi(\tau)\|_{\mathrm{H}^{\sigma-1}(\mathbf{R})}\, d\tau \end{aligned} \tag{4.10}$$

for any $\sigma \geqslant 1$ and a suitable constant $C > 0$. Now we use the fact that in our case $\Phi = |v|^p$ and we can apply the inequality

$$\| \, |v|^p \, \|_{\mathrm{H}^{\sigma-1}(\mathbf{R})} \leqslant c \|v\|_{\mathrm{H}^{\sigma-1}(\mathbf{R})} \, \|v\|^{p-1}_{L^\infty(\mathbf{R})}$$

for $\sigma - 1 < p$ and a positive constant c (see Theorem 1 of Section 5.4.3, page 363 of [16]) to get

$$\begin{aligned} \|v(t)\|_{\mathrm{H}^\sigma(\mathbf{R})} \leqslant C\|v_0\|_{\mathrm{H}^\sigma(\mathbf{R})} + C(1+t)\|v_1\|_{\mathrm{H}^{\sigma-1}(\mathbf{R})} \\ + cC \int_0^t (1 + |t-\tau|) \|v\|_{\mathrm{H}^{\sigma-1}(\mathbf{R})} \, \|v\|^{p-1}_{L^\infty(\mathbf{R})} \, d\tau. \end{aligned} \tag{4.11}$$

Since $\sigma \geqslant 1$, we have

$$||v||_{\mathrm{H}^{\sigma-1}} \leqslant C||v||_{\mathrm{H}^\sigma}, \qquad ||v||_{\mathrm{L}^\infty} \leqslant C||v||_{\mathrm{H}^\sigma} \tag{4.12}$$

for a suitable constant $C > 0$, hence (4.11) becomes

$$\begin{aligned} \|v(t)\|_{\mathrm{H}^\sigma(\mathbf{R})} \leqslant C\|v_0\|_{\mathrm{H}^\sigma(\mathbf{R})} + C(1+t)\|v_1\|_{\mathrm{H}^{\sigma-1}(\mathbf{R})} \\ + C \int_0^t (1 + |t-\tau|) \|v\|^p_{\mathrm{H}^\sigma(\mathbf{R})} \, d\tau \end{aligned} \tag{4.13}$$

(with a new positive constant C).

To conclude, it is sufficient to combine this energy estimate with the Sobolev embedding $\mathrm{H}^\sigma(\mathbf{R}) \subset \mathrm{L}^\infty(\mathbf{R})$, which holds for $\sigma > 1/2$, obtaining easily the desired local existence result for $\sigma \in [1, p+1)$ (i.e., Theorem 1.1). □

5. Appendix II: Estimates for some associated elliptic linear problems

Our first step in this section is to consider the problem

$$\begin{cases} -\varphi''(s) + W(s)\varphi(s) = 0 & s \in \mathbf{R}, \\ |\varphi(s) - bs| \lesssim \log(2+s) & \text{for } s \geqslant 0, \\ 0 < \varphi(s) \lesssim 1 & \text{for } s < 0, \end{cases} \tag{5.1}$$

where the potential $W(s)$ is assumed to satisfy

$$0 < W(s) \lesssim (1 + |s|)^{-a} \tag{5.2}$$

for some $a \geqslant 3$. Note that condition (5.2) is weaker than the assumptions (1.14), (1.15) and (1.16).

Our first result is the following.

Lemma 5.1. *There exists a real number $b > 0$ such that problem* (5.1) *has a non–negative solution $\varphi_0 \in \mathcal{C}^2(\mathbf{R})$ so that the limit*

$$D = \lim_{s \to -\infty} \varphi_0(s)$$

exists, $D \geqslant 0$ and the following relation

$$0 \leqslant \varphi_0(s) - D \lesssim |s|^{2-a} \qquad \textit{for } s \to -\infty \tag{5.3}$$

holds.

Proof. Consider the Cauchy problem

$$\begin{cases} -y''(s) + W(s)y(s) = 0 & s \in \mathbf{R}, \\ y(0) = 1, \quad y'(0) = 0. \end{cases} \tag{5.4}$$

This Cauchy problem has a unique solution $y(s) \in \mathcal{C}^2(\mathbf{R})$. A qualitative analysis of the equation and the assumption $W(s) > 0$ show that the solution satisfies

$$\begin{aligned} y'(s) &> 0 \qquad \text{for } s > 0, \\ y'(s) &< 0 \qquad \text{for } s < 0, \end{aligned}$$

and hence $y(s) \geqslant 1$ for all real s.

One can rewrite problem (5.4) in the form of the following integral equation

$$y(s) = 1 + I(y)(s), \tag{5.5}$$

where

$$I(y)(s) = \int_0^s \int_0^\sigma W(\vartheta)y(\vartheta)\, d\vartheta\, d\sigma = \int_0^s (s - \vartheta)W(\vartheta)y(\vartheta)\, d\vartheta. \tag{5.6}$$

We shall show that

$$|y(s) - d_+ s| \lesssim \log(2 + s) \qquad \text{for } s \geqslant 0, \tag{5.7}$$

$$|y(s) - d_- s| \lesssim \log(2 + |s|) \qquad \text{for } s < 0, \tag{5.8}$$

where

$$d_\pm = \int_0^{\pm\infty} W(\vartheta)y(\vartheta)\, d\vartheta.$$

Assumption (5.2) shows that the integral operator $I(y)(s)$ is well defined (in particular $d_\pm \in \mathbf{R}$) and satisfies the estimate

$$0 \leqslant I(y)(s) \leqslant s \int_0^s W(\vartheta)y(\vartheta)\, d\vartheta; \tag{5.9}$$

hence (5.5) implies the inequality

$$y(s) \leqslant 1 + s \int_0^s W(\vartheta)y(\vartheta)\, d\vartheta. \tag{5.10}$$

Now, let us consider the case $s \geqslant 0$. The previous inequality yields

$$y(s) \leqslant 1 + s \int_0^\infty W(\vartheta) y(\vartheta)\, d\vartheta = 1 + d_+ s. \tag{5.11}$$

On the other hand, combining (5.5) and (5.11), we get

$$\begin{aligned} y(s) - 1 - d_+ s &= s \int_0^\infty W(\vartheta) y(\vartheta)\, d\vartheta - s \int_s^\infty W(\vartheta) y(\vartheta)\, d\vartheta \\ &\quad - \int_0^s \vartheta W(\vartheta) y(\vartheta)\, d\vartheta - d_+ s \\ &= -s \int_s^\infty W(\vartheta) y(\vartheta)\, d\vartheta - \int_0^s \vartheta W(\vartheta) y(\vartheta)\, d\vartheta \\ &\gtrsim -s \int_s^\infty \frac{d\vartheta}{(1+\vartheta)^{a-1}} - \int_0^s \frac{d\vartheta}{(1+\vartheta)^{a-2}}. \end{aligned}$$

But, for each $a \geqslant 3$, we have also

$$\int_s^\infty \frac{d\vartheta}{(1+\vartheta)^{a-1}} \lesssim (1+s)^{2-a}, \qquad \int_0^s \frac{d\vartheta}{(1+\vartheta)^{a-2}} \lesssim \log(2+s),$$

and thus we deduce

$$y(s) - 1 - d_+ s \gtrsim -\log(2+s).$$

From this equation and (5.11), we finally obtain the precise asymptotic estimate (5.7). Analogously, we get the parallel result for $s < 0$. It is important to note that $d_+ > 0$ and

$$d_- = -\int_{-\infty}^0 W(\vartheta) y(\vartheta)\, d\vartheta < 0.$$

In a similar way, we can consider the Cauchy problem

$$\begin{cases} -z''(s) + W(s) z(s) = 0 & s \in \mathbf{R}, \\ z(0) = 0, \quad z'(0) = 1. \end{cases} \tag{5.12}$$

Obviously, this Cauchy problem has a unique solution $z(s) \in \mathcal{C}^2(\mathbf{R})$. The assumption $W(s) > 0$ guarantees that the solution satisfies

$$z'(s) > 0 \qquad \forall s \in \mathbf{R},$$

so

$$z(s) > 0 \qquad \text{for } s > 0$$

and

$$z(s) < 0 \qquad \text{for } s < 0.$$

Equation (5.5) has to be replaced by

$$z(s) = s + I(z)(s) \tag{5.13}$$

and the argument given in the proof of estimates (5.7) and (5.8) leads to

$$|z(s) - e_+ s| \lesssim \log(2+s) \qquad \text{for } s \geqslant 0, \tag{5.14}$$

$$|z(s) - e_- s| \lesssim \log(2+|s|) \qquad \text{for } s < 0, \tag{5.15}$$

where

$$e_\pm = 1 + \int_0^{\pm\infty} W(\vartheta) z(\vartheta)\, d\vartheta \in \mathbf{R}.$$

Note that $e_+ > 0$ and

$$e_- = 1 - \int_{-\infty}^0 W(\vartheta) z(\vartheta)\, d\vartheta > 0.$$

Setting

$$\varphi(s) = e_- y(s) - d_- z(s), \qquad b = e_- d_+ - d_- e_+ > 0,$$

we take advantage of (5.8) and (5.15), and conclude that

$$|\varphi(s)| \lesssim \log(2+|s|) \qquad \text{for } s < 0, \tag{5.16}$$

while from (5.7) and (5.14), we deduce

$$|\varphi(s) - bs| \lesssim \log(2+s) \qquad \text{for } s > 0. \tag{5.17}$$

To improve estimate (5.16), we note that $\varphi(s)$ satisfies the integral equation

$$\varphi(s) = \varphi(0) + \varphi'(0)s + I(\varphi)(s). \tag{5.18}$$

As before, we have (for any $s < 0$)

$$\begin{aligned}\varphi(s) - \varphi(0) - \varphi'(0)s = -s \int_{-\infty}^0 W(\vartheta)\varphi(\vartheta)\, d\vartheta + s \int_{-\infty}^s W(\vartheta)\varphi(\vartheta)\, d\vartheta \\ + \int_{-\infty}^0 \vartheta W(\vartheta)\varphi(\vartheta)\, d\vartheta - \int_{-\infty}^s \vartheta W(\vartheta)\varphi(\vartheta)\, d\vartheta\end{aligned} \tag{5.19}$$

and then a combination of (5.16) and assumption (5.2) implies

$$s \int_{-\infty}^s W(\vartheta)|\varphi(\vartheta)|\, d\vartheta \lesssim (1+|s|)^{2-a} \log(2+|s|) \lesssim 1, \tag{5.20}$$

$$\int_{-\infty}^s \vartheta W(\vartheta)|\varphi(\vartheta)|\, d\vartheta \lesssim (1+|s|)^{2-a} \log(2+|s|) \lesssim 1, \tag{5.21}$$

so (5.19) yields

$$\left| \varphi(s) - \left(\varphi'(0) - \int_{-\infty}^0 W(\vartheta)\varphi(\vartheta)\, d\vartheta \right) s \right| \lesssim 1.$$

Comparing this estimate with (5.16), we see that

$$\varphi'(0) - \int_{-\infty}^{0} W(\vartheta)\varphi(\vartheta)\, d\vartheta = 0;$$

this implies

$$|\varphi(s)| \lesssim 1 \qquad \text{for } s < 0. \tag{5.22}$$

We set

$$D = \varphi(0) + \int_{-\infty}^{0} \vartheta W(\vartheta)\varphi(\vartheta)\, d\vartheta \tag{5.23}$$

and observe that the function $\varphi(s)$ is positive near $s = 0$. Moreover, for $s > 0$, $\varphi(s)$ increases and is positive. It is easy to show that $\varphi(s) \geqslant 0$ for all $s < 0$. Indeed, if $\varphi(s_0) < 0$ for some $s_0 < 0$, then $\varphi(s_1) < 0$, $\varphi'(s_1) > 0$ for some $s_1 < 0$, thus the equation

$$\varphi''(s) = W(s)\varphi(s)$$

would imply that

$$\varphi(s) < 0, \qquad \varphi'(s) > 0, \qquad \varphi''(s) < 0 \qquad \text{for } s < s_1.$$

This contradicts (5.22) and shows that $\varphi(s) \geqslant 0$ for all $s < 0$. Hence D is non–negative, because $\varphi(s) \geqslant 0$ for all $s \in \mathbf{R}$ and

$$\lim_{s \to -\infty} (\varphi(s) - D) = 0.$$

We have indeed $\varphi \geqslant D$, since

$$\varphi(s) - D = \int_{-\infty}^{s} (s - \vartheta) W(\vartheta)\varphi(\vartheta)\, d\vartheta \geqslant 0; \tag{5.24}$$

so $\varphi(s) \geqslant D \geqslant 0$.

From these relations and (5.24), we also get

$$\varphi(s) - D \lesssim \int_{-\infty}^{s} \frac{s - \vartheta}{(1 + |\vartheta|)^a}\, d\vartheta \sim |s|^{2-a} \qquad \text{for } s \to -\infty \tag{5.25}$$

and this proves (5.3). □

If we make a stronger assumption on the potential for negative values of s, then we can prove the positiveness of D.

Lemma 5.2. *If $W(s)$ satisfies estimate* (5.2) *and*

$$0 < W(s) \lesssim (1 + |s|)^{-4} \qquad \textit{for } s < 0, \tag{5.26}$$

then there exists a real number $b > 0$ such that problem (5.1) *has a positive solution $\varphi_0 \in \mathcal{C}^2(\mathbf{R})$ so that the limit*

$$D = \lim_{s \to -\infty} \varphi_0(s)$$

is strictly positive.

Proof. The previous lemma guarantees that $D \geqslant 0$. Let us suppose that $D = 0$. Using (5.25) together with the assumption $a \geqslant 3$, we can get the better decay estimate

$$|\varphi(s)| + |\varphi'(s)| \lesssim |s|^{-N} \qquad \text{for } s \to -\infty \tag{5.27}$$

for any integer $N \geqslant 1$. Turning back to the equation satisfied by φ, we make the change $s \to \sigma = 1/s$, $\varphi \to \psi = s^{-1}\varphi$, and see that ψ satisfies the equation

$$\psi''(\sigma) = \sigma^{-4} W \psi;$$

moreover, estimate (5.27) guarantees that $\psi(0) = \psi'(0) = 0$. Therefore, our assumption (5.26) that $s^4 W(s)$ is bounded as $s \to -\infty$ implies that $\sigma^{-4} W$ is bounded, so the classical Cauchy uniqueness theorem implies that ψ is identically 0 and this contradiction implies $D > 0$. □

Lemma 5.3. *If W satisfies estimates* (1.14), (1.15) *and* (1.16)*, then there exists a positive function $\varphi_0 \in \mathcal{C}^2(\mathbf{R})$ such that $(-\partial_{ss}^2 + W(s))\varphi_0 = 0$ in $\mathbf{R}$ and for some positive constants b and D we have*

$$\begin{cases} |\varphi_0(s) - bs| \lesssim \log(2 + |s|) & \textit{for } s \geqslant 0, \\ \varphi_0(s) - D \sim \mathrm{e}^{s/2M} & \textit{for } s \to -\infty. \end{cases}$$

Proof. Let φ satisfy

$$\varphi''(s) - W(s)\varphi(s) = 0.$$

Using the asymptotic estimates (1.14), (1.15) and (1.16), we find for $s < 0$

$$\begin{aligned} \varphi(s) - D &= \int_{-\infty}^{s} (s - \vartheta) W(\vartheta) \varphi(\vartheta)\, d\vartheta \\ &\sim \int_{-\infty}^{s} (s - \vartheta)\, \mathrm{e}^{\vartheta/2M}\, d\vartheta = 4M^2\, \mathrm{e}^{s/(2M)} \end{aligned} \tag{5.28}$$

as $s \to -\infty$, and this leads to $\varphi_0(s) - D \sim \mathrm{e}^{s/2M}$ for $s \to -\infty$. The rest of the claim follows directly from the assertion of the previous lemma. □

Now we state a corollary of the Levinson theorem (see [4], page 49, Chapter 2 Section 5.4), which we are going to apply to get the estimate for φ_1.

Proposition 5.1. *Consider the equation*

$$y^{(n)} + \sum_{k=1}^{n} \alpha_k(s) y^{(n-k)} = 0, \qquad s \in \mathbf{R}^+, \tag{5.29}$$

where $\alpha_k(s) \in \mathcal{C}^\infty(\mathbf{R}^+)$ *are complex–valued functions such that*

$$\alpha_k(s) = \beta_k + \gamma_k(s), \qquad \int_{\mathbf{R}^+} |\gamma_k(s)|\, ds < \infty,$$

and let $q_1, q_2, \ldots, q_n$ *be the distinct roots of the equation*

$$q^n + \sum_{k=1}^{n} \beta_k q^{n-k} = 0.$$

Then equation (5.29) *has* n *linearly independent solutions* $y_j(s)$, $j = 1, 2, \ldots, n$, *having the asymptotic expansion*

$$y_j^{(k-1)}(s) = q_j^{k-1}\, \mathrm{e}^{q_j s}[1 + o(1)] \qquad \text{as } s \to \infty,$$

where $j, k = 1, 2, \ldots, n$.

Lemma 5.4. *Given any* $A > 0$, *the equation*

$$(-\partial_{ss}^2 + W(s) + A^2)\varphi(s) = 0, \qquad s \in \mathbf{R} \tag{5.30}$$

admits a positive solution $\varphi_1 \in \mathcal{C}^2(\mathbf{R})$ *such that* $\varphi_1(s) \sim \mathrm{e}^{As}$ *as* $|s|$ *approaches* ∞.

Proof. Proposition 5.1 guarantees that there exists a solution φ_1 of (5.30) such that $\varphi_1(s) \sim \mathrm{e}^{As}$ as $s \to -\infty$. From

$$\varphi_1'' = \left(W(s) + A^2\right) \varphi_1, \qquad W(s) + A^2 > 0$$

and a qualitative study, we get

$$\varphi_1''(s) > 0 \qquad \text{and thus } \varphi_1(s) > 0, \quad \varphi_1'(s) > 0 \tag{5.31}$$

for each $s \in \mathbf{R}$. Now, from Proposition 5.1 for $s \to +\infty$, we deduce $\varphi_1(s) \sim \lambda\, \mathrm{e}^{As} + \mu\, \mathrm{e}^{-As}$ for suitable λ, $\mu \in \mathbf{R}$ and $s \to +\infty$. Property (5.31) guarantees that $\lambda > 0$, so necessarily $\varphi_1(s) \sim \mathrm{e}^{As}$ for $s \to +\infty$ and the proof is finished. □

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