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SOME OBSERVATIONS ON THE LOCAL BEHAVIOUR OF INFINITY-HARMONIC FUNCTIONS

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Abstract. We study some aspects of the local behaviour of infinity-harmonic functions. We study one-sided radial derivatives at points of extrema on the boundary of concentric balls and relate it to some issues in differentiability. In particular we show differentiability of these functions at such points of extrema. We use convexity and the Harnack inequality to obtain our results.

1. Introduction.

In this work we discuss some aspects of the local behaviour of infinity-harmonic functions. The works in [1, 3, 8, 10] contain discussions of the origins, history and motivation for studying these functions. These functions are turning out to be increasingly important of late due to the applications they are beginning to find in other areas. The recent works [3, 11] are particularly interesting as they describe the strong role played by these functions in certain aspects of game theory and the Monge-Kantarovich problem. In this work, we look at some aspects of differentiability in higher dimensions and bring together results about local behaviour which highlight the special nature of these functions.

Our effort in this work will be essentially to study properties of one-sided radial derivatives at the points of extrema on balls, referred to as Hopf derivatives, and show connections to the question of differentiability. See Theorem 2 in Section 2. The results and the methods developed will then be used to prove differentiability at points of extrema on balls. These will also imply that the Hopf derivatives are the gradients. This forms Theorem 1 in this work. A second consequence of our methods will be a proof of a version of the Crandall-Evans result about blow-up limits [9]. See Theorem 3 in Section 3. Our proof is based on the Harnack inequality and uses a device that is also used in the proof of Theorem 2.

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To describe our results more precisely, we introduce notation which will be used throughout this work. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a domain. We say a function $u = u(x)$ is infinity-harmonic in Ω if u solves

$$\Delta_\infty u(x) = \sum_{i,j=1}^{n} D_i u(x) D_j u(x) D_{ij} u(x) = 0, \quad x \in \Omega, \tag{1}$$

in the viscosity sense. For a more detailed description we refer to [1, 5, 10]. We call Δ_∞ the infinity-Laplacian. It is well known that u is locally Lipschitz continuous, and the comparison principle [2] and the Harnack inequality holds [1, 4, 5, 12]. Recently, the regularity of such functions has been improved to C^1 when Ω is a planar domain [15].

Let $B_\rho(x)$ denote the open ball in $\mathbb{R}^n$ with center x and radius ρ. By o we denote the origin in $\mathbb{R}^n$. Since Δ_∞ is invariant under translations, rotations and reflections, we assume from hereon that $o \in \Omega$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and $\omega,\ \nu \in S^{n-1}$ will denote directions in $\mathbb{R}^n$. For $B_\rho(x) \subset\subset \Omega$ and $0 \leq r \leq \rho$, let $M_x(r) = \sup_{B_r(x)} u(y)$ and $m_x(r) = \inf_{B_r(x)} u(y)$. Since u satisfies the maximum principle it follows that for $0 < r \leq \rho$ there are directions $\omega = \omega_x(r)$ and $\nu = \nu_x(r)$, not necessarily unique, such that

$$M_x(r) = u(x + r\omega_x(r)) \quad \text{and} \quad m_x(r) = u(x + r\nu_x(r)), \ \ \forall\, 0 < r \leq \rho.$$

The left-hand and the right-hand derivatives of M_x will be denoted by $M_x'(r-)$ and $M_x'(r+)$. We will drop the subscript o when working in $B_r(o)$. Theorems 1 and 2 constitute the main contribution of this work. We will state Theorem 1 in this section and postpone a statement of Theorem 2 to Section 2 due to its detailed nature.

Theorem 1. *Let $B_r(o) \subset\subset \Omega$ and $q \in \partial B_r(o)$ be such that $u(q) = M(r)$ and $\omega = q/|q|$. Then u is differentiable at q,*

$$|Du(q)| = \lim_{t \to r} \frac{M(r) - u(t\omega)}{r - t}, \ \ t < r, \ \ \text{and} \ \ Du(q) = |Du(q)|\omega.$$

Moreover, for every $r > 0$, we have that $M'(r-) \leq |Du(q)| \leq M'(r+)$, and there exists a point $q' \in \partial B_r(o)$ with $u(q') = M(r)$ and $|Du(q')| = M'(r+)$. An analogous result holds at points of minima.

A proof appears in Section 4. A consequence of Theorem 1 is that one may obtain differentiability in sets with special geometry. Thus, we have an improvement of Corollary 8.2 in [13] and obtain everywhere differentiability of solutions, since level sets of solutions are convex. In particular, the infinity-capacitary potential of a convex ring is differentiable everywhere, also see [14]. See Remarks 5 and 6 in Section 4. We also comment that the works in [13, 14] utilize approximating equations using the p-Laplacian. The convexity of level sets follows from a study of solutions of these equations. For the case of convex rings, an attempt based on the viscosity approach was made in [5]. We proved directly a lower bound for difference quotients. Theorem 1 will then yield that the gradients of infinity-capacitary potentials on convex sets are non-vanishing everywhere. We thank the referee for pointing out the work in [13] and the consequences of Theorem 1.

We have divided our work as follows. In Section 2 we recall some previously known facts and prove Theorem 2. Roughly stated we relate the Hopf derivatives at points of extrema to the derivatives of $M'(r)$ and $m'(r)$, also see Remark 2. These derivatives are increasing and are closely related to the gradients at the points of maxima and at o. This will then lead to the result in Theorem 1 that u is differentiable and the Hopf derivative at q is $|Du(q)|$. While there are points of maxima where $M'(r+)$ is attained it is unclear if this is true of every such point. Also see Lemma 1. Section 3 contains a proof of a version of the Crandall-Evans result based on the Harnack inequality. In particular, we will show if directional derivatives exist along a certain sequence $r_k \downarrow 0$, then the same result holds along a sequence which is a scalar multiple of r_k, namely, θr_k. See Theorem 3. This will imply readily that blow-up limits are linear. All this follows from our direct use of the Harnack inequality. However, it is unclear to us at this time if the Harnack inequality alone will yield differentiability. Also see the work [11].

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2. Properties of the Hopf derivatives

In this work, $u = u(x)$ will always denote an infinity-harmonic function in a domain $\Omega \subset R^n$, $n \geq 2$. We first recall some previously known facts. Let $\rho > 0$, $B_\rho(z) \subset\subset \Omega$ and $\eta \in S^{n-1}$ be a fixed direction. If $u > 0$ in $B_r(z)$, $r \leq \rho$, then for $0 \leq t < r$,

(i) $\quad \dfrac{u(z+t\eta)}{r-t}$ is increasing in t, and $u(z+t\eta)(r-t)$ is decreasing in t,

(ii) $\quad M_z(r)$ and $-m_z(r)$ are convex in r. Thus, $\dfrac{M_z(r)-u(z)}{r}$ (2)

$$\text{and} \quad \frac{u(z) - m_z(r)}{r} \text{ are both increasing in } r.$$

See [4, 5, 6, 7]. We use the following version of the Harnack inequality. Let $y,\ z \in \Omega$ and suppose that the distance of the segment yz to $\partial\Omega$ is at least $\delta > 0$. If $u(x) > 0$ in Ω, then

$$u(y) \geq u(z)e^{-|y-z|/\delta}. \tag{3}$$

See [1, 4, 5, 7, 12]. Suppose that $B_\rho(z) \subset\subset \Omega$. For every $0 \leq r \leq \rho$, let $I_r(z)$ denote the set of all $\omega \in S^{n-1}$ such that $M_z(r) = u(z + r\omega)$, and $J_r(z)$ be the set of all $\nu \in S^{n-1}$ such that $m_z(r) = u(z + r\nu)$. We now define the Hopf derivatives of u. For each $\omega \in I_r(z)$ and $\nu \in J_r(z)$, set

$$\begin{aligned} H_{z,r}(\omega) &= \lim_{t \uparrow r} \frac{M_z(r) - u(z + t\omega)}{r - t} \geq \frac{M_z(r) - u(z)}{r}, \\ h_{z,r}(\nu) &= \lim_{t \uparrow r} \frac{u(z + t\nu) - m_z(r)}{r - t} \geq \frac{u(z) - m_z(r)}{r}. \end{aligned} \tag{4}$$

We refer to these one-sided radial derivatives $H_{z,r}(\omega)$ and $h_{r,z}(\omega)$ as the Hopf derivatives of u at points of extrema on $\partial B_r(z)$. By (2) and local Lipschitz continuity, these exist and are positive. We use $\langle \xi, \zeta \rangle$ to denote the Euclidean inner product of vectors ξ and ζ in $\mathbb{R}^n$. For convenience we will often work in a ball centered at o and will suppress the notation o when there is no possibility of confusion. Let $B_\rho(o) \subset \Omega$; for small $0 < r \leq \rho$ and $0 < r' \leq \rho$, set

$$\begin{aligned} L_1(r, r') &= \sup\{\langle \omega, \nu \rangle : \omega \in I_r \text{ and } \nu \in J_{r'}\}, \\ L_2(r, r') &= \sup\{\langle \omega, \omega' \rangle : \omega \in I_r \text{ and } \omega' \in I_{r'}\}, \\ L_3(r, r') &= \sup\{\langle \nu, \nu' \rangle : \nu \in J_r \text{ and } \nu' \in J_{r'}\}. \end{aligned} \tag{5}$$

The quantities $l_1(r, r')$, $l_2(r, r')$ and $l_3(r, r')$ represent the infima of the corresponding inner products. In Theorem 2, we will calculate limits of these quantities when r' is a fixed scalar multiple of r as $r,\ r' \to 0$. For $z \in \Omega$, set

$$\Lambda(z) = \lim_{t \downarrow 0} \frac{M_z(t) - u(z)}{t}, \quad \text{and} \quad \lambda(z) = \lim_{t \downarrow 0} \frac{u(t) - m_z(t)}{t}. \tag{6}$$

These exist by (2)(i). We will also work in balls $B_s(r\omega)$, $\omega \in I_r$ and $B_s(r\nu)$, $\nu \in J_r$ which are centered at the points of extrema of u on $\partial B_r(o)$. For each $\omega \in I_r$ and $\nu \in J_r$, the expressions in (6) will then be $\Lambda(r\omega)$ and $\lambda(r\nu)$. We now state Theorem 2.

Theorem 2. *Let u be a non-constant infinity-harmonic in Ω, $B_\rho(o) \subset \Omega$ and $r \in (0, \rho)$. Then the following hold.*
(i) $0 < \inf_{\nu \in J_r} h_r(\nu)$, $\sup_{\nu \in J_r} h_r(\nu) < \infty$, $\inf_{\omega \in I_r} H_r(\omega), \sup_{\omega \in I_r} H_r(\omega) < \infty$.
(ii) For $0 < r_1 < r_2$,

$$\sup_{\omega \in I_{r_1}} H_{r_1}(\omega) \leq \frac{M(r_2) - M(r_1)}{r_2 - r_1} \leq \inf_{\omega \in I_{r_2}} H_{r_2}(\omega) \quad \textit{and}$$

$$\sup_{\nu \in J_{r_1}} h_{r_1}(\nu) \leq \frac{m(r_1) - m(r_2)}{r_2 - r_1} \leq \inf_{\nu \in J_{r_2}} h_{r_2}(\nu).$$

(iii) For each $\omega \in I_r$, $\nu \in J_r$, $H_r(\omega) \geq \Lambda(o)$, $h_r(\nu) \geq \lambda(o)$ *and*

$$\lim_{r \to 0} \sup_{\omega \in I_r} H_r(\omega) = \lim_{r \to 0} \inf_{\omega \in I_r} H_r(\omega) = \Lambda(o),$$

$$\lim_{r \to 0} \sup_{\nu \in J_r} h_r(\nu) = \lim_{r \to 0} \inf_{\nu \in J_r} h_r(\nu) = \lambda(o).$$

Moreover, $\Lambda(o) = \lambda(o)$, *and if u is differentiable at o, then* $|Du(o)| = \Lambda(o)$.
(iv) For every $r_1 < r < r_2 \leq \rho$, $\omega \in I_r$ *and* $\nu \in J_r$,

$$\sup_{\omega' \in I_{r_1}} H_{r_1}(\omega') \leq H_r(\omega) \leq \Lambda(r\omega) \leq \frac{M(r_2) - M(r)}{r_2 - r} \leq \inf_{\omega' \in I_{r_2}} H_{r_2}(\omega'),$$

$$\sup_{\nu' \in J_{r_1}} h_{r_1}(\nu') \leq h_r(\nu) \leq \lambda(r\nu) \leq \frac{m(r) - m(r_2)}{r_2 - r} \leq \inf_{\nu' \in J_{r_2}} h_{r_2}(\nu').$$

(v) For fixed $\alpha > 0$ *and* $\beta > 0$,

$$\lim_{r \to 0} l_1(\alpha r, \beta r) = \lim_{r \to 0} L_1(\alpha r, \beta r) = -1, \quad \textit{and}$$

$$\lim_{r \to 0} l_i(\alpha r, \beta r) = \lim_{r \to 0} L_i(\alpha r, \beta r) = 1, \quad i = 2, 3.$$

First we make a few remarks. While $\omega(r)$ and $\nu(r)$ need not be unique, part (ii) shows that the Hopf derivatives are related to the derivatives of the convex function $M(r)$. Part (v) uses the Harnack inequality and a variant of the method will be used in the proof of Theorem 3.

Proof. We will work the proof for $H_r = H_{o,r}$. The proofs for h_r are analogous. Firstly, by (2)(i), the strong maximum principle and (4) we have that for $0 \leq t < r$, every $\omega \in I_r$ and $\nu \in J_r$,

$$0 < \frac{M(r) - u(o)}{r} \leq \frac{M(r) - u(t\omega)}{r - t} \leq H_r(\omega) \text{ and}$$

$$0 < \frac{u(o) - m(r)}{r} \leq \frac{u(t\nu) - m(r)}{r - t} \leq h_r(\nu).$$

Clearly, $h_r(\nu)$ and $H_r(\omega)$ are bounded by the local Lipschitz constant for u. This proves part (i).

We now show part (ii). Fix $0 < r_1 < r_2 < \rho$ and $\omega \in I_{r_1}$; let $0 < t < r_1$. Note that $u(r_1\omega) = M(r_1)$, $M_{t\omega}(s) = sup_{B_s(t\omega)} u(x)$ and $v(x) = u(x) - u(t\omega)$ is infinity-harmonic in $B_{r_2-t}(t\omega)$. Set $\sigma(s, t\omega) = sup_{B_s(t\omega)} v(x) = M_{t\omega}(s) - u(t\omega)$. Clearly $\sigma(0, t\omega) = 0$; by (2)(ii), $\sigma(s, t\omega)$ is convex in s and $\sigma(s, t\omega)/s$ is increasing in s. By the maximum principle, $M_{t\omega}(r_1 - t) = M(r_1)$; thus, (2) and the fact that $B_{r_2-t}(t\omega) \subset B_{r_2}(o)$ imply that

$$\frac{\sigma(r_1 - t, t\omega)}{r_1 - t} = \frac{M(r_1) - u(t\omega)}{r_1 - t} \le \frac{\sigma(r_2 - t, t\omega)}{r_2 - t} \le \frac{M(r_2) - u(t\omega)}{r_2 - t}. \tag{7}$$

It follows from (2)(i), (4) and (7) that

$$H_{r_1}(\omega) = \lim_{t \to r_1} \frac{M(r_1) - u(t\omega)}{r_1 - t} \le \frac{M(r_2) - M(r_1)}{r_2 - r_1}. \tag{8}$$

By the maximum principle $u(t\omega) \le M(t)$ when $0 < t < r_1$. Thus, from (7) and (2)(i)

$$\frac{M(r_1) - M(t)}{r_1 - t} \le \frac{M(r_1) - u(t\omega)}{r_1 - t} \le H_{r_1}(\omega). \tag{9}$$

Thus, by (8) and (9), we see that

$$\frac{M(r_1) - M(t)}{r_1 - t} \le H_{r_1}(\omega) \le \frac{M(r_2) - M(r_1)}{r_2 - r_1}.$$

Since $\omega \in I_{r_1}$ is arbitrary, by using the convexity of $M(r)$, it follows that for $0 < r_1 < r_2 \le \rho$,

$$\sup_{\omega \in I_{r_1}} H_{r_1}(\omega) \le \frac{M(r_2) - M(r_1)}{r_2 - r_1} \le \inf_{\omega \in I_{r_2}} H_{r_2}(\omega). \tag{10}$$

This proves part (ii).

For part (iii), we first observe from part (ii) that the supremum and the infimum of H_r and h_r are increasing in r. Set $\Lambda_1 = \lim_{r\to 0} \sup_{\omega \in I_r} H_r(\omega)$, $\Lambda_2 = \lim_{r\to 0} \inf_{\omega \in I_r} H_r(\omega)$, $\lambda_1 = \lim_{r \to 0} \inf_{\nu \in J_r} h_r(\nu)$, and $\lambda_2 = \lim_{r\to 0} \inf_{\nu \in J_r} h_r(\nu)$. From (10) it follows that

$$\frac{M(r) - u(o)}{r} \le \inf_{\omega \in I_r} H_r(\omega) \le \sup_{\omega \in I_r} H_r(\omega) \le \frac{M(r+\delta) - M(r)}{\delta}, \ \ \forall\, \delta > 0.$$

Using the convexity of $M(r) - u(o)$ and (6) we see that

$$\Lambda(o) = \lim_{r \to 0} \frac{M(r) - u(o)}{r} \le \Lambda_1 \le \Lambda_2 \le \frac{M(\delta) - u(o)}{\delta}.$$

Letting $\delta \to 0$ yields

$$\Lambda_i = \lim_{r \downarrow 0} \frac{M(r) - u(o)}{r}, \text{ and } \lambda_i = \lim_{r \downarrow 0} \frac{u(o) - m(r)}{r}, \quad i = 1, 2. \tag{11}$$

Thus, $\Lambda(o) = \Lambda_i$ and $\lambda(o) = \lambda_i$. Since $v = u(x) - m(\rho) \geq 0$ in $B_\rho(o)$, we now apply (2)(i) to obtain that, for $0 < r < \rho$ and any $\omega \in I_r$,

$$(M(r) - m(\rho))(\rho - r) = (u(r\omega)) - m(\rho))(\rho - r) \leq (u(o) - m(\rho))\rho.$$

Rearranging and using convexity and (6), we see that

$$\Lambda(o) \leq \frac{M(r) - u(o)}{r} \leq \frac{u(o) - m(\rho)}{\rho - r}.$$

Clearly, $\Lambda(o) \leq \lambda(o)$. Now apply (2)(i) to $w(x) = M(\rho) - u(x)$ in $B_\rho(o)$ to obtain

$$\frac{M(\rho) - u(o)}{\rho - r} \geq \frac{u(o) - m(r)}{r} \geq \lambda(o).$$

Thus, $\Lambda(o) = \lambda(o)$. We now show that $\Lambda(o) = |Du(o)|$, if u is differentiable at o. Also see [9]. Let $e \in S^{n-1}$ and $y = re \in \partial B_r(o)$. Then $M(r) - u(o) \geq u(y) - u(o) = r\langle Du(o), e\rangle + o(r)$, $r \to 0$. By (11), $\Lambda(o) = \lim_{r\downarrow 0}(M(r) - u(o))/r \geq \langle Du(o), e\rangle$, for all $e \in S^{n-1}$. Hence $\Lambda(o) \geq |Du(o)|$. On the other hand for any $\omega \in I_r$,

$$\frac{M(r) - u(o)}{r} = \frac{u(r\omega) - u(o)}{r} = \langle Du(o), \omega\rangle + o(1) \leq |Du(o)| + o(1).$$

Thus, $\Lambda(o) = |Du(o)|$.

We now prove part (iv). Fix $\omega = \omega_o \in I_r(o)$ and set $q = r\omega$. We work in the ball $B_s(q)$. Recall (6) and note that $M_q(s) = sup_{B_s(q)} u(x)$. We now apply parts (ii) and (iii) to the ball $B_s(q)$. Recall from (4) the definitions of $I_s(q)$ and the Hopf derivatives $H_{q,s}$ at the points of maxima on $\partial B_s(q)$. Applying part (iii) to $H_{q,s}$, we see that

$$\lim_{s \to 0} \sup_{\omega_q \in I_s(q)} H_{q,s}(\omega_q) = \lim_{s \to 0} \inf_{\omega_q \in I_s(q)} H_{q,s}(\omega_q) = \Lambda(q).$$

Next we show that $\Lambda(q) = \Lambda(r\omega) \geq H_r(\omega)$. If x is on the line segment oq and $s = r - |x|$, then by the maximum principle

$$\frac{u(q) - m_q(s)}{s} = \frac{M(r) - m_q(s)}{s} \geq \frac{M(r) - u(x)}{r - |x|}.$$

Note that $x = |x|\omega$; letting $s \to 0$ ($x \to q$), using (4), (6) and part (iii) we see that $\Lambda(q) \geq H_r(\omega) > 0$. We work in a ball $B_{r_2}(o)$, $r < r_2$. Once again applying (6) to q, 2(ii), (10) and the maximum principle it follows that

$$H_r(\omega) \le \Lambda(r\omega) = \lim_{t\to 0}\frac{M_q(t)-u(q)}{t} \le \frac{M_q(r_2-r)-u(q)}{\rho - r} \le \frac{M(r_2)-M(r)}{r_2-r} \le \inf_{\omega'\in I_{r_2}} H_{r_2}(\omega'). \tag{12}$$

An application of (10) to (12) yields the conclusion. An analogous statement for h may now be derived by arguing as above.

We now prove part (v). We assume $\Lambda(o)>0$, see Remark 1. Fix $\alpha>0$ and $\beta>0$; let $c=\max(\alpha,\beta)$, $0<r<\rho$ and $s>1$. Select any $\omega\in I_{\alpha r}$ and $\nu \in J_{\beta r}$; note that $M(\alpha r)=u(\alpha r\omega)$ and $m(\beta r)=u(\beta r\nu)$. The idea here is to use the Harnack inequality (3) as follows. Fix $s>c$ and choose $r<\rho/s$, then

$$m(\beta r)-m(sr) \ge \{M(\alpha r)-m(sr)\}\exp\Big(-\frac{|\alpha\omega-\beta\nu|}{s-c}\Big). \tag{13}$$

Since $|\alpha\omega-\beta\nu| = \sqrt{\alpha^2+\beta^2-2\alpha\beta\langle\nu,\omega\rangle}$, we see that (13) continues to hold with $L_1(\alpha r,\beta r)$ or $l_1(\alpha r,\beta r)$ (see (5)) in place of $\langle \nu,\omega\rangle$. We reserve $A(r)$ for either quantity. We divide (13) by r and write $m(\beta r)-m(sr) = m(\beta r)-u(o)+u(o)-m(sr)$. Now letting $r\to 0$ and using (6), we see that

$$(s-\beta)\Lambda(o) \ge (s+\alpha)\Lambda(o)\exp\Big(-\liminf_{r\to 0}\frac{\sqrt{\alpha^2+\beta^2-2\alpha\beta B(r)}}{s-c}\Big).$$

Setting $U=\limsup_{r\to 0} A(r)$ and rearranging terms we see that

$$e^{(\sqrt{\alpha^2+\beta^2-2\alpha\beta U})} \ge \Big(1+\frac{\alpha+\beta}{s-\beta}\Big)^{s-c}.$$

Letting $s\to\infty$ we find that $\sqrt{\alpha^2+\beta^2-2\alpha\beta U} \ge \alpha+\beta$. Clearly, $2\alpha\beta U \le -2\alpha\beta$. Thus, $U\le -1$. Since $L_1,\ l_1\ \ge -1$, it follows that $U=-1$ and hence the claim. We now show that this implies the remaining assertions. Let $\nu\in J_r$; by $\nu(r)^{\perp}$ we denote a unit vector orthogonal to ν. Let $\nu(r)^{\perp}_i,\ i=1,2$ be such that for $\omega\in I_{\alpha r}$ and $\omega'\in I_{\beta r}$, $\omega = a_\alpha(r)\nu + b_\alpha(r)\nu(r)^{\perp}_1$ and $\omega' = a_\beta(r)\nu + b_\beta(r)\nu(r)^{\perp}_2$, where $a^2_\alpha(r)+b^2_\alpha(r) = a^2_\beta(r)+b^2_\beta(r)=1$. Using the above result it is clear that $\sup a_\alpha(r)$, $\inf a_\alpha(r)$, $\sup a_\beta(r)$ and $\inf a_\beta(r)$ all tend to -1 as $r\to 0$. Since

$$\begin{aligned}\langle\omega,\omega'\rangle &= \langle a_\alpha(r)\nu + b_\alpha(r)\nu(r)^{\perp}_1, a_\beta(r)\nu + b_\beta(r)\nu(r)^{\perp}_2\rangle\\ &= a_\alpha a_\beta + b_\alpha b_\beta\langle\nu(r)^{\perp}_1,\nu(r)^{\perp}_2\rangle,\end{aligned}$$

the assertion for L_2 and l_2 now follows. A proof for L_3 and l_3 is analogous. This concludes the proof of Theorem 2. □

Remark 1. In part (v) of Theorem 2, we assumed that $\Lambda(o) > 0$. We show that if $\Lambda(o) = 0$, then u is differentiable at o and $Du(o) = 0$ (also see [9]). Let $e \in S^{n-1}$ and $0 < r < R$; using the maximum principle, we see that

$$\Big|\frac{u(re) - u(o)}{r}\Big| = \Big|\frac{u(re) - M(r)}{r} + \frac{M(r) - u(o)}{r}\Big| \le \frac{M(r) - m(r)}{r} + \frac{M(r) - u(o)}{r}.$$

Clearly, $u(re) = u(o) + o(r)$, as $r \to 0$. The claim follows. □

Remark 2. Observe that, being convex, $M(r)$ is differentiable almost everywhere. By (2)(i) the limits $M'(r-) = \lim_{a\uparrow r}(M(r) - M(a))/(r - a)$ and $M'(r+) = \lim_{a\downarrow r}(M(a) - M(r))/(a - r)$ exist. By Theorem 2(iv), $M'(r-) \le H_r(\omega) \le \Lambda(r\omega) \le M'(r+)$, for all $\omega \in I_r$. Thus, for each $\omega \in I_r$, $H_r(\omega) = \Lambda(r\omega) = M'(r)$, almost everywhere r. A partial improvement appears in Theorem 1. □

Lemma 1. *Let u be infinity-harmonic in $B_r(o) \subset\subset \Omega$. Fix $0 < s < r$. Let $y \in B_s(o)$ and $0 < \varepsilon < r - s$. Then the following holds.*

(*i*) $\Lambda(x)$ *is upper semicontinuous in* Ω,

(*ii*) $\displaystyle\lim_{r\to 0}\inf_{\omega\in I_r}\Lambda(r\omega) = \lim_{r\to 0}\sup_{\omega\in I_r}\Lambda(r\omega) = \Lambda(o)$,

(*iii*) $\displaystyle\sup_{|y|=s}\Lambda(y) \le \sup_{|y|=s}\sup_{\omega\in I_\varepsilon(y)} H_{y,s}(\omega) \le \frac{M(r) - u(o)}{r}\,\frac{r^2}{[r - (s+\varepsilon)]^2}$.

Proof. Let $y \in \partial B_s(o)$ and $B_\delta(y) \subset B_r(o)$. From (2), we see that $[M(r) - u(y)](r - s) \le [M(r) - u(o)]r$. Using this together with convexity and the maximum principle, it follows that

$$\frac{M_y(\delta) - u(y)}{\delta} \le \frac{M_y(r-s) - u(y)}{r - s} \le \frac{M(r) - u(y)}{r - s} \le \frac{(M(r) - u(o))r}{(r - s)^2}. \tag{14}$$

Applying (6) to y we see that $\Lambda(y) \le r(M(r) - u(o))/(r - s)^2$. We show part (i). Fix $x \in \Omega$, let $\mu > 0$ and $t > 0$ be such that $[M_x(t) - u(x)]/t \le \Lambda(x) + \mu$. Let $z \in B_t(x)$ be such that $|x - z| << t$. Using (14), we see that

$$\Lambda(z) \le \frac{M_x(t) - u(x)}{t}\Big(\frac{t}{t - |x - z|}\Big)^2 \le (\Lambda(x) + \mu)\Big(\frac{t}{t - |x - z|}\Big)^2.$$

Clearly, $\limsup_{z\to x}\Lambda(z) \le \Lambda(x)$. To show part (ii), we first note from Theorem 2 (iii) and (iv) that $\lim_{r\to 0}\inf_{\omega\in I_r}\Lambda(r\omega) \ge \lim_{r\to 0}\inf_{\omega\in I_r} H_r(\omega) = \Lambda(o)$.

Part (i) implies $\lim_{r\to 0}\sup_{\omega\in I_r}\Lambda(r\omega) \le \Lambda(o)$. We now show (iii). Let $\omega\in I_y(\varepsilon)$ and $y_\varepsilon\in\partial B_\varepsilon(y)$ be such that $y_\varepsilon = y+\varepsilon\omega$. Note that $u(y_\varepsilon)=M_y(\varepsilon)$ and for every $t\in[0,\varepsilon)$, $y+t\omega\in B_r(o)$. We apply (2), (6) and (14), with $\delta=\varepsilon-t$ and $y_t=y+t\omega$, to conclude

$$\frac{M_y(\varepsilon)-u(y_t)}{\varepsilon-t} \le \frac{M_{y_t}(r-|y_t|)-u(y_t)}{r-|y_t|} \le \frac{M(r)-u(o)}{r}\Big(\frac{r}{r-|y_t|}\Big)^2.$$

The above holds for any y_ε. Recall from Theorem 2(iii) that $\Lambda(y)\le H_{y,\varepsilon}(\omega)$. Letting $t\to\varepsilon$ and recalling (4), we see that

$$\sup_{|y|=s}\Lambda(y) \le \sup_{|y|=s}\sup_{\omega\in I_\varepsilon(y)} H_{y,\varepsilon}(\omega) \le \frac{M(r)-u(o)}{r}\Big(\frac{r}{r-(s+\varepsilon)}\Big)^2.$$

A similar estimate holds for h. □

3. An extension of the Crandall-Evans result

We now state and prove a version of the Crandall-Evans result [9]. Our proof uses the Harnack inequality and we will adapt the device used in proving Theorem 2(v).

Theorem 3. *Let $\Omega\subset\mathbb{R}^n$ be a domain and u be infinity-harmonic in Ω. Let $B_\rho(o)\subset\subset\Omega$ and let $\Lambda=\Lambda(o)\ge 0$ be as in (6) and Theorem 2(iii). Then there exist a direction $\omega\in S^{n-1}$ and a decreasing sequence r_k with $r_k\to 0$ as $k\to\infty$, such that*

$$(i)\quad -\Lambda \le \liminf_{r\to 0}\frac{u(r\eta)-u(o)}{r} \le \limsup_{r\to 0}\frac{u(r\eta)-u(o)}{r} \le \Lambda,\ \forall\,\eta\in S^{n-1},$$

and

$$(ii)\quad \text{for each } \theta>0,\ \lim_{k\to\infty}\frac{u(\theta r_k\eta)-u(o)}{\theta r_k} = \Lambda\langle\omega,\eta\rangle,\quad \forall\,\eta\in S^{n-1}.$$

Moreover, if $x\in\mathbb{R}^n$, $\eta=x/|x|$ and $\theta=|x|$, then

$$\lim_{k\to\infty}\frac{u(r_k x)-u(o)}{r_k} = \Lambda\langle x,\omega\rangle.$$

Proof. We will assume that $0<r<t$, where $B_{2t}(o)\subset\Omega$ and $\Lambda=\Lambda(o)>0$, see Remark 1. Firstly, if $\eta\in S^{n-1}$, then

$$\frac{m(r)-u(o)}{r} \le \frac{u(r\eta)-u(o)}{r} \le \frac{M(r)-u(o)}{r}.$$

Part (i) then follows by letting $r\to 0$, using (6) and Theorem 2(iii). We achieve part (ii) in four steps.

Step 1. We show that Λ is attained along a specific direction for a specific sequence. For $r > 0$, recall the definition of I_r. By the compactness of S^{n-1} we may find a sequence r_k, $\omega_k \in I_{r_k}$ and an $\omega \in S^{n-1}$ such that $lim_{k\to\infty}\omega_k = \omega$ and $lim_{k\to\infty} r_k = 0$. For $k = 1, 2, ...$, set $\theta(k) = \cos^{-1}\langle \omega_k, \omega\rangle$. Since $u(r\omega) \leq M(r)$, it follows by part (i) that

$$\limsup_{r_k\to 0} \frac{u(r_k\omega) - u(o)}{r_k} \leq \limsup_{r\to 0} \frac{u(r\omega) - u(o)}{r} \leq \Lambda.$$

Using the Harnack inequality (3) in $B_{2r_k}(o)$, we have that

$$\begin{aligned} M(2r_k) - M(r_k) &= M(2r_k) - u(r_k\omega_k) \\ &\geq \exp\Big(- \frac{r_k\theta(k)}{r_k}\Big)[M(2r_k) - u(r_k\omega)]. \end{aligned}$$

Writing $M(2r_k) - M(r_k) = M(2r_k) - u(o) + u(o) - M(r_k)$ and rearranging we see that

$$M(r_k) - u(o) \leq [u(r_k\omega) - u(o)]e^{-\theta(k)} + (M(2r_k) - u(o))\left(1 - e^{-\theta(k)}\right).$$

Dividing by r_k, using (6) and Theorem 2(iii), we see that $\liminf_{r_k\to 0}[u(r_k\omega) - u(o)]/r_k \geq \Lambda$. Thus, by part (i),

$$\limsup_{r\to 0} \frac{u(r\omega) - u(o)}{r} = \Lambda = \lim_{r_k\to 0} \frac{u(r_k\omega) - u(o)}{r_k}.$$

Thus, part (ii) holds for r_k when $\theta = 1$ and $\eta = \omega$.

Step 2. We show that part (ii) holds for the sequence r_k, as in Step 1, when $\theta = 1$ and $\eta = -\omega$. We assume to the contrary. Then there is a subsequence r_m of r_k with $r_m \to 0$ such that $\lim_{r_m\to 0} \frac{u(o)-u(-r_m\omega)}{r_m} < \Lambda$. We now proceed as in Step 1. Let $\nu_m \in J_{r_m}$; then there are a subsequence r_l of r_m, decreasing to 0, and a $\nu \in S^{n-1}$ such that $\lim_{l\to\infty} \nu_l \to \nu$. Arguing as in Step 1, we see that $\lim_{r_l\to 0}[u(o) - u(r_l\nu)]/r_l = \Lambda$. We now employ the Harnack inequality in $B_{sr_l}(o)$ as follows. For $s > 1$ and l large

$$u(r_l\nu) - m(sr_l) \geq (u(r_l\omega) - m(sr_l)) \exp\Big(- \frac{|\omega - \nu|}{s - 1}\Big).$$

Writing $u(r_l\nu) - m(sr_l) = u(r_l\nu) - u(o) + u(o) - m(sr_l)$, dividing both sides of the inequality by r_l, letting $r_l \to 0$ and using part (i), we get

$$(s - 1) \geq (s + 1) \exp\Big(- \frac{|\omega - \nu|}{s - 1}\Big) \text{ and } e^{|\omega-\nu|} \geq \Big(\frac{s + 1}{s - 1}\Big)^{s-1}.$$

Letting $s \to \infty$ we obtain $|\omega - \nu| \geq 2$. Thus, $\omega = -\nu$ and

$$\lim_{r_l \to 0} \frac{u(o) - u(r_l \nu)}{r_l} = \lim_{r_l \to 0} \frac{u(o) - u(-r_l \omega)}{r_l} = \Lambda,$$

thus leading to a contradiction. Hence the claim holds.

Step 3. Let r_k be as in Step 1. We will now prove part (ii) for θr_k, $\eta = \omega$ and any $\theta > 0$. The conclusion for $\eta = -\omega$ will follow as in Step 2. We first discuss the case $0 < \theta < 1$. We employ (2) in $B_{sr_k}(o)$ to see that for $s > 1$ we have

$$[u(\theta r_k \omega) - m(sr_k)](s - \theta) r_k \geq [u(r_k \omega) - m(sr_k)](s-1) r_k.$$

Setting $L = \liminf_{r_k \to 0} [u(\theta r_k \omega) - u(o)]/(\theta r_k)$, dividing by r_k^2, letting $r_k \to 0$ and using part (6) and Theorem 2(iii), we see that

$$(\theta L + s\Lambda)(s - \theta) \geq (s^2 - 1)\Lambda.$$

Thus, $\frac{s\theta L}{\Lambda} \geq s\theta + \frac{\theta^2 L}{\Lambda} - 1$. Divide by s and let $s \to \infty$ to see that $L \geq \Lambda$. Using part (i) we obtain that $L = \Lambda$. For $\theta > 1$ we use contradiction. Fix $\theta > 1$ and assume that $\liminf_{r_k \to 0} [u(\theta r_k \omega) - u(o)]/(\theta r_k) < \Lambda$. Call t_m, with $t_m \downarrow 0$, a subsequence of θr_k for which the liminf is attained. Let $\omega_m \in I_{t_m}$; there are a subsequence t_l of t_m, a subsequence ω_l of ω_m and an $\omega_\theta \in S^{n-1}$ such that $\omega_l \to \omega_\theta$ as $t_l \to 0$. By Step 1 we see that $\lim_{t_l \to 0} \{u(t_l \omega_\theta) - u(o)\}/t_l = \Lambda$. Now set $s_l = t_l/\theta$. We use the Harnack inequality in $B_{ss_l}(o)$ as follows. For $s > \theta$,

$$u(-s_l \omega) - m(ss_l) \geq (u(s_l \omega_\theta) - m(ss_l)) \exp\Big(- \frac{|\omega_\theta + \omega|}{s-1} \Big). \tag{15}$$

Note: (a) using the result for $\theta < 1$, we see $\lim_{s_l \to 0} \{u(s_l \omega_\theta) - u(o)\}/s_l = \Lambda$, and (b) from Step 2, $\lim_{s_l \to 0} \{u(-s_l \omega) - u(o)\}/s_l = -\Lambda$, since $s_l = t_l/\theta$ is a subsequence of r_k. Dividing (15) by s_l, letting $s_l \to 0$ and using Step 2 along $-\omega$, we see that

$$(s-1) \geq (s+1) \exp\Big(- \frac{|\omega_\theta + \omega|}{s-1} \Big) \ \text{ and } \ e^{|\omega_\theta + \omega|} \geq \Big(\frac{s+1}{s-1} \Big)^{s-1}.$$

Now letting $s \to \infty$, we see that $|\omega_\theta + \omega| \geq 2$. Thus, $\omega_\theta = \omega$, and hence $\lim_{t_l \to 0} \{u(t_l \omega) - u(o)\}/t_l = \Lambda$, contradicting our assumption since t_l is a subsequence of t_m. Thus, the claim holds.

Step 4. We now employ Step 3 to prove part (ii) for the sequence r_k. Let $\eta \in S^{n-1}$. Select $s > \alpha > 1$. We apply the Harnack inequality as follows

$$u(r_k \eta) - m(sr_k) \geq (u(\alpha r_k \omega) - m(sr_k)) \exp\Big(- \frac{|\alpha\omega - \eta|}{s - \alpha} \Big). \tag{16}$$

Set $L = \liminf_{r_k \to 0} \frac{u(r_k\eta)-u(o)}{r_k}$ and note that $L \le \Lambda$. Dividing (16) by r_k and letting $r_k \to 0$, we get

$$e^{|\alpha\omega-\eta|} \ge \Big(\frac{s+\alpha}{s+L/\Lambda}\Big)^{s-\alpha} = \Big(1+\frac{\alpha-L/\Lambda}{s+L/\Lambda}\Big)^{s-\alpha}.$$

Letting $s \to \infty$ we see that $|\alpha\omega-\eta| \ge \alpha - L/\Lambda \ge 0$. Hence $\alpha^2+1-2\alpha\langle\omega,\eta\rangle \ge \alpha^2 + L^2/Q^2 - 2\alpha L/\Lambda$. Dividing by α and letting $\alpha \to \infty$ to see that $L \ge \Lambda\langle\omega,\eta\rangle$. Set that $U = \limsup_{r_k\to 0}(u(r_k\eta)-u(o))/r_k$, we show that $U \le \Lambda\langle\omega,\eta\rangle$. Select $s > \alpha > 1$. The Harnack inequality implies

$$u(-\alpha r_k\omega) - m(sr_k) \ge (u(r_k\eta)-m(sr_k))\exp\Big(-\frac{|\alpha\omega+\eta|}{s-\alpha}\Big).$$

Dividing by r_k, letting $r_k \to 0$ and using Step 3, we see that

$$e^{|\alpha\omega+\eta|} \ge \Big(\frac{s+U/\Lambda}{s-\alpha}\Big)^{s-\alpha} = \Big(1+\frac{\alpha+U/\Lambda}{s-\alpha}\Big)^{s-\alpha}.$$

Since $-\Lambda \le U$ and $\alpha > 1$, we have $\alpha + U/\Lambda \ge 0$. Letting $s \to \infty$, we get $|\alpha\omega+\eta| \ge \alpha + U/\Lambda$. Thus, $\langle\omega,\eta\rangle \ge U/\Lambda - 1/\alpha + U^2/(\alpha\Lambda^2)$. Now $U \le \Lambda\langle\omega,\eta\rangle$ follows by letting $\alpha \to \infty$. Thus $L = U$. Using Step 3, we may now replace r_k by θr_k to obtain part (ii). □

Remark 3. If the set $\{\omega(r) : r > 0\}$ as $r \downarrow 0$ has only one limit ω, then directional derivatives exist at o. As a matter of fact if part (ii) holds for any sequence along ω, then ω would be the only limit point. This would then imply differentiability [9].

4. Proof of Theorem 1

We prove Theorem 1 by using Theorems 2, 3 and Lemma 1. Let $B_{2s}(O) \subset \Omega$ and $0 < r < s$. Recall from (4), Theorem 2 and Remark 2 that if $\omega \in I_r$ and $q = r\omega \in \partial B_r(o)$, then the Hopf derivative $H_r(\omega)$ at q exists and

$$\begin{aligned} 0 < M'(r-) \le H_r(\omega) &= \lim_{\delta\to r}\frac{M(r)-u(\delta\omega)}{r-\delta} = \frac{u(q)-u(\delta\omega)}{r-\delta} \\ &\le M'(r+) < \infty. \end{aligned}$$

We will now show that u is differentiable at q. Fix q and for small $\rho > 0$ consider the ball $B_\rho(q)$. Let $\omega' \in I_\rho(q)$. By the maximum principle, $q+\rho\omega' \in \partial B_\rho(q) \setminus B_r(o)$. Let T_q be the hyperplane tangential to $\partial B_r(o)$ at q. Let $T_q^+ = \{x \in \Omega : \langle x-q,\omega\rangle \ge 0\}$ and $T_q^- = \{x \in \Omega : \langle x-q,\omega\rangle < 0\}$ denote the half-spaces. Note that $B_r(o) \subset T_q^-$.

Step 1. Let $\rho_k \downarrow 0$, $\omega_k \in I_{\rho_k}(q)$ and $\omega_0 \in S^{n-1}$ be such that $\omega_k \to \omega_0$. Clearly $\omega_0 \in T_q^+$ and $\langle \omega_0, \omega\rangle \geq 0$. To see this note that the farthest $\rho_k\omega_k$ can be from T_q^+ is to be on $\partial B_r(o) \cap \partial B_{\rho_k}(q)$. Since T_q is the tangent plane to the sphere $\partial B_r(o)$ it follows that $\lim_{\rho_k \to 0}\langle \omega_k, \omega\rangle = \langle \omega_0, \omega\rangle \geq 0$.

Step 2. Let $\omega^\perp \in S^{n-1}$ be such that $\omega^\perp \perp \omega$. Let $q_k = q + \rho_k\omega^\perp \in T_q$ and $\Pi(q)$ be the 2-dimensional plane that contains q, ω and $\omega^\perp$. For each k, let $z_k \in \Pi(q) \cap \partial B_{\rho_k}(q) \cap \partial B_r(o)$ such that $\langle z_k - q, \omega^\perp\rangle \geq 0$. Note that $|q_k - q| = |z_k - q| = \rho_k$ and $u(z_k) \leq u(q) = M(r)$. By Theorem 2(iii), (iv) and Step 1 of Theorem 3, we see that

$$\Lambda(q) = \lim_{\rho_k \to 0} \frac{u(q + \rho_k\omega_0) - u(q)}{\rho_k} = \lim_{t \to 0} \frac{M_q(t) - u(q)}{t} = \Lambda(r\omega) \geq H_r(\omega).$$

By Theorem 3(ii),

$$\lim_{\rho_k \to 0} \frac{u(q_k) - u(q)}{\rho_k} = \lim_{\rho_k \to 0} \frac{u(q + \rho_k\omega^\perp) - u(q)}{\rho_k} = \Lambda(q)\langle \omega^\perp, \omega_0\rangle. \tag{17}$$

We now apply the Harnack inequality as follows. For small ρ_k, $M_q(2\rho_k) - u(q_k) \geq (M_q(2\rho_k) - u(z_k))e^{-\theta_k}$, where $\theta_k = \cos^{-1}\{\langle z_k - q, \omega\rangle/\rho_k\}$. A rearrangement yields

$$\frac{M_q(2\rho_k) - u(q)}{\rho_k}\left(1 - e^{-\theta_k}\right) + \frac{u(z_k) - u(q)}{\rho_k}e^{-\theta_k} \geq \frac{u(q_k) - u(q)}{\rho_k}.$$

Letting $\rho_k \to 0$, using (17) and noting that $\theta_k \to 0$, we see that

$$\lim_{\rho_k \to 0} \frac{u(q_k) - u(q)}{\rho_k} = \Lambda(q)\langle \omega^\perp, \omega_0\rangle \leq 0. \tag{18}$$

Clearly (18) holds for any $\omega^\perp \in T_q$. From Step 1 it follows that $\omega_0 \perp T_q$ and that $\omega_0 = \omega$. This being true of any limit point ω_0, Remark 3 implies that u is differentiable at q. Clearly, $Du(q) = \Lambda(q)\omega$. Moreover by Theorems 2, 3 and Remark 2,

$$M'(r-) \leq H_r(\omega) = \lim_{t \downarrow 0} \frac{u(q) - u(q - t\omega)}{t} = \Lambda(q) = |Du(q)| \leq M'(r+). \tag{19}$$

An analogous argument leads to differentiability at points of minimum on $\partial B_r(O)$.

Step 3. Let $r_k > r$ be a sequence with $r_k \downarrow r$ as $k \to \infty$. Let $\omega_k \to\in I_{r_k}$. We may choose these sequences such that $\omega_k \to \omega \in I_r$. The latter follows from the maximum principle. Set $q' = r\omega$ and $q'_k = r_k\omega_k$. Applying Theorem 2(ii) and (iv) we see that

$$\Lambda(q') \leq \frac{M(r_k) - M(r)}{r_k - r} \leq H_{r_k}(\omega_k) \leq \Lambda(q'_k).$$

Clearly, Lemma 1(i) yields that $\Lambda(q') = M'(r+) = |Du(q')|$. □

Remark 4. By Theorem 1, Theorem 2(iii) and Lemma 1(ii), we see that $\lim_{r\to 0} \sup_{\omega\in I_r} |Du(r\omega)| = \lim_{r\to 0} M'(r+) = \Lambda(o) = M'(0)$.

Remark 5. We may extend Theorem 1 as follows. Let $o \in \Omega$ and S be a connected C^1 surface passing through o. Let $n(o)$ be a unit normal to S at o. By a local parametrization of S, we may assume that the x_n axis coincides with $n(o)$ and S is the graph of a C^1 function $x_n = f(x_1, x_2, ..x_{n-1})$, in a ball B, centered at o. Set $B^- = \{x :\ x_n \leq f(x_1, x_2, .., x_{n-1})\} \cap B$. Assume that $u(x) \leq u(o),\ x \in B^-$. By adapting Steps 1 and 2 in the proof of Theorem 1, we can show that u is differentiable at o and $Du(o) = |Du(o)|n(o)$. This result then extends Corollary 8.2 in [13] and we obtain everywhere differentiability of solutions u_∞. Also see Lemma 8.1 in the same work.

Remark 6. A possible approach to the question of differentiability would be to show that either (i) there exists $\delta > 0$ and $\omega \in S^{n-1}$ such that $u(o) = \sup_{B_\delta(\delta\omega)} u(x)$ or $u(o) = \inf_{B_\delta(\delta\omega)} u(x)$, or (ii) $\Lambda(o) = 0$. Theorem 1 will then imply that u is differentiable at o. However, it is unclear to us how this may be proved if true. Note that the function $x^{4/3} - y^{4/3}$ satisfies (ii) at o but not (i).

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