

## CONVERGENCE OF SCATTERING OPERATORS FOR THE KLEIN-GORDON EQUATION WITH A NONLOCAL NONLINEARITY

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**Abstract.** We consider the scattering problem for two types of nonlinear Klein-Gordon equations. One is the equation of the Hartree type, and the other one is the equation with power nonlinearity. We show that the scattering operator for the equation of the Hartree type converges to that for the one with power nonlinearity in some sense. Our proof is based on some inequalities in the Lorentz space, and a strong limit of Riesz potentials.

### 1. INTRODUCTION

This paper is concerned with the scattering problem for the nonlinear Klein-Gordon equation of the form

$$\partial_t^2 u - \Delta u + u = \gamma F_\sigma(u)$$

in space-time  $\mathbb{R} \times \mathbb{R}^n$ , where  $u$  is a real or complex-valued unknown function of  $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ ,  $\partial_t = \partial/\partial t$ ,  $\Delta$  is the Laplacian in  $\mathbb{R}^n$  and  $\gamma$  is a real number. The nonlinearity  $F_\sigma$  has the form

$$F_\sigma(u) = (V_\sigma * |u|^{p-1})u \tag{1.1}$$

with  $p > 1$  and

$$|V_\sigma(x)| \leq C_V |x|^{-\sigma}, \quad 0 < \sigma < n. \tag{1.2}$$

The constant  $C_V$  is independent of  $\sigma$ , and  $*$  denotes the convolution in space variables. Furthermore, we suppose that there exists some real number  $\lambda_0$  such that

$$\|V_\sigma|\cdot|^\sigma - \lambda_0\|_{L^\infty} \rightarrow 0 \quad \text{as } \sigma \rightarrow n. \tag{1.3}$$

The term  $F_\sigma(u)$  is the generalization of the Hartree term  $(|\cdot|^{-\sigma} * |u|^2)u$  which is an approximate expression of the nonlocal interaction with potential  $|x|^{-\sigma}$ .

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Menzala and Strauss started to study the Klein-Gordon equation with the Hartree term in [3].

In order to treat the scattering problem, we define the scattering operator for the nonlinear Klein-Gordon equation  $\partial_t^2 u - \Delta u + u = f(u)$ . First, we list some notation to give the definition. Put  $\omega = \sqrt{1 - \Delta}$ . Let  $H^s$  be the Sobolev space  $\omega^{-s} L^2(\mathbb{R}^n)$ . We denote  $H^1 \oplus L^2$  by  $X$ . For a positive number  $\delta$  and a Banach space  $A$ , we denote the set  $\{a \in A; \|a\|_A \leq \delta\}$  by  $B(\delta; A)$ .

The scattering operator is defined as the mapping  $S : B(\delta; X) \ni (\phi_-^1, \phi_-^2) \mapsto (\phi_+^1, \phi_+^2) \in X$  if the following condition holds for some  $\delta > 0$ :

Let  $Z$  be a suitable subspace of  $C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)$ , and  $u_\pm(t)$  solutions of linear Klein-Gordon equations whose initial data are  $(\phi_\pm^1, \phi_\pm^2)$ , respectively. For any  $(\phi_-^1, \phi_-^2) \in B(\delta; X)$ , there uniquely exist  $u \in Z$  and data  $(\phi_+^1, \phi_+^2) \in X$  such that

$$u(t) = u_-(t) + \int_{-\infty}^t \frac{\sin(t-t')\omega}{\omega} f(u(t')) dt'$$

and

$$\|u(t) - u_\pm(t)\|_{H^1} + \|\partial_t u(t) - \partial_t u_\pm(t)\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty.$$

We say that “ $(S, X)$  is well defined” if we can define the scattering operator  $S : B(\delta; X) \rightarrow X$  for some  $\delta > 0$ .

For simplicity, for the present, we assume that  $F_\sigma$  has the form  $(|\cdot|^{-\sigma} * |u|^{p-1})u$ , which clearly satisfies the conditions (1.2) and (1.3). For  $0 < \sigma < n$  and a suitable function  $v$ , the term  $|\cdot|^{-\sigma} * v$  is expressed by the Riesz potential. In fact, we have

$$(-\Delta)^{(-n+\sigma)/2} v = \gamma(\sigma)^{-1} |\cdot|^{-\sigma} * v, \tag{1.4}$$

where

$$\gamma(\sigma) = \frac{\pi^{n/2} 2^{n-\sigma} \Gamma((n-\sigma)/2)}{\Gamma(\sigma/2)} \tag{1.5}$$

and  $\Gamma$  is the usual Gamma function. Thus, it seems natural to expect that

$$\gamma(\sigma)^{-1} F_\sigma(v) \rightarrow |v|^{p-1} v \quad \text{as } \sigma \rightarrow n$$

in some sense. From the above expectation, we shall prove the following properties:

Suppose that  $F_\sigma$  and  $V_\sigma$  satisfy (1.1)–(1.3). Consider the nonlinear Klein-Gordon equations

$$\partial_t^2 u - \Delta u + u = \gamma(\sigma)^{-1} F_\sigma(u) \tag{NLKG}_\sigma$$

and

$$\partial_t^2 u - \Delta u + u = \lambda_0 |u|^{p-1} u. \tag{NLKG}$$

Let  $S_\sigma$  and  $S_n$  be the scattering operators for  $(NLKG_\sigma)$  and for  $(NLKG)$ , respectively. Then, for sufficiently small data  $\phi_- = (\phi_-^1, \phi_-^2) \in X$ , we have

$$\|S_\sigma(\phi_-) - S_n(\phi_-)\|_X \rightarrow 0 \quad \text{as } \sigma \rightarrow n. \tag{1.6}$$

Before we state our main result, we note some results on the scattering problem for  $(NLKG_\sigma)$  and for  $(NLKG)$ .

There is a large literature on the scattering problem for  $(NLKG)$ . Especially, it was proved by Strauss [9] and Pecher [7] that if

$$1 + \frac{4}{n} \leq p \begin{cases} < \infty & \text{if } n = 1, 2, \\ \leq 1 + \frac{4}{n-2} & \text{if } n \geq 3, \end{cases} \tag{1.7}$$

then  $(S_n, X)$  is well defined. On the other hand, for  $(NLKG_\sigma)$ , we can easily see from Mochizuki [4] that if  $p \geq 2$ ,  $0 < \sigma < n$  and

$$2 - \frac{n(p-3)}{2} \leq \sigma \leq p + 1 - \frac{n(p-3)}{2},$$

then  $(S_\sigma, X)$  is well defined.

Therefore, if we assume that  $p \geq 2$  and

$$2 - \frac{n(p-3)}{2} < n \leq p + 1 - \frac{n(p-3)}{2}, \tag{1.8}$$

then it follows that  $(S_\sigma, X)$  is well defined if  $2 - n(p-3)/2 \leq \sigma < n$  and  $\sigma > 0$ . Moreover,  $(S_n, X)$  is also well defined because (1.8) is equivalent to

$$1 + \frac{4}{n} < p \begin{cases} < \infty & \text{if } n = 1, 2, \\ \leq 1 + \frac{4}{n-2} & \text{if } n \geq 3, \end{cases} \tag{1.9}$$

which implies (1.7).

Again, suppose that  $p \geq 2$  and (1.8) (or equivalently, (1.9)) holds. Then, the dimensional number  $n$  is automatically restricted to  $1 \leq n \leq 6$ . For  $0 < \sigma < n$ , we again denote the scattering operator for  $(NLKG_\sigma)$  by  $S_\sigma$ . Let  $S_n$  be the scattering operator for  $(NLKG)$ . As we stated before, the existence of the scattering operator  $S_\kappa$  has been already known if  $\kappa \in (0, n]$  is sufficiently close to  $n$ . In this paper, we prove that for sufficiently small data  $\phi_- \in X$ , (1.6) holds.

In order to state our result precisely, we give some notation which will be used later. For function spaces  $A(\mathbb{R})$  and  $B(\mathbb{R}^n)$ , we denote  $A(\mathbb{R}; B(\mathbb{R}^n))$  by  $AB$ . For  $l = 0, 1$ , let  $\mathcal{B}^l(\mathbb{R}) = C^l(\mathbb{R}) \cap W_\infty^l(\mathbb{R})$ , where  $W_\infty^l(\mathbb{R})$  is the Sobolev space. We define  $\mathcal{H}$  by  $\mathcal{B}H^1 \cap \mathcal{B}^1L^2$  with the norm  $\|v\|_{\mathcal{H}} = \|v\|_{L^\infty H^1} + \|\partial_t v\|_{L^\infty L^2}$ . We denote the Fourier transform and its inverse by  $\mathcal{F}$  and  $\mathcal{F}^{-1}$ ,

respectively. Let  $\{\widetilde{\psi}_j\}_{j=0}^\infty$  be the Littlewood-Paley decomposition on  $\mathbb{R}^n$ . For  $s \in \mathbb{R}$ ,  $1 \leq q \leq \infty$  and  $1 \leq r < \infty$ , the inhomogeneous Besov space  $B_{q,r}^s$  is defined by  $B_{q,r}^s = \{w \in \mathcal{S}' ; \|w\|_{B_{q,r}^s} < \infty\}$ . Here,

$$\|w\|_{B_{q,r}^s} = \left\{ \sum_{j=0}^\infty 2^{sjr} \|\psi_j * w\|_{L^q}^r \right\}^{\frac{1}{r}}$$

and  $\psi_j = \mathcal{F}^{-1}\widetilde{\psi}_j$ . For  $s \in \mathbb{R}$  and  $1 \leq q < \infty$ , we put  $B_q^s = B_{q,2}^s$ . For  $n \geq 3$ , let  $\vartheta_p$  be a real number satisfying

$$p = \vartheta_p \left(1 + \frac{4}{n}\right) + (1 - \vartheta_p) \left(1 + \frac{4}{n-2}\right).$$

Set

$$\theta = \begin{cases} 1 & \text{if } n = 1, 2, \\ \vartheta_p & \text{if } 3 \leq n \leq 6, \end{cases}$$

and

$$\frac{1}{r(p)} = \frac{1}{2} - \frac{2}{p(n-1+\theta)}, \quad \rho(p) = \frac{n+1+\theta}{p(n-1+\theta)}.$$

We denote  $L^p B_{r(p)}^{1-\rho(p)} \cap \mathcal{H}$  by  $Z(p)$ . Put

$$f_\kappa(v) = \begin{cases} \gamma(\kappa)^{-1} F_\kappa(v) & \text{if } 0 < \kappa < n, \\ \lambda_0 |v|^{p-1} v & \text{if } \kappa = n. \end{cases}$$

We are ready to state our main result.

**Theorem 1.1.** *Let  $1 \leq n \leq 6$ . Assume that  $p \geq 2$  satisfies (1.9), and that  $F_\sigma$  and  $V_\sigma$  satisfy (1.1)–(1.3). Put  $u_\star(t) = \cos(t\omega)\phi_\star^1 + \omega^{-1} \sin(t\omega)\phi_\star^2$ , where  $\star$  denotes either  $+$  or  $-$ . Then there exist some  $\delta_0 > 0$  and  $\sigma(p) \in (0, n)$  satisfying the following properties:*

*If  $\kappa \in [\sigma(p), n]$  and  $(\phi_-^1, \phi_-^2) \in B(\delta_0; X)$ , then there uniquely exist  $u^\kappa \in Z(p)$  and  $(\phi_+^1, \phi_+^2) \in X$  such that  $u^\kappa$  satisfies*

$$u^\kappa(t) = u_-(t) + \int_{-\infty}^t \frac{\sin(t-t')\omega}{\omega} f_\kappa(u^\kappa(t')) dt', \tag{1.10}$$

*and we have*

$$\|u^\kappa(t) - u_\pm(t)\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty. \tag{1.11}$$

*Furthermore, the scattering operator  $S_\kappa$  is well defined on  $B(\delta_0; X)$  and we have (1.6).*

**Remark 1.** *The radius  $\delta_0$  is dependent on  $p$ , but independent of  $\kappa$ .*

The contents of this paper is as follows. In Section 2, we show some inequalities in the Lorentz space, and we give a strong limit of  $(-\Delta)^{(-n+\sigma)/2}$  as  $\sigma$  tends to  $n$ . In Section 3, we first show an embedding inequality for the Besov spaces. Using the inequality and the Strichartz estimate, we prove that the radius  $\delta_0$  is given independently with respect to  $\sigma \in [\sigma(p), n)$ . Finally, applying a strong limit of  $(-\Delta)^{(-n+\sigma)/2}$ , we prove (1.6).

2. PRELIMINARIES

In this section, we first state some properties of the Lorentz space. We define the Lorentz space as follows: We denote  $n$ -dimensional Lebesgue measure by  $\mu$ . For a measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ , and for real numbers  $a, y$  and  $z$ , we define  $m(f, a) = \mu(\{x; |f(x)| > a\})$ ,  $f^*(y) = \inf\{a; m(f, a) \leq y\}$  and  $f^{**}(z) = z^{-1} \int_0^z f^*(y) dy$ . For  $1 \leq q, r \leq \infty$ , set

$$\|f\|_{q,r} = \begin{cases} \left[ \int_0^\infty (z^{1/q} f^{**}(z))^r \frac{dz}{z} \right]^{1/r} & \text{if } 1 \leq r < \infty, \\ \sup_{z>0} z^{1/q} f^{**}(z) & \text{if } r = \infty. \end{cases}$$

For  $1 \leq q, r \leq \infty$ , the Lorentz space  $L^{q,r} = L^{q,r}(\mathbb{R}^n)$  is the set of all measurable functions satisfying  $\|f\|_{q,r} < \infty$ .

Put  $\|\cdot\|_p = \|\cdot\|_{L^p}$ . For  $1 \leq q \leq \infty$ ,  $q'$  denotes the Hölder conjugate of  $q$ . The following proposition is shown by O’Neil [6]:

**Proposition 2.1.** *Let  $1 \leq q, r, q_j, r_j \leq \infty, j = 1, 2$ . Then, we have the following properties:*

(1) *If  $1 < q < \infty$ , then*

$$\|f\|_q \leq \|f\|_{q,q} \leq q' \|f\|_q. \tag{2.1}$$

(2) *If  $1 < q < \infty$  and  $1 \leq r_1 < r_2 \leq \infty$ , then*

$$\|f\|_{q,r_2} \leq \left(\frac{r_1}{q}\right)^{\frac{1}{r_1} - \frac{1}{r_2}} \|f\|_{q,r_1}. \tag{2.2}$$

(3) *If  $1/q_1 + 1/q_2 > 1, 1 + 1/q = 1/q_1 + 1/q_2, r \geq 1$  and  $1/r_1 + 1/r_2 \geq 1/r$ , then*

$$\|f * g\|_{q,r} \leq 3q \|f\|_{q_1,r_1} \|g\|_{q_2,r_2}. \tag{2.3}$$

The Hardy-Littlewood-Sobolev inequality implies that the operator  $(-\Delta)^{(-n+\sigma)/2}$  is a bounded operator from some  $L^{q_2}$ -space into some  $L^{q_1}$ -space. In general, the operator norm of  $(-\Delta)^{(-n+\sigma)/2}$  is dependent on  $q_1, q_2$  and  $\sigma$ . Applying the following proposition, we can see the concrete dependency.

**Proposition 2.2.** *Let  $0 < \sigma < n, 1 < q_2 < q_1 < \infty$ . If  $1+1/q_1 = \sigma/n+1/q_2$ , then we have*

$$\| |\cdot|^{-\sigma} * f \|_{q_1} \leq 3q_1 q_2' n \alpha(n)^{\frac{\sigma}{n}} (n - \sigma)^{-1} \|f\|_{q_2}, \tag{2.4}$$

$$\| (-\Delta)^{\frac{-n+\sigma}{2}} f \|_{q_1} \leq C_H(q_1, q_2, \sigma) \|f\|_{q_2}, \tag{2.5}$$

where

$$C_H(q_1, q_2, \sigma) = \frac{3q_1 q_2' n \alpha(n)^{\sigma/n} \Gamma(n/2)}{2^{n-\sigma+1} \pi^{n/2} \Gamma(1 + (n - \sigma)/2)}$$

and

$$\alpha(n) = \mu(B_1(0)) = \pi^{\frac{n}{2}} \left(\frac{n}{2} \Gamma\left(\frac{n}{2}\right)\right)^{-1}.$$

**Proof.** For  $0 < \sigma < n$ , we obtain

$$\| |\cdot|^{-\sigma} \|_{\frac{n}{\sigma}, \infty} = n \alpha(n)^{\frac{\sigma}{n}} (n - \sigma)^{-1}.$$

Thus, it follows from (2.1)–(2.3) that

$$\begin{aligned} \| |\cdot|^{-\sigma} * f \|_{q_1} &\leq \| |\cdot|^{-\sigma} * f \|_{q_1, q_1} \leq 3q_1 \| |\cdot|^{-\sigma} \|_{n/\sigma, \infty} \|f\|_{q_2, q_1} \\ &\leq 3q_1 n \alpha(n)^{\sigma/n} (n - \sigma)^{-1} \|f\|_{q_2, q_2} \leq 3q_1 q_2' n \alpha(n)^{\sigma/n} (n - \sigma)^{-1} \|f\|_{q_2}. \end{aligned}$$

Hence (2.4) holds. By (1.4), (1.5), (2.4) and the equivalence  $y\Gamma(y) = \Gamma(y+1)$ , we can easily see (2.5).  $\square$

**Remark 2.** Let  $0 < \sigma_0 < n$ . Assume that  $K$  is a compact subset of  $(1, \infty)$ . Then, from (2.5), we see that

$$C_H = \sup_{\sigma \in [\sigma_0, n), q_1, q_2 \in K} C_H(q_1, q_2, \sigma) < \infty.$$

Next, we consider a strong limit of the operator  $(-\Delta)^{(-n+\sigma)/2}$ . For this purpose, we introduce some function spaces. Set

$$\widehat{\mathcal{D}}_0 = \{ \phi \in \mathcal{S}; \mathcal{F}\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \},$$

where  $C_0^\infty(\mathbb{R}^n \setminus \{0\})$  is the set of all compactly supported and smooth functions. Let  $\{\tilde{\phi}_j\}_{j \in \mathbb{Z}}$  be the Littlewood-Paley decomposition on  $\mathbb{R}^n \setminus \{0\}$ . For  $s \in \mathbb{R}, 1 \leq q < \infty$  and  $1 \leq r < \infty$ , the homogeneous Triebel-Lizorkin space  $\dot{F}_{q,r}^s$  is defined by  $\dot{F}_{q,r}^s = \{w \in \mathcal{S}'; \|w\|_{\dot{F}_{q,r}^s} < \infty\} / \mathcal{P}$ . Here,

$$\|w\|_{\dot{F}_{q,r}^s} = \left\| \left\{ \sum_{j=-\infty}^{\infty} 2^{sjr} |\phi_j * w|^r \right\}^{\frac{1}{r}} \right\|_q,$$

$\phi_j = \mathcal{F}^{-1} \widehat{\phi}_j$ , and  $\mathcal{P}$  is the set of all polynomials.

**Proposition 2.3.** *For  $1 < q_2 < q_1 < \infty, \widehat{\mathcal{D}}_0$  is dense in  $L^{q_1} \cap L^{q_2}$ .*

**Proof.** For  $v \in \mathcal{S}$ , set

$$v_N = \sum_{j=-N}^N \phi_j * v, \quad N = 1, 2, \dots$$

Clearly, we have  $v_N \in \widehat{\mathcal{D}}_0$ . If  $1 < q < \infty$ , then  $L^q$  is equivalent to  $\dot{F}_{q,2}^0$  (see, e.g., Ginibre-Velo [1]). Therefore, we obtain

$$\begin{aligned} \|v - v_N\|_q &\leq C \|v - v_N\|_{\dot{F}_{q,2}^0} = C \left\| \left\{ \sum_{k=-\infty}^{\infty} |\phi_k * (v - \sum_{j=-N}^N \phi_j * v)|^2 \right\}^{\frac{1}{2}} \right\|_q \\ &= C \left\| \left\{ \sum_{k=-\infty}^{\infty} |(\phi_k - \sum_{j=-N}^N \phi_k * \phi_j) * v|^2 \right\}^{\frac{1}{2}} \right\|_q. \end{aligned}$$

Since

$$\phi_k - \sum_{j=-N}^N \phi_k * \phi_j = \begin{cases} 0 & \text{if } |k| \leq N - 1, \\ \phi_{-N-1} * (\phi_{-N-2} + \phi_{-N-1}) & \text{if } k = -N - 1, \\ \phi_{-N} * \phi_{-N-1} & \text{if } k = -N, \\ \phi_N * \phi_{N+1} & \text{if } k = N, \\ \phi_{N+1} * (\phi_{N+1} + \phi_{N+2}) & \text{if } k = N + 1, \\ \phi_k & \text{if } |k| > N + 1 \end{cases}$$

and

$$\|\phi_k * \phi_j * v\|_q \leq \|\phi_0\|_1 \|\phi_k * v\|_q,$$

we see that

$$\begin{aligned} \|v - v_N\|_q &\leq C \left\| \left\{ \left| \phi_{-N-1} * (\phi_{-N-2} + \phi_{-N-1}) * v \right|^2 \right. \right. \\ &\quad \left. \left. + \left| \phi_{N+1} * (\phi_{N+1} + \phi_{N+2}) * v \right|^2 + \left| \phi_{-N} * \phi_{-N-1} * v \right|^2 \right. \right. \\ &\quad \left. \left. + \left| \phi_N * \phi_{N+1} * v \right|^2 + \sum_{|k| > N+1} \left| \phi_k * v \right|^2 \right\}^{\frac{1}{2}} \right\|_q \\ &\leq C \left\{ \sum_{|k|=N, N+1} \|\phi_k * v\|_q + \left\| \left\{ \sum_{|k| > N+1} |\phi_k * v|^2 \right\}^{\frac{1}{2}} \right\|_q \right\} \\ &\leq C \left\| \left\{ \sum_{|k| \geq N} |\phi_k * v|^2 \right\}^{\frac{1}{2}} \right\|_q. \end{aligned}$$

It follows from  $v \in \dot{F}_{q,r}^0$  and the Lebesgue dominated convergence theorem that

$$\left\| \left\{ \sum_{|k| \geq N} |\phi_k * v|^2 \right\}^{\frac{1}{2}} \right\|_q \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Hence, we have  $\lim_{N \rightarrow \infty} \|v - v_N\|_{q_j} = 0$  for  $j = 1, 2$ . Therefore, we obtain

$$\|v - v_N\|_{L^{q_1} \cap L^{q_2}} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Thus,  $\widehat{\mathcal{D}}_0$  is dense in  $L^{q_1} \cap L^{q_2}$  because  $\mathcal{S}$  is dense in the same space.  $\square$

Using the above proposition and Remark 2, we have the following strong limit of  $(-\Delta)^{(-n+\sigma)/2}$ :

**Lemma 2.4.** *For  $1 < q < \infty$  and  $0 < \sigma < n$ , define*

$$\frac{1}{q_\sigma} = 1 + \frac{1}{q} - \frac{\sigma}{n}. \quad (2.6)$$

*Let  $0 < \sigma_0 < n$ . If  $1 < q_{\sigma_0} < q < \infty$ ,  $q \geq 2$  and  $f \in L^{q_{\sigma_0}} \cap L^q$ , then we have*

$$\left\| (-\Delta)^{\frac{-n+\sigma}{2}} f - f \right\|_q \rightarrow 0 \quad \text{as } \sigma \rightarrow n. \quad (2.7)$$

**Proof.** From Proposition 2.3, for all  $\varepsilon > 0$ , there exists some  $f_\varepsilon \in \widehat{\mathcal{D}}_0$  such that

$$\|f - f_\varepsilon\|_{L^{q_{\sigma_0}} \cap L^q} \leq \varepsilon. \quad (2.8)$$

We can take some  $\eta \in (0, 1)$  satisfying

$$\text{supp } \widehat{f}_\varepsilon \subset \{x \in \mathbb{R}^n; |x| > \eta\}.$$

Using the Hausdorff-Young inequality, we obtain

$$\left\| (-\Delta)^{\frac{-n+\sigma}{2}} f_\varepsilon - f_\varepsilon \right\|_q \leq C \left\| (|\cdot|^{-n+\sigma} - 1) \widehat{f}_\varepsilon \right\|_{q'} \leq C_\varepsilon |\eta^{-n+\sigma} - 1|,$$

where  $C_\varepsilon$  is independent of  $\sigma$ . Therefore, we see from (2.5) and Remark 2 that

$$\begin{aligned} & \left\| (-\Delta)^{\frac{-n+\sigma}{2}} f - f \right\|_q \\ & \leq \left\| (-\Delta)^{\frac{-n+\sigma}{2}} (f - f_\varepsilon) \right\|_q + \left\| (-\Delta)^{\frac{-n+\sigma}{2}} f_\varepsilon - f_\varepsilon \right\|_q + \|f_\varepsilon - f\|_q \\ & \leq C_H \|f - f_\varepsilon\|_{q_\sigma} + C_\varepsilon |\eta^{-n+\sigma} - 1| + \|f_\varepsilon - f\|_q \\ & \leq C_\varepsilon |\eta^{-n+\sigma} - 1| + (C_H + 1) \|f - f_\varepsilon\|_{L^{q_{\sigma_0}} \cap L^q}, \end{aligned}$$



where  $C_H$  is independent of  $\sigma$  and  $\varepsilon$ . Thus, if  $n - \sigma$  is sufficiently small, then we see from (2.8) that

$$\|(-\Delta)^{\frac{-n+\sigma}{2}} f - f\|_q \leq (C_H + 2)\varepsilon.$$

Hence, we have (2.7). □

### 3. PROOF OF THEOREM

In this section, we prove Theorem 1.1. Throughout this section, we assume that  $1 \leq n \leq 6$ ,  $p \geq 2$  and (1.8) (or equivalently, (1.9)). We set  $r = r(p)$ ,  $\rho = \rho(p)$  and  $Z = Z(p)$ .

For  $0 \leq \kappa \leq n$ , let  $p_\kappa$  be a real number which satisfies

$$\frac{3}{2} = \frac{\kappa}{n} + \frac{p-1}{2p_\kappa} + \frac{1}{2p}. \tag{3.1}$$

We can easily see that

$$p_\kappa < p \quad \text{if } 0 < \kappa < n, \tag{3.2}$$

$$\lim_{\sigma \rightarrow n} p_\sigma = p. \tag{3.3}$$

The space  $L^{2p_\sigma}$  has the following property:

**Proposition 3.1.** *Given  $p$ , there exists some  $\sigma(p) \in (0, n)$  such that*

$$\|f\|_{2p_\kappa} \leq C_E \|f\|_{B_r^{1-\rho}} \quad \text{if } \sigma(p) \leq \kappa \leq n, \tag{3.4}$$

where  $C_E$  is independent of  $f$  and  $\kappa$ .

**Proof.** If

$$\frac{1}{r} \geq \frac{1}{2p_\kappa} \geq \frac{1}{r} - \frac{1-\rho}{n}, \tag{3.5}$$

then we have  $B_r^{1-\rho} \hookrightarrow L^{2p_\kappa}$  (see, e.g., Triebel [10]). The condition (3.5) is equivalent to

$$p \geq \frac{p}{p_\kappa} + \frac{4}{n-1+\theta} \tag{3.6}$$

and

$$\frac{p}{p_\kappa} \geq \frac{p(n-2)}{n} + \frac{2(-n+1+\theta)}{n(n-1+\theta)}. \tag{3.7}$$

For  $n \leq 2$ , (3.6) and (3.7) obviously hold if  $\kappa$  is sufficiently close to  $n$ . Suppose that  $3 \leq n \leq 6$ . Since the function  $1 + 4/(n - s)$  is convex with respect to  $s \in (-\infty, n)$ , it follows from  $\vartheta_p \in [0, 1]$  that

$$\begin{aligned} p &= \vartheta_p \left(1 + \frac{4}{n}\right) + (1 - \vartheta_p) \left(1 + \frac{4}{n-2}\right) \\ &\geq 1 + \frac{4}{n-2(1-\vartheta_p)} > 1 + \frac{4}{n-1+\vartheta_p}. \end{aligned}$$

Therefore, we see from (3.2) and (3.3) that (3.6) holds if  $\kappa$  is sufficiently close to  $n$ . By elementary calculations, we have (3.7).

For all  $\kappa \in [\sigma(p), n]$ ,  $p_\kappa$  belongs to some compact set of  $(1, \infty)$ . Then we can take  $C_E$  which is independent of  $f$  and  $\kappa$  (see [1]). □

Let us give a proof of Theorem 1.1 divided into three steps.

(Step I.) Put  $\phi_- = (\phi_-^1, \phi_-^2) \in X$ . Let  $\sigma(p)$  be a real number which satisfies (3.4) and such that

$$\frac{2p\sigma(p)}{p-1} > 1. \tag{3.8}$$

For  $\kappa \in [\sigma(p), n]$ , we define a mapping

$$\Phi^\kappa : u \mapsto u_- + \int_{-\infty}^t \frac{\sin(t-t')\omega}{\omega} f_\kappa(u(t')) dt'.$$

We prove that  $\Phi^\kappa$  is a contraction mapping on a suitable complete metric space. By using the Strichartz estimate shown in Machihara-Nakanishi-Ozawa [2], we have

$$\|u_-\|_Z \leq C_S \|\phi_-\|_X, \quad \|\Phi^\kappa u - u_-\|_Z \leq C_S \|f_\kappa(u)\|_{L^1 L^2}.$$

Here, the constant  $C_S$  is dependent only on  $n$  and  $p$ .

Let  $q$  be a real number which satisfies

$$\frac{1}{2} = \frac{1}{q} + \frac{1}{2p}. \tag{3.9}$$

We immediately see from (3.1) that

$$1 + \frac{1}{q} = \frac{\sigma}{n} + \frac{p-1}{2p_\sigma}. \tag{3.10}$$

By (1.2), (3.9), the Hölder inequality and (1.4), we obtain

$$\|f_\kappa(u(t))\|_{L^2} \leq \max\{C_V, \lambda_0\} \|(-\Delta)^{(-n+\kappa)/2} |u(t)|^{p-1}\|_q \|u(t)\|_{2p}.$$

From (3.10), (2.5) and Remark 2, we see that

$$\|(-\Delta)^{(-n+\kappa)/2}|u(t)|^{p-1}\|_q \leq C_H \| |u(t)|^{p-1} \|_{2p\kappa/(p-1)} \leq C_H \|u(t)\|_{2p\kappa}^{p-1},$$

where the constant  $C_H$  is dependent only on  $n$  and  $p$ . Accordingly, it follows from (3.4) that there exists a positive constant  $C_0$  depending only on  $n$  and  $p$ , such that

$$\begin{aligned} \|\Phi^\kappa u - u_-\|_Z &\leq C_0 \|u\|_{L^p B_r^{1-\rho}}^p, \\ \|\Phi^\kappa u\|_Z &\leq C_0 \left( \|\phi_-\|_X + \|u\|_{L^p B_r^{1-\rho}}^p \right). \end{aligned}$$

We can choose  $C_0$  satisfying

$$\|\Phi^\kappa u - \Phi^\kappa v\|_Z \leq C_0 \|u - v\|_{L^p B_r^{1-\rho}} \left( \|u\|_{L^p B_r^{1-\rho}}^{p-1} + \|v\|_{L^p B_r^{1-\rho}}^{p-1} \right).$$

Hence we obtain

$$\|\Phi^\kappa u - u_-\|_Z \leq C_0 \|u\|_Z^p, \tag{3.11}$$

$$\|\Phi^\kappa u\|_Z \leq C_0 \left( \|\phi_-\|_X + \|u\|_Z^p \right), \tag{3.12}$$

$$\|\Phi^\kappa u - \Phi^\kappa v\|_Z \leq C_0 \|u - v\|_Z \left( \|u\|_Z^{p-1} + \|v\|_Z^{p-1} \right). \tag{3.13}$$

(Step II.) Set  $Z_0 = B(2C_0\|\phi_-\|_X, Z)$  and  $d(u, v) = \|u - v\|_Z$ . Clearly,  $(Z_0, d)$  is a complete metric space.

From (3.11) and elementary calculations, it follows that  $\Phi^\kappa u \in \mathcal{H}$  if  $u \in Z$ . Hence, by (3.12) and (3.13), we see that  $\Phi^\kappa$  is a contraction mapping from  $Z_0$  into itself if  $\|\phi_-\|_X \leq \delta$  and  $\delta \leq (4C_0)^{-1/(p-1)}$ . Consequently, we obtain a unique fixed point  $u^\kappa$  of  $\Phi^\kappa$ . Since the fixed point is also unique in  $Z$  (see Nakamura [5]), we have a unique solution  $u^\kappa \in Z$  of (1.10).

Put

$$\begin{aligned} \phi_+^1 &= \phi_-^1 + \int_{\mathbb{R}} \frac{\sin(-t\omega)}{\omega} f_\kappa(u^\kappa(t)) dt, \\ \phi_+^2 &= \phi_-^2 + \int_{\mathbb{R}} \cos(-t\omega) f_\kappa(u^\kappa(t)) dt. \end{aligned}$$

By the analogous argument on (3.11), we see that (1.11) holds. We remark that we can easily show the uniqueness of  $u_\pm(t)$  satisfying (1.11) (see Sasaki [8]). Thus, the scattering operator  $S_\kappa$  is well defined.

(Step III.) Since  $u^\kappa$  satisfies  $\Phi^\kappa u^\kappa = u^\kappa$ , we see that, for  $\sigma \in [\sigma(p), n)$ ,

$$u^\sigma - u^n = \int_{-\infty}^t \frac{\sin(t-t')\omega}{\omega} \left\{ f_\sigma(u^\sigma(t')) - f_n(u^n(t')) \right\} dt',$$

$$\begin{aligned} f_\sigma(u^\sigma) - f_n(u^n) &= \gamma(\sigma)^{-1} \left( V_\sigma * |u^\sigma|^{p-1} \right) (u^\sigma - u^n) \\ &\quad + \left( \gamma(\sigma)^{-1} V_\sigma * |u^\sigma|^{p-1} - \lambda_0 |u^n|^{p-1} \right) u^n. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} &\gamma(\sigma)^{-1} V_\sigma * |u^\sigma|^{p-1} - \lambda_0 |u^n|^{p-1} \\ &= \gamma(\sigma)^{-1} V_\sigma * |u^\sigma|^{p-1} - \gamma(\sigma)^{-1} \lambda_0 \cdot |\cdot|^{-\sigma} * |u^\sigma|^{p-1} \\ &\quad + \gamma(\sigma)^{-1} \lambda_0 \cdot |\cdot|^{-\sigma} * |u^\sigma|^{p-1} - \gamma(\sigma)^{-1} \lambda_0 \cdot |\cdot|^{-\sigma} * |u^n|^{p-1} \\ &\quad + \gamma(\sigma)^{-1} \lambda_0 \cdot |\cdot|^{-\sigma} * |u^n|^{p-1} - \lambda_0 |u^n|^{p-1}. \end{aligned}$$

Therefore, following the line of Step I, we obtain

$$\begin{aligned} &\|u^\sigma - u^n\|_{L^p B_r^{1-\rho}} \\ &\leq C_0 \|u^\sigma - u^n\|_{L^p B_r^{1-\rho}} \left( \|u^\sigma\|_{L^p B_r^{1-\rho}}^{p-1} + \|u^n\|_{L^p B_r^{1-\rho}}^{p-1} \right) \\ &\quad + C_0 \|V_\sigma \cdot |\cdot|^\sigma - \lambda_0\|_\infty \|u^\sigma\|_{L^p B_r^{1-\rho}}^p \\ &\quad + C_0 \left\| (-\Delta)^{(-n+\sigma)/2} |u^n|^{p-1} - |u^n|^{p-1} \right\|_{L^{p/(p-1)} L^{2p/(p-1)}} \|u^n\|_{L^p B_r^{1-\rho}} \\ &\leq \frac{1}{4} \|u^\sigma - u^n\|_{L^p B_r^{1-\rho}} + \|V_\sigma \cdot |\cdot|^\sigma - \lambda_0\|_\infty \\ &\quad + C_0 \left\| (-\Delta)^{(-n+\sigma)/2} |u^n|^{p-1} - |u^n|^{p-1} \right\|_{L^{p/(p-1)} L^{2p/(p-1)}}, \end{aligned} \tag{3.14}$$

where we have used

$$\|u^\kappa\|_Z \leq 2C_0 \|\phi_-\|_X \leq 2C_0 \delta.$$

By  $u^n \in Z$  and (3.4), we see that

$$\begin{aligned} &|u^n|^{p-1}(t) \in L^{2p_{\sigma(p)}/(p-1)}(\mathbb{R}^n) \cap L^{2p/(p-1)}(\mathbb{R}^n) \quad \text{for a.e. } t, \\ &1 + \frac{p-1}{2p} = \frac{\sigma}{n} + \frac{p-1}{2p_{\sigma(p)}}. \end{aligned}$$

Thus, it follows from (3.8) and Lemma 2.4 that

$$\lim_{\sigma \rightarrow n} \left\| (-\Delta)^{(-n+\sigma)/2} |u^n|^{p-1}(t) - |u^n|^{p-1}(t) \right\|_{2p/(p-1)} = 0 \quad \text{for a.e. } t.$$

From (2.5) and (3.4), we have

$$\left\| (-\Delta)^{(-n+\sigma)/2} |u^n|^{p-1}(t) - |u^n|^{p-1}(t) \right\|_{2p/(p-1)} \leq 2C_H \|u^n(t)\|_{B_r^{1-\rho}}^{p-1}.$$

Hence, by  $\|u^n(\cdot)\|_{B_r^{1-\rho}}^{p-1} \in L^{p/(p-1)}(\mathbb{R})$  and the Lebesgue dominated convergence theorem with respect to  $t$ , it follows that

$$\lim_{\sigma \rightarrow n} \left\| (-\Delta)^{(-n+\sigma)/2} |u^n|^{p-1} - |u^n|^{p-1} \right\|_{L^{p/(p-1)} L^{2p/(p-1)}} = 0. \tag{3.15}$$

From (1.3), (3.14) and (3.15), we have

$$\|u^\sigma - u^n\|_{L^p B_r^{1-\rho}} \rightarrow 0 \quad \text{as } \sigma \rightarrow n. \tag{3.16}$$

Similarly, we obtain

$$\begin{aligned} \|S_\sigma(\phi_-) - S_n(\phi_-)\|_X &\leq \frac{1}{2} \|u^\sigma - u^n\|_{L^p B_r^{1-\rho}} + 2 \|V_\sigma \cdot |\cdot|^\sigma - \lambda_0\|_\infty \\ &\quad + 2C_0 \left\| (-\Delta)^{(-n+\sigma)/2} |u^n|^{p-1} - |u^n|^{p-1} \right\|_{L^{p/(p-1)} L^{2p/(p-1)}}. \end{aligned}$$

By (1.3), (3.15) and (3.16), We obtain (1.6). This completes the proof.

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