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FRACTIONAL HEAT EQUATIONS ON A SEGMENT

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Abstract. We study the initial-boundary-value problem for the fractional heat equation on a segment $(0,a)$

$$\begin{cases} u_t+\lambda|u|^{\rho}\ u+C_{\alpha}\partial_x^{\alpha}u=0,\ t>0, \\ u(x,0)=u_0(x), \\ u(a,t)=h_1(t),u_x(0,t)=h_2(t),t>0, \end{cases} \tag{0.1}$$

where $\lambda\in R$, $\rho\geq 0$, $\alpha\in(1,\frac{3}{2}]$, the constant C_{α} is chosen by a dissipative condition, such that $\mathrm{Re}C_{\alpha}p^{[\alpha]+1-\{\alpha\}}>0$ for $\mathrm{Re}p=0$ and

$$\partial_x^{\alpha}u=\int_0^x\frac{\partial_s^{[\alpha]+1}u(s,t)}{(x-s)^{\{\alpha\}}}ds.$$

Here $[\alpha]$ and $\{\alpha\}$ are integer and fractional parts of α.

The aim of this paper is to prove the global existence of solutions to the initial-boundary-value problem (0.1) and to find the main term of the asymptotic representation of solutions.

1. Introduction

We study the initial-boundary-value problem for the nonlinear fractional heat equation on a segment

$$\begin{cases} u_t+\lambda|u|^{\rho}\ u+C_{\alpha}\partial_x^{\alpha}u=0,\ t>0,x\in(0,a), \\ u(x,0)=u_0(x),\ x\in(0,a), \\ u(a,t)=h_1(t),\ u_x(0,t)=h_2(t),t>0, \end{cases} \tag{1.1}$$

where $\lambda\in R$, $\rho\geq 0$, $\alpha\in(1,\frac{3}{2}]$, the constant C_{α} is chosen by a dissipative condition, such that $\mathrm{Re}C_{\alpha}p^{[\alpha]+1-\{\alpha\}}>0$ for $\mathrm{Re}p=0$ and the fractional derivative on a segment is defined as follows

$$\partial_x^{\alpha}u=\int_0^x\frac{\partial_s^{[\alpha]+1}u(s,t)}{(x-s)^{\{\alpha\}}}ds.$$

Here $[\alpha]$ and $\{\alpha\}$ are the integer and fractional parts of α.

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A great number of publications have dealt with asymptotic representations of solutions to the Cauchy problem for nonlinear evolution equations in the last twenty years. While not attempting to provide a complete review of these publications, we do list some known results [1] - [8], [10], [11], [13] -[16], [19], [20], [21], [25]–[36], where there were obtained optimal time decay estimates and asymptotic formulas of solutions to different nonlinear local and nonlocal dissipative equations. In the case of the Cauchy problem the nonlinear equations were divided into three general types: asymptotically weak nonlinearity, critical nonlinearity and strong (or subcritical) nonlinearity.

For the general theory of nonlinear pseudodifferential equations on a half-line we refer to the book [18].

Up to now the theory of nonlinear nonlocal initial-boundary-value problems on a segment is not developed well due to it's difficulty. There are many open natural questions which we need to study. The first of them is how many boundary data should be posed in the initial-boundary-value problems for their correct solvability. There are some results in the case of nonlinear differential equations [9], [12]. However, as far as we know there are few results in the case of nonlinear pseudodifferential equations ([22], [23], [24]).

This paper is the first attempt to give a systematic approach for obtaining the large time asymptotic representations for solutions to the nonlinear nonlocal equations on a segment with nonhomogeneous boundary data in the different cases of the order ρ of the nonlinearity. A description of the large time asymptotic behavior of solutions for the initial-boundary-value problem requires new approaches and the reorientation of points of view compared with the Cauchy problem.

Note that the nonlinear fractional heat equation (1.1) has a huge number of applications. For example, in the case of $\alpha = \frac{3}{2}$, the nonlocal equation (1.1) is a famous Ott-Sudan-Ostrovskiy equation (see [27])

$$u_t + \lambda |u|^\rho \ u + \int_0^x \frac{u_{ss}(s,t)}{\sqrt{x-s}} ds = 0.$$

We adopt here an approach based on estimates of the Green function. The difficulty for nonlocal equations on a segment is that the symbol $K(p)$ is nonanalytic in the left half-complex plane . Therefore we can not apply the Laplace theory directly, so we use the methods of paper [23] to construct the Green's function.

To state the results of the present paper precisely we give some notation. Let $\mathbf{B}$ be a Banach space; we then denote

$$\mathbf{C}([0,T],\mathbf{B})=\Big\{f(t)\in\mathbf{B}:\lim_{t_1\to t,t_1\in[0,T]}\|f(t_1)-f(t)\|_{\mathbf{B}}=0,\forall t\in[0,T]\Big\}.$$

Now we define well posedness of the problem (1.1).

Definition 1. *Problem* (1.1) *is called well posed in a semiclassical sense if the following properties are fulfilled. Firstly, there exists a unique solution* $u(x,t)$ *belonging to a metric space* $\mathbf{C}([0,T],\mathbf{L}^{\infty}(0,a))\cap\mathbf{C}((0,T],\mathbf{C}^1([0,a]))$, *which satisfies the equation* $u_t+\lambda|u|^{\rho}\ u+C_{\alpha}\partial_x^{\alpha}u=0$ *in the generalized sense. Secondly, boundary and initial conditions are fulfilled in the classical sense*

$$\begin{aligned}\lim_{t\to 0}u(x,t)&=u_0(x)\ \textit{in}\ \mathbf{L}^{\infty}(0,a)\\ \lim_{x\to a-0}u(x,t)&=h_1(t)\ \textit{in}\ \mathbf{C}([0,T])\\ \lim_{x\to +0}\partial_x u(x,t)&=h_2(t)\ \textit{in}\ \mathbf{C}([0,T]).\end{aligned}$$

The function $u(x,t)$ *we call a semiclassical solution. If* $T=+\infty$ *the function* $u(x,t)$ *is called a global semiclassical solution.*

We denote metric spaces

$$\mathbf{Z}=\Big\{\phi\in\mathbf{L}^1([0,\infty)):\|\phi\|_{\mathbf{Z}}=\sup_{t>0}\{t\}^{1-\gamma}\langle t\rangle^{1+\gamma}\|\phi\|_{\mathbf{L}^{\infty}}<\infty\Big\},\gamma>0,$$

and

$$\mathbf{X}=\{\phi(x,t)\in\mathbf{C}([0,\infty);\mathbf{L}^{\infty}(0,a)):\|\phi\|_{\mathbf{X}}<+\infty\},$$

where

$$\|\phi\|_{\mathbf{X}}=\sup_{t>0}\Big(\langle t\rangle^{\frac{\{\alpha\}}{2-\{\alpha\}}}\|\phi(t)\|_{\mathbf{L}^{\infty}}\Big).$$

We introduce constants

$$\Lambda_0=\frac{e^{i\frac{\pi}{4}}\sqrt{2}}{2\pi}\int_0^{+i\infty}e^{sz-K(z)}dz,\quad K(z)=C_{\alpha}z^{2-\{\alpha\}},\quad \delta=\frac{1-\{\alpha\}}{2-\{\alpha\}}>0.$$

Now we state the main results.

First we study large time asymptotic behavior of solutions to the problem (1.1) in the so-called subcritical case, when the time decay rate of the nonlinear term is slower than that for the linear behavior.

Denote $g(t)=1+\eta\theta^{\rho}t^{\frac{2-(\rho+1)\{\alpha\}}{2-\{\alpha\}}}$, where $\eta=\lambda\frac{2-\{\alpha\}}{2-(\rho+1)\{\alpha\}}\rho\Lambda_0^{\rho}>0$ and the constant θ is the solution of the following equation

$$\theta=\int_0^{+\infty}(1+\eta\theta^{\rho}\tau^{\frac{2-(\rho+1)\{\alpha\}}{2-\{\alpha\}}})^{\frac{1}{\rho}}h_2(\tau)d\tau.$$

We prove the following theorem.

Theorem 1. *Let* $\{\alpha\}^{-1} - 2 < \rho < \{\alpha\}^{-1} - 1$ *and* $\theta > 0$. *Suppose that the initial data* $u_0 \in \mathbf{L}^\infty$ *and* $h_1 \in \mathbf{Z}, h_2 \in \mathbf{Z}$ *are such that* $\|u_0\|_{\mathbf{L}^\infty} + \|g^{\frac{1}{\rho}} h_1\|_{\mathbf{Z}} \leq \varepsilon^{1+\gamma_1}$ *and* $\|g^{\frac{1}{\rho}} h_2\|_{\mathbf{Z}} \leq \varepsilon$ *are sufficiently small. Then there exists a unique global semiclassical solution* $u \in \mathbf{X}_0$ *to the initial-boundary-value problem* (1.1) *such that*

$$\|(u - Ag^{-\frac{1}{\rho}} \langle t \rangle^{-\frac{\{\alpha\}}{2-\{\alpha\}}}) g^{\frac{1}{\rho}} \langle t \rangle^{\delta} \|_{\mathbf{X}} \leq C,$$

where

$$A = \frac{\sqrt{2}}{2\pi i} \int_0^{+i\infty} e^{-K(z)} z^{\{\alpha\}-1} dz \int_0^{+\infty} g^{\frac{1}{\rho}}(\tau) h_2(\tau) d\tau.$$

In the second theorem we consider the case of the critical nonlinearity, that is, when the time decay rate of the nonlinear term is balanced with that of the linear part of the equation. We obtain the following result.

Theorem 2. *Let* $\rho = \{\alpha\}^{-1} - 1$.*Suppose that* $u_0 \in \mathbf{L}^\infty$, h_1, $h_2 \in \mathbf{Z}$ *are such that the norm* $\|u_0\|_{\mathbf{L}^\infty} + \sum_{j=1,2} \|h_j\|_{\mathbf{Z}}$ *is sufficiently small. Then there exists a unique global semiclassical solution* $u \in \mathbf{X}$ *of the problem* (1.1). *Furthermore, there exists a constant* V *such that the asymptotic formula*

$$u(t) = Vt^{-\frac{\{\alpha\}}{2-\{\alpha\}}} + O(t^{-\frac{\{\alpha\}}{2-\{\alpha\}}-\delta})$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in [0, a]$, *where the constant* V *is a small solution of the equation*

$$V = A \int_0^{+\infty} h_2(\tau) d\tau - BV^{\rho+1},$$

where

$$A = \frac{\sqrt{2} e^{-i(\frac{\pi}{4} + \frac{\{\alpha\}-1}{2})}}{2\pi i} \int_0^{+i\infty} e^{-K(z)} z^{\{\alpha\}-1} dz$$

and

$$B = a\lambda\Lambda_0 \int_0^1 (1-z)^{-\frac{1}{2-\{\alpha\}}} z^{-(\rho+1)\frac{\{\alpha\}}{2-\{\alpha\}}} dz.$$

In the third theorem we find the large time asymptotic representations in the supercritical case; that is, when the nonlinear term decays in time faster than the linear part of the equation. This type of nonlinearity we call the asymptotically weak one. We prove the following theorem.

Theorem 3. *Let* $\rho > \{\alpha\}^{-1} - 1$. *Suppose that* $u_0 \in \mathbf{L}^{\infty}$, h_1, $h_2 \in \mathbf{Z}$ *are such that the norm* $\|u_0\|_{\mathbf{L}^{\infty}} + \sum_{j=1,2} \|h_j\|_{\mathbf{Z}}$ *is sufficiently small. Then there exists a unique global semiclassical solution* $u \in \mathbf{X}$ *of the problem* (1.1). *Furthermore, there exists a constant* A *such that the asymptotic formula*

$$u(t) = At^{-\frac{\{\alpha\}}{2-\{\alpha\}}} + O(t^{-\frac{\{\alpha\}}{2-\{\alpha\}}-\delta})$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in [0, a]$, *where*

$$A = \frac{e^{-i(\frac{\pi}{4}+\frac{\{\alpha\}-1}{2})}\sqrt{2}}{2\pi i} \int_0^{+i\infty} e^{sz-K(z)} z^{\{\alpha\}-1} dz \int_0^{\infty} h_2(\tau) d\tau.$$

Remark 1. In the case $h_2(t) = 0$ Theorems 1 -3 give us only a decay rate estimate for the solutions. To obtain the main term of the asymptotics another approach is necessary. We will study this case in a separate paper.

Remark 2. In this work we pay attention to the case of $\alpha \in (1, \frac{3}{2}]$. Certainly we do not claim that we could embrace all equations and all cases $\alpha > 0$. However we expect that a sufficiently wide class of nonlinear nonlocal equations on a segment could be treated by the same approach of the present paper.

2. Preliminaries

We consider the following linear initial-boundary-value problem

$$\begin{cases} u_t + \partial_x^{\alpha} u = f, \ t > 0, x \in (0, a), \\ u(x, 0) = u_0(x), \ x \in (0, a), \\ u(a, t) = h_1(t), u_x(0, t) = h_2(t), t > 0, \end{cases} \tag{2.1}$$

where

$$\partial_x^{\alpha} u = \int_0^x \frac{\partial_s^{[\alpha]+1} u(s, t)}{(x - s)^{\{\alpha\}}} ds, \quad \alpha \in (1, \tfrac{3}{2}].$$

Let $K(p) = C_{\alpha} p^{[\alpha]+1-\{\alpha\}} = C_{\alpha} p^{2-\{\alpha\}}$. Denote

$$\mathcal{G}\phi = \int_0^a G(x, y, t)\phi(y) dy,$$

$$\mathcal{G}_{1-\{\alpha\}}\phi = \int_0^t G_1(x, t - \tau)\phi(\tau) d\tau \quad \text{and} \quad \mathcal{G}_{\{\alpha\}-1}\phi = \int_0^t G_2(x, t - \tau)\phi(\tau) d\tau,$$

where the Green's function $G(x, y, t)$ is defined by

$$G(x, y, t) \ = \ \theta_a(x) \frac{1}{2\pi i} \Big(\int_{-i\infty}^{i\infty} e^{-K(p)t + p(x-y)} \tag{2.2}$$

$$-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\frac{\phi e^{-\phi(\xi)y}}{\xi}\int_{-i\infty}^{i\infty}dpe^{px}\frac{K(p)}{p(K(p)+\xi)}\Big),$$

$$\begin{aligned} G_1(x,t) &= -\frac{1}{2\pi i}\theta_a(x)\Big(\int_{-i\infty}^{i\infty}e^{p(x-a)}e^{-K(p)t}\frac{K(p)}{p}dp \\ &\quad -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t-\phi(\xi)a}\int_{-i\infty}^{i\infty}dpe^{px}\frac{K(p)}{p(K(p)+\xi)}\Big) \\ &= C_1\lim_{y\to a}\partial_y^{1-\{\alpha\}}G(x,y,t), \end{aligned}$$

and

$$\begin{aligned} G_2(x,t) &= \frac{1}{2\pi i}\theta_a(x)\Big(\int_{-i\infty}^{i\infty}e^{px}e^{-K(p)t}\frac{1}{p^{\{\alpha\}}}dp \\ &\quad -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\phi^{\{\alpha\}}\xi^{-1}\int_{-i\infty}^{i\infty}dpe^{px}\frac{K(p)}{p(K(p)+\xi)}\Big) \\ &= C_1\lim_{y\to 0}\partial_y^{-(1-\{\alpha\})}G(x,y,t). \end{aligned}$$

Here for $w\in(-1,1)$

$$\partial_x^w f=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{px}(-p)^w\widehat{f}dp,$$

and the function $\phi(\xi)$ is defined as

$$\begin{cases} K(\phi(\xi))=-\xi, \\ \mathrm{Re}\phi(\xi)>0 \text{ for all } \mathrm{Re}\xi>0. \end{cases}$$

Now we prove the following result.

Proposition 1. *Let the initial data* $u_0\in L^1(0,a)$*, a source*

$$f(x,t)\in\mathbf{L}^1_{loc}(0,\infty;\mathbf{L}^1(0,a))$$

and boundary data $h_j(t)\in\mathbf{L}^1_{loc}(0,\infty)$*. Then there exists a unique semiclassical solution* $u(x,t)$ *of the initial-boundary-value problem* (2.1), *which has the representation*

$$u(x,t)=\mathcal{G}u_0+\int_0^t\mathcal{G}(t-\tau)f(\tau)d\tau+\mathcal{G}_{1-\{\alpha\}}h_1+\mathcal{G}_{\{\alpha\}-1}h_2. \tag{2.3}$$

Proof. To derive an integral representation for solutions of the problem (2.1) we suppose that there exists a solution $u(x,t)$ of problem (2.1), which is extendede to zero outside of the interval $(0,a)$; that is,

$$u(x,t)=0 \text{ for all } x\notin[0,a].$$

However everywhere below we denote by

$$\begin{aligned}\partial_x^k u(0,t) &= \lim_{x\to+0}\partial_x^k u(x,t)\\ \partial_x^k u(a,t) &= \lim_{x\to a-0}\partial_x^k u(x,t),\quad k=0,1.\end{aligned}$$

We denote the operator

$$\mathbb{P}[g(p,t)] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{(q-p)a}-1}{q-p}g(q,t)dq.$$

We have for the Laplace transform

$$\begin{aligned}&\mathcal{L}\Big\{\int_0^x \frac{u_{ss}(s,t)}{\sqrt{x-s}}ds\Big\}\\ &= \mathbb{P}\Big[\frac{p^2}{\sqrt{p}}\Big(\widehat{u}(p,t)) - \frac{u(0,t)-e^{-pa}u(a,t)}{p} - \frac{\partial_x u(0,t)}{p^2}\Big)\Big]\end{aligned}$$

and since $\widehat{u}(p,t)$ and $\widehat{f}(p,t)$ are analytic for all p

$$\widehat{u}(p,t) = \mathbb{P}[\widehat{u}(p,t)],\ \ \widehat{f}(p,t) = \mathbb{P}[\widehat{f}(p,t)].$$

Applying the Laplace transformation with respect to x to problem (2.1) we obtain

$$\begin{cases}\mathbb{P}\Big[\widehat{u}_t + K(p)(\widehat{u}(p,t)) - \frac{u(0,t)-e^{-pa}u(a,t)}{p} - \frac{\partial_x u(0,t)}{p^2}) - \widehat{f}(p,t)\Big] = 0,\\ t>0, x\in(0,a),\\ \widehat{u}(p,0) = u_0(p),\\ u(a,t) = h_1(t), u_x(0,t) = h_2(t), t>0.\end{cases} \tag{2.4}$$

We look for the solution of (2.4) in the form

$$\widehat{u}(p,t) = \mathbb{P}[u_1(p,t)], \tag{2.5}$$

where the function $u_1(p,t)$ is the solution to the following problem (see paper [23])

$$\begin{cases}\widehat{u}_{1t} + K(p)(\widehat{u}_1(p,t) - \frac{u(0,t)-e^{-pa}u(a,t)}{p} - \frac{\partial_x u(0,t)}{p^2}) = \widehat{f}(p,t),\\ u_1(p,0) = \widehat{u}_0(p),\\ u(a,t) = h_1(t), u_x(0,t) = h_2(t), t>0,\\ |\hat{u}_1(p,t)| \le M(1+|p|)^{-\delta}(1+|e^{-pa}|) \text{ for all } |p|\ge 1,\end{cases} \tag{2.6}$$

with some M, $\delta > 0$. Integrating equation (2.6) with respect to time, we write $u_1(p,t)$ as

$$\widehat{u}_1(p,t) = e^{-K(p)t}\widehat{u}_0(p) + \int_0^t e^{-K(p)(t-\tau)}f_1(p,\tau)d\tau, \tag{2.7}$$

where

$$f_1(p,\tau) = \widehat{f}(p,t) + K(p)\big(\frac{u(0,t) - e^{-pa}u(a,t)}{p} + \frac{\partial_x u(0,t)}{p^2}\big).$$

In order to get the integral formula for solutions, we need to know the boundary values $u(0,t)$, $u(a,t)$ and $u_x(0,t)$. We will find them using the growth condition

$$|\hat{u}_1(p,t)| \le M(1+|p|)^{-\delta}(1+|e^{-pa}|) \text{ for all } |p| \ge 1. \quad (2.8)$$

It is easy to prove that condition (2.8) is fulfilled in domains $\mathrm{Re}K(p) > 0$. In domains where $\mathrm{Re}K(p) < 0$, we rewrite formula (2.7) as

$$\widehat{u}_1(p,t) = e^{-K(p)t}\Big(\widehat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau} f_1(p,\tau)d\tau\Big) - \int_t^{+\infty} e^{-K(p)(t-\tau)} f_1(p,\tau)d\tau.$$

Clearly the last integral

$$\int_t^{+\infty} e^{-K(p)(t-\tau)} f_1(p,\tau)d\tau$$

satisfies condition (2.8) for all $|p| \ge 1$ such that Re $K(p) < 0$. However, the first summand with exponentially growing factor $e^{-K(p)t}$ does not satisfy condition (2.8), therefore we have to put the following necessary and sufficient condition

$$\hat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau} f_1(p,\tau)d\tau = 0 \quad (2.9)$$

for all $|p| > 1$ in the domains where Re $K(p) < 0$. We use the equation (2.9) to find the boundary values involved in formula (2.7).There exists only one root of equation $K(p) = -\xi$ (see paper [23]) such that in the right-half complex plane $\mathrm{Re}\xi > 0$

$$\mathrm{Re}\phi > 0.$$

Taking this root $\phi(\xi)$ we transform the half complex plane Re $\xi > 0$ to the domain where Re $K(p) < 0$. The condition (2.9) can be written as the equation

$$\widehat{u}_0(\phi) + \widehat{\widehat{f}}(\phi,\xi) + \xi\frac{\widehat{u}(0,\xi) - e^{-\phi a}\widehat{u}(a,\xi)}{\phi} + \frac{\partial_x\widehat{u}(0,\xi)}{\phi^{\{\alpha\}}} = 0 \quad (2.10)$$

for Re $\xi > 0$, where the functions $\widehat{u}(0,\xi)$, $\widehat{u}(a,\xi)$ and $\partial_x\widehat{u}(0,\xi)$ are the Laplace transforms of the boundary data $u(0,t)$, $u(a,t)$ and $u_x(0,t)$ with respect to time, and

$$\widehat{u}_0(\phi) = \int_0^a e^{-\phi y}u_0(y)dy, \ \widehat{\widehat{f}}(\phi,\xi) = \int_0^{+\infty}\int_0^a e^{-(\phi y + \xi t)} f(y,t)dydt.$$

From (2.10) we obtain

$$\widehat{u}(0,\xi) = -\frac{\phi(\widehat{u}_0(\phi) + \widehat{\widehat{f}}(\phi,\xi))}{\xi} + e^{-\phi a}\widehat{u}(a,\xi) - \frac{\phi^{\{\alpha\}}\partial_x \widehat{u}(0,\xi)}{\xi} \tag{2.11}$$

and the Laplace transform $\widehat{u}(0,\xi)$ satisfies the growth condition

$$|\hat{u}(\cdot,\xi)| \leq M(1+|\xi|)^{\gamma} \text{ for all } |\xi| \geq 1 \tag{2.12}$$

with some M, $\gamma > 0$, which is sufficient for the existence of the inverse Laplace transform $u(\cdot,t)$. Taking the inverse Laplace transform of (2.11) we obtain

$$u(0,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}\Big(-\frac{\phi(\widehat{u}_0(\phi) + \widehat{\widehat{f}}(\phi,\xi))}{\xi} + e^{-\phi a}\widehat{u}(a,\xi) - \frac{\phi^{\{\alpha\}}\partial_x \widehat{u}(0,\xi)}{\xi}\Big)d\xi. \tag{2.13}$$

Thus under the supposition that there exists a solution of the problem (2.1) we get the integral representation for this solution

$$\begin{aligned} u(x,t) &= \theta_a(x)\mathcal{L}^{-1}\{u_1\} \\ &= \theta_a(x)\frac{1}{2\pi i}\Big(\int_{-i\infty}^{i\infty} dpe^{px}e^{-K(p)t}\widehat{u}_0(p) + \int_{-i\infty}^{i\infty} dpe^{px}\int_0^t e^{-K(p)(t-\tau)} \\ &\quad \times (f(p,\tau) + \frac{K(p)}{p}u(0,\tau) - \frac{K(p)}{p}e^{-pa}h_1(\tau) + \frac{1}{p^{\{\alpha\}}}h_2(\tau))d\tau\Big), \end{aligned}$$

where the functions $u(0,\tau)$ were defined by formula (2.13).Using representations (2.13) we have (for simplicity we put $f(x,t) = 0$)

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}\int_0^t e^{-K(p)(t-\tau)}K(p)\frac{u(0,\tau)}{p}d\tau = I_1 + I_2 + I_3, \tag{2.14}$$

where

$$\begin{aligned} I_1 &= \frac{1}{4\pi^2}\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p}e^{-K(p)t}\int_{-i\infty}^{i\infty}\frac{\phi\widehat{u}_0(\phi)}{\xi}d\xi\int_0^t d\tau e^{(K(p)+\xi)\tau}, \\ I_2 &= -\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p}e^{-K(p)t}\int_{-i\infty}^{i\infty} e^{-\phi a}\widehat{h}_1(\xi)d\xi\int_0^t d\tau e^{(K(p)+\xi)\tau}, \\ I_3 &= \frac{1}{4\pi^2}\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p}e^{-K(p)t}\int_{-i\infty}^{i\infty}\frac{\phi^{\{\alpha\}}\widehat{h}_2(\xi)}{\xi}d\xi\int_0^t d\tau e^{(K(p)+\xi)\tau}. \end{aligned}$$

Integrating with respect to τ, substituting the Laplace transform $\widehat{u}_0(\phi)$ and using

$$\int_{-i\infty}^{i\infty}\frac{\phi e^{-\phi(\xi)y}}{\xi}\frac{1}{K(p)+\xi}d\xi = 0$$

we obtain

$$I_1 = \frac{1}{4\pi^2}\int_0^a dy u_0(y)\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{\phi e^{-\phi(\xi)y}}{\xi}\int_{-i\infty}^{i\infty} dp e^{px}\frac{K(p)}{p(K(p)+\xi)}d\xi. \quad (2.15)$$

Also in the same way we obtain

$$I_2 = -\frac{1}{4\pi^2}\int_0^t d\tau h_1(\tau)\int_{-i\infty}^{i\infty} d\xi e^{\xi(t-\tau)-\phi(\xi)a}\int_{-i\infty}^{i\infty} dp e^{px}\frac{K(p)}{p(K(p)+\xi)} \quad (2.16)$$

and

$$I_3 = \frac{1}{4\pi^2}\int_0^t d\tau h_2(\tau)\int_{-i\infty}^{i\infty} d\xi e^{\xi(t-\tau)}\phi^{\{\alpha\}}\xi^{-1}\int_{-i\infty}^{i\infty} dp e^{px}\frac{K(p)}{p(K(p)+\xi)}. \quad (2.17)$$

Thus from (2.14) and (2.15)-(2.17) we obtain the integral representation (2.3) for solutions $u(x,t)$ of problem (2.1). Proposition 1 is proved. □

Denote

$$\Lambda_\mu(s) = \frac{e^{-i(\frac{\pi}{4}+\frac{\mu}{2})}\sqrt{2}}{2\pi i}\int_0^{+i\infty} e^{sz-K(z)}z^\mu dz, \quad (2.18)$$

where

$$K(z) = C_\alpha z^\beta, \beta = 2-\{\alpha\}.$$

In the next lemma we obtain some necessary estimates of the Green's function. We prove the following result.

Lemma 1. *The following estimates are true:*

$$\sup_{t>0}\langle t\rangle^{\frac{1}{\beta}}\|\mathcal{G}\phi\|_{\mathbf{L}^\infty} \le C\|\phi\|_{\mathbf{L}^\infty},$$

$$\sup_{t>0}\langle t\rangle^{\frac{1}{\beta}+\delta}\left\|\mathcal{G}\phi - \langle t\rangle^{-\frac{1}{\beta}}\Lambda_0(0)\int_0^a \phi(y)dy\right\|_{\mathbf{L}^\infty} \le C\|\phi\|_{\mathbf{L}^\infty},$$

$$\sup_{t>0}\langle t\rangle^{\frac{1}{\beta}(2-\{\alpha\})}\|\mathcal{G}_{1-\{\alpha\}}\phi\|_{\mathbf{L}^\infty} \le C\sup_{t>0}\|\{\tau\}^{1-\gamma}\langle\tau\rangle^{1+\gamma}\phi\|_{\mathbf{L}^\infty},$$

$$\sup_{t>0}\langle t\rangle^{\frac{\{\alpha\}}{\beta}}\|\mathcal{G}_{\{\alpha\}-1}\phi\|_{\mathbf{L}^\infty} \le C\sup_{t>0}\|\{\tau\}^{1-\gamma}\langle\tau\rangle^{1+\gamma}\phi\|_{\mathbf{L}^\infty}$$

and

$$\langle t\rangle^{\frac{\{\alpha\}}{\beta}+\delta}\left\|\mathcal{G}_{\{\alpha\}-1}\phi - \Lambda_{\{\alpha\}-1}(0)\langle t\rangle^{-\frac{\{\alpha\}}{\beta}}\int_0^{+\infty}\phi(\tau)d\tau\right\|_{\mathbf{L}^\infty}$$
$$\le C\sup_{t>0}\left\|\{t\}^{1-\gamma}\langle t\rangle^{1+\gamma}\phi\right\|_{\mathbf{L}^\infty},$$

where $\delta = \frac{1-\{\alpha\}}{2-\{\alpha\}}$.

Proof. We write the Green's function (see (2.2))

$$G(x,y,t) = F_1(x-y,t) + F_2(x,y,t).$$

First we prove the following asymptotic behaviour for $t \to \infty$ uniformly with respect to $x, y \in (0,a)$:

$$\partial_y^\mu G(x,y,t) = t^{-\frac{1}{\beta}(1+\mu)} \Lambda_\mu (xt^{-\frac{1}{\beta}}) + O(t^{-\frac{1(1+\mu)}{\beta}-\delta}) \tag{2.19}$$

is true, where $\mu \in (-1,1)$. We write the representation for the function $F_1(x-y,t)$ as follows:

$$\partial_y^\mu F_1(x-y,t) = \partial_y^\mu F_1(x,t) + \partial_y^\mu (F_1(x-y,t) - F_1(x,t)). \tag{2.20}$$

Making a change of variables $K(p)t = K(z)$ we easily find that

$$\partial_y^\mu F_1(x,t) = t^{-\frac{1}{\beta}(1+\mu)} \Lambda_\mu (xt^{-\frac{1}{\beta}}). \tag{2.21}$$

Using the estimate $|e^{-py} - 1| \le C|py|^\delta$ for $y > 0$ and $p \in (-i\infty, i\infty)$ and making the same change of variables we get for $y \in (0,a)$

$$\begin{aligned} |\partial_y^\mu (F_1(x-y,t) - F_2(x,t))| &\le \Big| \int_{-i\infty}^{+i\infty} e^{-K(p)t+px}(1-e^{-py})p^\mu dp \Big| \\ &\le Ct^{-\frac{1}{\beta}(1+\mu)-\delta} \int_{-i\infty}^{+i\infty} e^{\mathrm{Re}K(z)} |z|^{\delta+\mu} |dz| \le Ct^{-\frac{1(1+\mu)}{\beta}+\delta}. \end{aligned} \tag{2.22}$$

Therefore, from (2.20)-(2.22) we obtain for large time $t \to \infty$

$$\partial_y^\mu F_1(x,y,t) = t^{-\frac{1}{\beta}(1+\mu)} \Lambda_\mu (xt^{-\frac{1}{\beta}}) + O(t^{-\frac{1(1+\mu)}{\beta}+\delta}). \tag{2.23}$$

We write the representation of the function $\partial_y^\mu F_2(x,y,t)$

$$\partial_y^\mu F_2(x,y,t) = \partial_y^\mu F_2(x,0,t) + \partial_y^\mu (F_2(x,y,t) - F_2(x,0,t)). \tag{2.24}$$

Making a change of variables $\xi t = q$ and $K(p)t = K(z)$ we get

$$\partial_y^\mu F(x,0,t) = \int_{-i\infty}^{i\infty} d\xi e^{\xi t} \frac{\phi^{1+\mu}}{\xi} \int_{-i\infty}^{i\infty} dp e^{px} \frac{K(p)}{p(K(p)+\xi)} = t^{-\frac{1}{\beta}(1+\mu)} \Lambda_2 (xt^{-\frac{1}{\beta}}),$$

where

$$\Lambda_2(s) = \int_{-i\infty}^{i\infty} e^q \phi^\mu \phi'(q) dq \int_{-i\infty}^{i\infty} dz e^{zs} \frac{K(z)}{z(K(z)+q)}. \tag{2.25}$$

Since $\mathrm{Re}K(z) > 0$ for $\mathrm{Re}z < 0$ via the Cauchy theorem we have for $s > 0$

$$\int_{-i\infty}^{i\infty} dz e^{zs} \frac{K(z)}{z(K(z)+q)} = \int_\Gamma dz e^{zs} \frac{K(z)}{z(K(z)+q)}, \tag{2.26}$$

where the contour Γ is defined as

$$\Gamma = \{z \in (e^{-i\pi}\infty, 0) \cup (0, e^{i\pi}\infty)\}.$$

Also using $\frac{1}{K'(\phi)} = -\phi'$ (by definition $K(\phi) = -q$) we obtain

$$\begin{aligned} \int_{-i\infty}^{i\infty} e^q \phi^{1+\mu}(q) dq \int_\Gamma dz \frac{1}{z(K(z)+q)} &= -2\pi i \int_{-i\infty}^{i\infty} dq e^q \frac{\phi^\mu(q)}{K'(\phi)} \\ &= 2\pi i \int_{-i\infty}^{i\infty} dq e^q \phi^\mu \phi'(q). \end{aligned}$$

Therefore we obtain the following estimate of the function $\Lambda_2(s)$:

$$\begin{aligned} \Lambda_2(s) &= \int_{-i\infty}^{i\infty} e^q \phi^\mu \phi'(q) dq \int_\Gamma dp e^{zs} \frac{K(z)}{z(K(z)+q)} = 2\pi i \int_{-i\infty}^{i\infty} e^q \phi^\mu \phi'(q) dq \\ &- \int_{-i\infty}^{i\infty} e^q \phi^{1+\mu}(q) dq \int_\Gamma dp e^{zs} \frac{1}{z(K(z)+q)} \qquad (2.27) \\ &= 2\pi i \int_{-i\infty}^{i\infty} e^q \phi^\mu \phi'(q) dq - \int_{-i\infty}^{i\infty} e^q \phi^{1+\mu}(q) dq \int_\Gamma dz (e^{zs}-1) \frac{1}{z(K(z)+q)} \\ &- \int_{-i\infty}^{i\infty} e^q \phi^{1+\mu}(q) dq \int_\Gamma dz \frac{1}{z(K(z)+q)} = O(t^{-\frac{\mu}{\beta}-\delta}). \end{aligned}$$

We have

$$\frac{z^{-1}K(z)}{K(z)+q} = \frac{1}{z+1} + \frac{-q}{(z+1)(K(z)+q)} + \frac{z^{-1}K(z)}{(z+1)(K(z)+q)}$$

and

$$\int_{-i\infty}^{i\infty} e^{zx}(z+1)^{-1} dz = 2\pi i e^{-x}.$$

Then changing the variables $K(p)t = K(z)$ and $\xi t = q$, using $\phi(q) = rq^{\frac{1}{\beta}}$, where r is some complex constant, we get

$$\begin{aligned} \partial_y^\mu F_2(x,y,t) &= Ct^{-\frac{1}{\beta}(1+\mu)} \Big(2\pi i e^{-\tilde{x}} \int_{-i\infty}^{+i\infty} e^{q - rq^{\frac{1}{\beta}}\tilde{y}} q^{\frac{1}{\beta}(1+\mu)-1} dq \\ &+ \int_{-i\infty}^{+i\infty} dz e^{z\tilde{x}} (z+1)^{-1} \int_{-i\infty}^{+i\infty} e^{q - rq^{\frac{1}{\beta}}\tilde{y}} q^{\frac{1}{\beta}(1+\mu)} (K(z)+q)^{-1} dq \qquad (2.28) \\ &+ \int_{-i\infty}^{+i\infty} dz e^{z\tilde{x}} z^{[\alpha]-\{\alpha\}} (z+1)^{-1} \int_{-i\infty}^{+i\infty} e^{q - rq^{\frac{1}{\beta}}\tilde{y}} q^{\frac{1}{\beta}(1+\mu)-1} (K(z)+q)^{-1} dq \Big), \end{aligned}$$

where $\tilde{x} = xt^{-\frac{1}{\beta}}$and $\tilde{y} = yt^{-\frac{1}{\beta}}$. Differentiating the representation (2.28) of the function $F_2(x,y,t)$ with respect to y we get

$$\partial_y^{\mu+1}F_2(x,y,t) = -Ct^{-\frac{1}{\beta}(2+\mu)}\Big(2\pi i e^{-\tilde{x}}\int_{-i\infty}^{+i\infty} e^{q-rq^{\frac{1}{\beta}}\tilde{y}} q^{\frac{1}{\beta}(2+\mu)-1}dq$$
$$+\int_{-i\infty}^{+i\infty} e^{z\tilde{x}}(z+1)^{-1}dz\int_{-i\infty}^{+i\infty} e^{q-rq^{\frac{1}{\beta}}\tilde{y}} q^{\frac{1}{\beta}(2+\mu)}(C_\alpha z^\beta+q)^{-1}dq$$
$$+\int_{-i\infty}^{+i\infty} e^{z\tilde{x}}z^{[\alpha]-\{\alpha\}}(z+1)^{-1}dz\int_{-i\infty}^{+i\infty} e^{q-q^{\frac{1}{\beta}}\tilde{y}} q^{\frac{1}{\beta}(1+\mu)-1}(C_\alpha z^\beta+q)^{-1}dq\Big).$$

It is easy to see that we can change the contour of integration into

$$\mathcal{C}_1 = \{z=\rho e^{\pm i\beta_1} : \rho \geq 0, \beta_1 = \frac{\pi}{2}+\epsilon_1\} \tag{2.29}$$

and

$$\mathcal{C}_2 = \{q=\rho e^{\pm i\beta_2} : \rho \geq 0, \beta_2 = \frac{\pi}{2}+\epsilon_2\}, \tag{2.30}$$

where $\epsilon_1$and ϵ_2 are fixed small positive constants. Then since $\mathrm{Re}q, \mathrm{Re}z < 0$ and $\mathrm{Re}rq^{\frac{2}{3}}\tilde{y} > 0$ for all $z \in \mathcal{C}_1$, $q \in \mathcal{C}_2$ we have

$$e^{\mathrm{Re}z\tilde{x}} \leq C|z|^{-\mu_2}\tilde{x}^{-\mu_2} \tag{2.31}$$

and

$$e^{-\mathrm{Re}rq^\beta\tilde{y}} \leq C|q|^{-\mu_1}\tilde{y}^{-\mu_1\beta}, \tag{2.32}$$

where $\mu_1, \mu_2, \gamma \geq 0$. Also it is easy to see that for all $z \in \mathcal{C}_1$, $q \in \mathcal{C}_2$ and $\nu \in [0,1]$

$$|K(z)+q|^{-1} \leq C|z|^{-\nu\beta}|q|^{\nu-1}. \tag{2.33}$$

Using the inequalities (2.31)–(2.33) with $\mu_1, \mu_2 = 0$ and $\nu \in [0,1]$ we get

$$\|\partial_y^{1+\mu}F_2(\cdot,\cdot,t)\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{\beta}(2+\mu)}\Big(\int_{\mathcal{C}_2} e^{-C|q|}|q|^{\frac{2}{3}(2+\mu)-1}|dq|$$
$$+\int_{-i\infty}^{+i\infty}|z|^{-\beta\nu}|z+1|^{-1}|dz|\int_{\mathcal{C}_2} e^{-C|q|}|q|^{\frac{1}{\beta}(2+\mu)-1+\nu}|dq|$$
$$+\int_{-i\infty}^{+i\infty}|z|^{[\alpha]-\{\alpha\}-\beta\nu}|z+1|^{-1}|dz|\int_{\mathcal{C}_2} e^{-C|q|}|q|^{\frac{1}{\beta}(2+\mu)-2+\nu}|dq|\Big)$$
$$\leq \quad Ct^{-\frac{1}{\beta}(2+\mu)}. \tag{2.34}$$

Also, for $\mu_1 \in [0,\beta)$ and $\mu_2 \in [0,\beta)$, choosing $\nu \in [0,1]$ such that $\frac{[\alpha]-\{\alpha\}}{\beta} < \mu_2+\nu < \beta$ we have

$$|\partial_y^\mu F_2(x,y,t)| \leq Ct^{-\frac{1}{\beta}(1+\mu-\mu_1-\mu_2)}y^{-\mu_1\beta}x^{-\mu_2\beta}\Big(\int_{\mathcal{C}_2} e^{-C|q|}|q|^{\frac{1}{\beta}(1+\mu)-1-\mu_1}|dq|$$

$$+\int_{\mathcal{C}_1}|dz||z+1|^{-1}|z|^{-\nu\beta-\mu_2\beta}\int_{\mathcal{C}_2}e^{-C|q|}|q|^{\frac{1}{\beta}(1+\mu)+\nu-1-\mu_1}|dq| \tag{2.35}$$
$$+\int_{\mathcal{C}_1}|dz||z|^{[\alpha]-\{\alpha\}-\nu\beta-\mu_2\beta}|z+1|^{-1}\int_{\mathcal{C}_2}e^{-C|q|}|q|^{\frac{1}{\beta}(1+\mu)-2+\nu-\mu_1}|dq|\Big)$$
$$\le Ct^{-\frac{1}{\beta}(1+\mu)+\mu_1+\mu_2}y^{-\mu_1\beta}x^{-\mu_2\beta}.$$

Therefore, using (2.35) with $\mu_1=0,\mu_2=0$ for $t>1$

$$\|\partial_y^{\mu}F_2(\cdot,\cdot,t)\|_{\mathbf{L}^\infty}\le Ct^{-\frac{1}{\beta}(1+\mu)}. \tag{2.36}$$

From (2.34) and (2.36) we get, for $\delta\in[0,1],y\in[0,a]$ and $t>1$,

$$|\partial_y^{\mu}(F_2(x,y,t)-F_2(x,0,t))| \tag{2.37}$$
$$\le\ C\|\partial_y^{1+\mu}F(\cdot,\cdot,t)\|^{\delta}_{\mathbf{L}^\infty}\|\partial_y^{\mu}F(\cdot,\cdot,t)\|^{1-\delta}_{\mathbf{L}^\infty}y^{\delta}\le Ct^{-\frac{1(1+\mu+\delta)}{\beta}}.$$

By virtue of (2.24), (2.25), (2.27) and (2.37) we estimate $F_2(x,y,t)$ as

$$\partial_y^{\mu}F_2(x,y,t)=O(t^{-\frac{1(1+\mu+\delta)}{\beta}}). \tag{2.38}$$

By formulas (2.23) and (2.38) we find

$$\partial_y^{\mu}G(x,y,t)=t^{-\frac{1}{\beta}(1+\mu)}\Lambda_{\mu}(xt^{-\frac{1}{\beta}})+O(t^{-\frac{1(1+\mu)}{\beta}-\delta}), \tag{2.39}$$

where

$$\Lambda_{\mu}(s)=\frac{e^{-i(\frac{\pi}{4}+\frac{\mu}{2})}\sqrt{2}}{2\pi i}\int_0^{+i\infty}e^{sz-K(z)}z^{\mu}dz.$$

The Laplace transform of the function $F_1(x,t)$ is equal to $\widehat{F_1}(p,t)=e^{-K(p)t}$. So making a change of variable $K(z)=K(p)t$ we get

$$\|F_1(t)\|_{\mathbf{L}^1}=\|\widehat{F}_1(t)\|_{\mathbf{L}^\infty(\mathrm{Re}p=0)}\le C.$$

Therefore, from (2.35) we have

$$\|\mathcal{G}\phi\|_{\mathbf{L}^\infty}=\|\int_0^a F_1(x-y,t)\phi(y)dy\|_{\mathbf{L}^\infty}+\|\int_0^a F_2(x,y,t)\phi(y)dy\|_{\mathbf{L}^\infty}$$
$$\le C\|F_1(t)\|_{\mathbf{L}^1}\|\phi\|_{L^\infty}+C\int_0^a y^{-1+\gamma}|\phi(y)|dy\le C\langle t\rangle^{-\frac{1}{\beta}}\|\phi\|_{\mathbf{L}^\infty}.$$

Also using (2.39) with $\mu=0$ we obtain

$$\langle t\rangle^{\frac{1}{\beta}+\delta}\|\mathcal{G}\phi-\langle t\rangle^{-\frac{1}{\beta}}\Lambda_0(x\langle t\rangle^{-\frac{1}{\beta}})\int_0^a\phi(y)dy\|_{\mathbf{L}^\infty}\le C\|\phi\|_{\mathbf{L}^\infty}.$$

Since, for $t < 1$,

$$\begin{aligned} &\|\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{p(x-a)-K(p)t}p^{1-\{\alpha\}}dp\|_{\mathbf{L}^\infty} \\ &= C\|e^{-pa-K(p)t}p^{1-\{\alpha\}}\|_{\mathbf{L}^1(\mathrm{Re}p=0)} \le C\langle t\rangle^{-\frac{2-\{\alpha\}}{\beta}}, \end{aligned}$$

from (2.35) we get

$$\begin{aligned} \|\mathcal{G}_{1-\{\alpha\}}\phi\|_{\mathbf{L}^\infty} &\le C\int_0^t \|\partial_y^{1-\{\alpha\}}G(x,a,t-\tau)\|_{\mathbf{L}^\infty}|\phi(\tau)|d\tau \\ &\le C\sup_{t>0}\|\{t\}^{1-\gamma}\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{L}^\infty}\int_0^t \langle t-\tau\rangle^{-\frac{2-\{\alpha\}}{\beta}}\{\tau\}^{-1+\gamma}\langle\tau\rangle^{-1-\gamma}d\tau \\ &\le C\sup_{t>0}\|\{t\}^{1-\gamma}\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{L}^\infty}\langle t\rangle^{-\frac{2-\{\alpha\}}{\beta}}, \\ \|\mathcal{G}_{\{\alpha\}-1}\phi\|_{\mathbf{L}^\infty} &\le C\int_0^t \|\partial_y^{\mu}G(x,0,t-\tau)\|_{\mathbf{L}^\infty}|\phi(\tau)|d\tau \\ &\le C\sup_{t>0}\|\{t\}^{1-\gamma}\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{L}^\infty}\int_0^t (t-\tau)^{-\frac{1}{3}}\{\tau\}^{-1+\gamma}\langle\tau\rangle^{-1-\gamma}d\tau \\ &\le C\sup_{t>0}\|\{t\}^{1-\gamma}\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{L}^\infty}\langle t\rangle^{-\frac{\{\alpha\}}{\beta}}. \end{aligned}$$

Also from (2.39) we easily get for $\delta \in (0, 1-\{\alpha\})$

$$\begin{aligned} &\|\mathcal{G}_{\{\alpha\}-1}\phi - \Lambda_{\{\alpha\}-1}(0)\langle t\rangle^{-\frac{\{\alpha\}}{\beta}}\int_0^{+\infty}\phi(\tau)d\tau\|_{\mathbf{L}^\infty} \\ &\le C\sup_{t>0}\|\{t\}^{1-\gamma}\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{L}^\infty}\int_0^t (t-\tau)^{-\frac{1}{\beta}(\{\alpha\}+\delta)}\{\tau\}^{-1+\gamma}\langle\tau\rangle^{-1-\gamma}d\tau \\ &\le \sup_{t>0}\|\{t\}^{1-\gamma}\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{L}^\infty}\langle t\rangle^{-\frac{\{\alpha\}}{\beta}-\delta}. \end{aligned}$$

Lemma 1 is proved. □

3. Proofs of Theorems

From Proposition 1 we write the solution $u(x,t)$ of the problem (1.1) in the form

$$u(x,t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)d\tau + \mathcal{G}_{1-\{\alpha\}}(t)h_1 + \mathcal{G}_{\{\alpha\}-1}(t)h_2, \quad (3.1)$$

where the operators $\mathcal{G}, \mathcal{G}_{1-\{\alpha\}}$ and $\mathcal{G}_{\{\alpha\}-1}$ were defined by (2.2). We denote metric spaces

$$\mathbf{Z}_1 = \{\phi \in L^\infty(0,a)\},$$

$$\mathbf{Z}_2 = \{\phi \in L^1([0,\infty)) : \|\phi\|_{\mathbf{Z}_2} = \sup_{t>0} \{t\}^{1-\gamma} \langle t \rangle^{1+\gamma} \|\phi\|_{\mathbf{L}^\infty} < \infty\}$$

and

$$\mathbf{X} = \{\phi(x,t) \in \mathbf{C}([0,\infty); \mathbf{L}^\infty(0,a)) : \|\phi\|_{\mathbf{X}} < +\infty\},$$

where

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0} (\langle t \rangle^{\frac{\{\alpha\}}{2-\{\alpha\}}} \|\phi(t)\|_{\mathbf{L}^\infty}).$$

Also we introduce

$$\Lambda_\mu(s) = \frac{e^{-i(\frac{\pi}{4}+\frac{\mu}{2})}\sqrt{2}}{2\pi i} \int_0^{+i\infty} e^{sz-K(z)} z^\mu dz.$$

From Lemma 1 we see that the operator

$$\mathcal{B}_0(t)\phi = t^{-\frac{\{\alpha\}}{2-\{\alpha\}}} \Lambda_{\{\alpha\}-1}(0) \int_0^{+\infty} \phi(\tau) d\tau \tag{3.2}$$

is the asymptotic self-similar operator for the Green's operator $\mathcal{G}_{\{\alpha\}-1}$ in spaces $\mathbf{Y}$, $\mathbf{Z}_2$ such that the following estimate is true:

$$\|\langle t \rangle^\delta (\mathcal{G}_{\{\alpha\}-1}(t)\phi - \mathcal{B}_0(t)\phi)\|_{\mathbf{Y}} \le C\|\phi\|_{\mathbf{Z}_2}, \tag{3.3}$$

where $\delta = \frac{1-\{\alpha\}}{2-\{\alpha\}}$,

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} (t^{\frac{\{\alpha\}}{2-\{\alpha\}}} \|\phi(t)\|_{\mathbf{L}^\infty}). \tag{3.4}$$

Also we define the operator

$$\mathcal{G}_0(t)\phi = t^{-\frac{1}{2-\{\alpha\}}} \Lambda_0(0) \int_0^a \phi(y) dy. \tag{3.5}$$

From Lemma 1 we see that the operator $\mathcal{G}_0(t)\phi$ is the asymptotic operator for the Green's operator $\mathcal{G}$ in spaces $\mathbf{Y}$, $\mathbf{Z}_1$ such that the following estimate is true:

$$\|\langle t \rangle^\delta (\mathcal{G}(t)\phi - \mathcal{G}_0(t)\phi)\|_{\mathbf{Y}} \le C\|\phi\|_{\mathbf{Z}_1}. \tag{3.6}$$

3.1. **Subcritical case (Proof of Theorem 1).** We consider the case $\{\alpha\}^{-1}-2<\rho<\{\alpha\}^{-1}-1$. By changing $\tau=tz$ we get for some constant θ

$$\begin{aligned}&\int_0^t \mathcal{G}_0(t-\tau)\mathcal{N}(\tau^{-\frac{\{\alpha\}}{2-\{\alpha\}}}\theta)d\tau \\ &= t^{-\frac{\{\alpha\}}{2-\{\alpha\}}+\mu}\theta^{\rho+1}\Lambda_0(0)a\lambda\int_0^1(1-z)^{-\frac{1}{2-\{\alpha\}}}z^{-(\rho+1)\frac{\{\alpha\}}{2-\{\alpha\}}}dz \\ &= A\theta^{\rho+1}t^{-\frac{\{\alpha\}}{2-\{\alpha\}}+\mu},\end{aligned} \tag{3.7}$$

where

$$0<\mu=\frac{1-(\rho+1)\{\alpha\}}{2-\{\alpha\}}<\frac{\{\alpha\}}{2-\{\alpha\}}$$

and

$$A=a\lambda\Lambda_0(0)\int_0^1(1-z)^{-\frac{1}{2-\{\alpha\}}}z^{-(\rho+1)\frac{\{\alpha\}}{2-\{\alpha\}}}dz.$$

We call the nonlinearity $\mathcal{N}$ in equation (1.1) a subcritical convective.

We make a change of the dependent variable $u(x,t)=v(x,t)e^{-\varphi(t)}$ in equation (3.1). Then we get the following equation for the new unknown function $v(x,t)$

$$v_t+\mathcal{L}v+e^{-\rho\varphi}\mathcal{N}(v)-\varphi' v=0.$$

We choose the auxiliary functions $\varphi(t)$ by the following condition

$$f(e^{-\rho\varphi}\mathcal{N}(v)-\varphi' v)=0,\phi(0)=0.$$

Thus we obtain the initial-boundary-value problem for the new dependent variable $v(t,x)$

$$\left\{\begin{array}{c} v_t+\mathcal{L}v=-e^{-\rho\varphi}(\mathcal{N}(v)-\frac{v}{f(v)}f(\mathcal{N}(v))),\ t>0,x\in(0,a),\\ v(0,x)=v_0(x)\equiv u_0(x),\ x\in(0,a),\\ v(a,t)=e^{\phi(t)}h_1(t),v_x(0,t)=e^{\phi(t)}h_2(t),\\ \varphi'=f^{-1}(v)f(e^{-\rho\varphi}\mathcal{N}(v)),\phi(0)=0.\end{array}\right. \tag{3.8}$$

We denote $\zeta(t)=e^{\rho\varphi(t)}$. Then we get

$$\zeta'=\frac{\rho}{f(v)}f(\mathcal{N}(v)),\ \zeta(0)=1,$$

hence integration with respect to time yields

$$\zeta(t)=1+\rho\int_0^t f^{-1}(v)f(\mathcal{N}(v(\tau)))d\tau.$$

According to Proposition 1 the integral equation associated with (3.8) can be written as

$$v(t) = \mathcal{G}(t)v_0 - \int_0^t \mathcal{G}(t-\tau)\frac{\mathcal{N}_1(v(\tau))}{\zeta_v(\tau)}d\tau + \mathcal{G}_{1-\{\alpha\}}\zeta^{\frac{1}{\rho}}h_1 + \mathcal{G}_{\{\alpha\}-1}\zeta^{\frac{1}{\rho}}h_1, \quad (3.9)$$

where the nonlinearity $\mathcal{N}_1$ and the functional ζ_ν were defined by

$$\mathcal{N}_1(v(\tau)) = \mathcal{N}(v(\tau)) - v(\tau)f^{-1}(v)f(\mathcal{N}(v(\tau))).$$

We now prove the existence of the solution $v(x,t)$ for the integral equation (3.9) by the contraction mapping principle. Denote by

$$\theta = \int_0^{+\infty}(1+\eta\theta^\rho\tau^{\frac{\{\alpha\}}{2-\{\alpha\}}})^{\frac{1}{\rho}}h_2(\tau)d\tau < C\varepsilon.$$

We define the transformation $\mathcal{M}(w)$ by the formula

$$\mathcal{M}(w) = \mathcal{G}(t)v_0 - \int_0^t \mathcal{G}(t-\tau)\frac{\mathcal{N}_1(w(\tau))}{\zeta_w(\tau)}d\tau + \mathcal{G}_{1-\{\alpha\}}\zeta_w^{\frac{1}{\sigma}}h_1 + \mathcal{G}_{\{\alpha\}-1}\zeta^{\frac{1}{\sigma}}h_1, \quad (3.10)$$

for any $w \in \mathbf{B}$, where $\mathbf{B} = \big\{w \in \mathbf{X} : f(w(t)) \geq \frac{1}{2}\theta a\Lambda_0\langle t\rangle^{-\frac{\{\alpha\}}{2-\{\alpha\}}},\ \|w\|_{\mathbf{X}} \leq C\varepsilon,\ \|\langle t\rangle^\delta(w-\mathcal{B}_0(t)g^{\frac{1}{\rho}}h_2)\|_{\mathbf{X}} \leq C\varepsilon^{1+\gamma_1}, |\zeta_w(t)-g(t)| \leq C\varepsilon^{1+\gamma_1}\langle t\rangle^{\frac{1}{2-\{\alpha\}}+\mu-\gamma}\big\}$, where $\zeta_w(t) = 1+\sigma\int_0^t f^{-1}(w)f(\mathcal{N}(w(\tau)))d\tau$ and

$$\begin{aligned} g(t) &= 1+\rho\int_0^t f^{-1}(\mathcal{B}_0(t)g^{\frac{1}{\rho}}h_2)f(\mathcal{N}(\mathcal{B}_0(t)g^{\frac{1}{\rho}}h_2))d\tau \qquad (3.11)\\ &= 1+\rho\theta^\rho\Lambda_0(0)^\rho\int_0^t\langle\tau\rangle^{\mu+\frac{1}{2-\{\alpha\}}-1}d\tau \\ &= 1+\eta\theta^\rho t^{\mu+\frac{1}{2-\{\alpha\}}}, \eta = \lambda(\mu+\frac{1}{2-\{\alpha\}})^{-1}\rho\Lambda_0^\rho > 0. \end{aligned}$$

From Lemma 1 we have

$$\|\langle t\rangle^\gamma\mathcal{G}(t)v_0\|_{\mathbf{X}} \leq C\int_0^a x^{\gamma(2-\{\alpha\})}|v_0(x)|dx \leq C\varepsilon^{1+\gamma_1} \quad (3.12)$$

and since $f(\mathcal{N}_1(w(\tau))) = 0$ using $\zeta_w(t) > \frac{1}{3}g(t) > \frac{1}{3}g^{\alpha_1}(t)$,where $\alpha_1 = \frac{\{\alpha\}}{2-\{\alpha\}}\mu < 1$, we get

$$\begin{aligned} &\Big\|\langle t\rangle^\gamma\int_0^t\mathcal{G}(t-\tau)\mathcal{N}_1(w(\tau))\frac{d\tau}{\zeta_w(\tau)}\Big\|_{\mathbf{X}} \qquad (3.13)\\ &\leq 3\Big\|\langle t\rangle^\gamma\int_0^t g^{-3\mu}(\tau)|\mathcal{G}(t-\tau)\mathcal{N}_1(w(\tau))|d\tau\Big\|_{\mathbf{X}} \leq \frac{C}{\eta\theta^{3\mu\rho}}\|w\|_{\mathbf{X}}^{\rho+1} \leq C\varepsilon^{1+\gamma_1}, \end{aligned}$$

where $\gamma_1 = (1-3\mu)\rho > 0$. Also by the condition of Theorem 1

$$\|\langle t\rangle^{\gamma} \mathcal{G}_{1-\{\alpha\}} g^{\frac{1}{\rho}} h_1\|_{\mathbf{X}} \le \varepsilon^{1+\gamma_1} \tag{3.14}$$

and

$$\|\mathcal{G}_{\{\alpha\}-1} g^{\frac{1}{\rho}} h_2\|_{\mathbf{X}} \le \varepsilon. \tag{3.15}$$

Therefore, from (3.12)-(3.15) we see that

$$\|\mathcal{M}(w)\|_{\mathbf{X}} \le \|\mathcal{G}(t)v_0\|_{\mathbf{X}} + \Big\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}_1(w(\tau)) \frac{d\tau}{\zeta_w(\tau)} \Big\|_{\mathbf{X}}$$
$$+ \|\mathcal{G}_{1-\{\alpha\}} \zeta_w^{\frac{1}{\rho}} h_1\|_{\mathbf{X}} + \|\mathcal{G}_{\{\alpha\}-1} g^{\frac{1}{\rho}} h_2\|_{\mathbf{X}} \le C\varepsilon.$$

By direct calculation we have

$$\Big\|\langle t\rangle^{\gamma} (\mathcal{G}_{\{\alpha\}-1} \zeta_w^{\frac{1}{\rho}} h_2 - \mathcal{B}_0(t) g^{\frac{1}{\sigma_1}} h_2)\Big\|_{\mathbf{X}} \le \Big\|\langle t\rangle^{\gamma} (\mathcal{G}_{-\frac{1}{2}} \zeta_w^{\frac{1}{\rho}} h_2 - \mathcal{B}_0(t) \zeta_w^{\frac{1}{\rho}} h_2)\Big\|_{\mathbf{X}}$$
$$+ C\Big\|\langle t\rangle^{\gamma} \int_0^{+\infty} |g^{\frac{1}{\rho}} - \zeta_w^{\frac{1}{\rho}}| h_2(\tau) d\tau \Big\|_{\mathbf{X}} \le C\varepsilon^{1+\gamma_1},$$

where we have used (3.3) and the following estimate with $\beta_1 = \frac{\{\alpha\}}{2-\{\alpha\}}$:

$$\begin{aligned}
&\Big| \int_0^{+\infty} |g^{\frac{1}{\rho}} - \zeta_w^{\frac{1}{\rho}}| h_2(\tau) d\tau \Big| \le \Big| \langle t\rangle^{\gamma} \int_0^t \langle \tau\rangle^{-\gamma} |g-\zeta_w|^{\frac{1}{\rho}} |h_2(\tau)| d\tau \Big| \\
&+ \Big| \langle t\rangle^{-\gamma} \int_t^{+\infty} \langle \tau\rangle^{\gamma} |g-\zeta_w|^{\frac{1}{\rho}} |h_2(\tau)| d\tau \Big| \\
\le\ & \varepsilon^{1+\gamma_1} \|h_2\|_{\mathbf{Z}} \Big(\Big| \langle t\rangle^{\gamma} \int_0^t \langle \tau\rangle^{-\gamma - \frac{1}{\beta_1\rho} - \gamma_2 - 1} \langle \tau\rangle^{\frac{1}{\beta_1\rho} - \frac{\gamma}{\rho}} \{\tau\}^{-1+\gamma_2} d\tau \Big| \\
&+ \Big| \langle t\rangle^{-\gamma} \int_0^{+\infty} \langle \tau\rangle^{\gamma - \frac{1}{\beta_1\rho} - \gamma_2 - 1} \langle \tau\rangle^{\frac{1}{\beta_1\rho} - \frac{\gamma}{\rho}} \{\tau\}^{-1+\gamma_2} d\tau \Big| \Big) \\
\le\ & \varepsilon^{1+\gamma_1} \langle t\rangle^{-\gamma} \|h_2\|_{\mathbf{Z}}.
\end{aligned}$$

Therefore, from (3.12)-(3.15) we obtain

$$\begin{aligned}
&\|\langle t\rangle^{\gamma} (\mathcal{M}(w) - \mathcal{G}_{\{\alpha\}-1} \zeta_w^{\frac{1}{\rho}} h_2)\|_{\mathbf{X}} \qquad (3.16)\\
\le\ & \|\langle t\rangle^{\gamma} \mathcal{G}(t)v_0\|_{\mathbf{X}} + \Big\| \langle t\rangle^{\gamma} \int_0^t \mathcal{G}(t-\tau)\mathcal{N}_1(w(\tau)) \frac{d\tau}{\zeta_w(\tau)} \Big\|_{\mathbf{X}} \\
&+ \|\langle t\rangle^{\gamma} \mathcal{G}_{1-\{\alpha\}} g^{\frac{1}{\rho}} h_1\|_{\mathbf{X}} + \|\langle t\rangle^{\gamma} (\mathcal{G}_{\{\alpha\}-1} \zeta_w^{\frac{1}{\rho}} h_2 - \mathcal{B}_0(t) g^{\frac{1}{\rho}} h_2)\|_{\mathbf{X}} \\
\le\ & C\varepsilon^{1+\gamma_1}.
\end{aligned}$$

Now we prove

$$f(\mathcal{M}(w)) \geq \frac{1}{2} f(\mathcal{B}_0(t) g^{\frac{1}{\rho}} h_2) = \frac{1}{2}\theta\Lambda_0(0)\langle t\rangle^{-\frac{\{\alpha\}}{2-\{\alpha\}}} a. \tag{3.17}$$

By definition we have

$$f(\mathcal{M}(w)) = f(\mathcal{G}v_0) - f\Big(\int_0^t \mathcal{G}(t-\tau)\frac{\mathcal{N}_1(w(\tau))}{\zeta_w(\tau)} d\tau\Big) \tag{3.18}$$
$$+ f(\mathcal{G}_{1-\{\alpha\}}\zeta_w^{\frac{1}{\rho}} h_1) + f(\mathcal{G}_{\{\alpha\}-1}\zeta_w^{\frac{1}{\rho}} h_2),$$

where

$$f(\mathcal{G}_{\{\alpha\}-1}\zeta_w^{\frac{1}{\rho}} h_2) = \langle t\rangle^{\frac{\{\alpha\}}{2-\{\alpha\}}}\theta\Lambda(0)a + f(\mathcal{G}_{\{\alpha\}-1}\zeta_w^{\frac{1}{\rho}} h_2 - \mathcal{B}_0(t) g^{\frac{1}{\rho}} h_2)$$

and

$$\Big| f(\mathcal{G}_{\{\alpha\}-1}\zeta_w^{\frac{1}{\rho}} h_2 - B_0(t)\int_0^{+\infty} g^{\frac{1}{\rho}} h_2 d\tau)\Big|$$
$$\leq \langle t\rangle^{-\frac{\{\alpha\}}{2-\{\alpha\}}+\gamma}\|\langle t\rangle^{\gamma}(\mathcal{G}_{\{\alpha\}-1}\zeta_w^{\frac{1}{\rho}} h_2 - \mathcal{B}_0(t) g^{\frac{1}{\rho}} h_2)\|_{\mathbf{X}} \leq \varepsilon^{1+\gamma_1}\langle t\rangle^{-\frac{\{\alpha\}}{2-\{\alpha\}}+\gamma}.$$

Therefore, from (3.12)-(3.15) we get

$$f(\mathcal{M}(w)) \geq \langle t\rangle^{-\frac{\{\alpha\}}{2-\{\alpha\}}}\theta\Lambda(0)a - C\varepsilon^{1+\gamma_1}\langle t\rangle^{-\frac{\{\alpha\}}{2-\{\alpha\}}+\gamma},$$

where $\theta\Lambda(0)a \leq \varepsilon$. The last estimate implies estimate (3.17). It remains to prove the estimate

$$|\zeta_{\mathcal{M}(w)}(t) - g(t)| \leq C\varepsilon^{1+\gamma_1}\langle t\rangle^{\frac{1}{2-\{\alpha\}}+\mu-\gamma}$$

for all $t > 0$. We have by (3.17)

$$|\zeta_{\mathcal{M}(w)}(t) - g(t)|$$
$$\leq \rho\int_0^t \Big| f^{-1}(\mathcal{B}_0 g^{\frac{1}{\rho}} h_2) f(\mathcal{N}(\mathcal{B}_0 g^{\frac{1}{\rho}} h_2)) - f^{-1}(\mathcal{M}(w)) f(\mathcal{N}(\mathcal{M}(w)))\Big| d\tau$$
$$\leq \sup_{t>0}\langle t\rangle^{1-\mu-\frac{1}{2-\{\alpha\}}+\gamma}\Big| f^{-1}(\mathcal{B}_0 g^{\frac{1}{\rho}} h_2) f(\mathcal{N}(\mathcal{B}_0 g^{\frac{1}{\rho}} h_2))$$
$$- f^{-1}(\mathcal{M}(w)) f(\mathcal{N}(\mathcal{M}(w)))\Big| \int_0^t \langle\tau\rangle^{-1+\frac{1}{2-\{\alpha\}}+\mu-\gamma} d\tau$$
$$\leq \varepsilon^{1+\gamma_1}\langle t\rangle^{\frac{1}{2-\{\alpha\}}+\mu-\gamma},$$

where we have used the following estimate

$$\sup_{t>0}\langle t\rangle^{1-\mu-\frac{1}{2-\{\alpha\}}+\gamma}\Big| f^{-1}(\mathcal{B}_0(t) g^{\frac{1}{\rho}} h_2) f(\mathcal{N}(\mathcal{B}_0(t) g^{\frac{1}{\rho}} h_2))$$

$$
\begin{aligned}
&\quad - f^{-1}(\mathcal{M}(w))f(\mathcal{N}(\mathcal{M}(w)))\Big| \\
&\le C\theta^{-1}\|\langle t\rangle^{\gamma}(\mathcal{N}(\mathcal{B}_0(t)g^{\frac{1}{\rho}}h_2) - \mathcal{N}(\mathcal{M}(w)))\|_{\mathbf{X}} \\
&+ \|\mathcal{N}(\mathcal{B}_0(t)g^{\frac{1}{\rho}}h_2)\|_{\mathbf{X}} \sup_{t>0}\langle t\rangle^{\frac{1}{2-\{\alpha\}}+\gamma}|f^{-1}(\mathcal{B}_0(t)g^{\frac{1}{\rho}}h_2) - f^{-1}(\mathcal{M}(w))| \\
&\le C\varepsilon^{1+\gamma_1}
\end{aligned}
$$

for all $t > 0$. Thus we see that $\mathcal{M}$ transforms $\mathbf{B}$ into itself. Now let us estimate the difference

$$
\begin{aligned}
& \|\mathcal{M}(v) - \mathcal{M}(w)\|_{\mathbf{X}} \\
= \ & \Big\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}_1(v(\tau))\frac{1}{\zeta_v(\tau)} - \mathcal{N}_1(w(\tau))\frac{1}{\zeta_w(\tau)})d\tau\Big\|_{\mathbf{X}} \\
& +C\|\mathcal{G}_{1-\{\alpha\}}(\zeta_v^{\frac{1}{\rho}} - \zeta_w^{\frac{1}{\rho}})h_1\|_{\mathbf{X}} + C\|\mathcal{G}_{\{\alpha\}-1}(\zeta_v^{\frac{1}{\rho}} - \zeta_w^{\frac{1}{\rho}})h_2\|_{\mathbf{X}} \\
\le \ & C\Big\| \int_0^t g^{-1}(\tau)|\mathcal{G}(t-\tau)(\mathcal{N}_1(v(\tau)) - \mathcal{N}_1(w(\tau)))|d\tau\Big\|_{\mathbf{X}} \\
& +C\| \int_0^t g^{-1}(\tau)|\mathcal{G}(t-\tau)\mathcal{N}_1(w(\tau))|\frac{|\zeta_v(\tau) - \zeta_w(\tau)|}{g(\tau)}d\tau\|_{\mathbf{X}} \\
& +C\|\mathcal{G}_{1-\{\alpha\}}(\zeta_v^{\frac{1}{\rho}} - \zeta_w^{\frac{1}{\rho}})h_1\|_{\mathbf{X}} + C\|\mathcal{G}_{\{\alpha\}-1}(\zeta_v^{\frac{1}{\rho}} - \zeta_w^{\frac{1}{\rho}})h_2\|_{\mathbf{X}} \\
\le \ & C\|v-w\|_{\mathbf{X}}(\varepsilon^{\rho} + \frac{1}{\theta}\| \int_0^t g^{-1}(\tau)|\mathcal{G}(t-\tau)\mathcal{N}_1(w(\tau))|d\tau\|_{\mathbf{X}}) \\
\le \ & C\varepsilon^{\rho}(1+\frac{\varepsilon}{\theta})\|v-w\|_{\mathbf{X}} \le \frac{1}{2}\|v-w\|_{\mathbf{X}},
\end{aligned}
$$

where we have used the estimate

$$
\begin{aligned}
& \frac{|\zeta_v(\tau) - \zeta_w(\tau)|}{g(\tau)} \le \frac{C}{\theta g(t)}\Big| \int_0^t f(\mathcal{N}(v(\tau)) - \mathcal{N}(w(\tau)))d\tau\Big| \\
\le \ & \frac{C\varepsilon^{\rho}\langle t\rangle^{\frac{1}{2-\{\alpha\}}+\mu}}{\theta g(t)}\|v-w\|_{\mathbf{X}} \le \frac{C}{\theta}\|v-w\|_{\mathbf{X}}.
\end{aligned}
$$

Therefore, $\mathcal{M}$ is a contraction mapping in the closed set $\mathbf{B}$ of a complete metric space $\mathbf{X}$. Hence there exists a unique global solution $v \in \mathbf{B}$ to the problem (3.8) such that

$$
\begin{aligned}
\|v\|_{\mathbf{X}} \le C\varepsilon, \qquad & \|\langle t\rangle^{\delta}(v - \mathcal{B}_0(t)g^{\frac{1}{\rho}}h_2)\|_{\mathbf{X}} \le C\varepsilon^{1+\gamma_1}, \\
|\zeta_v(t) - g(t)| \ \le \ & C\varepsilon^{1+\gamma_1}\langle t\rangle^{\frac{1}{2-\{\alpha\}}+\mu-\gamma}.
\end{aligned}
$$

Using the relation $u(t,x) = v(t,x)\zeta_v^{-\frac{1}{\rho}}(t)$ we obtain the existence of solutions to the problem (1.1) satisfying the following time decay estimates

$$\|g^{\frac{1}{\rho}} u\|_{\mathbf{X}} \leq C\varepsilon.$$

We now prove the asymptotic behavior for solutions. By representation (3.17) we have

$$\zeta_v(t) = g(t) + O(\langle t\rangle^{\frac{1}{2-\{\alpha\}}+\mu-\gamma}).$$

Then via formula $u(x,t) = e^{-\varphi(t)}v(x,t) = \zeta_v^{-\frac{1}{\rho}}(t)v(x,t)$ we find the estimate

$$\|\langle t\rangle^{\gamma} g^{\frac{1}{\rho}}(u - g^{-\frac{1}{\rho}}\mathcal{B}_0 g^{\frac{1}{\rho}} h_2)\|_{\mathbf{X}} \leq C,$$

since

$$\begin{aligned} &\|\frac{g^{1+\frac{1}{\rho}}}{\langle t\rangle^{\frac{1}{2-\{\alpha\}}+\mu-\gamma}}((\zeta_v^{-\frac{1}{\rho}} - g^{-\frac{1}{\rho}})\mathcal{B}_0 g^{\frac{1}{\rho}} h_2)\|_{\mathbf{X}} \\ \leq\ & C\sup_{t>0} |\frac{g^{1+\frac{1}{\rho}}}{\langle t\rangle^{\frac{1}{2-\{\alpha\}}+\mu-\gamma}}(\zeta_v^{-\frac{1}{\rho}} - g^{-\frac{1}{\rho}})|\|\mathcal{B}_0 g^{\frac{1}{\rho}} h_2\|_{\mathbf{X}} \leq C. \end{aligned}$$

This completes the proof of Theorem 1.

3.2. **Critical case (Proof of Theorem 2).** Now we consider the case of $\rho = \{\alpha\}^{-1} - 1$.By changing $\tau = tz$ we get for some constant θ

$$\begin{aligned} &\int_0^t \mathcal{G}_0(t-\tau)\mathcal{N}(\tau^{-\frac{\{\alpha\}}{2-\{\alpha\}}}\theta)d\tau \qquad (3.19)\\ &= t^{-\frac{\{\alpha\}}{2-\{\alpha\}}}\theta^{\rho+1}\Lambda_0(0)a\lambda\int_0^1 (1-z)^{-\frac{1}{2-\{\alpha\}}} z^{-(\rho+1)\frac{\{\alpha\}}{2-\{\alpha\}}}dz = A\theta^{\rho+1}t^{-\frac{\{\alpha\}}{2-\{\alpha\}}}, \end{aligned}$$

where

$$A = a\lambda\Lambda_0(0)\int_0^1 (1-z)^{-\frac{1}{2-\{\alpha\}}} z^{-(\rho+1)\frac{\{\alpha\}}{2-\{\alpha\}}}dz.$$

We call the nonlinearity $\mathcal{N}$ in equation (1.1) a critical convective.

Now we prove global existence of the solution of the problem (1.1) in the space $\mathbf{X}$. Via Lemma 2.1, we have

$$\|\int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(w) - \mathcal{N}(v))d\tau\|_{\mathbf{X}} \leq C\|w-v\|_{\mathbf{X}}(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}). \qquad (3.20)$$

Also we have

$$\|\mathcal{G}\phi\|_{\mathbf{X}} \leq C\|\phi\|_{\mathbf{Z}_1}. \qquad (3.21)$$

We apply the contraction mapping principle in a ball $\mathbf{X}_\varepsilon = \{\phi \in \mathbf{X}: \ \|\phi\|_{\mathbf{X}} \leq \varepsilon\}$ in the space $\mathbf{X}$ of radius $\varepsilon = \frac{1}{2C}\|u_0\|_{\mathbf{Z}_1} > 0$. For $v \in \mathbf{X}_\varepsilon$ we define the mapping $\mathcal{M}(v)$ by the formula

$$\mathcal{M}(v) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)d\tau + \mathcal{G}_{1-\{\alpha\}}(t)h_1 + \mathcal{G}_{\{\alpha\}-1}(t)h_2. \quad (3.22)$$

We first prove that $\|\mathcal{M}(v)\|_{\mathbf{X}} \leq \varepsilon$, where $\varepsilon > 0$ is sufficiently small. From (3.20), (3.21) and the integral formula (3.22) we have

$$\begin{aligned} \|\mathcal{M}(v)\|_{\mathbf{X}} &\leq \|\mathcal{G}u_0\|_{\mathbf{X}} + \left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau \right\|_{\mathbf{X}} \\ &\quad + \|\mathcal{G}_{1-\{\alpha\}}(t)h_1\|_{\mathbf{X}} + \|\mathcal{G}_{\{\alpha\}-1}(t)h_2\|_{\mathbf{X}} \\ &\leq C\|u_0\|_{\mathbf{Z}_1} + \sum_{j=1}^{2} \|h_j\|_Z + C\|v\|_{\mathbf{X}}^{\rho+1} \leq \frac{\varepsilon}{2} + C\varepsilon^{\rho+1} < \varepsilon, \end{aligned}$$

since $\varepsilon > 0$ is sufficiently small. Hence the mapping $\mathcal{M}$ transforms a ball $\mathbf{X}_\varepsilon$ into itself. In the same manner we estimate the difference

$$\|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbf{X}} \leq \frac{1}{2}\|w - v\|_{\mathbf{X}},$$

which shows that $\mathcal{M}$ is a contraction mapping. Therefore there exists a unique solution $u \in \mathbf{X}$ to the problem (1.1) such that

$$\|u\|_{\mathbf{X}} \leq \varepsilon. \quad (3.23)$$

We now compute the asymptotic behavior of the solution. First we prove the following:

Lemma 2. *There exists a unique small solution V of the equation*

$$V = V_0 - V^{\rho+1}A \quad (3.24)$$

where $V_0 = \Lambda_{-\frac{1}{2}}(0)\int_0^{+\infty} h_2(\tau)d\tau$, $|V_0| < \varepsilon$ and

$$A = a\lambda\Lambda_0(0)\int_0^1 (1-z)^{-\frac{1}{2-\{\alpha\}}} z^{-(\rho+1)\frac{\{\alpha\}}{2-\{\alpha\}}} dz.$$

Moreover, we have the following estimate:

$$|\ V| \leq C\varepsilon, \ |V - V_0| \leq C\varepsilon^2.$$

Proof. We prove the existence of the solution V by the contraction mapping principle. We define the transformation $\mathcal{R}(V)$ by the formula $\mathcal{R}(V) = V_0 - AV^{\rho+1}$, for any V, where $\mathbf{W} = \{|V| \leq C\varepsilon: \ |V - V_0| \leq C\varepsilon^2\}$. First we check that the mapping $\mathcal{R}$ transforms the set $\mathbf{W}$ into itself. We have by definition of the operator $\mathcal{G}_0$ (see formula (3.5) and (3.19))

$$|\mathcal{R}(V)-V_0| \leq C|V|^{\rho+1} \leq C\varepsilon^{\rho+1}.$$

In the same manner we obtain

$$|\mathcal{R}(V_1)-\mathcal{R}(V_2)| \leq \frac{1}{2}|V_1-V_2.|$$

Therefore $\mathcal{R}$ is a contraction mapping in the closed set $\mathbf{W}$. Hence there exists a unique solution $V \in \mathbf{W}$ to the equation such that

$$|V| \leq C\varepsilon,\ |V-V_0| \leq C\varepsilon.$$

The lemma is proved. □

We are now in a position to prove the asymptotic behavior of solutions u. By definition of the operator $\mathcal{G}_0$ (see formula (3.5) and (3.19)) we have

$$\begin{aligned} t^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V^{\rho+1}A &= t^{-\frac{\{\alpha\}}{2-\{\alpha\}}}\int_0^1 \mathcal{G}_0(1-z)\mathcal{N}(z^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V)dz \\ &= \int_0^t \mathcal{G}_0(t-\tau)\mathcal{N}(\tau^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V)d\tau \end{aligned}$$

and therefore,

$$\begin{aligned} &\|\langle t\rangle^{\delta}(\tau^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V-u(t))\|_{\mathbf{Y}} \leq C\|\langle t\rangle^{\delta}(t^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V-\mathcal{G}(t)u_0 \\ &+\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)d\tau-\mathcal{G}_{1-\{\alpha\}}h_1-\mathcal{G}_{\{\alpha\}-1}h_2)\|_{\mathbf{Y}} \leq C\|\langle t\rangle^{\delta}\mathcal{G}(t)u_0\|_{\mathbf{Y}} \\ &+C\|\langle t\rangle^{\delta}\int_0^t \mathcal{G}_0(t-\tau)(\mathcal{N}(u(t))-\mathcal{N}(\tau^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V))d\tau\|_{\mathbf{Y}} \\ &+C\|\langle t\rangle^{\delta}\int_0^t (\mathcal{G}(t-\tau)-\mathcal{G}_0(t-\tau))\mathcal{N}(u(t))d\tau\|_{\mathbf{Y}} \\ &+C(\|\langle t\rangle^{\delta}(\mathcal{G}_{\{\alpha\}-1}h_2-\mathcal{B}_0h_2)\|_{\mathbf{Y}}+C\|\langle t\rangle^{\gamma}\mathcal{G}_{1-\{\alpha\}}h_1\|_{\mathbf{Y}}) \\ &\equiv I_1+I_2+I_3+I_4. \end{aligned}$$

From Lemma 1 we obtain

$$I_1 \leq \|\langle t\rangle^{\delta}\mathcal{G}(t)u_0\|_{\mathbf{Y}} \leq C\|u_0\|_{\mathbf{Z}_1}.$$

Using (3.20) we gain

$$\begin{aligned} I_2 &= C\|\langle t\rangle^{\delta}\int_0^t \mathcal{G}_0(t-\tau)(\mathcal{N}(u(t))-\mathcal{N}(\tau^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V))d\tau\|_{\mathbf{Y}} \\ &\leq C\|\langle t\rangle^{\delta}(u(t)-t^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V)\|_{\mathbf{Y}}(\|u\|_{\mathbf{Y}}+|V|) \leq C\varepsilon\|\langle t\rangle^{\delta}(u(t)-t^{\frac{\{\alpha\}}{2-\{\alpha\}}}V)\|_{\mathbf{Y}} \end{aligned}$$

since by virtue of (3.23) and Lemma 2 $\|u\|_{\mathbf{Y}} + |V| < \varepsilon$. Also from estimate (3.6) we get

$$I_3 = C\|\langle t\rangle^\delta \int_0^t (\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau))\mathcal{N}(u(t))d\tau\|_{\mathbf{Y}} \le C.$$

Via Lemma 1 and the estimate (3.3) we obtain

$$I_4 = C(\|\langle t\rangle^\delta (\mathcal{G}_{\{\alpha\}-1}h_2 - \mathcal{B}_0 h_2)\|_{\mathbf{Y}} + \|\langle t\rangle^\delta \mathcal{G}_{1-\{\alpha\}}h_2\|_{\mathbf{Y}}) \le C.$$

Hence,

$$\|\langle t\rangle^\delta (u(t) - t^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V)\|_{\mathbf{Y}} \le C + \varepsilon\|\langle t\rangle^\delta (u(t) - t^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V)\|_{\mathbf{Y}}$$

and therefore

$$\|\langle t\rangle^\delta (u(t) - t^{-\frac{\{\alpha\}}{2-\{\alpha\}}}V)\|_{\mathbf{Y}} \le C.$$

This completes the proof of Theorem 2.

3.3. **Super critical case (Proof of Theorem 3).** Let $\rho > \{\alpha\}^{-1} - 1$. By changing $\tau = tz$ we get for some constant θ

$$\int_0^t \mathcal{G}_0(t-\tau)\mathcal{N}(\tau^{-\frac{\{\alpha\}}{2-\{\alpha\}}}\theta)d\tau$$
$$= t^{-\frac{\{\alpha\}}{2-\{\alpha\}}-\gamma}\theta^{\rho+1}\Lambda_0(0)a\lambda\int_0^1 (1-z)^{-\frac{1}{2-\{\alpha\}}} z^{-(\rho+1)\frac{\{\alpha\}}{2-\{\alpha\}}}dz = A\theta^{\rho+1}t^{-\frac{\{\alpha\}}{2-\{\alpha\}}-\gamma},$$

where

$$A = a\lambda\Lambda_0(0)\int_0^1 (1-z)^{-\frac{1}{2-\{\alpha\}}} z^{-(\rho+1)\frac{\{\alpha\}}{2-\{\alpha\}}}dz.$$

We call the nonlinearity $\mathcal{N}$ in equation (1.1) a supercritical type.

Now we prove global existence of the solution of the problem (1.1) in the space $\mathbf{X}$.

We apply the contraction mapping principle in a ball $\mathbf{X}_\varepsilon = \{\phi \in \mathbf{X} : \|\phi\|_{\mathbf{X}} \le \varepsilon\}$ in the space $\mathbf{X}$ of radius $\varepsilon = \frac{1}{2C}\|u_0\|_{\mathbf{Z}_1} > 0$. For $v \in \mathbf{X}_\varepsilon$, we define the mapping $\mathcal{M}(v)$ by the formula

$$\mathcal{M}(v) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)d\tau + \mathcal{G}_{1-\{\alpha\}}(t)h_1 + \mathcal{G}_{\{\alpha\}-1}(t)h_2.$$

We first prove that $\|\mathcal{M}(v)\|_{\mathbf{X}} \le \varepsilon$, where $\varepsilon > 0$ is sufficiently small. From (3.20), (3.21) and the integral formula (3.22) we have

$$\|\mathcal{M}(v)\|_{\mathbf{X}} \le \|\mathcal{G}u_0\|_{\mathbf{X}} + \|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau\|_{\mathbf{X}}$$
$$+ \|\mathcal{G}_{1-\{\alpha\}}(t)h_1\|_{\mathbf{X}} + \|\mathcal{G}_{\{\alpha\}-1}(t)h_2\|_{\mathbf{X}}$$

$$\leq C\|u_0\|_{\mathbf{Z}_1} + \sum_{j=1}^{2} \|h_j\|_Z + C\|v\|_{\mathbf{X}}^{\rho+1} \leq \frac{\varepsilon}{2} + C\varepsilon^{\rho+1} < \varepsilon,$$

since $\varepsilon > 0$ is sufficiently small. Hence the mapping $\mathcal{M}$ transforms a ball $\mathbf{X}_\varepsilon$ into itself. In the same manner we estimate the difference

$$\|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbf{X}} \leq \frac{1}{2}\|w - v\|_{\mathbf{X}},$$

which shows that $\mathcal{M}$ is a contraction mapping. Therefore, there exists a unique solution $u \in \mathbf{X}$ to the problem (1.1) such that $\|u\|_{\mathbf{X}} \leq \varepsilon$. We now compute the asymptotic behavior of the solution. We introduce

$$\Lambda_\mu(s) = \frac{e^{-i(\frac{\pi}{4}+\frac{\mu}{2})}\sqrt{2}}{2\pi i} \int_0^{+i\infty} e^{sz-K(z)} z^\mu dz.$$

From Lemma 1 we see that the operator

$$\mathcal{B}_0(t)\phi = t^{-\frac{\{\alpha\}}{2-\{\alpha\}}} \Lambda_{\{\alpha\}-1}(0) \int_0^{+\infty} \phi(\tau)d\tau$$

is the asymptotic self-similar operator for the Green's operator $\mathcal{G}_{\{\alpha\}-1}$ in the spaces $\mathbf{Y}$, $\mathbf{Z}_2$ such that the following estimate is true:

$$\|\langle t\rangle^\delta (\mathcal{G}_{\{\alpha\}-1}(t)\phi - \mathcal{B}_0(t)\phi)\|_{\mathbf{Y}} \leq C\|\phi\|_{\mathbf{Z}_2},$$

where $\delta = \frac{1-\{\alpha\}}{2-\{\alpha\}}$ and $\|\phi\|_{\mathbf{Y}} = \sup_{t>0} \left(t^{\frac{\{\alpha\}}{2-\{\alpha\}}} \|\phi(t)\|_{\mathbf{L}^\infty}\right)$. Now we prove that the solution has the following large time asymptotic behavior:

$$\left\|\langle t\rangle^\delta \left(u(t) - At^{-\frac{\{\alpha\}}{2-\{\alpha\}}}\right)\right\|_{\mathbf{Y}} \leq C\|u_0\|_{\mathbf{Z}} + C\|u\|_{\mathbf{X}}^2, \tag{3.25}$$

where the constant

$$A = \frac{e^{-i(\frac{\pi}{4}+\frac{\{\alpha\}-1}{2})}\sqrt{2}}{2\pi i} \int_0^{+i\infty} e^{sz-K(z)} z^{\{\alpha\}-1} dz \int_0^\infty h_2(\tau)d\tau.$$

Indeed, by virtue of the integral equation (3.1) we get

$$\begin{aligned} &\left\|\langle t\rangle^\delta (u(t) - \mathcal{B}_0(t)h_2)\right\|_{\mathbf{X}} \qquad (3.26)\\ &\leq \|\langle t\rangle^\gamma (\mathcal{G}(t)u_0\|_{\mathbf{X}} + \left\|\langle t\rangle^\delta \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau\right\|_{\mathbf{X}} \\ &+ \|\langle t\rangle^\delta \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau\|_{\mathbf{X}} + \|\langle t\rangle^\delta (\mathcal{G}_{\{\alpha\}-1}(t)\phi - \mathcal{B}_0(t)\phi)\|_{\mathbf{Y}} \leq C. \end{aligned}$$

All summands in the right-hand side of (3.26) are estimated by

$$C\|u_0\|_{\mathbf{Z}_1} + \sum_{j=1,2} \|h_j\|_{\mathbf{Z}_2} + C\|u\|_{\mathbf{X}}^{\rho+1}$$

via estimates of Lemma 1 and (3.23).Thus by (3.26) the asymptotic (3.25) is valid. This completes the proof of Theorem 3.

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