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ON THE STABILIZATION OF DYNAMIC ELASTICITY EQUATIONS WITH UNBOUNDED LOCALLY DISTRIBUTED DISSIPATION

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Abstract. We consider the dynamic elasticity equations with a locally distributed damping in a bounded domain. The local dissipation of the form $a(x)y_t$ allows coefficients a that lie in some $L^r(\Omega)$, with $(r > 2)$. Using multiplier techniques, interpolation inequalities, and a judicious application of the Hölder inequality, we prove sharp energy decay estimates for all $r > Max(2, N)$, where N denotes the space dimension. All space dimensions are considered; the results obtained generalize and improve earlier works where r is required to satisfy $r \geq \frac{3N+\sqrt{9N^2-16N}}{4}$, for $N \geq 3$.

1. Introduction and statements of main results

Unlike the stabilization of the wave equation with locally distributed damping for which the literature is well documented beginning with the pioneering work of Dafermos [5], then the papers of Bardos-Lebeau-Rauch [3], Haraux [8], Slemrod [25], Zuazua [32], and many other papers [4, 9, 21, 22, 24, 27, 28, 29, 30, 33], the stabilization of locally damped elastodynamic systems is poorly covered; there are only a few papers including [2, 6, 22] in the literature. However it should be noted that the situation is completely different for the boundary stabilization of elastodynamic systems; beginning with the work of Lagnese [16], numerous other papers discuss this topic [17, 13, 1, 26, 10, 18, 23] just to name a few. In this project we are interested in generalizing and improving the results in [2, 24, 30]. For the sequel we need some notation. Let Ω be a bounded nonempty subset of $\mathbb{R}^N$ with boundary Γ of class C^2. Let $x^0 \in \mathbb{R}^N$, and set $m(x) = x - x^0$, $R = \sup\{|m(x)|; x \in \Omega\}$, and $\Gamma_+ = \{x \in \Gamma; m(x) \cdot \nu(x) \geq 0\}$, where ν denotes the unit normal vector pointing into the exterior of Ω. Throughout

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the paper subscripts following a comma stand for differentiation, and we use the Einstein summation convention on repeated indices.

Consider the damped elasticity system

$$\begin{cases} y_{i,tt} - \sigma_{ij,j} + a y_{i,t} = 0 \text{ in } \Omega \times (0,\infty) \\ y_i = 0 \text{ on } \Gamma \times (0,\infty) \\ y_i(0) =, \quad y_{i,t}(0) = y_i^1, \quad i = 1,\ 2,\ ...,\ N, \end{cases} \tag{1.1}$$

where $a : \Omega \to \mathbb{R}$ is a nonnegative function satisfying

$$a \in L^r(\Omega),\ 2 < r < \infty,\ \text{and } \exists\ p > 0 : \int_\omega \frac{dx}{(a(x))^p} < \infty. \tag{1.2}$$

In (1.1) the elasticity stress tensor (σ_{ij}) is given by

$$\sigma_{ij} = \sigma_{ij}(y) = a_{ijkl}\varepsilon_{kl},$$

where (ε_{kl}) defined by

$$\varepsilon_{kl} = \varepsilon_{kl}(y) = \frac{1}{2}(y_{k,l} + y_{l,k})$$

is the strain tensor. The a_{ijkl} are the elasticity coefficients. They satisfy the symmetry properties

$$a_{ijkl} = a_{jilk} = a_{klij}, \quad \forall i,j,k,l.$$

Throughout the paper we assume that the a_{ijkl} depend on the space variable x but not on time, and that they are continuously differentiable, and satisfy the ellipticity condition

$$\exists a_0 > 0 : a_{ijkl}u_{ij}u_{kl} \geq a_0 u_{ij}u_{kl} \tag{1.3}$$

for all second-order symmetric tensors (u_{ij}).

Under the above assumptions on the coefficients, when r is large enough, and for all i, $(y_i^0, y_i^1) \in H_0^1(\Omega) \times L^2(\Omega)$, it is well known that System (1.1) has a unique weak solution

$$y \in \mathcal{C}([0,\infty); [H_0^1(\Omega)]^N) \cap \mathcal{C}^1([0,\infty); [L^2(\Omega)]^N). \tag{1.4}$$

Similarly, if for all i, $(y_i^0, y_i^1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, then it can be shown that the unique solution of System (1.1) satisfies

$$y \in \mathcal{C}([0,\infty); [H^2(\Omega)]^N \cap [H_0^1(\Omega)]^N) \cap \mathcal{C}^1([0,\infty); [H_0^1(\Omega)]^N). \tag{1.5}$$

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{|y_t(x,t)|^2 + (\sigma_{ij}\varepsilon_{ij})(x,t)\}\,dx, \quad \forall t \geq 0. \tag{1.6}$$

The energy E is a nonincreasing function of the time variable t and its derivative satisfies

$$E'(t) = -\int_{\Omega} a|y_t|^2\,dx, \quad \forall t \geq 0. \tag{1.7}$$

Remark 1.1. It should be noted that when (1.2) holds the dissipation law (1.7) is valid only for strong solutions of (1.1).

Before stating our main result we need some additional notation. In the sequel, c denotes various positive constants independent of the initial data, $|u|_q$ denotes the $L^q(\Omega)$ norm of u when $q \geq 1$, and $|(1/a)|_p$ stands for $\left(\int_\omega \frac{dx}{a^p}\right)^{\frac{1}{p}}$. We also set

$$\begin{aligned} r_N &= Max(2, N),\\ \mu_2 &= Max\left\{\frac{1}{r-2-\delta}, \frac{1+\delta p}{p(1-\delta)}\right\}, \quad 0 < \delta < Min(1, r-2) \\ \mu_N &= Max\left\{\frac{N}{2(r-r_N)}, \frac{N}{2p}\right\}, \quad \text{for } N \neq 2, \end{aligned} \tag{1.8}$$

and

$$F_0 = (\|y^1\|^2_{[H^1_0(\Omega)]^N} + \|y^0\|^2_{[H^2(\Omega)]^N})^{\frac{1}{2}}. \tag{1.9}$$

One can show that for some constant C

$$\|y_t(.,t)\|^2_{[H^1_0(\Omega)]^N} + \|y(.,t)\|^2_{[H^2(\Omega)]^N} \leq CF_0^2, \quad \forall t \geq 0. \tag{1.10}$$

Theorem 1.2. *Let $\{y^0, y^1\} \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$. Assume that $a \in L^r(\Omega)$ with*

$$r > Max(2, N), \tag{1.11}$$

and a satisfies (1.2). *Also assume that the elasticity coefficients a_{ijkl} satisfy, for all symmetric tensors ξ_{ij},*

$$\big(3a_{ijkl} - 2m_n a_{ijkl,n}\big)\xi_{ij}\xi_{kl} \geq 0. \tag{1.12}$$

Then for $N \neq 2$, we have the decay estimate

$$E(t) \leq K_0(1+t)^{-\frac{1}{\mu_N}}, \quad \forall t \geq 0 \tag{1.13}$$

with

$$K_0 = c\Big(E(0)^{\mu_N} + |\frac{1}{a}|_p F_0^{\frac{N}{p}} E(0)^{\frac{2p\mu_N - N}{2p}} + |a|_r^{\frac{r}{r-r_N}} F_0^{\frac{N}{r-r_N}} E(0)^{\frac{2(r-r_N)\mu_N - N}{2(r-r_N)}}\Big)^{\frac{1}{\mu_N}}. \tag{1.14}$$

When $N = 2$, we have the decay estimate

$$E(t) \le K_1(1+t)^{-\frac{1}{\mu_2}}, \quad \forall t \ge 0 \tag{1.15}$$

with

$$K_1 = c_\delta\Big(E(0)^{\mu_2} + |\frac{1}{a}|_p F_0^{\frac{2(1+\delta p)}{p(1-\delta)}} E(0)^{\frac{\mu_2 p(1-\delta)-(1+\delta p)}{p(1-\delta)}} + |a|_r^{\frac{r-\delta}{r-2-\delta}} F_0^{\frac{2}{r-2-\delta}} E(0)^{\frac{\mu_2(r-2-\delta)-1}{r-2-\delta}}\Big)^{\frac{1}{\mu_2}}. \tag{1.16}$$

Remark 1.3. Hypothesis (1.11) is almost optimal since, for System (1.1) to be well posed, we need $r \ge 2$ when $N = 1$, and $r \ge N$ if $N \ge 3$, and $r > 2$ for $N = 2$. Hypothesis (1.12) is required in general when we are dealing with variable coefficients (e.g. [33]); however it should be noted that the results of [19, 31] make that hypothesis unnecessary in the case of the wave equation when the coefficients are smooth enough; in particular hypothesis (1.12) is not needed in the proof of Theorem 1.12 when $N = 1$. To the best of our knowledge, getting rid of (1.12) for elastodynamic systems remains an open problem.

Remark 1.4. When $N = 3$, and

$$a_{ijkl} = 2c^2\delta_{ik}\delta_{jl} + (d^2 - 2c^2)\delta_{ij}\delta_{kl}, \tag{1.17}$$

where d and c are positive constants related to Lamé constants with $d \ge c$, and δ_{ij} denotes the Kronecker symbol, our system reduces to the one considered in [2]. More precisely, the authors of [2] show that the energy satisfies a decay estimate of the type we have above but for

$$p \ge 1, \quad r \ge \frac{9+\sqrt{33}}{4}, \quad \mu_3 = Max(3/p, (9r-6)/2r(r-2)). \tag{1.18}$$

Requiring that p be large enough in (1.2) -as they do- means that we limit the degeneracy of the localizing coefficient; we have managed to get rid of this restriction in Theorem 1.2 above. On the other hand the value of r is also greater in [2] than in Theorem 1.2. Even with those restrictions their energy decay is slower than ours as their μ_3 given in (1.18) is greater than ours; so our result not only generalizes the one provided in [2], it also strongly improves that result by removing all the restrictions imposed by its authors, and getting better energy decay estimates.

Remark 1.5. When $c = d$ in (1.17), we get a system of uncoupled wave equations; so our result also improves Theorem 1 of [24] and Theorem 1.2 of [30].

The rest of the paper is organized as follows: Section 2 is devoted to some preliminary technical lemmas while in Section 3 we provide the proof of Theorem 1.2.

2. Some technical Lemmas

The proof of Theorem 1.2 relies essentially on the following lemmas.

Lemma 2.1. *Let $E : [0,\infty) \to [0,\infty)$ be a nonincreasing locally absolutely continuous function such that there are positive constants β and A with*

$$\int_S^\infty E(t)^{\beta+1}dt \le AE(S),\ \forall S \ge 0. \tag{2.1}$$

Then we have

$$E(t) \le (A(1+\frac{1}{\beta}))^{\frac{1}{\beta}}(t+1)^{-\frac{1}{\beta}},\ \forall t \ge 0. \tag{2.2}$$

This lemma is due to Haraux and its proof can be found in [7, 11, 12, 14, 22]. This lemma reduces the proof of Theorem 1.2 to the proof of an estimate of type (2.1).

From now on, S and T denote two real numbers such that $0 \le S < T < \infty$, and Ω_{ST} stands for $\Omega \times (S,T)$. We write E instead of $E(t)$.

Lemma 2.2. *Let $\mu \ge 0$, $q \in (W^{1,\infty}(\Omega))^N$, $\alpha \in \mathbb{R}$ and $\xi \in W^{1,\infty}(\Omega)$. We have the identities*

$$\begin{aligned}
&\int_\Omega y_{i,t}\{2q_n y_{i,n} + \alpha y_i\}\,dx\ E^\mu\Big|_S^T + \int_{\Omega_{ST}} (div(q)-\alpha)\{y_{i,t}y_{i,t} - \sigma_{ij}\varepsilon_{ij}\}E^\mu\,dxdt\\
&-\mu\int_{\Omega_{ST}} E^{\mu-1}E' y_{i,t}\{2q_n y_{i,n} + \alpha y_i\}\,dxdt + 2\int_{\Omega_{ST}} E^\mu \sigma_{ij} q_{n,j} y_{i,n}\,dxdt\\
&-\int_{\Omega_{ST}} q_n a_{ijkl,n}\varepsilon_{kl}\varepsilon_{ij}E^\mu\,dxdt + \int_{\Omega_{ST}} a y_{i,t}\{2q_n y_{i,n} + \alpha y_i\}E^\mu\,dxdt\\
&= \int_{\Gamma\times(S,T)} E^\mu (q\cdot\nu)\sigma_{ij}\varepsilon_{ij}\,d\Gamma dt.
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\int_\Omega y_{i,t}\xi y_i\,dx\ E^\mu\Big|_S^T - \int_{\Omega_{ST}} \xi\{y_{i,t}y_{i,t} - \sigma_{ij}\varepsilon_{ij}\}E^\mu\,dxdt\\
&-\mu\int_{\Omega_{ST}} E^{\mu-1}E' y_{i,t}y_i\xi\,dxdt + \int_{\Omega_{ST}} \sigma_{ij}\xi_{,j}y_iE^\mu\,dxdt\\
&+\int_{\Omega_{ST}} a y_{i,t}\xi y_i E^\mu\,dxdt = 0.
\end{aligned} \tag{2.4}$$

The proof of Lemma 2.2 is based on standard multipliers technique; the interested reader should refer to Lions [20] or Komornik [11].

Lemma 2.3. *Under the hypotheses of Theorem* 1.2, *we have for all* $t \geq 0$,

$$\int_\omega |y_t|^2\,dx \leq \begin{cases} c|(1/a)|_p^{\frac{p}{p+1}} F_0^{\frac{1}{p+1}} |E'|^{\frac{p}{p+1}} E^{\frac{1}{2(p+1)}}, \text{ if } N=1,\\ c_\delta|(1/a)|_p^{\frac{p(1-\delta)}{p+1}} F_0^{\frac{2(1+\delta p)}{p+1}} |E'|^{\frac{p(1-\delta)}{p+1}}, \ \forall 0<\delta<1, \text{ if } N=2,\\ c|(1/a)|_p^{\frac{2p}{N+2p}} F_0^{\frac{2N}{N+2p}} |E'|^{\frac{2p}{N+2p}}, \text{ if } N\geq 3, \end{cases} \tag{2.5}$$

and

$$\int_\Omega a^2 y_{i,t} y_{i,t}\,dx \leq \begin{cases} c|a|_r^{\frac{r}{r-1}} F_0^{\frac{1}{r-1}} |E'|^{\frac{r-2}{r-1}} E^{\frac{1}{2(r-1)}}, \text{ if } N=1,\\ c_\delta|a|_r^{\frac{r-\delta}{r-1-\delta}} F_0^{\frac{2}{r-1-\delta}} |E'|^{\frac{r-2-\delta}{r-1-\delta}}, \ \forall 0<\delta<1, \text{ if } N=2,\\ c|a|_r^{\frac{2r}{2r-N}} F_0^{\frac{2N}{2r-N}} |E'|^{\frac{2(r-N)}{2r-N}}, \text{ if } N\geq 3. \end{cases} \tag{2.6}$$

Proof of Lemma 2.3. We begin with the proof of (2.5). For this proof we need different approaches for the three cases involved, so we will proceed by cases.

Case 1: $N=1$**.** Since y is a strong solution of (1.3), it is known that $y_t(.,t) \in H_0^1(\Omega)$, for all $t \geq 0$. On the other hand, a Sobolev imbedding theorem shows that $H_0^1(\Omega)$ is continuously embedded in $L^\infty(\Omega)$ for $N=1$. Applying the Hölder inequality and this imbedding result, we get

$$\begin{aligned} \int_\omega |y_t|^2\,dx &= \int_\omega (1/a)^{\frac{p}{p+1}} a^{\frac{p}{p+1}} |y_t|^2 dx \leq |(1/a)|_p^{\frac{p}{p+1}} \Big(\int_\omega a|y_t|^{\frac{2(p+1)}{p}}\,dx\Big)^{\frac{p}{p+1}} \\ &\leq c|(1/a)|_p^{\frac{p}{p+1}} |y_t(.,t)|_\infty^{\frac{2}{p+1}} \Big(\int_\Omega a|y_t|^2\,dx\Big)^{\frac{p}{p+1}} \\ &\leq c|(1/a)|_p^{\frac{p}{p+1}} |y_t(.,t)|_\infty^{\frac{2}{p+1}} |E'(t)|^{\frac{p}{p+1}}. \end{aligned} \tag{2.7}$$

Thanks to an elementary interpolation inequality and (1.10), we have

$$|y_t(.,t)|_\infty \leq c||y_t(.,t)||_{H^1(\Omega)}^{\frac{1}{2}} |y_t(.,t)|_2^{\frac{1}{2}} \leq cF_0^{\frac{1}{2}} E^{\frac{1}{4}}. \tag{2.8}$$

Reporting (2.8) in (2.7), we get the claimed estimate, and we are done with this case.

Case 2: $N = 2$**.** Let $s > 1$, and $\tau \in (0, 2)$. We have by a twofold application of the Hölder inequality

$$\begin{aligned}
&\int_\omega |y_t|^2\,dx = \int_\omega |y_t|^{2-\tau}|y_t|^\tau\,dx\\
&\le \Big(\int_\Omega |y_t|^{(2-\tau)s}\,dx\Big)^{\frac{1}{s}}\Big(\int_\omega (1/a)^{\frac{p}{p+1}}(a)^{\frac{p}{p+1}}|y_t|^{\frac{\tau s}{s-1}}\,dx\Big)^{\frac{s-1}{s}}\\
&\le |(1/a)|_p^{\frac{p(s-1)}{s(p+1)}}\Big(\int_\Omega |y_t|^{\frac{2(s+p)}{p+1}}\,dx\Big)^{\frac{1}{s}}\Big(\int_\Omega a|y_t|^2\,dx\Big)^{\frac{p(s-1)}{s(p+1)}},\ \text{with } \tau = \frac{2p(s-1)}{s(p+1)}\\
&\le c_s|(1/a)|_p^{\frac{p(s-1)}{s(p+1)}}||y_t(.,t)||_{H^1(\Omega)}^{\frac{2(s+p)}{s(p+1)}}|E'|^{\frac{p(s-1)}{s(p+1)}}\\
&\le c_s|(1/a)|_p^{\frac{p(s-1)}{s(p+1)}}F_0^{\frac{2(s+p)}{s(p+1)}}|E'|^{\frac{p(s-1)}{s(p+1)}},\ \text{by (1.10).} \qquad (2.9)
\end{aligned}$$

Setting $s = 1/\delta$ in (2.9) we get the claimed estimate. It should be noted that, in (2.9), we use in an essential manner the Sobolev imbedding theorem: $H^1(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $1 \le q < \infty$, when $N = 2$. To complete the proof of (2.5), it remains to deal with the last case.

Case 3: $N \ge 3$**.** First of all, we note the Sobolev imbedding theorem: $H^1(\Omega)$ is continuously embedded in $L^q(\Omega)$ for $1 \le q \le \frac{2N}{N-2}$ when $N \ge 3$. Choosing $s = \frac{N+2p}{N-2}$ in (2.9), we get the claimed estimate, and we are done with the proof of (2.5). We now turn to the proof of (2.6). Here we also proceed by cases.

Case 1: $N = 1$**.** We have by the Sobolev imbedding theorem and the Hölder and interpolation inequalities

$$\begin{aligned}
&\int_\Omega a^2|y_t|^2\,dx = \int_\Omega a^{\frac{r}{r-1}}a^{\frac{r-2}{r-1}}|y_t|^2\,dx \le |a|_r^{\frac{r}{r-1}}\Big(\int_\Omega a|y_t|^{\frac{2(r-1)}{r-2}}\,dx\Big)^{\frac{r-2}{r-1}}\\
&\le c|a|_r^{\frac{r}{r-1}}|y_t(.,t)|_\infty^{\frac{2}{r-1}}\Big(\int_\Omega a|y_t|^2\,dx\Big)^{\frac{r-2}{r-1}}\\
&\le c|a|_r^{\frac{r}{r-1}}|y_t(.,t)|_2^{\frac{1}{r-1}}||y_t(.,t)||_{H^1(\Omega)}^{\frac{1}{r-1}}|E'(t)|^{\frac{r-2}{r-1}}\\
&\le c|a|_r^{\frac{r}{r-1}}E^{\frac{1}{2(r-1)}}F_0^{\frac{1}{r-1}}|E'(t)|^{\frac{r-2}{r-1}},\ \text{by (1.10).} \qquad (2.10)
\end{aligned}$$

Case 2: $N \ge 2$**.** Let $1 < s < r - 1$. We have by the Sobolev imbedding theorem and a twofold application of the Hölder inequality

$$\int_\Omega a^2|y_t|^2\,dx = \int_\Omega a^{\frac{s+1}{s}}|y_t|^{\frac{2}{s}}a^{\frac{s-1}{s}}|y_t|^{\frac{2(s-1)}{s}}\,dx$$

$$\leq \Big(\int_\Omega a^{s+1}|y_t|^2\,dx\Big)^{\frac{1}{s}}\Big(\int_\Omega a|y_t|^2\,dx\Big)^{\frac{s-1}{s}}$$
$$\leq |a|_r^{\frac{s+1}{s}}\Big(\int_\Omega |y_t|^{\frac{2r}{r-s-1}}\,dx\Big)^{\frac{r-s-1}{rs}}|E'|^{\frac{s-1}{s}}$$
$$\leq c|a|_r^{\frac{s+1}{s}}||y_t(.,t)||_{H^1(\Omega)}^{\frac{2}{s}}|E'|^{\frac{s-1}{s}} \leq c|a|_r^{\frac{s+1}{s}}F_0^{\frac{2}{s}}|E'(t)|^{\frac{s-1}{s}},\ \text{by (1.10)}. \tag{2.11}$$

To get the claimed estimates for $N \geq 2$, it is enough to choose $s = r-1-\delta$ when $N=2$ and $s = (2r-N)/N$ when $N \geq 3$ in (2.11); this completes the proof of Lemma 2.3. □

3. Proof of Theorem 1.2

Several steps will be needed to complete this proof.

Step 1. Applying (2.3) with $\alpha = N-(1/4), q(x) = m(x)$, observing that $div(m) = N$ and using (1.6), we find

$$\begin{aligned}
&\frac{1}{8}\int_S^T E^{\mu+1}dt + (3/2)\int_{\Omega_{ST}}\sigma_{ij}\varepsilon_{ij}E^\mu\,dxdt \\
&= -\int_\Omega y_{i,t}\{2m_n y_{i,n} + (N-1)y_i\}\,dx\ E^\mu\Big]_S^T \\
&+ \mu\int_{\Omega_{ST}} E^{\mu-1}E' y_{i,t}\{2m\cdot\nabla y_i + (N-1)y_i\}\,dxdt \\
&- \int_{\Omega_{ST}} a y_{i,t}\{2m\cdot\nabla y_i + (N-1)y_i\}E^\mu\,dxdt + \int_{\Omega_{ST}} m_n a_{ijkl,n}\varepsilon_{kl}\varepsilon_{ij}E^\mu\,dxdt \\
&+ \int_{\Gamma\times(S,T)} E^\mu(m\cdot\nu)\sigma_{ij}\varepsilon_{ij}d\Gamma dt.
\end{aligned} \tag{3.1}$$

Since the energy is nonincreasing, it follows from the Young and Korn inequalities that

$$\begin{aligned}
&\Big|-\int_\Omega y_{i,t}\{2m_n y_{i,n} + (N-1)y_i\}\,dx\ E^\mu\Big]_S^T\Big| \\
&+\Big|\mu\int_{\Omega_{ST}} E^{\mu-1}E' y_{i,t}\{2m\cdot\nabla y_i + (N-1)y_i\}\,dxdt\Big| \leq cE(0)^\mu E(S).
\end{aligned} \tag{3.2}$$

By the Hölder and Korn inequalities we have

$$\Big|\int_{\Omega_{ST}} a y_{i,t}\{2m\cdot\nabla y_i + (N-1)y_i\}E^\mu\,dxdt\Big|$$

$$\leq c\int_S^T E^\mu\Big(\int_\Omega a^2|y_t|^2\,dx\Big)^{\frac12}\Big(\int_\Omega|\nabla y|^2\,dx\Big)^{\frac12}dt$$
$$\leq c\int_S^T E^{\frac{2\mu+1}{2}}|ay_t|_2dt,\text{ by the Korn inequality and (1.6)} \tag{3.3}$$

Combining (3.1)-(3.3) and using (1.12), we get

$$\int_S^T E^{\mu+1}dt\leq cE(0)^\mu E(S)+c\int_S^T E^{\frac{2\mu+1}{2}}|ay_t|_2dt+c\int_{\Gamma_+\times(S,T)}E^\mu\sigma_{ij}\varepsilon_{ij}\,d\Gamma dt. \tag{3.4}$$

At this stage, we observe, thanks to Lemma 2.3, that it suffices to obtain judicious estimates of the last term in the right-hand side of (3.4) in terms of $E(S)$ and $\int_S^T E^{\mu+1}\,dt$ to complete the proof of Theorem 1.2.

Step 2. Let $h\in(W^{2,\infty}(\Omega))^N$ such that

$$h=\nu\quad on\quad\Gamma_+,\quad h\cdot\nu\geq 0\quad on\quad\Gamma,\quad h=0\quad in\quad\Omega\setminus\omega_1, \tag{3.5}$$

where ω_1 is another neighborhood of Γ_+ strictly contained in ω.

Choose $\alpha=0$ and $q=h$ in (2.3). Following Zuazua [32], and using the Korn inequality, we can show that there exists a positive constant c_0 depending only on Ω and ω such that

$$\begin{aligned}\bar c\int_{\Gamma_+\times(S,T)}E^\mu\sigma_{ij}\varepsilon_{ij}\,d\Gamma dt&\leq\bar c\int_{\Gamma\times(S,T)}E^\mu(h\cdot\nu)\sigma_{ij}\varepsilon_{ij}\,d\Gamma dt\leq\frac12\int_S^T E^{\mu+1}\,dt\\&+c_0\int_{\omega_1\times(S,T)}\big\{|y_t|^2+\sigma_{ij}\varepsilon_{ij}\big\}E^\mu\,dxdt+2\bar c\int_\Omega y_{i,t}h\cdot\nabla y_i\,dx\ E^\mu\Big]_S^T\\&-2\mu\bar c\int_{\Omega_{ST}}E^{\mu-1}E'y_{i,t}h\cdot\nabla y_i\,dxdt+2\bar c\int_{\Omega_{ST}}ay_{i,t}h\cdot\nabla y_iE^\mu\,dx\,dt,\end{aligned} \tag{3.6}$$

where $\bar c$ is the constant in (3.4).

Simple calculations using the Young and Korn inequalities show that

$$\Big|2\bar c\int_\Omega y_{i,t}h\cdot\nabla y_i\,dx\ E^\mu\Big]_S^T\Big|+\Big|2\mu\bar c\int_{\Omega_{ST}}E^{\mu-1}E'y_th\cdot\nabla y\,dxdt\Big|\leq cE(0)^\mu E(S). \tag{3.7}$$

In the last term of the right-hand side of (3.6), we proceed as in (3.3) to get

$$\Big|2\bar c\int_{\Omega_{ST}}ay_th\cdot\nabla yE^\mu\,dxdt\Big|\leq c\int_S^T E^{\frac{2\mu+1}{2}}|ay_t|_2dt. \tag{3.8}$$

Combining (3.6) with (3.8) and reporting the obtained result in (3.4) yield

$$\int_S^T E^{\mu+1}dt \leq cE(0)^\mu E(S) + c\int_{\omega_1\times(S,T)} \{|y_t|^2 + \sigma_{ij}\varepsilon_{ij}\}E^\mu \, dx\, dt$$
$$+ \int_S^T E^{\frac{2\mu+1}{2}} |ay_t|_2 dt. \tag{3.9}$$

Thanks to Lemma 2.3, it remains to get rid of the term involving ε_{ij} in the right-hand side of (3.9) to complete the proof of Theorem 1.2.

Step 3. Let η be a function satisfying

$$\eta \in W^{2,\infty}(\Omega), \quad 0 \leq \eta \leq 1, \quad \eta = 1 \quad in \quad \omega_1, \quad \eta = 0, \text{ in } \Omega\setminus\omega. \quad (3.10).$$

Applying (2.4) with $\xi = \eta^2$, (we choose $\xi = \eta^2$ to make our computations easy to understand) we find

$$\int_{\Omega_{ST}} \eta^2 \sigma_{ij}\varepsilon_{ij} E^\mu \, dxdt = -\int_\Omega \eta^2 y_{i,t} y_i \, dx \; E^\mu\Big]_S^T + \int_{\Omega_{ST}} \eta^2 y_{i,t} y_{i,t} E^\mu \, dxdt$$
$$-\mu \int_{\Omega_{ST}} E^{\mu-1}E' y_{i,t} y_i \eta^2 \, dxdt + 2\int_{\Omega_{ST}} \sigma_{ij}\eta_{,j}\eta y_i E^\mu \, dx\, dt$$
$$+ \int_{\Omega_{ST}} a y_{i,t}\eta^2 y_i E^\mu \, dxdt = 0. \tag{3.11}$$

It follows from (3.11) and some simple computations using the Korn inequality that

$$\tilde{c}\int_{\Omega_{ST}} \sigma_{ij}\eta^2\varepsilon_{ij}E^\mu \, dxdt \leq cE(0)^\mu E(S) + \frac{\tilde{c}}{2}\int_{\Omega_{ST}} \eta^2\sigma_{ij}\varepsilon_{ij}E^\mu \, dx\, dt$$
$$+ c\int_S^T E^{\frac{2\mu+1}{2}}|ay_t|_2 dt + c\int_{\omega\times(S,T)} \{|y_t|^2 + |y|^2\} E^\mu \, dxdt, \tag{3.12}$$

where $\tilde{c}$ is the constant appearing in the right-hand side of (3.9).

Therefore,

$$\tilde{c}\int_{\Omega_{ST}} \eta^2\sigma_{ij}\varepsilon_{ij}E^\mu \, dxdt \leq cE(0)^\mu E(S) + c\int_S^T E^{\frac{2\mu+1}{2}}|ay_t|_2 dt$$
$$+ c\int_{\omega\times(S,T)} |y_t|^2 E^\mu \, dxdt + c\int_{\omega\times(S,T)} |y|^2 E^\mu \, dxdt. \tag{3.13}$$

Using (3.13) in (3.9), we find

$$\int_S^T E^{\mu+1}dt \leq cE(0)^\mu E(S) + c\int_S^T E^{\frac{2\mu+1}{2}}|ay_t|_2 dt + c\int_{\omega\times(S,T)} |y_t|^2 E^\mu \, dxdt$$

$$+ c \int_{\omega \times (S,T)} |y|^2 E^\mu \, dxdt. \tag{3.14}$$

Now, we are going to use a special multiplier to absorb the term in the lower line of (3.14).

Step 4. Introduce $z(t) \in H_0^1(\Omega)$ to be a solution of

$$\begin{cases} -\sigma_{ij,j}(z) = \chi_\omega y_i \text{ in } \Omega, \\ z = 0 \text{ on } \Gamma, \end{cases} \tag{3.15}$$

where χ_ω is the characteristic function of ω.

It is easy to check that $z' = z_t$ satisfies

$$\begin{cases} -\sigma_{ij,j}(z') = \chi_\omega y_{i,t} \text{ in } \Omega, \\ z' = 0 \text{ on } \Gamma. \end{cases} \tag{3.16}$$

Some elementary calculations show that

$$\int_\Omega \sigma_{ij}(z)\varepsilon_{ij}(z)\, dx \le c \int_\omega |y|^2\, dx, \quad \int_\Omega \sigma_{ij}(z')\varepsilon_{ij}(z')\, dx \le c \int_\omega |y_t|^2\, dx$$
$$\int_\Omega \sigma_{ij}(z)\varepsilon_{ij}(y)\, dx = \int_\Omega \sigma_{ij}(y)\varepsilon_{ij}(z)\, dx = \int_\omega |y|^2\, dx. \tag{3.17}$$

Now multiplying the first equation of (1.1) by $z_i E^\mu$, integrating by parts over $\Omega \times (S,T)$ and using the second line of (3.17) we find

$$\int_{\omega \times (S,T)} |y|^2 E^\mu \, dxdt = - \int_\Omega y_{i,t} z_i \, dx \; E^\mu \Big]_S^T + \int_{\Omega_{ST}} E^\mu y_{i,t} z_i' \, dxdt$$
$$+ \mu \int_{\Omega_{ST}} E^{\mu-1} E' y_{i,t} z_i \, dxdt - \int_{\Omega_{ST}} a y_{i,t} z_i E^\mu \, dxdt, \tag{3.18}$$

from which we derive that

$$\check{c} \int_{\omega \times (S,T)} |y|^2 E^\mu \, dxdt \le cE(0)^\mu E(S) + \frac{1}{2} \int_S^T E^{\mu+1} dt$$
$$+ c \int_{\omega \times (S,T)} |y_t|^2 E^\mu \, dxdt + c \int_S^T E^{\frac{2\mu+1}{2}} |a y_t|_2 \, dt, \tag{3.19}$$

where $\check{c}$ stands for the constant in (3.14).

Using (3.19) in (3.14) we get

$$\int_S^T E^{\mu+1}\, dt \le cE(0)^\mu E(S) + c \int_{\omega \times (S,T)} |y_t|^2 E^\mu \, dxdt + c \int_S^T E^{\frac{2\mu+1}{2}} |a y_t|_2 \, dt. \tag{3.20}$$

To complete the proof of Theorem 1.2. we shall absorb the last two terms in the right-hand side of (3.20). Applying Lemma 2.3, we find

$$\int_S^T E^{\mu+1}\,dt \le cE(0)^\mu E(S) + c|(1/a)|_p^{\frac{p}{p+1}} F_0^{\frac{1}{p+1}} \int_S^T |E'|^{\frac{p}{p+1}} E^{\mu+\frac{1}{2(p+1)}}\,dt$$
$$+ c|a|_r^{\frac{r}{2r-2}} F_0^{\frac{1}{2r-2}} \int_S^T |E'|^{\frac{r-2}{2r-2}} E^{\mu+\frac{1}{2}+\frac{1}{4r-4}}\,dt,\ \text{if } N=1, \tag{3.21}$$

$$\int_S^T E^{\mu+1}\,dt \le cE(0)^\mu E(S) + c_\delta|(1/a)|_p^{\frac{p(1-\delta)}{p+1}} F_0^{\frac{2(1+\delta p)}{p+1}} \int_S^T E^\mu |E'|^{\frac{p(1-\delta)}{p+1}}\,dt$$
$$+ c_\delta|a|_r^{\frac{r-\delta}{2r-2-2\delta}} F_0^{\frac{1}{r-1-\delta}} \int_S^T |E'|^{\frac{r-2-\delta}{2r-2-\delta}} E^{\mu+\frac{1}{2}}\,dt,$$
$$\forall 0<\delta<Min(1,r-2),\ \text{if } N=2, \tag{3.22}$$

and

$$\int_S^T E^{\mu+1}\,dt \le cE(0)^\mu E(S) + c|(1/a)|_p^{\frac{2p}{N+2p}} F_0^{\frac{2N}{N+2p}} \int_S^T E^\mu |E'|^{\frac{2p}{N+2p}}\,dt$$
$$+ c|a|_r^{\frac{r}{2r-N}} F_0^{\frac{N}{2r-N}} \int_S^T E^{\mu+\frac{1}{2}} |E'|^{\frac{r-N}{2r-N}}\,dt,\ \text{if } N\ge 3. \tag{3.23}$$

Choosing $\mu=\mu_N$ in (3.21) and (3.23), and using the Young inequality, we find

$$\int_S^T E^{\mu_N+1}dt \le cE(0)^{\mu_N}E(S) + \frac{r+1}{p+r+1}\int_S^T E^{\mu_N+1}dt$$
$$+ c|(1/a)|_p F_0^{\frac{N}{p}} E(0)^{\frac{2\mu_N p-N}{2p}} E(S)$$
$$+ c|a|_r^{\frac{r}{r-r_N}} F_0^{\frac{1}{r-r_N}} E(0)^{\frac{2(r-r_N)\mu_N-N}{2(r-r_N)}} E(S),\ \text{if } N\neq 2. \tag{3.24}$$

Therefore, letting $T\to\infty$, and applying Lemma 2.1, one gets (1.13). It remains to prove (1.15). For this purpose, choosing $\mu=\mu_2$ in (3.22), and using the Young inequality, we get

$$\int_S^T E^{\mu_2+1}dt \le cE(0)^{\mu_2}E(S) + c_\delta|(1/a)|_p F_0^{\frac{2(1+\delta p)}{p(1-\delta)}} E(0)^{\frac{p(1-\delta)\mu_2-(1+\delta p)}{p(1-\delta)}} E(S)$$
$$+ \frac{r+p}{r+p+1}\int_S^T E^{\mu_2+1}dt \tag{3.25}$$
$$+ c_\delta|a|_r^{\frac{r-\delta}{r-2-\delta}} F_0^{\frac{2}{r-2-\delta}} E(0)^{\frac{(r-2-\delta)\mu_2-1}{r-2-\delta}} E(S),\ \forall 0<\delta<Min(1,r-2),\ \text{if } N=2,$$

from which one derives (1.15) by the application of Lemma 2.1, and the proof of Theorem 1.2 is complete. □

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