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L^q SPECTRAL ASYMPTOTICS FOR NONLINEAR STURM-LOUIVILLE PROBLEMS

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Abstract. We consider the nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t)) = \lambda u(t), \ \ u(t) > 0, \quad t \in I := (0,1), \ \ u(0) = u(1) = 0,$$

where $\lambda > 0$ is an eigenvalue parameter. For better understanding of the global behavior of the branch of positive solutions in $\mathbf{R}_+ \times L^q(I)$ $(1 \le q \le \infty)$, we establish precise asymptotic formulas for the eigenvalue λ with respect to $\|u_\lambda\|_q$, where u_λ is the unique solution associated with given $\lambda > \pi^2$.

1. Introduction

In this paper, we are concerned with the following nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I := (0,1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(0) = u(1) = 0, \tag{1.3}$$

where $\lambda > 0$ is an eigenvalue parameter. We assume that $f(u)$ satisfies the following conditions (A.1)–(A.3).
(A.1) $f(u)$ is a C^1 function for $u \ge 0$ satisfying $f(0) = f'(0) = 0$.
(A.2) $g(u) := f(u)/u$ is strictly increasing for $u \ge 0$ $(g(0) := 0)$.
(A.3) $g(u) \to \infty$ as $u \to \infty$.

Let $1 \le q \le \infty$ be fixed. Then under the conditions (A.1)–(A.3), we know from [1] that for each given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda(q,\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{I})$ of (1.1)–(1.3) which satisfies $\|u_\alpha\|_q = \alpha$, where $\|\cdot\|_q$ denotes the usual L^q-norm. Besides, the set $\{(\lambda(q,\alpha), u_\alpha) : \alpha > 0\}$ gives all solutions of (1.1)–(1.3) and is an unbounded curve of class C^1

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in $\mathbf{R}_+ \times L^q(I)$ emanating from $(\pi^2, 0)$. Furthermore, $\lambda(q,\alpha)$ is a strictly increasing function of $\alpha > 0$.

The purpose of this paper is to study precisely the global behavior of this branch of positive solutions in $\mathbf{R}_+ \times L^q(I)$. To this end, we establish a precise asymptotic formula for $\lambda(q,\alpha)$ as $\alpha \to \infty$. Then we find that the asymptotic formula for $\lambda(\infty,\alpha)$ is regarded as the $\lim_{q\to\infty} \lambda(q,\alpha)$.

The equation (1.1)–(1.3) has been studied extensively by many authors regarding local and global L^∞ bifurcation theory. We refer to [1], [2], [5], [6], and [7]. Especially, it is known from [1] that for $\alpha \gg 1$

$$\lambda(\infty,\alpha) = g(\alpha) + O(1). \tag{1.4}$$

Furthermore, one of the chief concern in this field is to investigate the local and global shape of the L^2-bifurcation curve $\lambda(2,\alpha)$, and the asymptotic behavior of $\lambda(2,\alpha)$ as $\alpha \to 0$ has been studied in [2], [3]. When $f(u) = u^p$ $(p > 1)$, the asymptotic behavior of $\lambda(2,\alpha)$ as $\alpha \to \infty$ has been studied in [8]. As far as the author knows, however, little is known about the asymptotic behavior of $\lambda(q,\alpha)$ as $\alpha \to \infty$ for general f and general $1 \le q < \infty$ $(q \ne 2)$.

The leading term of $\lambda(q,\alpha)$ can be obtained easily as follows. Since it is known from [1] that

$$\frac{u_\lambda(t)}{g^{-1}(\lambda)} \to 1 \tag{1.5}$$

locally uniformly on I as $\lambda \to \infty$, we obtain that for $\lambda \gg 1$

$$\alpha = \|u_\lambda\|_q = (1 + o(1))g^{-1}(\lambda(q,\alpha)).$$

This implies that, for many cases, as $\alpha \to \infty$,

$$\lambda(q,\alpha) = g(\alpha) + o(g(\alpha)). \tag{1.6}$$

We know from (1.4) and (1.6) that the leading term $g(\alpha)$ of $\lambda(q,\alpha)$ as $\alpha \to \infty$ does not depend on q with $1 \le q \le \infty$. Therefore, to investigate the asymptotic behavior of $\lambda(q,\alpha)$ as $\alpha \to \infty$, it is natural to ask whether $\lambda(q,\alpha)$ is affected by q or not. To this end, we represent the exact second term of $\lambda(q,\alpha)$ by q to understand well the effect of q on the global shape of the branch of positive solutions in the L^q-framework.

For the case $f(u) = u^p$ and $q = 2$, the following asymptotic formula for $\lambda(q,\alpha)$ as $\alpha \to \infty$ has been established in [8]:

Theorem 1.1. *Let $f(u) = u^p$ $(p > 1)$ in (1.1). Furthermore, let $q = 2$. Then the following asymptotic formula holds as $\alpha \to \infty$:*

$$\lambda(2,\alpha) = \alpha^{p-1} + (p+3)C_0\alpha^{(p-1)/2} + O(1), \tag{1.7}$$

where

$$C_0 = \int_0^1 \sqrt{\frac{p-1}{p+1} - s^2 + \frac{2}{p+1}s^{p+1}}ds. \tag{1.8}$$

We note that, since we are interested in the second term of $\lambda(2,\alpha)$ for $\alpha \gg 1$ here, we represent the third term of $\lambda(2,\alpha)$ as $O(1)$ in (1.7). In [8], the asymptotic expansion formula for $\lambda(2,\alpha)$ for $\alpha \gg 1$ has been given.

When we compare (1.4) and (1.6) with (1.7), it is natural to consider the following problems.

(a) *Assume that $f(u)$ satisfies (A.1)–(A.3). Then for $1 \le q < \infty$, find $g_q(u)$ such that as $u \to \infty$*

$$g_q(u) \to \infty, \quad \frac{g_q(u)}{g(u)} \to 0,$$

and as $\alpha \to \infty$

$$\lambda(q,\alpha) = g(\alpha) + g_q(\alpha) + o(g_q(\alpha)). \tag{1.9}$$

(b) *Is* $\lim_{q\to\infty} g_q(u) = g_\infty(u)$ $(:= 0)$?

When we consider the problems above, we encounter the following difficulty. The argument used in [8] to prove Theorem 1.1 is based on the clear relationship between $\lambda(\|u_\lambda\|_2)$ and the critical value $\gamma(\|u_\lambda\|_2)$ associated with u_λ, which holds only for the case $f(u) = u^p$ $(p > 1)$ and $q = 2$. So the argument in [8] is not able to be applied to our problems here.

We overcome this difficulty and answer these questions (i) and (ii) above affirmatively. To this end, we assume the following additional conditions.
(A.4) There exists a constant $1 < r \le \infty$ such that as $u \to \infty$

$$\frac{f'(u)u}{f(u)} \to r. \tag{1.10}$$

Furthermore, if $1 < r < \infty$, then for a fixed $0 < s < 1$, as $u \to \infty$,

$$\frac{f(us)}{f(u)} \to s^r. \tag{1.11}$$

(A.5) If $r = \infty$, then $f'(u)(u/f(u))^{3/2} \to 0$ as $u \to \infty$.
(A.6) $f'(u)$ is non-decreasing for $u \ge 0$ and there exists $u_0 > 0$ such that $f \in C^2((u_0,\infty))$. Furthermore, there exists a constant $C > 0$ such that

$$f'(u_2) - f'(u_1) \le C(f''(u_1) + f''(u_2))(u_2 - u_1), \quad u_0 < u_1 < u_2, \tag{1.12}$$

$$f''(u)f(u) \le Cf'(u)^2, \quad u_0 < u. \tag{1.13}$$

We note that $f'(u)(u/f(u))^{3/2} \to 0$ as $u \to \infty$ for $1 < r < \infty$ holds, since we have (A.3) and (A.4). Therefore, in what follows, (A.5) implies $f'(u)(u/f(u))^{3/2} \to 0$ as $u \to \infty$ for $1 < r \le \infty$.

Let us briefly explain the meaning of (A.4)–(A.6). Roughly speaking, (1.10) in (A.4) implies that, if $1 < r < \infty$, then $f(u) \sim u^r$ at ∞, and if $r = \infty$, then $f(u)/u^\beta \to \infty$ for any $\beta \gg 1$ as $u \to \infty$. (A.5) implies that $f(u)/u \to \infty$ as $u \to \infty$. (A.6) consists of the technical conditions to obtain the estimates on, for instance, $f(\|u_\alpha\|_\infty) - f(\alpha)$, $f'(\|u_\alpha\|_\infty) - f'(\alpha)$ and so on when $\alpha \gg 1$.

The typical examples of $f(u)$ which satisfy (A.1)–(A.6) are $f(u) = u^p$ ($p > 1$) and $f(u) = u^p e^u$ ($p > 1$). If $f(u) = u^p$ (respectively $f(u) = u^p e^u$), then $r = p$ (respectively $r = \infty$) in (1.10).

Now we state our results.

Theorem 1.2. *Assume* (A.1)–(A.6). *Let* $C_2 := 1/(1+r)$ ($C_2 := 0$ *if* $r = \infty$). *Then as* $\alpha \to \infty$

$$\lambda(q,\alpha) = \frac{f(\alpha)}{\alpha} + \frac{C_3(q)}{q}\Big(f'(\alpha)\sqrt{\frac{\alpha}{f(\alpha)}}(1+o(1)) - \sqrt{\frac{f(\alpha)}{\alpha}}(1+o(1))\Big) + O(1), \tag{1.14}$$

where

$$C_3(q) := 2\int_0^1 \frac{1-s^q}{\sqrt{1-s^2-2C_2(1-s^{r+1})}}ds. \tag{1.15}$$

With reference to (1.9), we see from Theorem 1.2 that for $1 \le q < \infty$

$$g_q(u) = \frac{C_3(q)}{q}\Big(f'(u)\sqrt{\frac{u}{f(u)}} - \sqrt{\frac{f(u)}{u}}\Big). \tag{1.16}$$

By (A.2), we see that $g_q(u) > 0$ for $u > 0$. We obtain by (A.3), (A.5) and (1.16) that the leading term of $\lambda(q,\alpha)$ as $\alpha \to \infty$ is $f(\alpha)/\alpha$. Furthermore, since it is clear that $g_q(u) \to 0$ as $q \to \infty$ for a fixed $u > 0$, we see from (1.6) that under the stated conditions on f, the answer to problems (a) and (b) is affirmative.

Remark. Let $f(u) = u^p$ ($p > 1$) and $q = 2$. Then it follows from (1.7), (1.8), (1.14) and (1.15) that $(p+3)C_0 = (p-1)C_3(2)/2$. Indeed, since $C_2 = 1/(p+1)$ in this case, by integration by parts,

$$C_0 = \int_0^1 \sqrt{1-s^2-\frac{2}{p+1}(1-s^{p+1})}ds \tag{1.17}$$

$$= \left[s\sqrt{1-s^2-\frac{2}{p+1}(1-s^{p+1})} \right]_0^1 - \int_0^1 \frac{s(-2s+2s^p)}{2\sqrt{1-s^2-\frac{2}{p+1}(1-s^{p+1})}} ds$$
$$= \int_0^1 \frac{s^2-s^{p+1}}{\sqrt{1-s^2-\frac{2}{p+1}(1-s^{p+1})}} ds$$
$$= \int_0^1 \frac{1-s^{p+1}}{\sqrt{1-s^2-\frac{2}{p+1}(1-s^{p+1})}} ds - \int_0^1 \frac{1-s^2}{\sqrt{1-s^2-\frac{2}{p+1}(1-s^{p+1})}} ds$$
$$= \frac{1}{2} C_3(p+1) - \frac{1}{2} C_3(2).$$

Clearly, we have

$$\begin{aligned} C_0 &= \int_0^1 \sqrt{1-s^2-\frac{2}{p+1}(1-s^{p+1})} ds \\ &= \int_0^1 \frac{1-s^2}{\sqrt{1-s^2-\frac{2}{p+1}(1-s^{p+1})}} ds \\ &\quad - \frac{2}{p+1} \int_0^1 \frac{1-s^{p+1}}{\sqrt{1-s^2-\frac{2}{p+1}(1-s^{p+1})}} ds \\ &= \frac{1}{2} C_3(2) - \frac{1}{p+1} C_3(p+1). \end{aligned} \tag{1.18}$$

By (1.17) and (1.18), we obtain $(p+3)C_0 = (p-1)C_3(2)/2$. □

We prove Theorem 1.2 by a quite elementary way and do not use any complicated tools. That is, we compare $\alpha = \|u_\alpha\|_q$ with $\|u_\alpha\|_\infty$ directly. Therefore, the arguments here do not depend on the ideas in [8] at all.

2. Proof of Theorem 1.2

Let $1 \le q < \infty$ be fixed. We begin with notation and the fundamental properties of $\lambda(q, \alpha)$ and u_α. In what follows, C denotes various positive constants independent of $\alpha \gg 1$ for simplicity. We know from [1] that for $\alpha \gg 1$

$$\frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} \le \lambda(q,\alpha) \le \frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} + \pi^2, \tag{2.1}$$
$$u_\alpha(t) = \|u_\alpha\|_\infty(1+o(1)) = \alpha(1+o(1)), \quad t \in I, \tag{2.2}$$
$$u_\alpha(t) = u_\alpha(1-t), \quad 0 \le t \le 1, \tag{2.3}$$

$$u_\alpha(\tfrac{1}{2}) = \max_{0\le t\le 1} u_\alpha(t) = \|u_\alpha\|_\infty, \tag{2.4}$$

$$u'_\alpha(t) > 0, \quad 0 \le t < \tfrac{1}{2}. \tag{2.5}$$

Lemma 2.1. *For $\alpha \gg 1$*

$$\|u_\alpha\|_\infty^q - \alpha^q = C_3(1+o(1))\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}}\|u_\alpha\|_\infty^q. \tag{2.6}$$

Proof. We have only to consider the case where $1 < r < \infty$, since the case $r = \infty$ can be treated similarly. Let $F(u) := \int_0^u f(s)ds$. It follows from (1.1) that

$$\frac{d}{dt}\left[\frac{1}{2}u'_\alpha(t)^2 - F(u_\alpha(t)) + \frac{1}{2}\lambda(q,\alpha)u_\alpha(t)^2\right] = 0 \quad \text{for } 0 \le t \le 1.$$

By this and (2.4), for $0 \le t \le 1$, we obtain

$$\begin{aligned} &\frac{1}{2}u'_\alpha(t)^2 - F(u_\alpha(t)) + \frac{1}{2}\lambda(q,\alpha)u_\alpha(t)^2 = \text{constant} \\ &= -F(\|u_\alpha\|_\infty) + \frac{1}{2}\lambda(q,\alpha)\|u_\alpha\|_\infty^2. \end{aligned} \tag{2.7}$$

We put

$$M_\alpha(\theta) := \lambda(q,\alpha)(\|u_\alpha\|_\infty - \theta^2) - 2(F(\|u_\alpha\|_\infty) - F(\theta)), \tag{2.8}$$

$$Q_\alpha(s) := \lambda(q,\alpha)\|u_\alpha\|_\infty^2(1-s^2) - 2(F(\|u_\alpha\|_\infty) - F(s\|u_\alpha\|_\infty)). \tag{2.9}$$

By (2.5), (2.7) and (2.8), for $0 \le t \le 1/2$,

$$u'_\alpha(t) = \sqrt{M_\alpha(u_\alpha(t))}. \tag{2.10}$$

By (2.3), (2.4), (2.8), (2.10) and putting $\theta := u_\alpha(t)$ and $s = \theta/\|u_\alpha\|_\infty$, we obtain

$$\begin{aligned} \|u_\alpha\|_\infty^q - \alpha^q &= 2\int_0^{1/2}(\|u_\alpha\|_\infty^q - u_\alpha^q(t))\frac{u'_\alpha(t)}{\sqrt{M_\alpha(u_\alpha(t))}}dt \\ &= 2\int_0^{\|u_\alpha\|_\infty}(\|u_\alpha\|_\infty^q - \theta^q)\frac{1}{\sqrt{M_\alpha(\theta)}}d\theta \\ &= 2\frac{\|u_\alpha\|_\infty^q}{\sqrt{\lambda(q,\alpha)}}\int_0^1 \frac{1-s^q}{\sqrt{Q_\alpha(s)/(\lambda(q,\alpha)\|u_\alpha\|_\infty^2)}}ds. \end{aligned} \tag{2.11}$$

We show that as $\alpha \to \infty$

$$\int_0^1 \frac{1-s^q}{\sqrt{Q_\alpha(s)/(\lambda(q,\alpha)\|u_\alpha\|_\infty^2)}}ds \to \int_0^1 \frac{1-s^q}{\sqrt{1-s^2-2C_2(1-s^{r+1})}}ds (= \tfrac{C_3}{2}). \tag{2.12}$$

Step 1. By (1.10) and l'Hospital's rule,

$$\lim_{u\to\infty} \frac{F(u)}{f(u)u} = \lim_{u\to\infty} \frac{f(u)}{f'(u)u + f(u)} = C_2 = \frac{1}{1+r} < \frac{1}{2}. \tag{2.13}$$

Let $0 < s < 1$ be fixed. By (1.11), (2.13) and l'Hospital's rule,

$$\begin{aligned} \lim_{u\to\infty} \frac{F(su)}{f(u)u} &= \lim_{u\to\infty} \frac{F(u)}{f(u)u}\frac{F(su)}{F(u)} = C_2 \lim_{u\to\infty} \frac{F(su)}{F(u)} \\ &= C_2 \lim_{u\to\infty} \frac{f(su)s}{f(u)} = C_2 s^{r+1}. \end{aligned} \tag{2.14}$$

By this, (2.1), (2.9) and (2.13), for $0 < s < 1$, as $\alpha \to \infty$

$$\begin{aligned} \frac{1-s^q}{\sqrt{Q_\alpha(s)/(\lambda(q,\alpha)\|u_\alpha\|_\infty^2)}} &= \frac{1-s^q}{\sqrt{Q_\alpha(s)/(f(\|u_\alpha\|_\infty)\|u_\alpha\|_\infty(1+o(1)))}} \\ &\to \frac{1-s^q}{\sqrt{1-s^2-2C_2(1-s^{r+1})}}. \end{aligned} \tag{2.15}$$

Note that for $0 \le s \le 1$,

$$\eta(s) := 1 - s^2 - 2C_2(1-s^{r+1}) \ge 0. \tag{2.16}$$

Indeed, for $0 \le s \le 1$,

$$\eta'(s) = -2s + 2(r+1)C_2 s^r = 2s(-1+s^{r-1}) \le 0. \tag{2.17}$$

This implies that $\eta(s)$ is non-increasing for $0 \le s \le 1$. Since $\eta(1) = 0$, we obtain (2.16). So the third term of (2.15) is well defined.

Step 2. To apply Lebesgue's convergence theorem to (2.12), we prove that for $\alpha \gg 1$ and $0 \le s \le 1$

$$\frac{1-s^q}{\sqrt{Q_\alpha(s)/(\lambda(q,\alpha)\|u_\alpha\|_\infty^2)}} \le C. \tag{2.18}$$

We put

$$m_\alpha(s) := \frac{(1-s^q)^2}{(1-s)^2} g_\alpha(s), \quad g_\alpha(s) := \frac{(1-s)^2}{Q_\alpha(s)}. \tag{2.19}$$

Then we have

$$\frac{1-s^q}{\sqrt{Q_\alpha(s)/(\lambda(q,\alpha)\|u_\alpha\|_\infty^2)}} = \sqrt{m_\alpha(s)\lambda(q,\alpha)\|u_\alpha\|_\infty^2}. \tag{2.20}$$

Clearly,

$$\frac{1-s^q}{1-s} < q, \quad (0 \le s < 1), \quad \lim_{s\to 1}\frac{1-s^q}{1-s} = q. \tag{2.21}$$

So we show that $g_\alpha(s)$ is bounded. To do this, we show that $g_\alpha(s)$ is non-increasing for $0 \le s \le 1$. We have

$$g'_\alpha(s) \quad = \quad (1-s)\frac{h_\alpha(s)}{Q^2_\alpha(s)}, \tag{2.22}$$

where

$$h_\alpha(s) \quad := \quad -2Q_\alpha(s) - (1-s)Q'_\alpha(s). \tag{2.23}$$

By (2.9), we have

$$Q'_\alpha(s) = -2\lambda(q,\alpha)\|u_\alpha\|^2_\infty s + 2\|u_\alpha\|_\infty f(\|u_\alpha\|_\infty s), \tag{2.24}$$

$$Q''_\alpha(s) = -2\lambda(q,\alpha)\|u_\alpha\|^2_\infty + 2\|u_\alpha\|^2_\infty f'(\|u_\alpha\|_\infty s). \tag{2.25}$$

Then

$$h'_\alpha(s) \quad = \quad -Q'_\alpha(s) + (s-1)Q''_\alpha(s) := I + II + III, \tag{2.26}$$

where

$$\begin{aligned} I \quad &:= \quad 2\lambda(q,\alpha)\|u_\alpha\|^2_\infty - 2\|u_\alpha\|_\infty f(\|u_\alpha\|), \\ II \quad &:= \quad 2\|u_\alpha\|_\infty\{f(\|u_\alpha\|) - f(\|u_\alpha\|s)\}, \\ III \quad &:= \quad 2(s-1)\|u_\alpha\|^2_\infty f'(\|u_\alpha\|s). \end{aligned}$$

By (2.1), we see that $I \ge 0$. Further, for a fixed $0 < s < 1$, by (A.6) and the mean value theorem, there exists $s < s_\alpha < 1$ such that

$$II + III = 2\|u_\alpha\|^2_\infty(1-s)(f'(s_\alpha\|u_\alpha\|_\infty) - f'(\|u_\alpha\|s)) \ge 0. \tag{2.27}$$

Consequently, we see that $h'_\alpha(s) \ge 0$ for $0 \le s \le 1$. This implies that $h_\alpha(s)$ is increasing for $0 \le s \le 1$. Then for $0 \le s \le 1$,

$$h_\alpha(s) \le h_\alpha(1) = -2Q_\alpha(1) = 0.$$

By this and (2.22), we see that $g_\alpha(s)$ is non-increasing for $0 \le s \le 1$. Therefore, by (2.9) and (2.19), for $0 \le s \le 1$,

$$g_\alpha(s) \le g_\alpha(0) = \frac{1}{Q_\alpha(0)} = \frac{1}{\lambda(q,\alpha)\|u_\alpha\|^2_\infty - 2F(\|u_\alpha\|_\infty)}. \tag{2.28}$$

By this, (2.1), (2.13), (2.20), (2.21) and (A.4), for $\alpha \gg 1$

$$\frac{1-s^q}{\sqrt{Q_\alpha(s)/(\lambda(q,\alpha)\|u_\alpha\|^2_\infty)}} = \sqrt{m_\alpha(s)\lambda(q,\alpha)\|u_\alpha\|^2_\infty} \le q\sqrt{g_\alpha(s)\lambda(q,\alpha)\|u_\alpha\|^2_\infty}$$

$$\leq q\sqrt{\frac{\lambda(q,\alpha)\|u_\alpha\|_\infty^2}{\lambda(q,\alpha)\|u_\alpha\|_\infty^2 - 2F(\|u_\alpha\|_\infty)}} < q\sqrt{\frac{1}{1-2(C_2+o(1))}} < C. \tag{2.29}$$

This along with (2.15) and Lebesgue's convergence theorem implies (2.12). By (2.1), (2.11) and (2.12), for $\alpha \gg 1$, we obtain (2.6). Thus the proof is complete. □

Lemma 2.2. *For* $\alpha \gg 1$

$$\|u_\alpha\|_\infty - \alpha = \frac{C_3}{q}(1+o(1))\sqrt{\frac{\alpha}{f(\alpha)}}\alpha. \tag{2.30}$$

Proof. By Lemma 2.1, for $\alpha \gg 1$, we have

$$\|u_\alpha\|_\infty^q\Big(1 - C_3(1+o(1))\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}}\Big) = \alpha^q. \tag{2.31}$$

This along with (2.2), (A.3) and a Taylor expansion implies that for $\alpha \gg 1$

$$\begin{aligned}\|u_\alpha\|_\infty &= \alpha\Big(1 - C_3(1+o(1))\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}}\Big)^{-1/q} \\ &= \alpha\Big(1 + \frac{C_3}{q}(1+o(1))\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}}\Big) \\ &= \alpha\Big(1 + \frac{C_3}{q}(1+o(1))\sqrt{\frac{\alpha}{f(\|u_\alpha\|_\infty)}}\Big).\end{aligned} \tag{2.32}$$

Therefore, to obtain (2.30), we have only to show that for $\alpha \gg 1$

$$f(\|u_\alpha\|_\infty) = f(\alpha)(1+o(1)). \tag{2.33}$$

By the mean value theorem,

$$f(\|u_\alpha\|_\infty) \quad = \quad f(\alpha) + f'(\alpha_1)(\|u_\alpha\|_\infty - \alpha), \tag{2.34}$$

where $\alpha < \alpha_1 < \|u_\alpha\|_\infty$. Since $f'(u)$ is non-decreasing by (A.6), it follows from (A.5), (2.32) and (2.34) that as $\alpha \to \infty$

$$\begin{aligned} 0 \quad &\leq \quad 1 - \frac{f(\alpha)}{f(\|u_\alpha\|_\infty)} = \frac{f'(\alpha_1)(\|u_\alpha\|_\infty - \alpha)}{f(\|u_\alpha\|_\infty)} \\ &\leq \quad \frac{f'(\|u_\alpha\|_\infty)(\|u_\alpha\|_\infty - \alpha)}{f(\|u_\alpha\|_\infty)} \\ &= \quad \frac{C_3}{q}(1+o(1))\frac{f'(\|u_\alpha\|_\infty)}{f(\|u_\alpha\|_\infty)}\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}}\|u_\alpha\|_\infty \to 0.\end{aligned} \tag{2.35}$$

This implies (2.33). By (2.32) and (2.33), we obtain (2.30). Thus the proof is complete. □

Lemma 2.3. *For* $\alpha \gg 1$

$$f(\|u_\alpha\|_\infty) - f(\alpha) = \frac{C_3}{q} f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}}(1+o(1)). \tag{2.36}$$

Proof. By (A.5), (A.6), (1.12), (1.13), (2.32) and Lemma 2.2, for $\alpha \gg 1$, we have

$$\begin{aligned} f'(\|u_\alpha\|_\infty) - f'(\alpha) &\le C(f''(\|u_\alpha\|_\infty) + f''(\alpha))(\|u_\alpha\|_\infty - \alpha) \\ &\le Cf''(\|u_\alpha\|_\infty)\|u_\alpha\|_\infty\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}} + Cf''(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}} \\ &= o(f'(\|u_\alpha\|_\infty)) + o(f'(\alpha)). \end{aligned} \tag{2.37}$$

Since $f'(\alpha) \le f'(\|u_\alpha\|_\infty)$ by (A.6), (2.37) implies that for $\alpha \gg 1$

$$f'(\|u_\alpha\|_\infty) = f'(\alpha)(1+o(1)). \tag{2.38}$$

By this, the mean value theorem, (A.6) and Lemma 2.2, we obtain

$$\begin{aligned} f(\|u_\alpha\|_\infty) - f(\alpha) &= f'(\alpha_1)(\|u_\alpha\|_\infty - \alpha) \\ &\le f'(\|u_\alpha\|_\infty)(\|u_\alpha\|_\infty - \alpha) = \frac{C_3}{q}(1+o(1))f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}}, \end{aligned} \tag{2.39}$$

$$\begin{aligned} f(\|u_\alpha\|_\infty) - f(\alpha) &= f'(\alpha_1)(\|u_\alpha\|_\infty - \alpha) \\ &\ge f'(\alpha)(\|u_\alpha\|_\infty - \alpha) = \frac{C_3}{q}(1+o(1))f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}}. \end{aligned} \tag{2.40}$$

By (2.39) and (2.40), we obtain (2.36). Thus the proof is complete. □

Proof of Theorem 1.2. By (2.1), Lemmas 2.2 and 2.3 and a Taylor expansion, for $\alpha \gg 1$,

$$\begin{aligned} \lambda(q,\alpha) &= \frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} + O(1) \\ &= \frac{f(\alpha) + \frac{C_3}{q}f'(\alpha)\alpha\sqrt{\alpha/f(\alpha)}(1+o(1))}{\alpha(1+\frac{C_3}{q}\sqrt{\alpha/f(\alpha)}(1+o(1)))} + O(1) \\ &= \frac{1}{\alpha}\Big(f(\alpha) + \frac{C_3}{q}f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}}(1+o(1))\Big) \\ &\quad \times \Big(1 - \frac{C_3}{q}\sqrt{\frac{\alpha}{f(\alpha)}}(1+o(1))\Big) + O(1). \end{aligned} \tag{2.41}$$

This implies Theorem 1.2. Thus the proof is complete. □

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