

LIMIT BEHAVIOR OF BLOW-UP SOLUTIONS FOR CRITICAL NONLINEAR SCHRÖDINGER EQUATION WITH HARMONIC POTENTIAL

XIAOGUANG LI AND JIAN ZHANG

College of Mathematics and Software Science, Sichuan Normal University
Chengdu, 610066, China

and

College of Economics, Sichuan Normal University, Chengdu, 610066, China

JIAN ZHANG

College of Mathematics and Software Science, Sichuan Normal University
Chengdu, 610066, China

(Submitted by: Takashi Suzuki)

Abstract. We consider the blow-up solutions of the Cauchy problem for the critical nonlinear Schrödinger equation with a harmonic potential $i\phi_t + \frac{1}{2} \Delta \phi - \frac{1}{2} \omega^2 |x|^2 \phi + |\phi|^{4/N} \phi = 0$, $x \in R^N$, $t \geq 0$, which models the Bose-Einstein condensate. We establish the lower bound of blow-up rate as $t \rightarrow T$. Furthermore, the L^2 -concentration property of the radially symmetric blow-up solutions is obtained.

1. INTRODUCTION AND MAIN RESULTS

This paper is concerned with the Cauchy problem of the following attractive nonlinear Schrödinger equation with a harmonic potential:

$$i\phi_t + \frac{1}{2} \Delta \phi - \frac{1}{2} \omega^2 |x|^2 \phi + |\phi|^{4/N} \phi = 0, \quad x \in R^N, \quad t \geq 0, \quad (1.1)$$

$$\phi(0, x) = \varphi(x), \quad (1.2)$$

where $\omega > 0$, N is the space dimension, $\phi = \phi(t, x): [0, T) \times R^N \rightarrow C$ and $0 < T \leq \infty$, $i = \sqrt{-1}$, Δ is the Laplace operator on R^N . If we replace the nonlinear term by $|\phi|^{p-1} \phi$, it is well known that the exponent $p = p_c = 1 + 4/N$ in dimension N is the critical value for nonexistence of global solutions (see e.g. [2], [16]). Equation (1.1) models the Bose-Einstein condensate

Accepted for publication: January 2006.

AMS Subject Classifications: 35F25 35Q55.

This work was supported by National Natural Science Foundation of P.R. China (10271084).

with attractive inter-particle interactions under a magnetic trap, which is also known as the Gross-Pitaevski equation (see [1],[12]). The harmonic potential $|x|^2$ models a magnetic field whose role is to confine the movement of particles, ω is the trap frequency.

We define the energy space in the course of nature by

$$H := \{\phi \in H^1(\mathbb{R}^N) : |x|\phi \in L^2(\mathbb{R}^N)\}, \quad (1.3)$$

$$\|\phi\|_H^2 = \int_{\mathbb{R}^N} |\nabla\phi|^2 + |\phi|^2 + |x|^2|\phi|^2 dx. \quad (1.4)$$

From [4] or [9], we note that the local well posedness for the Cauchy problem (1.1), (1.2) holds in H in the following sense:

For any $\varphi \in H$, there exist $T > 0$ and a unique solution $\phi(t, x)$ of the Cauchy problem (1.1), (1.2) in $C([0, T]; H)$ such that either $T = \infty$ (global existence), or $T < \infty$ and $\lim_{t \rightarrow T} \|\phi(t)\|_H = \infty$ (blow up).

Furthermore, for all $t \in [0, T)$, $\phi(t)$ satisfies the following two conservation laws of mass and energy:

$$\|\phi(t)\|_{L^2} = \|\varphi\|_{L^2}, \quad (1.5)$$

$$E(\phi(t)) := \frac{1}{2} \|\nabla\phi(t)\|_{L^2}^2 + \frac{1}{2} \omega^2 \|x\phi(t)\|_{L^2}^2 - \frac{1}{1 + 2/N} \|\phi(t)\|_{L^{2+4/N}}^{2+4/N} = E(\varphi). \quad (1.6)$$

Let Q denote the ground state, which is the unique, positive, radially symmetric solution of the elliptic equation (see [6])

$$-\frac{1}{2} \Delta u + u - |u|^{4/N} u = 0, \quad u \in H^1(\mathbb{R}^N). \quad (1.7)$$

The ground state Q of equation (1.7) plays an important role in the formation of singularities for the solutions of the Cauchy problem (1.1), (1.2). Precisely, Zhang [16] and Carles [2] proved the following results: if $\|\varphi\|_{L^2} < \|Q\|_{L^2}$, the solution of the Cauchy problem (1.1), (1.2) is globally defined; if $\|\varphi\|_{L^2} \geq \|Q\|_{L^2}$, the solution of the Cauchy problem (1.1), (1.2) may blow up.

In this paper, we are interested in the limit behavior of the blow-up solution for the Cauchy problem (1.1), (1.2) as $t \rightarrow T$ (T is the blow-up time), and in particular, we are interested in the relation between the limit behavior and the ground $Q(x)$ of (1.7). Our results are the following theorems.

Theorem 1.1. (Blow-up rate). *Assume that ϕ is a blow-up solution of the Cauchy problem (1.1), (1.2) in $C([0, T), H)$ for some $T \in (0, \frac{\pi}{2\omega})$, where T*

is the maximal existence time. Then there exists a constant $L > 0$ such that

$$\|\nabla\phi(t)\|_{L^2} \geq \frac{L}{\sqrt{(T-t)}}, \quad \text{as } t \rightarrow T. \tag{1.8}$$

Theorem 1.2. (L^2 -concentration of blow-up solutions). *Let $N \geq 2$, $0 < T < \frac{\pi}{2\omega}$, and $\phi(t)$ be a radially symmetric solution of the Cauchy problem (1.1), (1.2) in $C([0, T]; H)$ such that $\phi(t)$ blows up at a finite time T .*

(i) *If $a(t)$ is a decreasing function from $[0, T)$ to \mathbb{R}^+ such that $a(t) \rightarrow 0(t \rightarrow T)$ and $\frac{(T-t)^{1/2}}{a(t)} \rightarrow 0(t \rightarrow T)$, then*

$$\liminf_{t \rightarrow T} \|\phi(t)\|_{L^2(|x| < a(t))} \geq \|Q\|_{L^2}, \tag{1.9}$$

where Q is a ground state solution of (1.7).

(ii) *For any $\varepsilon > 0$, there exists a $K > 0$ such that*

$$\liminf_{t \rightarrow T} \|\phi(t)\|_{L^2(|x| < K(T-t)^{1/2})} > (1 - \varepsilon)\|Q\|_{L^2}. \tag{1.10}$$

The following result is an immediate consequence of Theorem 1.2.

Corollary 1.3. *If all the assumptions in Theorem 1.2 are satisfied, then for any q with $2 < q \leq \infty$ there exists an $M > 0$ such that*

$$\frac{M}{(T-t)^\alpha} \leq \|u(t)\|_{L^q}, \quad t \in [0, T), \tag{1.11}$$

where $\alpha = N(q - 2)/4q$.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, we state the proofs of Theorem 1.2 and Corollary 1.3.

We conclude this section with some notation. We denote $L^q(\mathbb{R}^N)$ and $\|\cdot\|_{L^q(\mathbb{R}^N)}$ by L^q and $\|\cdot\|_{L^q}$, respectively. The various positive constants will be simply denoted by C .

2. PROOF OF THEOREM 1.1

Consider the Cauchy problem of the nonlinear Schrödinger equation without harmonic potential

$$i\psi_t + \frac{1}{2} \Delta \psi + |\psi|^{4/N} \psi = 0, \quad t \geq 0, x \in \mathbb{R}^N, \tag{2.1}$$

$$\psi(0, x) = \varphi(x). \tag{2.2}$$

It is well known (see e.g. [4]) that the Cauchy problem (2.1),(2.2) is locally well posed in H . In addition, let $\psi(t, x)$ be a solution of (2.1)-(2.2) in

$C([0, T], H)$ for some $T \in [0, +\infty)$ (maximal existence time), then $\psi(t, x)$ satisfies the following two conservation laws

$$\|\psi(t)\|_{L^2} = \|\varphi\|_{L^2}, \quad (2.3)$$

$$E^*(\psi(t)) := \frac{1}{2} \|\nabla \psi(t)\|_{L^2}^2 - \frac{1}{1 + 2/N} \|\psi(t)\|_{L^4}^4 = E^*(\varphi). \quad (2.4)$$

It is also well known [8, 13] that, for some initial data φ , the solution $\psi(t, x)$ of (2.1)-(2.2) blows up in finite time. And the lower bound of the blow-up rate rate is known as follows:

Lemma 2.1. ([13]). *Assume that ψ is a blow-up solution of the Cauchy problem (2.1), (2.2) in $C([0, T], H)$ for some $T \in (0, +\infty)$, where $[0, T]$ is the maximal existence time. Then there is a constant $M > 0$, such that*

$$\|\nabla \psi(t)\|_{L^2} \geq \frac{M}{\sqrt{T-t}}, \quad t \in (0, \tau). \quad (2.5)$$

Carles [3] builds the relation between the solutions of (1.1)-(1.2) and that of (2.1)-(2.2) as follows:

Lemma 2.2. (Carles [3]). (1) *Assume that ψ is a solution of the Cauchy problem (2.1), (2.2) in $C([0, t_0], H)$ where $t_0 > 0$. Let*

$$\phi(t, x) = \frac{1}{(\cos \omega t)^{\frac{N}{2}}} e^{-i \frac{\omega}{2} x^2 \tan \omega t} \psi\left(\frac{\tan \omega t}{\omega}, \frac{x}{\cos \omega t}\right), \quad (2.6)$$

then $\phi(t, x) \in C([0, \frac{\arctan \omega t_0}{\omega}], H)$ is a solution of the Cauchy problem (1.1), (1.2). In particular, if $\psi(t) \in C([0, +\infty), H)$ (global solution),

$$\phi(t) \in C([0, \frac{\pi}{2\omega}], H)$$

is a local solution of the Cauchy problem (1.1), (1.2).

(2) *Assume that ϕ is a solution of the Cauchy problem (1.1), (1.2) in $C([0, t'_0], H)$ where $t'_0 \in (0, \frac{\pi}{2\omega}]$. Let*

$$\psi(t, x) = \frac{1}{(1 + (\omega t)^2)^{\frac{N}{4}}} e^{i \frac{x^2}{2} \frac{\omega^2 t}{1 + \omega^2 t^2}} \phi\left(\frac{\arctan \omega t}{\omega}, \frac{x}{(1 + (\omega t)^2)^{\frac{1}{2}}}\right), \quad (2.7)$$

then $\psi(t, x) \in C([0, \frac{\tan \omega t'_0}{\omega}], H)$ is a solution of the Cauchy problem (2.1), (2.2). In particular, if $\phi \in C([0, \frac{\pi}{2\omega}], H)$, $\psi(t, x) \in C([0, +\infty), H)$ is a global solution of the Cauchy problem (2.1), (2.2).

Remark. The transform of Lemma 2.2 is based on the fact that $\phi(0, x) = \psi(0, x) = \varphi$. Moreover, the solution obtained by Lemma 2.2 is usually a local solution. The maximal existence time of the local solution for the Cauchy problem relies on the initial data φ .

Based on Lemma 2.2, we get following two lemmas directly.

Lemma 2.3. *Assume that ϕ is a blow-up solution of the Cauchy problem (1.1), (1.2) in $C([0, T), H)$ where $T \in (0, \frac{\pi}{2\omega}]$ (maximal existence time). Let $\psi(t, x)$ be defined by (2.7), then*

- (1) $\psi(t, x) \in C([0, \frac{\tan \omega T}{\omega}), H)$ is a blow-up solution of the Cauchy problem (2.1), (2.2), where $[0, \frac{\tan \omega T}{\omega})$ is the maximal existence time.
- (2)

$$\phi(t, x) = \frac{1}{(\cos \omega t)^{\frac{N}{2}}} e^{-i\frac{\omega}{2}x^2 \tan \omega t} \psi\left(\frac{\tan \omega t}{\omega}, \frac{x}{\cos \omega t}\right), \quad t \in [0, T).$$

With the lemmas above, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.3, let $\psi(t, x)$ be defined by (2.7), then $\psi(t, x) \in C([0, \frac{\tan \omega T}{\omega}), H)$ is a blow-up solution of the Cauchy problem (2.1), (2.2), where $[0, \frac{\tan \omega T}{\omega})$ is the maximal existence time. And (2.6) holds true. That is,

$$\phi(t, x) = \frac{1}{(\cos \omega t)^{\frac{N}{2}}} e^{-i\frac{\omega}{2}x^2 \tan \omega t} \psi\left(\frac{\tan \omega t}{\omega}, \frac{x}{\cos \omega t}\right), \quad t \in [0, T).$$

Then for $t \in [0, T), T \in (0, \frac{\pi}{2\omega})$, we have

$$\begin{aligned} \|\nabla \phi(t)\|_{L^2} &= \frac{1}{\cos \omega t} \left\| -i\omega x \sin(\omega t) \psi\left(\frac{\tan \omega t}{\omega}, \cdot\right) + \nabla \psi\left(\frac{\tan \omega t}{\omega}, \cdot\right) \right\|_{L^2} \\ &\geq \frac{1}{\cos \omega t} (\|\nabla \psi\left(\frac{\tan \omega t}{\omega}, \cdot\right)\|_{L^2} - \omega \|x \psi\left(\frac{\tan \omega t}{\omega}, \cdot\right)\|_{L^2}) \quad (2.8) \\ &\geq \|\nabla \psi\left(\frac{\tan \omega t}{\omega}, \cdot\right)\|_{L^2} - \omega \|x \psi\left(\frac{\tan \omega t}{\omega}, \cdot\right)\|_{L^2}. \end{aligned}$$

Then we claim that there is a constant $C > 0$, such that

$$\omega \|x \psi\left(\frac{\tan \omega t}{\omega}, \cdot\right)\|_{L^2} \leq C, \quad t \in [0, T). \quad (2.9)$$

In fact, consider

$$J(t^*) \equiv \int |x|^2 |\psi(t^*, x)|^2 dx = \|x \psi(t^*)\|_{L^2}^2, \quad t^* \in [0, \frac{\tan \omega T}{\omega}).$$

By Weinstein [13], we have that $\frac{d^2}{dt^2} J(t^*) = 4E^*(\varphi)$, $E^*(\varphi)$ a constant. By an analytical identity

$$\begin{aligned} J(t^*) &= J(0) + J'(0)t^* + \int_0^{t^*} J''(s)(t^* - s)ds \\ &= J(0) + J'(0)t^* + 2E^*(\varphi)(t^*)^2. \end{aligned} \tag{2.10}$$

Letting $t^* = \frac{\tan \omega t}{\omega}, t \in [0, T), 0 < T < \frac{\pi}{2\omega}$, we have

$$\|x\psi(\frac{\tan \omega t}{\omega}, \cdot)\|_{L^2}^2 = J(0) + J'(0)\frac{\tan \omega t}{\omega} + 2E^*(\varphi)(\frac{\tan \omega t}{\omega})^2. \tag{2.11}$$

This equation implies (2.9).

By Lemma 2.1, there is a constant $C > 0$, such that

$$\|\nabla\psi(\frac{\tan \omega t}{\omega}, \cdot)\|^2 \geq \frac{C}{\sqrt{\tan \omega T - \tan \omega t}}, \quad t \in [0, T). \tag{2.12}$$

From (2.8), (2.9), (2.12),

$$\|\nabla\phi(t, x)\|_{L^2} + C \geq \frac{C}{\sqrt{\tan \omega T - \tan \omega t}}, \quad t \in [0, T).$$

Since $\cos \omega t \geq \cos \omega T$, for $t \in [0, T), T \in (0, \frac{\pi}{2\omega})$, we have

$$\|\nabla\phi(t, x)\|_{L^2} + C \geq \frac{C}{\sqrt{\sin(\omega(T - t))}}.$$

It follows that

$$\|\nabla\phi(t, x)\|_{L^2} \geq \frac{C}{\sqrt{\sin(\omega(T - t))}}, \quad \text{as } t \rightarrow T.$$

Since $\frac{C}{\sqrt{\sin(\omega(T-t))}} \sim \frac{C}{\sqrt{\omega(T-t)}}$, as $t \rightarrow T$, we have

$$\|\nabla\phi(t, x)\|_{L^2} \geq \frac{C}{\sqrt{(T - t)}}, \quad \text{as } t \rightarrow T.$$

3. PROOF OF THEOREM 1.2

In order to state our proof precisely, we need an auxiliary function $\rho(x)$: let $\rho(x) = \rho(|x|)$ be a radially symmetric nonnegative function in $C_0^1(\mathbb{R}^N)$ such that

$$\rho(r) = \begin{cases} 1, & r = |x| < 1/2, \\ 0, & r = |x| > 1, \end{cases} \tag{3.1}$$

and $0 \geq \rho_r(r) \geq -8$.

Now, we give several lemmas and propositions.

Lemma 3.1. (Strauss[10]). *Assume that $N \geq 2$. Let $V(x)$ be a radially symmetric function in $H^1(\mathbb{R}^N)$. Then for any $R > 0$*

$$\|V\|_{L^\infty(|x|>R)}^2 \leq C_0 R^{-(N-1)} \|\nabla V\|_{L^2(|x|>R)} \|V\|_{L^2(|x|>R)}, \tag{3.2}$$

where C_0 does not depend on R and $V(x)$.

Lemma 3.2. (Weinstein[13]). *For any $f \in H^1(\mathbb{R}^N)$,*

$$\|f\|_{L^{2+4/N}}^{2+4/N} \leq \left(\frac{1}{2} + \frac{1}{N}\right) \left(\frac{\|f\|_{L^2}}{\|Q\|_{L^2}}\right)^{4/N} \|\nabla f\|_{L^2}^2,$$

where Q is the unique ground state solution of equation (1.7).

Lemma 3.3. *Assume that all the assumptions in Theorem 1.2 are satisfied. We put $\lambda(t) = \|\nabla\phi(t)\|_{L^2}$. Then, there exist two positive constants M_1 and M_2 such that*

$$\limsup_{t \rightarrow T} \frac{\|\nabla\phi(t)\|_{L^2}}{\|\nabla\phi(t)\|_{L^2(|x|<M_1/\lambda(t))}} \leq M_2. \tag{3.3}$$

Proof. Let M_1 be a large positive constant to be determined later. From (1.6), we have

$$\begin{aligned} \lambda(t)^2 &\leq \frac{2N}{N+2} \left(\|\rho\left(\frac{\lambda(t)}{M_1}x\right)\phi(t)\|_{L^{2+\frac{4}{N}}} \right. \\ &\quad \left. + \|\{1 - \rho\left(\frac{\lambda(t)}{M_1}x\right)\}\phi(t)\|_{L^{2+\frac{4}{N}}}\right)^{2+\frac{4}{N}} + 2E(\varphi) \\ &\leq C \|\rho\left(\frac{\lambda(t)}{M_1}x\right)\phi(t)\|_{L^{2+4/N}}^{2+4/N} + C \|\{1 - \rho\left(\frac{\lambda(t)}{M_1}x\right)\}\phi(t)\|_{L^{2+\frac{4}{N}}}^{2+\frac{4}{N}} + 2E(\varphi), \end{aligned} \tag{3.4}$$

where $\rho(x)$ is defined as in (3.1).

The Gagliardo-Nirenberg inequality implies that

$$\begin{aligned} \|\rho\left(\frac{\lambda(t)}{M_1}x\right)\phi(t)\|_{L^{2+4/N}}^{2+4/N} &\leq C \|\rho\left(\frac{\lambda(t)}{M_1}x\right)\phi(t)\|_{L^2}^{4/N} \|\nabla\{\rho\left(\frac{\lambda(t)}{M_1}x\right)\phi(t)\}\|_{L^2}^2 \\ &\leq C \|\phi(t)\|_{L^2}^{4/N} \|\nabla\rho\left(\frac{\lambda(t)}{M_1}x\right) \cdot \phi(t) + \rho\left(\frac{\lambda(t)}{M_1}x\right) \cdot \nabla\phi(t)\|_{L^2}^2 \\ &\leq C \|\phi(t)\|_{L^2}^{4/N} \|\nabla\rho\left(\frac{\lambda(t)}{M_1}x\right) \cdot \phi(t)\|_{L^2}^2 + C \|\phi(t)\|_{L^2}^{4/N} \|\rho\left(\frac{\lambda(t)}{M_1}x\right) \cdot \nabla\phi(t)\|_{L^2}^2 \\ &\leq C \|\phi(t)\|_{L^2}^{4/N} \left\| \frac{\lambda(t)}{M_1} \cdot \nabla\rho(x) \cdot \phi(t) \right\|_{L^2}^2 + C \|\phi(t)\|_{L^2}^{4/N} \|\nabla\phi(t)\|_{L^2(|x|<\frac{M_1}{\lambda(t)})}^2 \\ &\leq C \|\phi(t)\|_{L^2}^{4/N} \frac{\lambda(t)^2}{M_1^2} \|\phi(t)\|_{L^2}^2 + C \|\phi(t)\|_{L^2}^{4/N} \|\nabla\phi(t)\|_{L^2(|x|<\frac{M_1}{\lambda(t)})}^2 \end{aligned}$$

$$\leq \frac{C\|\phi(t)\|_{L^2}^{2+4/N}}{M_1^2} \lambda(t)^2 + C\|\phi(t)\|_{L^2}^{4/N} \|\nabla\phi(t)\|_{L^2(|x| < \frac{M_1}{\lambda(t)})}^2. \quad (3.5)$$

By Lemma 3.1, we have

$$\begin{aligned} C\|\{1 - \rho(\frac{\lambda(t)}{M_1}x)\}\phi(t)\|_{L^{2+4/N}}^{2+4/N} &\leq \|\phi(t)\|_{L^{2+4/N}(|\frac{\lambda(t)}{M_1}x| > \frac{1}{2})}^{2+4/N} \\ &\leq C\|\phi(t)\|_{L^\infty(|x| > \frac{M_1}{2\lambda(t)})}^{4/N} \|\phi(t)\|_{L^2(|x| > \frac{M_1}{2\lambda(t)})}^2 \\ &\leq C(\frac{\lambda(t)}{M_1})^{2(N-1)/N} \|\nabla\phi(t)\|_{L^2(|x| > \frac{M_1}{2\lambda(t)})}^{2/N} \|\phi(t)\|_{L^2(|x| > \frac{M_1}{2\lambda(t)})}^{2+2/N} \\ &\leq C(\frac{\lambda(t)}{M_1})^{2(N-1)/N} \lambda(t)^{2/N} \|\phi(t)\|_{L^2(|x| > \frac{M_1}{2\lambda(t)})}^{2+2/N} \leq \frac{C\|\phi(t)\|_{L^2}^{2+2/N}}{M_1^{2(N-1)/N}} \lambda(t)^2. \end{aligned} \quad (3.6)$$

From (2.10)-(3.6), we have

$$\begin{aligned} \lambda(t)^2 &\leq C_1\|\phi(t)\|_{L^2}^{4/N} \|\nabla\phi(t)\|_{L^2(|x| < \frac{M_1}{\lambda(t)})}^2 \\ &\quad + \left\{ \frac{C_2\|\phi(t)\|_{L^2}^{2+4/N}}{M_1^2} + \frac{C_3\|\phi(t)\|_{L^2}^{2+2/N}}{M_1^{2(N-1)/N}} \right\} \lambda(t)^2 + 2E(\varphi). \end{aligned} \quad (3.7)$$

If we choose M_1 so large that

$$\frac{C_2\|\phi(t)\|_{L^2}^{2+4/N}}{M_1^2} + \frac{C_3\|\phi(t)\|_{L^2}^{2+2/N}}{M_1^{2(N-1)/N}} \leq \frac{1}{2}, \quad (3.8)$$

then we have by (3.7)

$$\lambda(t)^2 \leq 2C_1\|\phi(t)\|_{L^2}^{4/N} \|\nabla\phi(t)\|_{L^2(|x| < \frac{M_1}{\lambda(t)})}^2 + 4E(\varphi). \quad (3.9)$$

Since $\|\nabla\phi(t)\|_{L^2} \rightarrow \infty$ as $t \rightarrow T$, (3.9) and (1.5) imply that

$$\|\nabla\phi(t)\|_{L^2(|x| < M_1/\lambda(t))} \rightarrow \infty (t \rightarrow T).$$

This fact and (3.9) show (3.3).

We next show the following proposition concerning the relation between the blow-up rate of $\|\nabla\phi(t)\|_{L^2}$ and the rate of L^2 -concentration.

Proposition 3.4. *Assume that all the assumptions in Theorem 1.2 are satisfied. We put $\lambda(t) = \|\nabla\phi(t)\|_{L^2}$.*

(i) If $a(t)$ is a decreasing function from $[0, T)$ to \mathbb{R}^+ such that $a(t) \rightarrow 0(t \rightarrow T)$ and $\frac{1}{\lambda(t)a(t)} \rightarrow 0(t \rightarrow T)$, then

$$\liminf_{t \rightarrow T} \|\phi(t)\|_{L^2(|x| < a(t))} \geq \|Q\|_{L^2}. \tag{3.10}$$

(ii) For any $\varepsilon > 0$, there exists a $K > 0$ such that

$$\liminf_{t \rightarrow T} \|\phi(t)\|_{L^2(|x| < K/\lambda(t))} \geq (1 - \varepsilon)\|Q\|_{L^2}. \tag{3.11}$$

Proof. Let $\rho(x)$ be defined as in (3.1). We put $\rho_a(x) = \rho(\frac{x}{a(t)})$ and $\lambda_a(t) = \|\nabla(\rho_a\phi(t))\|_{L^2}$.

(i) By (1.6) and Lemma 3.1, we have

$$\begin{aligned} & \lambda(t)^2 - \frac{2N}{N+2} \|\phi(t)\|_{L^{2+4/N}(|x| < \frac{a(t)}{2})}^{2+4/N} \\ &= \frac{2N}{N+2} \|\phi(t)\|_{L^{2+4/N}(|x| > \frac{a(t)}{2})}^{2+4/N} + 2E(\varphi) - \omega^2 \|x\phi(t)\|_{L^2}^2 \\ &\leq \frac{2N}{N+2} \|\phi(t)\|_{L^\infty(|x| > \frac{a(t)}{2})}^{4/N} \|\phi(t)\|_{L^2(|x| > \frac{a(t)}{2})}^2 + 2E(\varphi) \\ &\leq Ca(t)^{-2(N-1)/N} \|\phi(t)\|_{L^2(|x| > \frac{a(t)}{2})}^{2+2/N} \lambda(t)^{2/N} + 2E(\varphi). \end{aligned} \tag{3.12}$$

A simple calculation gives us

$$-\frac{2N}{N+2} \|\rho_a\phi(t)\|_{L^{2+4/N}}^{2+4/N} \leq -\frac{2N}{N+2} \|\phi(t)\|_{L^{2+4/N}(|x| < \frac{a(t)}{2})}^{2+4/N}, \tag{3.13}$$

$$\begin{aligned} \lambda_a(t)^2 &\leq (\|\rho_a\nabla\phi\|_{L^2} + \|\nabla\rho_a \cdot \phi\|_{L^2})^2 \leq (\lambda(t) + \frac{C\|\phi(t)\|_{L^2}}{a(t)})^2 \\ &\leq \lambda(t)^2 + \frac{C\|\phi(t)\|_{L^2}\lambda(t)}{a(t)} + \frac{C\|\phi(t)\|_{L^2}^2}{a(t)^2}. \end{aligned} \tag{3.14}$$

On the other hand, from Lemma 3.2, we have

$$\frac{2N}{N+2} \|\rho_a\phi(t)\|_{L^{2+4/N}}^{2+4/N} \leq (\frac{\|\rho_a\phi(t)\|_{L^2}}{\|Q\|_{L^2}})^{4/N} \lambda_a(t)^2. \tag{3.15}$$

By (3.12)-(3.15), we obtain

$$\begin{aligned} 1 - (\frac{\rho_a\phi(t)}{\|Q\|_{L^2}})^{4/N} &\leq C\|\phi(t)\|_{L^2}^{2+2/N} \frac{\lambda(t)^{2/N}}{\lambda_a(t)^{2/N} (a(t)\lambda_a(t))^{2(N-1)/N}} \\ &+ \frac{2E(\varphi)}{\lambda_a(t)^2} + \frac{C\|\phi(t)\|_{L^2}^2\lambda(t)}{a(t)\lambda_a(t)^2} + \frac{C\|\phi(t)\|_{L^2}^2}{(a(t)\lambda_a(t))^2}. \end{aligned} \tag{3.16}$$

Letting $t \rightarrow T$ in (3.16), we obtain by (1.5), Lemma 3.3, and the assumption of $a(t)$

$$\limsup_{t \rightarrow T} \left\{ 1 - \left(\frac{\|\rho_a \phi(t)\|_{L^2}}{\|Q\|_{L^2}} \right)^{4/N} \right\} \leq 0, \quad (3.17)$$

which proves Proposition 3.4(i).

(ii) We use the same argument as in the proof of Proposition 3.4(i) to prove Proposition 3.4(ii).

Let K be a sufficiently large constant to be determined later. If we insert $a(t) = \frac{K}{\lambda(t)}$ into (3.16), we have by Lemma 3.3

$$\begin{aligned} & 1 - \left(\|\rho\left(\frac{\lambda(t)}{K}x\right)\phi(t)\|_{L^2} / \|Q\|_{L^2} \right)^{4/N} \\ & \leq C \|\phi(t)\|_{L^2}^{2+\frac{2}{N}} K^{-\frac{2(N-1)}{N}} + E(\varphi)\lambda(t)^{-2} + C\|\phi(t)\|_{L^2} K^{-1} + C\|\phi(t)\|_{L^2}^2 K^{-2}. \end{aligned} \quad (3.18)$$

Letting $t \rightarrow T$ in (3.18), we obtain

$$\begin{aligned} & \limsup_{t \rightarrow T} \left\{ 1 - \left(\|\rho\left(\frac{\lambda(t)}{K}x\right)\phi(t)\|_{L^2} / \|Q\|_{L^2} \right)^{4/N} \right\} \\ & \leq C \|\phi(t)\|_{L^2}^{2+2/N} K^{-2(N-1)/N} + C\|\phi(t)\|_{L^2} K^{-1} + C\|\phi(t)\|_{L^2}^2 K^{-2}. \end{aligned} \quad (3.19)$$

(3.19) implies that if we choose K sufficiently large, then (3.11) holds.

Now, we are in a position to prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. By Theorem 1.1, we have

$$\lambda(t) \geq L(T-t)^{-1/2}, \quad t \in [0, T], \quad (3.20)$$

for some $L > 0$. Therefore, by Proposition 3.4 we obtain Theorem 1.2.

Proof of Corollary 1.3. For any $q > 2$, (1.10) and the Hölder inequality with $1/q + (q-2)/2q = 1/2$ gives us

$$C_1 \leq \|\phi(t)\|_{L^2(|x| < C_2(T-t)^{\frac{1}{2}})} \leq C_3(T-t)^{N(q-2)/4q} \|\phi(t)\|_{L^q}, \quad t \in [0, T] \quad (3.21)$$

for some C_1, C_2 and $C_3 > 0$, which completes the proof of Corollary 1.3.

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