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GLOBAL EXISTENCE IN SUB-CRITICAL CASES AND FINITE TIME BLOW-UP IN SUPER-CRITICAL CASES TO DEGENERATE KELLER-SEGEL SYSTEMS

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Abstract. We consider the degenerate Keller-Segel system (KS) of Nagai type below. We prove that when $m > 2-\frac{2}{N}$, the problem (KS) is solvable globally in time without any restriction on the size of the initial data and that when $1 < m \le 2-\frac{2}{N}$, the problem (KS) evolves in a finite time blow-up for some large initial data. Hence, we completely classify the existence and non-existence of the time global solution by means of the exponent $m = 2 - \frac{2}{N}$, which generalizes the Fujita exponent for (KS).

1. **Introduction**

We consider the degenerate Keller-Segel system of Nagai type:

$$\begin{cases} u_t = \nabla \cdot \Big(\nabla u^m - \chi u \nabla v\Big), & x \in \mathbb{R}^N,\ t > 0, \\ 0 = \Delta v - \gamma v + \alpha u, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{KS}$$

where $m > 1$, $\alpha, \gamma, \chi > 0$ and $N \ge 1$. Keller-Segel [19] proposed the mathematical model including our problem (KS). Their equation is often called the Keller-Segel system describing the motion of the chemotaxis molds.

In this paper, we consider the case $m > 1$ and show that the exponent $m = 2 - \frac{2}{N}$ divides the behaviour of the solution (u, v) into global existence and finite time blow-up. Specifically, we prove that

(I) when $m > 2-\frac{2}{N}$, the problem (KS) is solvable globally in time without any restriction on the size of the initial data, and that

(II) when $1 < m \le 2 - \frac{2}{N}$, the problem (KS) can evolve in a finite time blow-up for some large initial data.

In [35]–[37], it was shown that

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(i) when $1 < m \le 2 - \frac{2}{N}$, the problem (KS) is solvable globally in time for small initial data.

By combining (i) with (I)–(II) above, we completely classify the existence and finite time blow-up of the solution (u, v) for the degenerate Keller-Segel model (KS) by the exponent $m = 2 - \frac{2}{N}$.

To study our problem (KS), we encounter the following difficulties. Firstly, concerning the Keller-Segel system, the comparison principles do not hold. Secondly, since our problem is a "degenerate" quasi-linear system, we can not expect solutions of (KS) to be classical solutions. However, surprisingly, we can observe the following Lyapunov function (which was found in [35]–[37]) with L^m norm of the solution:

$$\begin{aligned} W(t) &= \frac{1}{m-1}\int_{\mathbb{R}^N} u^m(t)\,dx - \frac{\chi}{2}\int_{\mathbb{R}^N} u(t)v(t)\,dx \\ &= \frac{1}{m-1}\int_{\mathbb{R}^N} u^m(t)\,dx - \chi\int_{\mathbb{R}^N} u(t)v(t)\,dx \\ &\quad + \frac{\chi}{2\alpha}\Big(\|\nabla v(t)\|^2_{L^2(\mathbb{R}^N)} + \gamma\|v(t)\|^2_{L^2(\mathbb{R}^N)}\Big) \\ &\le W(0). \end{aligned} \tag{1.1}$$

By virtue of the above Lyapunov function, if we choose u_0 such that

$$W(0) = \frac{1}{m-1}\int_{\mathbb{R}^N} u_0^m(t)\,dx - \frac{\chi}{2}\int_{\mathbb{R}^N} u_0 v_0\,dx < 0 \text{ and } u_0|x|^2 \in L^1(\mathbb{R}^N), \tag{H1}$$

then the moment argument works, where $v_0 = G * (\alpha u_0)$ with the Bessel potential G. In consequence, we can show that a solution (u, v) of (KS) blows up in a finite time. (As for the moment argument, see also Nagai [24], [25], Biler [4], Biler and Nadzieja [5].) Furthermore, we take the initial function $u_0(x)$ to be $A\big(1 - \frac{|x|^N}{b^N}\big)_+$, where A and b control the maximum and the size of the support of u_0, respectively. If we assume that b is small (*i.e.,* the moment of u_0 is small), then we can prove that $u_0 = A\big(1 - \frac{|x|^N}{b^N}\big)_+$ satisfies (H1) for $1 < m \le 2 - \frac{2}{N}$. In consequence, we observe that smallness of "the moment of u_0 and m" is necessary to prove the blow-up of solutions for (KS).

The finite time blow-up of a solution for a quasi-linear Keller-Segel system was first formally obtained by Biler-Nadzieja-Stanczy [3] for the Neumann problem. They consider the second equation as $0 = \Delta v + u$ and gave a proof by using Riesz potential under an assumption similar to (H1). On the other hand, we give a rigorous complete proof for the Cauchy problem (KS) using the Bessel potential under the assumption (H1). Those results were obtained independently of each other.

In what follows, we give the definition of the weak solution (u, v) for (KS).

Definition 1. *For $m > 1$, let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ with $u_0^m \in H^1(\mathbb{R}^N)$. A pair (u, v) of non-negative functions defined in $\mathbb{R}^N \times [0, T)$ is called a weak solution of* (KS) *on* $[0, T)$ *if*

i) $u \in L^\infty(0, T; L^2(\mathbb{R}^N))$, $u^m \in L^2(0, T; H^1(\mathbb{R}^N))$,

ii) $v \in L^\infty(0, T; H^1(\mathbb{R}^N))$,

iii) (u, v) *satisfies the equations in the sense of distribution, i.e., that*

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} \left(\nabla u^m \cdot \nabla \varphi - \chi u \nabla v \cdot \nabla \varphi - u\varphi_t\right)\, dxdt &= \int_{\mathbb{R}^N} u_0(x)\varphi(x, 0)\, dx, \\ \int_{\mathbb{R}^N} \left(\Delta v \cdot \Delta \varphi - \gamma v \varphi + \alpha u \varphi\right)\, dx &= 0 \qquad a.a.\ t \in (0, T) \end{aligned}$$

for any continuously differentiable function φ with compact support in $\mathbb{R}^N \times [0, T)$.

The following theorem gives the existence of a time "local" weak solution to (KS) and the uniform bound of the weak solution when $u_0 \in L^\infty(\mathbb{R}^N)$. The proof is based on the L^∞-energy method which was employed in [32].

Theorem 1.1 (local existence of weak solution and its L^∞ uniform bound). *Let $N \geq 1, m > 1$, $\alpha, \gamma, \chi > 0$ and suppose that the initial data u_0 is non-negative everywhere. Then,* (KS) *has a non-negative weak solution (u, v) on $(0, T_0)$ with $T_0 = (\alpha\chi)^{-1}\big(\|u_0\|_{L^\infty(\mathbb{R}^N)} + 2\big)^{-2}$. Moreover, $u(t)$ satisfies the following a priori estimate*

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \;\leq\; \|u_0\|_{L^\infty(\mathbb{R}^N)} + 2 \qquad \text{for all } t \in [0, T_0]. \tag{1.2}$$

Furthermore, if the maximal existence time $T_{\max}$ of the above weak solution (u, v) is finite, then we have

$$\text{limsup}_{t \to T_{\max}-0} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \;=\; \infty.$$

Next, we consider the case of $m > 2 - \frac{2}{N}$. The following theorem gives the existence of a time "global" weak solution to (KS) and the uniform bound of its weak solution when $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$.

Theorem 1.2. (global existence of weak solution for $m > 2 - \frac{2}{N}$). *Let $N \geq 1, m > \max\{1, 2 - \frac{2}{N}\}, \alpha, \gamma, \chi > 0$ and suppose that the initial data u_0 is non-negative everywhere. Then, the weak solution (u, v) of* (KS) *obtained* *from Theorem* 1.1 *exists globally in time. Moreover, it satisfies a uniform estimate, i.e., there exists a positive constant*

$$K = K(\|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, m, N, \alpha, \gamma, \chi)$$

such that

$$\sup_{0<t<\infty} \big(\|u(t)\|_{L^r(\mathbb{R}^N)} + \|v(t)\|_{L^r(\mathbb{R}^N)} + \|\Delta v(t)\|_{L^r(\mathbb{R}^N)}\big) \le K \textit{ for all } r \in [1,\infty].$$

On the other hand, for the case of $1 < m \le 2 - \frac{N}{2}$, we find different phenomena from the previous case. Indeed, the decay property and the finite time blow-up of a weak solution (u, v) for (KS) are obtained under the appropriate initial condition.

Theorem 1.3. (decay for small data; blow-up for large data for $1 < m \le 2 - \frac{2}{N}$). *Let* $N \ge 3,\ 1 < m \le 2 - \frac{2}{N},\ \alpha, \gamma, \chi > 0$ *and suppose that the initial data* u_0 *is non-negative everywhere.*

(i) *We assume that the initial data is sufficiently small in the sense that* $\|u_0\|_{L^{\frac{N(2-m)}{2}}(\mathbb{R}^N)} << 1$, *then the weak solution* (u, v) *of* (KS) *obtained from Theorem* 1.1 *exists globally in time with the following property:*

$$\sup_{0<t<\infty} (1+t)^d (\|u(t)\|_{L^r(\mathbb{R}^N)} + \|v(t)\|_{L^r(\mathbb{R}^N)}) < \infty \quad \textit{for } r \in [1,\infty), \tag{1.3}$$

where

$$d = \frac{N}{\sigma}\Big(1 - \frac{1}{r}\Big), \quad \sigma = N(m-1) + 2.$$

(ii) *We assume that the initial data* $u_0 \in L^1 \cap L^2(\mathbb{R}^N)$ *satisfies the following condition:*

$$\int_{\mathbb{R}^N} u_0^m(x)\,dx \ < \ \frac{(m-1)\chi}{2} \int_{\mathbb{R}^N} u_0 v_0(x)\,dx \quad \text{and} \quad u_0|x|^2 \in L^1(\mathbb{R}^N), \tag{H1}$$

where $v_0 = G * (\alpha u_0)$ *with the Bessel potential* G, *i.e.,* $G * f = (-\Delta + \gamma)^{-1} f$. *Then, the weak solution* (u, v) *of* (KS) *obtained from Theorem* 1.1 *blows up in a finite time* $T_b (\le T_{\max})$ *in the following sense:*

$$\mathrm{limsup}_{t \to T_b - 0} \|u(t)\|_{L^\infty(\mathbb{R}^N)} \ = \ \infty,$$

where $T_{\max}$ *is the maximal existence time of the weak solution* (u, v) *obtained from Theorem* 1.1.

Remark 0.

$$T_b \le \frac{\int_{\mathbb{R}^N} u_0 |x|^2 dx}{2N(\frac{(m-1)\chi}{2} \int_{\mathbb{R}^N} u_0 v_0 dx - \int_{\mathbb{R}^N} u_0^m dx)} \to \infty \quad \text{as} \ \ \gamma \to \infty.$$

Hence, the life span might get larger as γ gets larger.

Finally, we give the relationship between the Fujita exponent and the initial condition (H1) by constructing the following example $u_0(x)$:

$$u_0(x) := c_N^* \frac{M}{b^N}\Big(1 - \frac{|x|^N}{b^N}\Big)_+ \quad \text{with} \quad c_N^* = \frac{2N}{\omega_N} \tag{1.4}$$

where M and b are positive parameters, ω_N is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$. It is easy to see that

$$\int_{\mathbb{R}^N} u_0(x)\, dx \;=\; M, \quad \max_{x\in\mathbb{R}^N} u_0(x) \;=\; c_N^* \frac{M}{b^N}, \quad \operatorname{supp} u_0 \;=\; \{x \in \mathbb{R}^N; |x| \le b\}.$$

Theorem 1.4 (Fujita exponent). *Let $N \ge 3$, $\alpha, \gamma, \chi > 0$. Suppose that $1 < m \le 2 - \frac{2}{N}$ and assume that the initial data u_0 is given by* (1.4). *There is a constant $C = C(m, N, \alpha, \gamma, \chi)$ such that if M and b satisfy*

$$b^{N(2-\frac{2}{N}-m)} e^{2b\sqrt{\gamma}} M^{-(2-m)} \;\le\; C, \tag{1.5}$$

then (KS) *has a finite time blow-up solution* (u, v).

Remark 1. (i) From (1.5) we see a different phenomena between the super critical case and the critical case for blow-up of solutions. Indeed, for the super critical case of $1 < m < 2 - \frac{N}{2}$, once the mass M of initial data is given, the blow-up solution can be obtained provided its moment is taken small enough according to M. On the other hand, in the critical case of $m = 2 - \frac{N}{2}$, even though we take the moment sufficiently small compared with the prescribed mass M, the hypothesis (1.5) does not hold. Hence, in such a case, we need to take the moment small and the mass large simultaneously.

(ii) Concerning the blow-up problem for the Keller-Segel model, the pioneering work was done by W.Jäger and S.Luckhaus in [17]. They constructed a blow-up solution and also gave the following conjecture: When $N = 2$ and $m = 1$, there exists a critical number $c > 0$ such that

(1) if $\int_\Omega u_0\, dx < c$, then the solution (u, v) exists globally in time, and

(2) if $\int_\Omega u_0\, dx > c$, then the solution u blows up in a finite time,

where Ω is a bounded domain in $\mathbb{R}^2$ with the C^1 boundary $\partial\Omega$. Their conjecture was proved rigorously by Nagai [24], [26] and [28]. In fact, for the 2-D semi-linear equation, *i.e.*, when $N = 2, m = 1$, we may take $c = \frac{8\pi}{\alpha\chi}$ in (1) provided Ω is a circle and u_0 has some symmetry. In (2) for a general domain Ω, we may also take $c = \frac{8\pi}{\alpha\chi}$ with some additional assumptions on u_0. For the semi-linear equation in the higher-dimensional case of $N \ge 3$, the blow-up solution was constructed by Nagai [25] under the assumption: $\int_{\mathbb{R}^N} u_0 |x|^N\, dx << \int_{\mathbb{R}^N} u_0\, dx$.

(iii) In Theorem 1.3 and Theorem 1.4, we show an aspect similar to the semi-linear case. *That is* that, in the critical case of $m = 2 - \frac{2}{N}$, if "$\|u_0\|_{L^1(\mathbb{R}^N)} << 1$", then a solution (u, v) exists globally in time, which corresponds to (1); if "$\|u_0\|_{L^1(\mathbb{R}^N)} >> 1$", then the blow-up solution is obtained under the balance of the mass and moment of u_0, which corresponds to (2). Since $\int_{\mathbb{R}^N} u_0(x)|x|^k\, dx \le b^k M$ holds for the initial data u_0 in (1.4), we see that the size of b^k controls the k-th moment of u_0. Hence, the hypotheses (1.5) in Theorem 1.4 imply that the moment of u_0 is small compared with $\int_{\mathbb{R}^N} u_0\, dx$. In such a situation, we obtain a blow-up solution in the case of $1 < m \le 2 - \frac{2}{N}$.

Remark 2. We rewrite the first equation of (KS) by substituting the second equation: $\Delta v = v - u$ (with $\alpha = \gamma = 1$) as follows:

$$u_t \;=\; \Delta u^m - \nabla u \cdot \nabla v - u\Delta v \;=\; \Delta u^m - \nabla u \cdot \nabla v - uv + u^2. \tag{E}$$

Since this equation (E) has three terms: $u_t, \Delta u^m$ and u^2, the first equation in (KS) is analogous to the following equation with $q = 2$.

$$\begin{cases} u_t = \Delta u^m + u^q & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{PS}$$

It is well known that the critical exponent $q = m + \frac{2}{N}$ divides the situation of the global existence and non-existence of the solution to the above equation (PS). This exponent is called the Fujita exponent [9]. Indeed, when $q > m + \frac{2}{N}$, it can be globally solvable for small initial data. When $q < m + \frac{2}{N}$ and $q = m + \frac{2}{N}$, it was proved that (all) non-negative solutions of (PS) blow up in a finite time without any restriction on the size of the initial data (see for example [10], [12], [18] and [23]). Consequently, we can see that (KS) has the same Fujita exponent as $q = m + \frac{2}{N}$ of (PS) in the case of $q = 2$.

In the following section, we prepare the preliminary lemmas. In Section 3, we introduce the approximated problem $(\mathrm{KS})_\varepsilon$. In Section 4, we give a proof of the time local existence of a weak solution. In Section 5, we show the global solvability of (KS) in the case of $m > 2 - \frac{2}{N}$. In Section 6, we discuss the finite time blow-up under the assumption (H1). Furthermore, in Section 7, we construct the initial function to satisfy the condition (H1) when $1 < m \le 2 - \frac{2}{N}$. In the Appendix, we give a proof for a power version of the Gagliardo-Nirenberg inequality.

We will use the following simplified notation:

1) $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \partial_i \partial_j$, $\nabla u = (\partial_1, \partial_2, \cdots)$, $\nabla^2 u = (\partial_{11}^2, \partial_{12}^2, \cdots)$,
$\|\cdot\|_{L^r} = \|\cdot\|_{L^r(\mathbb{R}^N)}, (1 \le r \le \infty)$, $\int \cdot\, dx := \int_{\mathbb{R}^N} \cdot\, dx$.

2) $Q_T := \mathbb{R}^N \times (0,T)$, $B_R(x) := \{y \in \mathbb{R}^N; |x-y| < R\}$.

3) When the weak derivatives $\nabla u, \nabla^2 u$ and u_t are in $L^p(Q_T)$ for some $p \geq 1$, we say that $u \in W_p^{2,1}(Q_T)$, *i.e.*, that

$$W_p^{2,1}(Q_T) := \Big\{ u \in L^p(0,T; W^{2,p}(\mathbb{R}^N)) \cap W^{1,p}(0,T; L^p(\mathbb{R}^N));$$
$$\|u\|_{W_p^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|\nabla^2 u\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)} < \infty \Big\}.$$

2. Preliminary Lemmas

The following representation is one from elliptic theory (we refer to E.M. Stein [34, Chapter V Section 6.5]. N. Aronszajn and K.T. Smith [2, page 415], S.T. Kuroda [20, page 58]).

Lemma 2.1. *Let $N \geq 2$, $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^N)$ and consider the following problem:*

$$(\mathrm{E}): \qquad -\Delta z + \gamma z = f \quad \text{for } x \in \mathbb{R}^N.$$

Then the function $z \in L^p(\mathbb{R}^N)$ given by

$$z(x) = \int_{\mathbb{R}^N} G(x-y) f(y)\, dy \tag{2.1}$$

is the strong solution of (E) *in $\mathbb{R}^N$, i.e.,* (E) *is satisfied almost everywhere.*

Here, $G(x)$ is the Bessel potential which can be expressed as

$$G(x) = \gamma^{\frac{N}{2}-1} \cdot a_N \cdot e^{-\sqrt{\gamma}|x|} \int_0^\infty e^{-\sqrt{\gamma}|x|s} \cdot \Big(s + \frac{s^2}{2}\Big)^{\frac{N-3}{2}} ds \tag{2.2}$$

with the constant a_N given by

$$a_N^{-1} = 2 \cdot (2\pi)^{\frac{N-1}{2}} \cdot \Gamma\Big(\frac{N-1}{2}\Big).$$

Proof of Lemma 2.1.

$$(\mathrm{E})_{\gamma=1}: \qquad -\Delta z + z = f \quad \text{for } x \in \mathbb{R}^N.$$

From Stein [34, Chapter V Section 6.5], the Bessel potential for $(\mathrm{E})_{\gamma=1}$ is given by

$$G_{\gamma=1}(x) = a_N \cdot e^{-|x|} \int_0^\infty e^{-|x|s} \cdot \Big(s + \frac{s^2}{2}\Big)^{\frac{N-3}{2}} ds. \tag{2.3}$$

On the other hand, from (3.37) in [20], the Bessel potential $G(x)$ for (E) is expressed as follows:

$$\begin{aligned} G(x) &= -\frac{1}{(4\pi)^{\frac{N}{2}}|x|^{N-2}}\int_0^\infty s^{-\frac{N}{2}}\cdot e^{-(4s)^{-1}-\gamma|x|^2 s}\,ds \\ &= -\frac{\sqrt{\gamma}^{N-2}}{(4\pi)^{\frac{N}{2}}|\sqrt{\gamma}x|^{N-2}}\int_0^\infty s^{-\frac{N}{2}}\cdot e^{-(4s)^{-1}-|\sqrt{\gamma}\cdot x|^2 s}\,ds \\ &= \gamma^{\frac{N}{2}-1}\cdot G_{\gamma=1}(\sqrt{\gamma}|x|) \qquad \text{for } N\geq 1. \end{aligned} \tag{2.4}$$

Combining (2.3) with (2.4), we obtain

$$G(x) = \gamma^{\frac{N}{2}-1}\cdot a_N\cdot e^{-\sqrt{\gamma}|x|}\int_0^\infty e^{-\sqrt{\gamma}|x|s}\cdot\left(s+\frac{s^2}{2}\right)^{\frac{N-3}{2}}\,ds.$$

Thus, we complete the proof of Lemma 2.1. □

Concerning the Bessel potential $G(x)$, we obtain the non-positivity property in the following lemma:

Lemma 2.2. *Let $N\geq 3$. Then, it holds that*

$$x\cdot\nabla G(x) \leq -(N-2)\cdot G(x) \leq 0 \qquad \text{for } x,y\in\mathbb{R}^N\ (x\neq y). \tag{2.5}$$

Proof of Lemma 2.2. We differentiate (2.3) with respect to x, then for $x\neq 0$ it holds that

$$\nabla G_{\gamma=1}(x) = -a_N\frac{x}{|x|}\cdot e^{-|x|}\int_0^\infty e^{-|x|s}(1+s)\left(s+\frac{s^2}{2}\right)^{\frac{N-3}{2}}\,ds. \tag{2.6}$$

By (2.6),

$$x\cdot\nabla G_{\gamma=1}(x) = -a_N|x|\cdot e^{-|x|}\int_0^\infty e^{-|x|s}(1+s)\left(s+\frac{s^2}{2}\right)^{\frac{N-3}{2}}\,ds.$$

For $N\geq 3$, integration by parts yields that

$$\begin{aligned} x\cdot\nabla G_{\gamma=1}(x) &= a_N\cdot e^{-|x|}\int_0^\infty \frac{de^{-|x|s}}{ds}\cdot(1+s)\left(s+\frac{s^2}{2}\right)^{\frac{N-3}{2}}\,ds \\ &= -a_N\cdot e^{-|x|}\int_0^\infty e^{-|x|s}\cdot\frac{d}{ds}\left[(1+s)\left(s+\frac{s^2}{2}\right)^{\frac{N-3}{2}}\right]\,ds. \end{aligned} \tag{2.7}$$

It is seen that

$$\begin{aligned} &\frac{d}{ds}\left[(1+s)\cdot\left(s+\frac{s^2}{2}\right)^{\frac{N-3}{2}}\right] \\ &= \left(s+\frac{s^2}{2}\right)^{\frac{N-5}{2}}\cdot\left(s+\frac{s^2}{2}+(N-3)\cdot\frac{(1+s)^2}{2}\right) \geq (N-2)\left(s+\frac{s^2}{2}\right)^{\frac{N-3}{2}}. \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7),

$$x \cdot \nabla G_{\gamma=1}(x) \le -(N-2) \cdot a_N \cdot e^{-|x|} \int_0^\infty e^{-|x|s} \cdot \Big(s + \frac{s^2}{2}\Big)^{\frac{N-3}{2}} ds \qquad (2.9)$$
$$= -(N-2) G_{\gamma=1}(x).$$

From (2.4) and (2.9),

$$x \cdot \nabla G = x \cdot \gamma^{\frac{N}{2}-1} \sqrt{\gamma} \nabla G_{\gamma=1}(y)|_{y=\sqrt{\gamma}|x|}$$
$$\le -\gamma^{\frac{N}{2}-1} \cdot (N-2) G_{\gamma=1}(\sqrt{\gamma}|x|) = -(N-2)G(x).$$

Thus, the proof of Lemma 2.2 is completed. □

The following lemma is a version of the Hölder inequality.

Lemma 2.3. (The moment inequality) *Let $p \ge 1$ and $|x|^p f \in L^1(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} |f(x)||x| \, dx \quad \le \quad \Big(\int_{\mathbb{R}^N} |f(x)| \, dx \Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^N} |f(x)||x|^p \, dx \Big)^{\frac{1}{p}}.$$

The following lemma gives us a version of the Gagliardo-Nirenberg inequality. The proof is given in the Appendix.

Lemma 2.4. *Let $N \ge 1$, $m \ge 1$, $a > 2$, $u \in L^{q_1}(\mathbb{R}^N)$ with $q_1 \ge 1$ and $u^{\frac{r+m-1}{2}} \in H^1(\mathbb{R}^N)$ with $r > 0$. If $q_1 \in [1, r+m-1]$, $q_2 \in [\frac{r+m-1}{2}, \frac{a(r+m-1)}{2}]$ and*

$$\begin{cases} 1 \le q_1 \le q_2 \le \infty & \text{when } N = 1, \\ 1 \le q_1 \le q_2 < \infty & \text{when } N = 2, \\ 1 \le q_1 \le q_2 \le \frac{(r+m-1)N}{N-2} & \text{when } N \ge 3, \end{cases}$$

then, it holds that

$$\|u\|_{L^{q_2}(\mathbb{R}^N)} \quad \le \quad C^{\frac{2}{r+m-1}} \|u\|^{1-\Theta}_{L^{q_1}(\mathbb{R}^N)} \cdot \|\nabla u^{\frac{r+m-1}{2}}\|^{\frac{2\Theta}{r+m-1}}_{L^2(\mathbb{R}^N)}$$

with

$$\Theta = \frac{r+m-1}{2} \cdot \Big(\frac{1}{q_1} - \frac{1}{q_2} \Big) \cdot \Big(\frac{1}{N} - \frac{1}{2} + \frac{r+m-1}{2q_1} \Big)^{-1},$$

where

$$\begin{cases} C \text{ depends only on } N \text{ and } a & \text{when } q_1 \ge \frac{r+m-1}{2}, \\ C = c_0^{\frac{1}{\beta}} \text{ with } c_0 \text{ depending only on } N \text{ and } a & \text{when } 1 \le q_1 < \frac{r+m-1}{2}, \end{cases}$$

and

$$\beta \quad = \quad \frac{q_2 - \frac{r+m-1}{2}}{q_2 - q_1} \Big[\frac{2q_1}{r+m-1} + \Big(1 - \frac{2q_1}{r+m-1}\Big) \frac{2N}{N+2} \Big].$$

We prepare a cut-off function in the following lemma:

Lemma 2.5. *We define the localized weight function* ψ *by*

$$\psi(r) = \begin{cases} 1 & \text{for} \quad 0 \le r \le 1, \\ 1-2(r-1)^2 & \text{for} \quad 1 < r \le \frac{3}{2}, \\ 2(2-r)^2 & \text{for} \quad \frac{3}{2} < r < 2, \\ 0 & \text{for} \quad r \ge 2 \end{cases}$$

and define ψ_ℓ *by* $\psi_\ell(x) := \psi(\frac{|x|}{\ell})$ *for* $x \in \mathbb{R}^N$ *and* $\ell = 1, 2, 3, \cdots$. *Then, there exist positive constants* c_1 *and* c_2 *depending only on* N *such that*

$$|\nabla \psi_\ell(x)| \ \le \ \frac{c_1}{\ell}(\psi_\ell(x))^{\frac{1}{2}}, \quad |\Delta \psi_\ell(x)| \ \le \ \frac{c_2}{\ell^2}. \qquad \text{for } x \in \mathbb{R}^N. \tag{2.10}$$

3. Approximated Problem

In order to justify the formal arguments, we introduce the following approximated equations of (KS):

$$(\mathrm{KS})_\varepsilon \quad \begin{cases} u_{\varepsilon t}(x,t) = \nabla \cdot \Big(\nabla (u_\varepsilon + \varepsilon)^m - \chi u_\varepsilon \nabla v_\varepsilon \Big), & (x,t) \in \mathbb{R}^N \times (0,T), \\ & \cdots (1), \\ 0 = \Delta v_\varepsilon - \gamma v_\varepsilon + \alpha u_\varepsilon, & (x,t) \in \mathbb{R}^N \times (0,T), \\ & \cdots (2), \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \mathbb{R}^N, \end{cases}$$

where ε is a positive parameter and $u_{0\varepsilon}$ is an approximation for the initial data u_0 such that

(A.1) $0 \le u_{0\varepsilon} \in L^1 \cap W^{2,p}(\mathbb{R}^N)$

$$\text{for all} \begin{cases} p \ \in \ [\frac{N}{N-1}, N+3], & \text{when } N \ge 2, \\ p \ \in \ [2,3], & \text{when } N = 1, \end{cases} \qquad \text{for all } \varepsilon \in (0,1],$$

(A.2) $\|u_{0\varepsilon}\|_{L^p(\mathbb{R}^N)} \le \|u_0\|_{L^p(\mathbb{R}^N)}$, for all $p \in [1,\infty]$, for all $\varepsilon \in (0,1]$,

(A.3) $\|\nabla u_{0\varepsilon}\|_{L^2(\mathbb{R}^N)} \le \|\nabla u_0\|_{L^2(\mathbb{R}^N)}$ for all $\varepsilon \in (0,1]$,

(A.4) $u_{0\varepsilon} \to u_0$, strongly in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$ for all $p \in [1,\infty)$.

Definition 2. We call $(u_\varepsilon, v_\varepsilon)$ *a strong solution* of $(\mathrm{KS})_\varepsilon$ if it belongs to $W_p^{2,1} \times W_p^{2,1}(Q_T)$ for some $p \ge 1$ and the equations (1),(2) in $(\mathrm{KS})_\varepsilon$ are satisfied almost everywhere.

For the strong solution, we consider the space $\mathbf{W}(\mathbf{Q_T})$ defined by

$$\mathbf{W}(Q_T) \ := \ \mathbf{W}_1(Q_T) \times \mathbf{W}_2(Q_T)$$

$$:= \begin{cases} \left(W^{2,1}_{\frac{N}{N-1}} \bigcap W^{2,1}_{N+3}(Q_T)\right) \times W^{2,1}_{N+2}(Q_T) & \text{for } N \geq 2, \\ W^{2,1}_3(Q_T) \times W^{2,1}_3(Q_T) & \text{for } N = 1. \end{cases}$$

In [35]–[37], the following proposition concerning the existence of a strong solution was proved:

Proposition 3.1 (Time local existence,[35]–[37]). *Let* $N \geq 1,\ m > 1,$ $\alpha, \gamma, \chi > 0$ *and suppose that* (A.1) *is satisfied. Then, there exists a number* $T_1 = T_1(\varepsilon, \|u_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)}, m, N) > 0$ *such that* $(\mathrm{KS})_\varepsilon$ *has the unique non-negative strong solution* $(u_\varepsilon, v_\varepsilon)$ *belonging to* $\mathbf{W}(Q_{T_1})$.

Proposition 3.2 (Extension criterion,[35]–[37]). *Let the same assumption as that in Proposition 3.1 hold and let* $T > 0$. *Suppose that* $(u_\varepsilon, v_\varepsilon)$ *is a strong solution of* $(\mathrm{KS})_\varepsilon$ *in the class* $\mathbf{W}(Q_T)$. *If it holds that*

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} < \infty,$$

then there is $T' > T$ *such that* $(u_\varepsilon, v_\varepsilon)$ *can be a strong solution of* $(\mathrm{KS})_\varepsilon$ *in* $\mathbf{W}(Q_{T'})$.

We prepare the energy inequality which will be often used in this paper.

Proposition 3.3. *Let the same assumption as that in Proposition* 3.1 *hold. Then, we have the following inequality:*

$$\frac{d}{dt}\|u_\varepsilon\|^r_{L^r} \leq -\frac{4m(r-1)r}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx + \alpha\chi(r-1)\int u_\varepsilon^{r+1}\,dx, \quad (3.1)$$

$$\frac{1}{r}\frac{d}{dt}\|u_\varepsilon\|^r_{L^r} \quad (3.2)$$
$$\leq -\frac{2m(r-1)}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\ dx + (r-1)\frac{\chi^2}{2m}\int u_\varepsilon^{r+1-m}|\nabla v|^2\,dx.$$

Proof of Proposition 3.3. We multiply (1) in $(\mathrm{KS})_\varepsilon$ by u_ε^{r-1} and integrate it over $\mathbb{R}^N$. Then, we have

$$\frac{1}{r}\frac{d}{dt}\|u_\varepsilon\|^r_{L^r} \leq -\frac{4m(r-1)}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\ dx - \frac{r-1}{r}\cdot\chi\int u_\varepsilon^r \Delta v_\varepsilon\ dx. \quad (3.3)$$

Substituting (2) of $(\mathrm{KS})_\varepsilon : \Delta v_\varepsilon = \gamma v_\varepsilon - \alpha u_\varepsilon$ into (3.3)

$$\frac{d}{dt}\|u_\varepsilon\|^r_{L^r} \leq -\frac{4m(r-1)r}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx - \gamma\chi(r-1)\int u_\varepsilon^r v_\varepsilon\,dx$$

$$+ \alpha\chi(r-1)\int u_\varepsilon^{r+1}\,dx.$$

Noting that u_ε and v_ε are non-negative, then we verify (3.1).

Similarly, we multiply (1) in $(\mathrm{KS})_\varepsilon$ by u_ε^{r-1} and integrate it over $\mathbb{R}^N$. Then, we have

$$\frac{1}{r}\frac{d}{dt}\|u_\varepsilon\|_{L^r}^r \le -m(r-1)\int u_\varepsilon^{m-1}u_\varepsilon^{r-2}|\nabla u_\varepsilon|^2\,dx + (r-1)\chi\int u_\varepsilon\nabla v_\varepsilon\cdot u_\varepsilon^{r-2}\nabla u_\varepsilon\,dx,$$

which yields (3.2) by the Young inequality. □

4. Proof of Theorem 1.1

Proposition 4.1 (Local existence of approximated solution)**.** *Let* $N \ge 1,\ m>1,\ \alpha,\gamma,\chi>0$ *and suppose that* (A.1) and (A.2) *are satisfied. Let* T_0 *be as in Theorem* 1.1. *Then,* $(\mathrm{KS})_\varepsilon$ *has the non-negative unique strong solution* $(u_\varepsilon, v_\varepsilon)$ *belonging to* $\mathbf{W}(Q_{T_0})$. *Moreover,* $u_\varepsilon(t)$ *satisfies the following a priori estimate*

$$\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)} + 2 \quad \text{for all } t\in[0,T_0] \text{ and } \varepsilon\in(0,1] \tag{4.1}$$

and if the maximal existence time $T_{\max}^{(\varepsilon)}$ *of the strong solution* $(u_\varepsilon, v_\varepsilon)$ *of* $(\mathrm{KS})_\varepsilon$ *is finite, then*

$$\operatorname{limsup}_{t\to T_{\max}^{(\varepsilon)}-0}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \infty.$$

Remark 3. By virtue of Proposition 4.1, there exists a positive constant C depending only on $m, N, \alpha, \gamma, \chi, \|u_0\|_{L^1}, \|u_0\|_{L^\infty}$ but not ε such that the strong solution $(u_\varepsilon, v_\varepsilon)$ of $(\mathrm{KS})_\varepsilon$ satisfies the following estimates:

$$\|u_\varepsilon(t)\|_{L^r} \le C \quad \text{for all } \ t\in[0,T_0], \quad \text{for all } r\in[1,\infty], \tag{4.2}$$

$$\|v_\varepsilon(t)\|_{L^r} + \|\nabla v_\varepsilon(t)\|_{L^r} \le C \quad \text{for a.a. } t\in(0,T_0), \quad \text{for all } r\in[1,\infty], \tag{4.3}$$

$$\|\Delta v_\varepsilon(t)\|_{L^r} \le C \quad \text{for a.a. } t\in(0,T_0), \quad \text{for all } r\in[1,\infty] \tag{4.4}$$

with $T_0 = (\alpha\chi)^{-1}(\|u_0\|_{L^\infty}+2)^{-2}$.

Here and in what follows, C denotes a general constant greater than 1 (not necessarily the same at different occurrences) but which depends only on $m, N, \alpha, \gamma, \chi, \|u_0\|_{L^1}, \|u_0\|_{L^\infty}$ but not ε.

Indeed, the above (4.2) follows by Proposition 4.1 and the mass conservation law for u_ε (*i.e.,* that $\|u_\varepsilon(t)\|_{L^1} = \|u_{0\varepsilon}\|_{L^1}$ for all $t\in[0,T_0]$). By (2.1) in Lemma 2.1, v_ε is represented as $v_\varepsilon(x,t) = \alpha G(x)*u_\varepsilon(x,t)$. Moreover, it holds that $\nabla G \in L^q(\mathbb{R}^N)$ with $q \in [1, \frac{N}{N-1})$ if $N \geq 2$ and with any $q \in [1,\infty]$ if $N = 1$. Therefore, it holds that for all $N \geq 1$

$$\begin{aligned} \|v_\varepsilon(t)\|_{L^r} &\leq \|G\|_{L^1}\|u_\varepsilon(t)\|_{L^r} < \infty, \\ \|\nabla v_\varepsilon(t)\|_{L^r} &\leq \|\nabla G\|_{L^1}\|u_\varepsilon(t)\|_{L^r} < \infty \end{aligned}$$

for almost all $t \in (0, T_0)$ and for all $r \in [1,\infty]$. Using (2) in $(\mathrm{KS})_\varepsilon$ and the above estimates for $(u_\varepsilon, v_\varepsilon)$, we have

$$\|\Delta v_\varepsilon(t)\|_{L^r} \leq \gamma\|v_\varepsilon(t)\|_{L^r} + \alpha\|u_\varepsilon(t)\|_{L^r} < \infty$$

for a.a. $t \in (0, T_0)$ and for all $r \in [1,\infty]$. Thus, we obtain (4.2)–(4.4).

Proof of Proposition 4.1. By virtue of Proposition 3.1 and 3.2, in order to complete the proof of Proposition 4.1, we have only to establish (4.1). To this end, we use the method called "L^∞-energy method" in [32] as follows:

By (3.1) in Proposition 3.3, we have

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{L^r}^r \leq \alpha\chi(r-1)\int u_\varepsilon^{r+1}(t)\ dx \leq \alpha\chi(r-1)\|u_\varepsilon(t)\|_{L^\infty}\|u_\varepsilon(t)\|_{L^r}^r$$

for all $\varepsilon \in (0,1]$ and $r \in [1,\infty)$, whence follows

$$\|u_\varepsilon(t)\|_{L^r} \leq \|u_{0\varepsilon}\|_{L^r} + \alpha\chi\int_0^t \|u_\varepsilon(s)\|_{L^\infty}\|u_\varepsilon(s)\|_{L^r}\ ds.$$

Thus, letting $r \to \infty$, we have

$$\|u_\varepsilon(t)\|_{L^\infty} \leq \|u_{0\varepsilon}\|_{L^\infty} + \alpha\chi\int_0^t \|u_\varepsilon(s)\|_{L^\infty}^2\ ds. \tag{4.5}$$

By (A.2) in Section 3, we derive

$$\|u_\varepsilon(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \alpha\chi\int_0^t \|u_\varepsilon(s)\|_{L^\infty}^2\ ds.$$

Here, we define a positive number T_0 by

$$T_0 = \frac{1}{\alpha\chi(\|u_0\|_{L^\infty} + 1 + \delta)^2}, \tag{4.6}$$

for any fixed number $\delta > 0$. Then $u_\varepsilon(t)$ satisfies the following *a priori* estimate:

$$\|u_\varepsilon(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + 1 + \delta = K_0 \tag{4.7}$$

for all $t \in [0, T_0]$ and $\varepsilon, \delta \in (0,1]$.

Indeed, suppose that (4.7) does not hold, then there exists a number $t_* \in [0, T_0]$ such that $\|u_\varepsilon(t_*)\|_{L^\infty} > K_0$. Since $\|u_0\|_{L^\infty} \leq \|u_0\|_{L^\infty} + 1 < K_0$ and $\|u_\varepsilon(t)\|_{L^\infty}$ is a continuous function on $[0, T_0]$, there exists $t_0 \in (0, t_*]$ such that $\|u_\varepsilon(t_0)\|_{L^\infty} = K_0$ and $\|u_\varepsilon(t)\|_{L^\infty} < K_0$ for all $t \in [0, t_0)$. Therefore, (4.5) and (4.6) give

$$\begin{aligned} K_0 = \|u_\varepsilon(t_0)\|_{L^\infty} &\le \|u_0\|_{L^\infty} + \alpha\chi(K_0)^2 \cdot t_0 \\ &\le \|u_0\|_{L^\infty} + \alpha\chi(K_0)^2 \cdot T_0 = \|u_0\|_{L^\infty} + 1 < K_0, \end{aligned}$$

which leads to a contradiction. Thus (4.7) is assured. By taking $\delta = 1$ and combining (4.7) with Proposition 3.1 and 3.2, the proof of Proposition 4.1 is completed. □

Proof of Theorem 1.1. If we assume that

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_{L^\infty} \le M_{u,\infty} \tag{4.8}$$

for some constant $M_{u,\infty}$ independently of ε, then the following convergence was given in [36]: There exists a subsequence $\{u_{\varepsilon_n}\}$ such that

$$\begin{aligned}
u_{\varepsilon_n}(x,t) \to u(x,t) & & & \text{a.a. } (x,t) \in \mathbb{R}^N \times (0,T), & (4.9)\\
u_{\varepsilon_n} \to u & & \text{strongly} & \text{ in } C((0,T); L^p_{loc}(\mathbb{R}^N)), & (4.10)\\
\nabla u^m_{\varepsilon_n} \rightharpoonup \nabla u^m & & \text{weakly star} & \text{ in } L^\infty(0,T; L^2(\mathbb{R}^N)), & (4.11)\\
v_{\varepsilon_n}(t) \to v(t) & & \text{strongly} & \text{ in } L^r_{loc}(\mathbb{R}^N) \text{ a.a. } t \in (0,T), & (4.12)\\
\nabla v_{\varepsilon_n}(t) \to \nabla v(t) & & \text{strongly} & \text{ in } L^r_{loc}(\mathbb{R}^N) \text{ a.a. } t \in (0,T), & (4.13)\\
\Delta v_{\varepsilon_n}(t) \rightharpoonup \Delta v(t) & & \text{weakly star} & \text{ in } L^r(\mathbb{R}^N) \text{ a.a. } t \in (0,T), & (4.14)
\end{aligned}$$

for any $p \in [2, \infty)$ and any $r \in (1, \infty]$ (see [36, Section 5] for details).

Since (4.8) is assured until T_0 by (4.1), we can obtain the above convergence (4.9)–(4.14) with T_0. By such convergence and the standard argument, we can construct a weak solution of (KS) until $T_0 = (\alpha\chi)^{-1}\big(\|u_0\|_{L^\infty} + 2\big)^{-2}$. Thus, the proof of Theorem 1.1 is completed. □

5. Proof of Theorem 1.2

As we noticed in Remark 3, in what follows C denotes a general constant greater than 1 (not necessarily the same at different occurrences) but which depends only on $m, N, \alpha, \gamma, \chi, \|u_0\|_{L^1}, \|u_0\|_{L^\infty}$ but not ε.

From Theorem 1.1, Lemma 2.4, we have

$$\|u_\varepsilon\|_{L^{r+1}} \le C^{\frac{1}{\beta_1}\cdot\frac{2}{r+m-1}} \|u_\varepsilon\|_{L^1}^{1-\theta_1} \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\theta_1}{r+m-1}} \quad \text{for } r \in [\max\{2, m-3\}, \infty), \tag{5.1}$$

where

$$\beta_1 := \frac{N}{N+2} \cdot \frac{(r+m-2+\frac{2}{N})(r-m+3)}{r(r+m-1)},$$
$$\theta_1 := \frac{r+m-1}{2} \cdot \left(1 - \frac{1}{r+1}\right) \cdot \frac{1}{\frac{1}{N} - \frac{1}{2} + \frac{r+m-1}{2}}.$$

It is easy to verify that $\frac{2\theta_1 \cdot (r+1)}{r+m-1} < 2$ if $m > 2 - \frac{2}{N}$ and $\frac{1}{\beta_1} \le \frac{2(N+2)}{N}$ if $r \in [\max\{2, 3m-7\}, \infty)$. Therefore, by the Young inequality,

$$\alpha\chi \|u_\varepsilon\|_{L^{r+1}}^{r+1} \le C(Cr)^{\frac{r}{m-2+\frac{2}{N}}} + \frac{2mr}{(r+m-1)^2} \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^2 \tag{5.2}$$
$$\text{for any } r \in [\max\{2, 3m-7\}, \infty) \text{ and } m > 2 - \frac{2}{N}.$$

From (3.1) in Proposition 3.3 and (5.2), we have

$$\frac{d}{dt}\|u_\varepsilon\|_{L^r}^r \le -\frac{2mr(r-1)}{(r+m-1)^2} \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^2 + (r-1)C(Cr)^{\frac{r}{m-2+\frac{2}{N}}} \tag{5.3}$$

for any $r \in [\max\{2, 3m-7\}, \infty)$, $m > 2 - \frac{2}{N}$ and for $t \in (0, T_0)$ with T_0 obtained from Theorem 1.1.

Again using Lemma 2.4, we have

$$\|u_\varepsilon\|_{L^r} \le C^{\frac{1}{\beta_2} \cdot \frac{2}{r+m-1}} \|u_\varepsilon\|_{L^1}^{1-\theta_2} \cdot \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\theta_2}{r+m-1}} \tag{5.4}$$
$$\text{for any } r \in [\max\{2, m-1\}, \infty),$$

where

$$\beta_2 := \frac{N}{N+2} \frac{(r+m-2+\frac{2}{N})(r-m+1)}{(r-1)(r+m-1)},$$
$$\theta_2 := \frac{r+m-1}{2} \cdot \left(1 - \frac{1}{r}\right) \cdot \frac{1}{\frac{1}{N} - \frac{1}{2} + \frac{r+m-1}{2}}.$$

Since $\frac{2\theta_2 \cdot r}{r+m-1} < 2$ by $m > 1 - \frac{2}{N}$, and $\frac{1}{\beta_2} \le \frac{2(N+2)}{N}$ if $r \ge 3(m-1)$, the Young inequality and (5.4) yield

$$\|u_\varepsilon\|_{L^r}^r \le C(Cr)^{\frac{r-1}{m-1+\frac{2}{N}}} + \frac{2m(r-1)}{(r+m-1)^2} \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^2 \tag{5.5}$$
$$\text{for any } r \in (\max\{2, 3(m-1)\}, \infty), \text{ and } m > 1 - \frac{2}{N}.$$

Substituting (5.5) into (5.3), we have

$$\frac{d}{dt}\|u_\varepsilon\|_{L^r}^r \leq -\|u_\varepsilon\|_{L^r}^r + (r-1)C(Cr)^{\frac{r}{m-2+\frac{2}{N}}} + C(Cr)^{\frac{r-1}{m-1+\frac{2}{N}}}$$

for any $r \in (\max\{2, 3(m-1)\}, \infty)$, $m > 2 - \frac{2}{N}$ and for $t \in (0, T_0)$. Hence, we have

$$\sup_{0<t<T_0} \|u_\varepsilon(t)\|_{L^r} \leq \|u_{0\varepsilon}\|_{L^r} + \Big((r-1)C(Cr)^{\frac{r}{m-2+\frac{2}{N}}}\Big)^{\frac{1}{r}} + \Big(C(Cr)^{\frac{r-1}{m-1+\frac{2}{N}}}\Big)^{\frac{1}{r}}$$

$$=: R_r \quad \text{for any } r \in (\max\{2, 3(m-1)\}, \infty) \text{ and } m > 2 - \frac{2}{N}. \tag{5.6}$$

We are now going to obtain the time global $L^\infty(\mathbb{R}^N)$-bound for u_ε by using (5.6).

From (3.2) in Proposition 3.3, Remark 3 and the Hölder and Young inequalities, we have

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|u_\varepsilon\|_{L^r}^r \\
&\leq -\frac{2m(r-1)}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx + (r-1)C\int u_\varepsilon^{r+1-m}\,dx \\
&\leq -\frac{2m(r-1)}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx + (r-1)C\|u_{0\varepsilon}\|_{L^1}^{\frac{m-1}{r-1}}\|u_\varepsilon\|_{L^r}^{r\cdot\frac{r-m}{r-1}} \\
&\leq -\frac{2m(r-1)}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx + C^r + r^2\|u_\varepsilon\|_{L^r}^r
\end{aligned} \tag{5.7}$$

for $r \geq m$ and $t \in (0, T_0)$.

On the other hand, by Lemma 2.4, we have

$$\|u_\varepsilon\|_{L^r} \leq C^{\frac{1}{\beta}\cdot\frac{2}{r+m-1}}\|u_\varepsilon\|_{L^{\frac{r}{4}}}^{1-\theta_3}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\theta_3}{r+m-1}} \tag{5.8}$$

for any $r \in (\max\{4, m-1\}, \infty)$ and $m \geq 1$, where

$$\begin{aligned}
\beta &= \frac{2(r-m+1)}{3(r+m-1)r}\Big[\frac{r}{2} + \frac{N}{N+2}(r+2m-2)\Big], \\
\theta_3 &= \frac{r+m-1}{2}\cdot\Big(\frac{4}{r} - \frac{1}{r}\Big)\cdot\frac{1}{\frac{1}{N} - \frac{1}{2} + \frac{2(r+m-1)}{r}}.
\end{aligned}$$

It is easy to verify that $\frac{2\theta_3 \cdot r}{r+m-1} < 2$ by $m \geq 1$, and $\frac{1}{\beta} \leq 6$ when $r \geq 3(m-1)$. Therefore, the Young inequality and (5.8) yield that

$$r^2\|u_\varepsilon\|_{L^r}^r \leq r^2 C^{\frac{12r}{r+m-1}}\|u_\varepsilon\|_{L^{\frac{r}{4}}}^{(1-\theta_3)r}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\theta_3}{r+m-1}\cdot r}$$

$$\leq \quad \frac{m(r-1)}{(r+m-1)^2}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^2 + (Cr)^{k_1}(Cr^2)^{k_2}\|u_\varepsilon\|_{L^{\frac{r}{4}}}^{(1-\theta_3)r\cdot k_2}$$
$$\text{for any } r \in [\max\{4, 3(m-1)\}, \infty) \text{ and } m \geq 1,$$

where $k_1 := \frac{\theta_3 r}{r+m-1-\theta_3 r}$ and $k_2 := \frac{r+m-1}{r+m-1-\theta_3 r}$. On the other hand, we easily see that

$$\frac{1}{(1-\theta_3)k_2} \geq 1 \text{ and } k_1 \to \frac{3N}{2}, \quad k_2 \to \frac{3N+2}{2}, \quad \frac{1}{(1-\theta_3)k_2} \to 1 \text{ as } r \to \infty.$$

Therefore, there exists a natural number p_0 depending only on m, N such that

$$r^2\|u_\varepsilon\|_{L^r}^r \leq \frac{m(r-1)}{(r+m-1)^2}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^2 + 1 + Cr^{6N}\|u_\varepsilon\|_{L^{\frac{r}{4}}}^r \tag{5.9}$$

for any $r \geq R_0 := 4^{p_0-1}$. Substituting (5.9) into (5.7),

$$\frac{1}{r}\frac{d}{dt}\|u_\varepsilon\|_{L^r}^r \leq -\frac{m(r-1)}{(r+m-1)^2}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^2 + C^r + Cr^{6N}\|u_\varepsilon\|_{L^{\frac{r}{4}}}^r \tag{5.10}$$

for any $r \geq R_0$ and $t \in (0, T_0)$. Moreover, substituting (5.9) into (5.10),

$$\frac{1}{r}\frac{d}{dt}\|u_\varepsilon\|_{L^r}^r + r^2\|u_\varepsilon\|_{L^r}^r \leq C^r + Cr^{6N}\|u_\varepsilon\|_{L^{\frac{r}{4}}}^r \tag{5.11}$$

for any $r \geq R_0$ and $t \in (0, T_0)$. By multiplying (5.11) by e^t and integrating it from 0 to t, we obtain the $L^r(\mathbb{R}^N)$ estimate for u_ε as follows:

$$\begin{aligned} \sup_{0<t<T_0}\|u_\varepsilon(t)\|_{L^r}^r &\leq \|u_{0\varepsilon}\|_{L^r}^r + rC^r + Cr^{6N+1}\sup_{0<t<T_0}\|u_\varepsilon(t)\|_{L^{\frac{r}{4}}}^r \\ &\leq \max\left\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}, C, \sup_{0<t<T_0}\|u_\varepsilon\|_{L^{\frac{r}{4}}}\right\}^r \cdot Cr^{6N+1} \end{aligned} \tag{5.12}$$

for any $r \geq R_0$. From (5.12), we have

$$\sup_{0<t<T_0}\|u_\varepsilon(t)\|_{L^r} \leq \max\left\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}, C, \sup_{0<t<T_0}\|u_\varepsilon\|_{L^{\frac{r}{4}}}\right\} \cdot C^{\frac{1}{r}} r^{\frac{C}{r}}$$

for any $r \geq R_0$.

From now on, we shall establish the time global $L^\infty(\mathbb{R}^N)$-bound for u_ε independently of ε. To this end, we employ Moser's iteration technique developed in Alikakos [1].

We take $r = 4^p$ with $p \geq 1$ and set

$$\alpha_p := \max\left\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}, C, \sup_{0<t<T_0}\|u_\varepsilon\|_{L^{4^p}}\right\}.$$

Then

$$\begin{aligned}\alpha_p &\le C^{1/4^p} 4^{Cp/4^p} \max\left\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}, C, \sup_{0<t<T_0} \|u_\varepsilon\|_{L^{4^{p-1}}}\right\}\\ &= C^{1/4^p} 4^{Cp/4^p} \cdot \alpha_{p-1}\\ &\le C^{\left(1/4^p+1/4^{p-1}+\cdots+1/4^{p_0}\right)} \cdot \alpha_{p_0-1} 4^{C\left(p/4^p+(p-1)/4^{p-1}+\cdots+p_0/4^{p_0}\right)}\\ &\le C^{4^{1-p_0}} 4^{C(p_0+1)4^{1-p_0}} \cdot \alpha_{p_0-1}.\end{aligned}$$

By (5.6), if $m > 2 - \frac{2}{N}$, then we have

$$\sup_{0<t<T_0} \|u_\varepsilon(t)\|_{L^{4^p}} \le C^{4^{1-p_0}} 4^{C(p_0+1)4^{1-p_0}} \max\left\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}, C, R_{R_0}\right\} \le C.$$

Consequently, by letting p tend to ∞, we see that $u_\varepsilon \in L^\infty(0,\infty; L^\infty(\mathbb{R}^N))$ and

$$\begin{aligned}&\sup_{0<t<T_0} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}\\ &\le C^{4^{1-p_0}} 4^{C(p_0+1)4^{1-p_0}} \max\left\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}, C, R_{R_0}\right\} \le C\end{aligned} \tag{5.13}$$

(see [31, Lemma 3.4] for instance). Thus, the time global $L^\infty(\mathbb{R}^N)$-bound for u_ε is obtained until T_0.

In the next step, we will extend the strong solution u_ε beyond T_0. We set $T_0' = (\alpha\chi)^{-1}(C+2)^{-2}$, then we can repeat the argument similar to that from (5.1) to (5.13) on $[T_0 - \frac{T_0'}{2}, T_0 + \frac{T_0'}{2}]$. Although the assumption (A.2) on $u_\varepsilon(t)$ is not exactly fulfilled at $t = T_0 - \frac{T_0'}{2}$, it can be easily seen that the *a priori* estimate (5.13) plays a substitutional role for the characterization of extensions of the strong solution u_ε. Hence, (5.13) enables us to continue the existence time interval of the strong solution onto $[T_0 - \frac{T_0'}{2}, T_0 + \frac{T_0'}{2}]$. Repeating this procedure, we see that our local solution can be extended globally in time. Furthermore, since C in (5.13) is independent of T_0, we see that our solution u_ε in $(\mathrm{KS})_\varepsilon$ neither blows up nor grows up on $[0,\infty)$.

Finally, by using (4.9)–(4.14) and the standard convergence argument as $\varepsilon \to 0$, we can construct a weak solution (u,v) of (KS). Moreover, by the mass conservation law and the lower semi-continuity of norm, the time global $L^1 \cap L^\infty(\mathbb{R}^N)$-bound for the weak solution (u,v) follows. Thus, we complete the proof of Theorem 1.2. □

6. Proof of Theorem 1.3

Definition 3 We call $T_b := \inf\{T > 0;\ \limsup_{t\to T-0}\|u(t)\|_{L^\infty(\mathbb{R}^N)} = \infty\}$ *the first blow-up time* of a weak solution of (KS).

We show the crucial inequality for a weak solution of (KS) in the following proposition:

Proposition 6.1 (L^m a priori estimate)**.** *Let* $m > 1$, $\alpha, \gamma, \chi > 0$ *and let the initial data* u_0 *be non-negative everywhere. Suppose that* (u, v) *is the weak solution of* (KS) *on* $[0, T)$ *as in Theorem* 1.1*. Then, we have*

$$\begin{aligned}&\int_{\mathbb{R}^N} u^m(t)\,dx - \frac{(m-1)\chi}{2}\int_{\mathbb{R}^N} uv(t)\,dx \\ &\quad\le \int_{\mathbb{R}^N} u_0^m\,dx - \frac{(m-1)\chi}{2}\int_{\mathbb{R}^N} u_0v_0\,dx \quad \text{for a.a. } t\in[0,T_b), \qquad (6.1)\end{aligned}$$

where $v_0 := G * u_0$ *with the Bessel potential* G *and* T_b *is the first blow-up time defined in Definition* 3.

Proof of Proposition 6.1. The following argument was first developed in Nagai-Senba-Yoshida [28] for the semi-linear Neumann problem to obtain the Lyapunov function. We modify it and obtain the Lyapunov function for our problem. We remark that since our problem (KS) is the quasi-linear "degenerate Cauchy" problem, the approximate procedure and the cut-off function ψ_ℓ as defined in Lemma 2.5 are needed to give the rigorous proof.

Firstly, we multiply (2) in $(\mathrm{KS})_\varepsilon$ by $\big(\frac{m(u_\varepsilon+\varepsilon)^{m-1}}{m-1}\psi_\ell - \chi v_\varepsilon\big)$. Then, we have

$$\begin{aligned}&\frac{1}{m-1}\cdot\frac{d}{dt}(u_\varepsilon+\varepsilon)^m\psi_\ell - \chi\frac{d}{dt}(u_\varepsilon v_\varepsilon) + \chi u_\varepsilon v_{\varepsilon t} \qquad (6.2)\\ &= \nabla\cdot\Big(\nabla(u_\varepsilon+\varepsilon)^m - \chi u_\varepsilon\nabla v_\varepsilon\Big)\times\Big(\frac{m(u_\varepsilon+\varepsilon)^{m-1}}{m-1}\psi_\ell - \chi v_\varepsilon\Big).\end{aligned}$$

By substituting $u_\varepsilon = \frac{1}{\alpha}(-\Delta v_\varepsilon + \gamma v_\varepsilon\ )$ into the third term of the left-hand side in (6.2),

$$\begin{aligned}&\frac{1}{m-1}\cdot\frac{d}{dt}(u_\varepsilon+\varepsilon)^m\psi_\ell - \chi\cdot\frac{d}{dt}(u_\varepsilon v_\varepsilon) + \frac{\gamma\chi}{2\alpha}\frac{d}{dt}v_\varepsilon^2 - \frac{\chi}{\alpha}\frac{dv_\varepsilon}{dt}\Delta v_\varepsilon\\ &= \nabla\cdot\Big(\nabla(u_\varepsilon+\varepsilon)^m - \chi u_\varepsilon\nabla v_\varepsilon\Big)\times\Big(\frac{m(u_\varepsilon+\varepsilon)^{m-1}}{m-1}\psi_\ell - \chi v_\varepsilon\Big). \qquad (6.3)\end{aligned}$$

We denote $W_\varepsilon(t)$ by

$$W_\varepsilon(t) := \int \frac{1}{m-1}(u_\varepsilon(t)+\varepsilon)^m\psi_\ell\ -\ \frac{\chi}{2}u_\varepsilon v_\varepsilon(t)\ dx \qquad (6.4)$$

$$= \int \frac{1}{m-1}(u_\varepsilon(t)+\varepsilon)^m\psi_\ell - \chi u_\varepsilon v_\varepsilon(t) + \frac{\gamma\chi}{2\alpha}v_\varepsilon^2(t) + \frac{\chi}{2\alpha}|\nabla v_\varepsilon(t)|^2\ dx.$$

Integrating (6.3) with respect to the space variable and using (6.4), we have

$$\begin{aligned}
\frac{d}{dt}W_\varepsilon(t) &:= \frac{d}{dt}\int\Big(\frac{1}{m-1}(u_\varepsilon(t)+\varepsilon)^m\psi_\ell \\
&\qquad - \chi u_\varepsilon v_\varepsilon(t) + \frac{\gamma\chi}{2\alpha}v_\varepsilon^2(t) + \frac{\chi}{2\alpha}|\nabla v_\varepsilon(t)|^2\Big)dx \\
&= \int \nabla\cdot\Big(\nabla(u_\varepsilon(t)+\varepsilon)^m - \chi u_\varepsilon\nabla v_\varepsilon(t)\Big)\Big(\frac{m(u_\varepsilon(t)+\varepsilon)^{m-1}}{m-1}\psi_\ell - \chi v_\varepsilon(t)\Big)dx \\
&= -\int\Big(\nabla(u_\varepsilon(t)+\varepsilon)^m - \chi u_\varepsilon\nabla v_\varepsilon(t)\Big)\nabla\Big(\frac{m(u_\varepsilon(t)+\varepsilon)^{m-1}}{m-1}\psi_\ell - \chi v_\varepsilon(t)\Big)\ dx \\
&=: -\int I\times II\ dx, \qquad (6.5)
\end{aligned}$$

where

$$I := \nabla(u_\varepsilon(t)+\varepsilon)^m - \chi u_\varepsilon\nabla v_\varepsilon(t), \quad II := \nabla\Big(\frac{m(u_\varepsilon(t)+\varepsilon)^{m-1}}{m-1}\psi_\ell - \chi v_\varepsilon(t)\Big).$$

It is easy to see that

$$\begin{aligned}
I &= (u_\varepsilon(t)+\varepsilon)\nabla\Big(\frac{m(u_\varepsilon(t)+\varepsilon)^{m-1}}{m-1} - \chi v_\varepsilon(t)\Big) + \varepsilon\chi\nabla v_\varepsilon(t), \\
II &= \nabla\Big(\frac{m(u_\varepsilon(t)+\varepsilon)^{m-1}}{m-1} - \chi v_\varepsilon(t)\Big)\psi_\ell + \Big(\frac{m(u_\varepsilon(t)+\varepsilon)^{m-1}}{m-1} - \chi v_\varepsilon(t)\Big)\nabla\psi_\ell \\
&\quad + \chi\nabla v_\varepsilon(t)(\psi_\ell - 1) + \chi v_\varepsilon(t)\nabla\psi_\ell.
\end{aligned}$$

We are now going to estimate $-\int I\times II\,dx$ from above. By denoting $\frac{m(u_\varepsilon+\varepsilon)^{m-1}}{m-1} - \chi v_\varepsilon$ by U_ϵ, we have

$$-\int I\times II\ dx \quad =: \quad -J_p + J_1 + J_2 + J_3 + J_4, \qquad (6.6)$$

where

$$\begin{aligned}
J_p &:= \int(u_\varepsilon+\varepsilon)|\nabla U_\varepsilon|^2\psi_\ell\ dx \geq 0, \\
J_1 &:= -\int(u_\varepsilon+\varepsilon)(U_\varepsilon + \chi v_\varepsilon)\nabla U_\varepsilon\cdot\nabla\psi_\ell\ dx, \\
J_2 &:= -\chi\int(u_\varepsilon+\varepsilon)\nabla U_\varepsilon\cdot\nabla v_\varepsilon(\psi_\ell - 1)\ dx,
\end{aligned}$$

$$J_3 := -\varepsilon\chi \int \nabla v_\varepsilon \cdot \Big(\nabla U_\varepsilon \psi_\ell + U \nabla \psi_\ell\Big)\ dx,$$
$$J_4 := -\varepsilon\chi^2 \int \nabla (v_\varepsilon(\psi_\ell - 1)) \cdot \nabla v_\varepsilon\ dx.$$

We denote D_ℓ by $D_\ell := \{x \in \mathbb{R}^N; \ell < |x| < 2\ell\}$. Then, since $|\nabla\psi_\ell| \leq \frac{c_1}{\ell}$ and $\mathrm{supp}|\nabla\psi_\ell| = \overline{D_\ell}$, it holds that

$$\begin{aligned} J_1 &\leq \frac{1}{4}J_p + \frac{c_1^2}{\ell^2}(\frac{m}{m-1})^2 \int_{D_\ell} (u_\varepsilon + \varepsilon)^{2m-1}\ dx \\ &\leq \frac{1}{4}J_p + \frac{c_1^2}{\ell^2}(\frac{m}{m-1})^2 \cdot 2^m \Big(\|u_\varepsilon\|_{L^{2m-1}}^{2m-1} + \varepsilon^{2m-1}|D_\ell|\Big). \end{aligned}$$

Moreover, using Proposition 4.1, (4.2) and (4.3), we have

$$J_1 \leq \frac{1}{4}J_p + \frac{1}{\ell^2}C\Big(1 + \varepsilon^{2m-1}|D_\ell|\Big) \qquad \text{for } t \in (0, T_0)$$

with $T_0 = (\alpha\chi)^{-1}(\|u_0\|_{L^\infty} + 2)^{-2}$. Next, we deal with J_2. Noting that $\psi_\ell - 1 \leq 0$ and $\Delta v_\varepsilon = \gamma v_\varepsilon + \alpha u_\varepsilon$, we have

$$\begin{aligned} J_2 &= -\frac{m\chi}{m-1}\int (u_\varepsilon + \varepsilon)\nabla(u_\varepsilon + \varepsilon)^{m-1} \cdot \nabla v_\varepsilon(\psi_\ell - 1)\ dx \\ &\quad + \chi^2 \int (u_\varepsilon + \varepsilon)|\nabla v_\varepsilon|^2(\psi_\ell - 1)\ dx \\ &\leq -\frac{m\chi}{m-1}\int (u_\varepsilon + \varepsilon)\nabla(u_\varepsilon + \varepsilon)^{m-1} \cdot \nabla v_\varepsilon(\psi_\ell - 1)\ dx \\ &= -\chi \int \nabla(u_\varepsilon + \varepsilon)^m \cdot \nabla v_\varepsilon(\psi_\ell - 1)\ dx \\ &= \chi \int (u_\varepsilon + \varepsilon)^m \Delta v_\varepsilon(\psi_\ell - 1)\ dx + \chi \int (u_\varepsilon + \varepsilon)^m \nabla v_\varepsilon \cdot \nabla\psi_\ell\ dx \\ &\leq \alpha\chi \int u_\varepsilon(u_\varepsilon + \varepsilon)^m(1 - \psi_\ell)\ dx + \frac{2^m\chi c_1}{\ell}\int (u_\varepsilon^m + \varepsilon^m)|\nabla v_\varepsilon|\ dx. \end{aligned}$$

Again using (4.2) and (4.3), we have

$$J_2 \leq C\int u_\varepsilon(1 - \psi_\ell)\ dx + \frac{1}{\ell}C \qquad \text{for } t \in (0, T_0). \tag{6.7}$$

By integrating by parts once,

$$J_3 = \varepsilon\chi \int \Delta v_\varepsilon U_\varepsilon \psi_\ell\ dx \leq \varepsilon\chi\|\Delta v_\varepsilon\|_{L^1}\|U_\varepsilon\|_{L^\infty} \leq C \tag{6.8}$$

and

$$J_4 \;=\; \varepsilon\chi^2\int v_\varepsilon\Delta v_\varepsilon(\psi_\ell-1)\;dx \;\le\; \varepsilon\chi^2\|v_\varepsilon\|_{L^2}\|\Delta v_\varepsilon\|_{L^2} \;\le\; C. \quad (6.9)$$

Gathering (6.6)–(6.9), we obtain

$$\begin{aligned}&-\int I\times II\;dx = -J_p \;+\; J_1 \;+\; J_2 \;+\; J_3 \;+\; J_4 \qquad (6.10)\\ &\le \frac{1}{\ell^2}C\Big(1+\varepsilon^{2m-1}|D_\ell|\Big) \;+\; C\int u_\varepsilon(1-\psi_\ell)\;dx + \frac{1}{\ell}C+\varepsilon C.\end{aligned}$$

We return to (6.4) and (6.5). By virtue of (6.10) we have

$$\begin{aligned}&\frac{d}{dt}\int\frac{1}{m-1}(u_\varepsilon(t)+\varepsilon)^m\psi_\ell-\frac{\chi}{2}u_\varepsilon v_\varepsilon(t)\,dx=\frac{d}{dt}W_\varepsilon(t)=-\int I\times II\;dx\\ &\le\frac{1}{\ell^2}C\Big(1+\varepsilon^{2m-1}|D_\ell|\Big)+C\int u_\varepsilon(t)(1-\psi_\ell)\;dx+\frac{1}{\ell}C+\varepsilon C. \qquad (6.11)\end{aligned}$$

Integrating (6.11) with respect to the time variable from 0 to t,

$$\begin{aligned}&\frac{1}{m-1}\int u_\varepsilon^m(t)\psi_\ell\,dx-\frac{\chi}{2}\int u_\varepsilon v_\varepsilon(t)\psi_\ell\,dx-\frac{\chi}{2}\int u_\varepsilon v_\varepsilon(t)(1-\psi_\ell)\,dx\\ &\le\frac{1}{m-1}\int(u_{0\varepsilon}+\varepsilon)^m\psi_\ell\;dx-\frac{\chi}{2}\int u_{0\varepsilon}v_{0\varepsilon}\,dx \qquad (6.12)\\ &+\frac{1}{\ell^2}C\Big(1+\varepsilon^{2m-1}|D_\ell|\Big)T_0+C\int_0^t\int u_\varepsilon(s)(1-\psi_\ell)\,dx\,ds+\frac{1}{\ell}CT_0+\varepsilon CT_0\end{aligned}$$

for $t\in(0,T_0)$. Let ε tend to 0 in (6.12). Then, by (4.10) and (4.12), we have

$$\int u_\varepsilon v_\varepsilon(t)\psi_\ell\;dx \;\;\to\;\; \int uv(t)\psi_\ell\;dx \qquad \text{a.a. } \; t\in(0,T_0).$$

In addition, by (4.10) and the mass conservation law for u_ε and u,

$$\int u_\varepsilon(t)\psi_\ell\;dx \;\;\to\;\; \int u(t)\psi_\ell\;dx \qquad \text{all } \; t\in(0,T_0),$$

$$\int u_\varepsilon(t)\;dx=\int u_{0\varepsilon}\;dx \;\;\to\;\; \int u_0\;dx\Big(=\int u(t)\;dx\Big) \qquad \text{all } \; t\in[0,T_0).$$

Therefore, by virtue of (4.3), (4.10), (4.12), and (A.4) in Section 3, we observe

$$\begin{aligned}&\frac{1}{m-1}\int u^m(t)\psi_\ell\;dx \;-\; \frac{\chi}{2}\int uv(t)\psi_\ell\;dx\\ &\le\;\; \frac{1}{m-1}\int u_0^m\psi_\ell\;dx \;-\; \frac{\chi}{2}\int u_0v_0\;dx \;+\; \frac{1}{\ell}CT(1+\frac{1}{\ell})\end{aligned}$$

$$+ C \int u(t)(1-\psi_\ell)\, dx + C \tag{6.13}$$

for almost all $t \in (0, T_0)$.

Next, letting ℓ tend to ∞ in (6.13), Lebesgue's dominated convergence theorem yields

$$\int u^m(t)\, dx - \frac{(m-1)\chi}{2} \int uv(t)\, dx \le \int u_0^m\, dx - \frac{(m-1)\chi}{2} \int u_0 v_0\, dx \tag{6.14}$$

for almost all $t \in (0, T_0)$.

Finally, we show that (6.14) can be continued to the right of $t = T_0$. To this end, put $T_0^{'} = (\alpha\chi)^{-1}(C+2)^{-2}$ and take the initial time to be $T_0 - \frac{T_0^{'}}{2}$. Then, we have

$$\|u(t)\|_{L^\infty} \le \|u\Big(T_0 - \frac{T_0^{'}}{2}\Big)\|_{L^\infty} + 2 \le \|u_0\|_{L^\infty(\mathbb{R}^N)} + 4 \tag{6.15}$$

for all $t \in [T_0 - \frac{T_0^{'}}{2}, T_0 + \frac{T_0^{'}}{2}]$.

Although the assumption (A.2) on $u_\varepsilon(t)$ is not exactly fulfilled at $t = T_0 - \frac{T_0^{'}}{2}$, it can be easily seen that the *a priori* estimate (6.15) plays a substitutional role for the characterization of extensions of the strong solution u_ε. Hence, (6.15) enables us to continue the existence time interval of the strong solution onto $[T_0 - \frac{T_0^{'}}{2}, T_0 + \frac{T_0^{'}}{2}]$. By an argument similar to that from (6.2) to (6.14) and using (6.14), we deduce

$$\begin{aligned} &\frac{1}{m-1}\int u^m(t)\, dx \ - \ \frac{\chi}{2}\int uv(t)\, dx \\ &\le \frac{1}{m-1}\int u^m\Big(T_0 - \frac{T_0^{'}}{2}\Big)\, dx \ - \ \frac{\chi}{2}\int uv\Big(T_0 - \frac{T_0^{'}}{2}\Big)\, dx \\ &\le \frac{1}{m-1}\int u_0^m\, dx \ - \ \frac{\chi}{2}\int u_0 v_0\, dx \end{aligned} \tag{6.16}$$

for almost all $t \in [0, T_0 + \frac{T_0^{'}}{2}]$.

Repeating this procedure, we see that our local solution can be extended to the right of $t = T$, as long as $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$ is bounded on $[0,T)$. In other words, we obtain (6.14) for almost all $t \in [0, T_b)$, where T_b is the first blow-up time defined in Definition 3. In consequence, we complete the proof of Proposition 6.1. □

The following lemma is a key tool which is essentially due to Theorem 1.3, which reads:

Lemma 6.2. *Let $N \geq 3,\ 1 < m \leq 2 - \frac{2}{N},\ \alpha, \gamma, \chi > 0$. Let the initial data u_0 be non-negative everywhere in $L^1 \cap L^2(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u_0(x)|x|^2\, dx < +\infty$. Suppose that (u, v) is the weak solution of* (KS) *on* $[0, T)$ *as in Theorem* 1.1. *Then, it holds that*

$$\int_{\mathbb{R}^N} u(t)|x|^2\, dx - \int_{\mathbb{R}^N} u_0|x|^2\, dx \leq 2Nt \int_{\mathbb{R}^N} \Big(u_0^m - \frac{(m-1)\chi}{2} u_0 v_0 \Big)\, dx \quad (6.17)$$

for almost all $t \in [0, T_b)$, where $v_0 := G * u_0$ with the Bessel potential G and T_b is the first blow-up time defined in Definition 3.

Proof of Lemma 6.2. We multiply (1) in $(\mathrm{KS})_\varepsilon$ by $|x|^2\psi_\ell(x)$ and integrate by parts. Then, we have

$$\frac{d}{dt}\int u_\varepsilon |x|^2 \psi_\ell\, dx = \int (u_\varepsilon + \varepsilon)^m \Delta\big(|x|^2\psi_\ell\big)\, dx + \chi \int u_\varepsilon \nabla v_\varepsilon \cdot \nabla\big(|x|^2\psi_\ell\big)\, dx, \quad (6.18)$$

where ψ_ℓ is the cut-off function in Lemma 2.5. It is easy to see that $\nabla|x|^2 = 2x$, and $\Delta|x|^2 = 2N$. In addition, $\psi' \leq 0$ by the definition of ψ in Lemma 2.5. Therefore, we have

$$\Delta(|x|^2\psi_\ell) = 2N\psi_\ell + 2x\psi'(\frac{|x|}{\ell})\frac{1}{\ell}\cdot\frac{x}{|x|} + |x|^2\Delta\psi_\ell \leq 2N\psi_\ell + |x|^2\Delta\psi_\ell. \quad (6.19)$$

We denote D_ℓ by $D_\ell := \{x \in \mathbb{R}^N; \ell < |x| < 2\ell\}$. From (4.2), (4.3) and (6.19), we have

$$\begin{aligned}
&\int (u_\varepsilon + \varepsilon)^m \Delta\big(|x|^2\psi_\ell\big)\ dx \\
&\leq\ 2N\int (u_\varepsilon+\varepsilon)^m \psi_\ell\ dx + \frac{c_2}{\ell^2}\int_{D_\ell} (u_\varepsilon+\varepsilon)^m |x|^2\ dx \\
&\leq\ 2^m N \int (u_\varepsilon^m + \varepsilon^m)\psi_\ell\ dx\ +\ 2^m c_2 \Big(\frac{2\ell}{\ell}\Big)^2 \Big(\int_{D_\ell} u_\varepsilon^m\ dx + \varepsilon^m |D_\ell|\Big) \\
&\leq\ C\ +\ \varepsilon^m C(|\mathrm{supp}\ \psi_\ell| + |D_\ell|) \qquad (6.20)
\end{aligned}$$

for all $\ell \geq 1$ and all $t \in [0, T_0]$, where $T_0 = (\alpha\chi)^{-1}\big(\|u_0\|_{L^\infty(\mathbb{R}^N)} + 2\big)^{-2}$ in Proposition 4.1 and c_2 is the constant defined in (2.10) of Lemma 2.5. Moreover, by the moment inequality in Lemma 2.3,

$$\int_{D_\ell} u_\varepsilon |x| \psi_\ell^{\frac{1}{2}}\ dx \ \ \leq\ \ \|u_0\|_{L^1(D_\ell)}^{\frac{1}{2}} \Big(\int_{D_\ell} u_\varepsilon |x|^2 \psi_\ell\ dx\Big)^{\frac{1}{2}}.$$

Therefore, from Lemma 2.5 and the Hölder and Young inequalities, we have

$$
\begin{aligned}
&\chi \int u_\varepsilon \nabla v_\varepsilon \cdot \nabla\big(|x|^2 \psi_\ell\big)\ dx \\
&= \ 2\chi \int u_\varepsilon \nabla v_\varepsilon \cdot x \psi_\ell \ dx + \ \chi \int_{D_\ell} u_\varepsilon \nabla v_\varepsilon |x|^2 \cdot \nabla \psi_\ell \ dx \\
&\leq \ 2\chi \int u_\varepsilon \nabla v_\varepsilon \cdot x \psi_\ell \ dx + \ \chi \|\nabla v_\varepsilon\|_{L^\infty} \frac{2c_1 \ell}{\ell} \int_{D_\ell} u_\varepsilon |x| \psi_\ell^{\frac{1}{2}} \ dx \\
&\leq \ 2\chi \int u_\varepsilon \nabla v_\varepsilon \cdot x \psi_\ell \ dx \\
&\qquad + 2c_1 \chi \|\nabla v_\varepsilon\|_{L^\infty} \|u_0\|_{L^1(D_\ell)}^{\frac{1}{2}} \Big(\int_{D_\ell} u_\varepsilon |x|^2 \psi_\ell \ dx \Big)^{\frac{1}{2}} \qquad (6.21) \\
&\leq \ 2\chi \|\nabla v_\varepsilon\|_{L^\infty} \int u_\varepsilon |x| \psi_\ell^{\frac{1}{2}} \ dx \\
&\qquad + 2c_1 \chi \|\nabla v_\varepsilon\|_{L^\infty} \|u_0\|_{L^1}^{\frac{1}{2}} \Big(\int_{D_\ell} u_\varepsilon |x|^2 \psi_\ell \ dx \Big)^{\frac{1}{2}} \\
&\leq \ 2(c_1+1)\chi \|\nabla v_\varepsilon\|_{L^\infty} \|u_0\|_{L^1}^{\frac{1}{2}} \Big(\int_{D_\ell} u_\varepsilon |x|^2 \psi_\ell \ dx \Big)^{\frac{1}{2}} \\
&\leq \ (c_1+1)\chi \|\nabla v_\varepsilon\|_{L^\infty} \Big(\|u_0\|_{L^1} + \int_{D_\ell} u_\varepsilon |x|^2 \psi_\ell \ dx \Big) \qquad (6.22)
\end{aligned}
$$

for all $\ell \geq 1$.

By virtue of (4.3) and (6.22), we have

$$
\chi \int u_\varepsilon \nabla v_\varepsilon \cdot \nabla\big(|x|^2 \psi_\ell\big)\ dx \ \leq \ C + C \int u_\varepsilon |x|^2 \psi_\ell \ dx. \qquad (6.23)
$$

Collecting (6.18), (6.20), and (6.23), we deduce

$$
\begin{aligned}
&\frac{d}{dt} \int u_\varepsilon(t) |x|^2 \psi_\ell \ dx \\
&\leq \ C + \ C \ + \ \varepsilon^m C(|\mathrm{supp}\ \psi_\ell| + |D_\ell|) \ + \ C \int u_\varepsilon |x|^2 \psi_\ell \ dx \\
&\qquad\qquad \text{for all } \ell \geq 1 \text{ and a.a. } t \in (0, T_0).
\end{aligned}
$$

Therefore, the Gronwall inequality yields that

$$
\int u_\varepsilon(t) |x|^2 \psi_\ell \ dx
$$

$$\leq \quad \Big(\int_{D_\ell} u_{0\varepsilon}|x|^2 \; dx \; + \; C \; + \; C \; + \; \varepsilon^m C(|\text{supp } \psi_\ell| + |D_\ell|) \Big) e^{CT_0}$$
$$\text{for all } \ell \geq 1 \text{ and a.a. } t \in (0, T_0). \qquad (6.24)$$

On the other hand, by (4.9) and the definition of ψ_ℓ in Lemma 2.5, it holds that

$$u_\varepsilon(x,t)|x|^2\psi_\ell \; \to \; u(x,t)|x|^2\psi_\ell, \quad u(x,t)|x|^2\psi_\ell \; \to \; u(x,t)|x|^2$$
$$\text{for a.a. } (x,t) \in \mathbb{R}^N \times (0, T_0).$$

Hence, applying Fatou's lemma to (6.24), we have

$$\int u(t)|x|^2 \; dx \leq \text{liminf}_{\ell\to\infty}\text{liminf}_{\varepsilon\to 0} \int u_\varepsilon(t)|x|^2\psi_\ell \, dx$$
$$\leq \quad \Big(\int u_0|x|^2 \; dx \; + \; C \; + \; C \Big) e^{CT_0} \qquad \text{for a.a. } t \in (0, T_0). \quad (6.25)$$

By virtue of (6.25), we are now going to establish (6.17). To this end, we integrate (6.18) with respect to the time variable from 0 to t. Then, noting Lemma 2.5, (4.9), (4.13), (6.19) and (6.21) and letting ε tend to 0, we have

$$\int u(t)|x|^2\psi_\ell \; dx - \int u_0|x|^2\psi_\ell \; dx$$
$$\leq \quad 2^m N \int_0^t \int u^m \psi_\ell \; dxds \; + \; 2^m c_2 \cdot \Big(\frac{2\ell}{\ell}\Big)^2 \int_0^t \int_{D_\ell} u^m \; dxds$$
$$+ \; 2\chi \int_0^t \int u\nabla v \cdot x\psi_\ell \; dxds$$
$$+ \; 2c_1\chi\|\nabla v\|_{L^\infty}\|u_0\|_{L^1(D_\ell)}^{\frac{1}{2}} \int_0^t \Big(\int_{D_\ell} u|x|^2\psi_\ell \; dx \Big)^{\frac{1}{2}} ds$$
$$\text{for all } \ell \geq 1 \text{ and a.a. } t \in (0, T_0). \quad (6.26)$$

Next, we shall let ℓ tend to ∞ in (6.26). By the mass conservation law for u (*i.e.*, $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$ for $t \in [0, T_0]$) and (4.2),

$$u(t) \in L^\infty(0, T_0; L^1 \cap L^\infty(\mathbb{R}^N)). \qquad (6.27)$$

From $v = G * u$, $\nabla G \in L^1(\mathbb{R}^N)$ and $u \in L^\infty(0, T_0; L^2(\mathbb{R}^N))$, we have

$$\nabla v \in L^\infty(0, T_0; L^2(\mathbb{R}^N)). \qquad (6.28)$$

In addition, from (6.25), we obtain

$$u|x|^2 \quad \in \quad L^\infty(0, T_0; L^1(\mathbb{R}^N)). \qquad (6.29)$$

Therefore, from (6.27)–(6.29), we find

$$|u\nabla v\cdot x\psi_\ell| \quad\le\quad u|x|^2 \;+\; u|\nabla v|^2 \;\in\; L^1(0,T_0;L^1(\mathbb{R}^N)).$$

Moreover, by (6.25) we observe

$$\sup_{\ell\ge 1}\int u(t)|x|^2\psi_\ell\,dx<\infty$$

for all $\ell\ge 1$ and almost all $t\in(0,T_0)$. Simultaneously, by (4.9) we obtain

$$u(x,t)|x|^2\psi_\ell \;\to\; u(x,t)|x|^2 \qquad \text{for all} \;\; (x,t)\in\mathbb{R}^N\times(0,T)$$

as $\ell\to\infty$. Therefore, Fatou's lemma, Lebesgue's dominated convergence theorem, and the absolute continuity of integration yield that

$$\begin{aligned}&\int u(t)|x|^2\,dx-\int u_0|x|^2\,dx \\ &\le 2N\int_0^t\int u^m\,dx\,ds+2\chi\int_0^t\int u\nabla v\cdot x\,dx\,ds \quad \text{for a.a.} \;\; t\in(0,T_0).\end{aligned} \tag{6.30}$$

We are now in a position to prove (6.17). From Proposition 6.1 and (6.30), we have

$$\begin{aligned}&\int u(t)|x|^2\,dx-\int u_0|x|^2\,dx \\ &\le\;\; 2N\int_0^t\int\Big(u^m-\frac{(m-1)\chi}{2}uv\Big)\,dxds \\ &\qquad\qquad\qquad +\chi\int_0^t\int\Big(N(m-1)uv+2u\nabla v\cdot x\Big)\,dxds \\ &\le\;\; 2Nt\int\Big(u_0^m-\frac{(m-1)\chi}{2}u_0v_0\Big)\,dx\;+\;\chi\int_0^t I(s)\,ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t\in(0,T_0),\end{aligned} \tag{6.31}$$

where

$$I(t)\quad:=\quad\int N(m-1)uv(t)+2u\nabla v(t)\cdot x\,dx.$$

We recall the representation for v given in (2.1),

$$v(x,t)=\alpha\int G(x-y)u(y,t)dy \tag{6.32}$$

with the Bessel potential G. Using (6.32), and (2.5) in Lemma 2.2, we have

$$I(t)\quad=\quad\alpha N(m-1)\int\int u(x,t)u(y,t)G(x-y)\,dxdy$$

$$+ \alpha \int\int u(x,t)u(y,t)(x-y)\cdot\nabla G(x-y)\ dxdy$$
$$\leq \alpha\Big(N(m-1)-(N-2)\Big)\int\int u(x,t)u(y,t)G(x-y)\ dxdy.$$

Therefore, by taking $m \leq 2 - \frac{2}{N}$, it holds that

$$I(t) \leq 0 \qquad \text{for a.a. } t \in (0, T_0). \tag{6.33}$$

Combining (6.31) with (6.33), we have

$$\int u(t)|x|^2\ dx - \int u_0|x|^2\ dx$$
$$\leq 2Nt\int\Big(u_0^m - \frac{(m-1)\chi}{2}u_0v_0\Big)\ dx \quad \text{for a.a. } t\in(0,T_0) \tag{6.34}$$

under the assumption $m \leq 2 - \frac{2}{N}$. By the standard continuity argument as from (6.15) to (6.16), we obtain (6.34) on $[0, T_b)$, where T_b is the first blow-up time defined in Definition 3. In consequence, we complete the proof of Lemma 6.2. □

Proof of Theorem 1.3. We are now in a position to prove Theorem 1.3. The proof proceeds by contradiction. Suppose that $T_b = \infty$. Then, from Lemma 6.2, it holds that

$$0 < M(t) := \int u(t)|x|^2\ dx < \infty \qquad \text{for a.a. } t \in (0,\infty). \tag{6.35}$$

On the other hand, from Lemma 6.2,

$$M(t) \leq M(0) + 2Nt\int\Big(u_0^m - \frac{(m-1)\chi}{2}u_0v_0\Big)\ dx$$
$$=: H(t) \qquad \text{for a.a. } t\in(0,\infty). \tag{6.36}$$

By virtue of the assumption (H1) in Theorem 1.3, *i.e.,*

$$k := \int (u_0^m - \frac{(m-1)\chi}{2}u_0v_0)\ dx < 0,$$

$$H^{'}(t) = 2N\cdot k < 0 \qquad \text{for a.a. } t\in(0,\infty). \tag{6.37}$$

Hence, the equation $H(t) = 0$ has a solution at $T_* := -\frac{M(0)}{2Nk}$, which gives that $M(T_*) \leq H(T_*) = 0$. This contradicts (6.35): $M(t) > 0$ for almost all $t \in (0,\infty)$. Thus, we conclude that $T_b < \infty$, *i.e.,* that the weak solution of (KS) obtained from Theorem 1.1 blows up in a finite time. Consequently, we complete the proof of (ii) in Theorem 1.3. As for (i) in Theorem 1.3, we refer to [35]–[37]. □

7. PROOF OF THEOREM 1.4

We are going to construct initial data which satisfy the following conditions in Theorem 1.3:

$$\int u_0^m \, dx \quad < \quad \frac{(m-1)\chi}{2} \int u_0 v_0 \, dx \quad \text{and} \quad u_0|x|^2 \in L^1(\mathbb{R}^N). \qquad (7.1)$$

To this end, we introduce the following function as initial data:

$$u_0(x) \quad = \quad A\Big(1 - \frac{|x|^N}{b^N}\Big)_+, \quad \text{and} \quad \int u_0 \, dx \;=\; \frac{\omega_N}{2N} \cdot Ab^N \;=:\; M, \qquad (7.2)$$

where ω_N is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$. We note that A and b control the maximum and the size of the support of u_0 respectively.

If the following condition is satisfied, then the above u_0 in (7.2) satisfies (7.1):

$$A^{m-1} \le \frac{\alpha\chi \cdot (m-1)}{2} \cdot M \cdot \min_{|x|,|y|\le b} G(x-y). \qquad (7.3)$$

We easily see that (7.3) is equivalent to the following inequality:

$$\Big(\frac{1}{b}\Big)^{N(m-1)} \le \frac{\alpha\chi(m-1)}{2} \cdot \Big(\frac{\omega_N}{2N}\Big)^{m-1} \cdot M^{2-m} \min_{|x|,|y|\le b} G(x-y). \qquad (7.4)$$

Hence, in order to construct the initial data u_0 to satisfy (7.1), we have only to find (b, M) to make (7.4) hold.

To find (b, M), firstly we estimate the Bessel potential $G(x-y)$ in $\{(x,y);\ |x|,|y| \le b\}$ from below.

$$\begin{aligned}
&\min_{|x|,|y|\le b} G(x-y) \\
&= \gamma^{\frac{N}{2}-1} a_N \min_{|x|,|y|\le b} e^{-\sqrt{\gamma}|x-y|} \int_0^\infty e^{-\sqrt{\gamma}|x-y|s} \cdot \Big(s + \frac{s^2}{2}\Big)^{\frac{N-3}{2}} \, ds \\
&\ge \gamma^{\frac{N}{2}-1} a_N e^{-2b\sqrt{\gamma}} \int_0^\infty e^{-2b\sqrt{\gamma}\cdot s} \cdot \Big(s + \frac{s^2}{2}\Big)^{\frac{N-3}{2}} \, ds \\
&= \gamma^{\frac{N}{2}-1} a_N e^{-2b\sqrt{\gamma}} \cdot \frac{1}{2b\sqrt{\gamma}} \int_0^\infty e^{-\xi} \cdot \Big(\frac{\xi}{2b\sqrt{\gamma}} + \frac{\xi^2}{8b^2\gamma}\Big)^{\frac{N-3}{2}} \, d\xi \\
&\ge a_N e^{-2b\sqrt{\gamma}} \cdot \frac{1}{2 \cdot (2\sqrt{2})^{N-3} \cdot b^{N-2}} \int_0^\infty e^{-\xi} \cdot \xi^{N-3} \, d\xi
\end{aligned}$$

$$=: \quad e^{-2b\sqrt{\gamma}} \cdot \left(\frac{1}{b}\right)^{N-2} \cdot C_N,$$

where $C_N := \frac{a_N}{2\cdot(2\sqrt{2})^{N-3}}\Gamma(N-2)$.

On the other hand, when $1 < m \leq 2 - \frac{2}{N}$, let us take M and b so that

$$\left(\frac{1}{b}\right)^{N(m-2+\frac{2}{N})} \quad \leq \quad \frac{\alpha\chi(m-1)}{2} \cdot \left(\frac{\omega_N}{2N}\right)^{m-1} \cdot M^{2-m} e^{-2b\sqrt{\gamma}} \cdot C_N,$$

which yields (7.4). *That is*, there is a constant $C = C(m, N, \alpha, \gamma, \chi)$ such that if M and b satisfy

$$b^{N(2-\frac{2}{N}-m)} e^{2b\sqrt{\gamma}} M^{-(2-m)} \ \leq \ C,$$

then (KS) has a finite time blow-up solution (u, v). Thus, we complete the proof of Theorem 1.4. □

8. Appendix: Proof of Lemma 2.4

In order to prove Lemma 2.4, we prepare the following lemmas:

Lemma 8.1. *Let $N \geq 1,\ a > 2,\ 1 \leq q_1 \leq 2$ and*

$$\begin{cases} 1 \leq q_1 \leq q_2 \leq \infty & \text{when } N = 1, \\ 1 \leq q_1 \leq q_2 < \infty & \text{when } N = 2, \\ 1 \leq q_1 \leq q_2 \leq \frac{2N}{N-2} & \text{when } N \geq 3. \end{cases}$$

Then, for all $u \in W^{1,2}(\mathbb{R}^N)$, it holds that

$$\|u\|_{L^{q_2}(\mathbb{R}^N)} \quad \leq \quad C\|u\|^{1-\theta}_{L^{q_1}(\mathbb{R}^N)} \cdot \|\nabla u\|^{\theta}_{L^2(\mathbb{R}^N)} \tag{8.1}$$

with $\theta = \left(\frac{1}{q_1} - \frac{1}{q_2}\right) \cdot \frac{1}{\frac{1}{N} - \frac{1}{2} + \frac{1}{q_1}}$, where

$$\begin{cases} C \text{ depends only on } N & \text{when } N = 1 \text{ or } N \geq 3, \\ C \text{ depends only on } a & \text{when } N = 2 \text{ and } q_1 \leq q_2 \leq a, \\ C = (c_0 \cdot q_2)^{\frac{1}{2}\cdot\frac{q_2-q_1}{q_2-2}} & \\ \qquad \text{with an absolute constant } c_0 & \text{when } N = 2 \text{ and } q_2 > a. \end{cases}$$

Lemma 8.2. *The above Lemma 8.1 holds true even for $0 < q_1 < 1$ by replacing C in (8.1) by $C^{1/\beta}$ with*

$$\beta = \frac{q_2 - 1}{q_2 - q_1}\left(q_1 + (1 - q_1)\frac{2N}{N+2}\right).$$

Proof of Lemma 8.1. To show Lemma 8.1, we divide the proof into three cases; (I) $N=1$; (II) $N\geq 3$; (III) $N=2$.

(I) Firstly, we consider the case of $N=1$. When $N=1$, by Morrey's theorem, there exists an absolute positive constant C_1 such that

$$\frac{|u(x)-u(y)|}{|x-y|^{\frac{1}{2}}} \;\leq\; C_1\|\nabla u\|_{L^2}.$$

By the above estimate, we have

$$|u(x)| \;\leq\; |u(x)-u(y)|+|u(y)| \;\leq\; C_1|x-y|^{\frac{1}{2}}\|\nabla u\|_{L^2}+|u(y)|. \tag{8.2}$$

From (8.2), for any fixed number $R>0$ and $q_1\geq 1$,

$$\begin{aligned}|u(x)| \;&=\; \frac{\int_{B_R(x)}|u(x)|dy}{|B_R(x)|} \;\leq\; C_1\|\nabla u\|_{L^2}\cdot R^{\frac{1}{2}}+\|u\|_{L^{q_1}(B_R(x))}\frac{|B_R(x)|^{1-\frac{1}{q_1}}}{|B_R(x)|}\\ &\leq\; C_1\|\nabla u\|_{L^2}\cdot R^{\frac{1}{2}}+\|u\|_{L^{q_1}}2^{-\frac{1}{q_1}}\cdot R^{-\frac{1}{q_1}}, \end{aligned}\tag{8.3}$$

where $B_R(x):=\{y\in\mathbb{R};|y-x|<R\}$. It is easy to see that

$$\min_{R>0}\Big(aR^{\alpha}+bR^{-\beta}\Big)=aR^{\alpha}+\frac{a\alpha}{\beta}R^{\alpha}=a\Big(1+\frac{\alpha}{\beta}\Big)\Big(\frac{b\beta}{a\alpha}\Big)^{\frac{\alpha}{\alpha+\beta}} \tag{8.4}$$

for positive constants a,b,α,β. From (8.3), (8.4) with $(\alpha,\beta)=(\frac{1}{2},\frac{1}{q_1})$, $a=C_1\|\nabla u\|_{L^2},\ b=2^{-\frac{1}{q_1}}\|u\|_{L^{q_1}}$, we have

$$|u(x)| \;\leq\; 3q_1^{\frac{2}{q_1+2}}C_1^{\frac{2}{q_1+2}}\|\nabla u\|_{L^2}^{\frac{2}{q_1+2}}\|u\|_{L^{q_1}}^{\frac{q_1}{q_1+2}}.$$

Therefore, for all (q_1,q_2) with $1\leq q_1\leq 2,\ 1\leq q_1\leq q_2\leq\infty$, we obtain

$$\begin{aligned}\|u\|_{L^{q_2}} \;&\leq\; \|u\|_{L^{q_1}}^{\frac{q_1}{q_2}}\|u\|_{L^{\infty}}^{1-\frac{q_1}{q_2}}\leq\|u\|_{L^{q_1}}^{\frac{q_1}{q_2}}\Big(6C_1\|\nabla u\|_{L^2}^{\frac{2}{q_1+2}}\|u\|_{L^{q_1}}^{\frac{q_1}{q_1+2}}\Big)^{1-\frac{q_1}{q_2}} \\ &\leq\; 6C_1\|u\|_{L^{q_1}}^{\frac{q_1}{q_1+2}+\frac{2q_1}{(q_1+2)q_2}}\|\nabla u\|_{L^2}^{\frac{2q_1}{q_1+2}(\frac{1}{q_1}-\frac{1}{q_2})}=6C_1\|u\|_{L^{q_1}}^{1-\theta}\|\nabla u\|_{L^2}^{\theta},\end{aligned}\tag{8.5}$$

where $\theta=\frac{2q_1}{q_1+2}(\frac{1}{q_1}-\frac{1}{q_2})=(\frac{1}{q_1}-\frac{1}{q_2})(1-\frac{1}{2}+\frac{1}{q_1})^{-1}$.

(II) When $N\geq 3$, by Talenti's theorem, we see that there exists a positive constant C_2 depending only on N such that

$$\|u\|_{L^{q_2}} \;\leq\; \|u\|_{L^{q_1}}^{1-\theta}\|u\|_{L^{\frac{2N}{N-2}}}^{\theta} \;\leq\; C_2\|u\|_{L^{q_1}}^{1-\theta}\|\nabla u\|_{L^2}^{\theta} \tag{8.6}$$

with $\theta=(\frac{1}{q_1}-\frac{1}{q_2})\cdot(\frac{1}{N}-\frac{1}{2}+\frac{1}{q_1})^{-1}$ for all $1\leq q_1\leq q_2\leq\frac{2N}{N-2}$.

(III) As for the case of $N = 2$, we use the following inequality by Ozawa [30, Proposition, page 261]: there exists an absolute positive constant c_0 such that

$$\|u\|_{L^{q_2}} \leq (c_0 \cdot q_2)^{\frac{1}{2}} \|u\|_{L^2}^{\frac{2}{q_2}} \|\nabla u\|_{L^2}^{1-\frac{2}{q_2}} \tag{8.7}$$

for any q_2 with $2 \leq q_2 < \infty$.

We assume that $1 \leq q_1 \leq 2,\ 1 \leq q_1 \leq q_2 < \infty$ and divide the proof into two steps. In the first step, we prove (8.1) for $q_2 \geq 2$ and in the second step, we prove (8.1) for $q_1 \leq q_2 \leq a$ with $a > 2$. (The case of $2 \leq q_2 \leq a$ is considered in both cases.)

Firstly, we treat the case of $q_2 \geq 2$. By the Hölder inequality and (8.7),

$$\|u\|_{L^{q_2}} \leq (c_0 \cdot q_2)^{\frac{1}{2}} \left(\|u\|_{L^{q_1}}^{\frac{q_1(q_2-2)}{2(q_2-q_1)}} \|u\|_{L^{q_2}}^{\frac{q_2(2-q_1)}{2(q_2-q_1)}} \right)^{\frac{2}{q_2}} \|\nabla u\|_{L^2}^{1-\frac{2}{q_2}}. \tag{8.8}$$

By (8.8), we have

$$\|u\|_{L^{q_2}} \leq (c_0 \cdot q_2)^{\frac{1}{2} \cdot \frac{q_2-q_1}{q_2-2}} \|u\|_{L^{q_1}}^{\frac{q_1}{q_2}} \|\nabla u\|_{L^2}^{1-\frac{q_1}{q_2}} = (c_0 \cdot q_2)^{\frac{1}{2} \cdot \frac{q_2-q_1}{q_2-2}} \|u\|_{L^{q_1}}^{1-\theta} \|\nabla u\|_{L^2}^{\theta} \tag{8.9}$$

with $\theta = 1 - \frac{q_1}{q_2} = \left(\frac{1}{q_1} - \frac{1}{q_2}\right) q_1 = \left(\frac{1}{q_1} - \frac{1}{q_2}\right)\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{q_1}\right)^{-1}$.

Secondly, we consider the case of $1 \leq q_2 \leq a$. By the Hölder inequality and (8.9) with $q_2 = a (> 2)$,

$$\begin{aligned}
\|u\|_{L^{q_2}} &\leq \|u\|_{L^{q_1}}^{\frac{q_1}{q_2}(\frac{a-q_2}{a-q_1})} \|u\|_{L^a}^{\frac{a}{q_2}(\frac{q_2-q_1}{a-q_1})} \\
&\leq \|u\|_{L^{q_1}}^{\frac{q_1}{q_2}(\frac{a-q_2}{a-q_1})} \left((c_0 \cdot a)^{\frac{1}{2} \cdot \frac{a-q_1}{a-2}} \|u\|_{L^{q_1}}^{\frac{q_1}{a}} \|\nabla u\|_{L^2}^{1-\frac{q_1}{a}} \right)^{\frac{a}{q_2}(\frac{q_2-q_1}{a-q_1})} \\
&= (c_0 \cdot a)^{\frac{1}{2} \cdot \frac{a}{a-2}(1-\frac{q_1}{q_2})} \|u\|_{L^{q_1}}^{\frac{q_1}{q_2}} \|\nabla u\|_{L^2}^{1-\frac{q_1}{q_2}} \\
&\leq (c_0 \cdot a)^{\frac{1}{2} \cdot \frac{a}{a-2}} \|u\|_{L^{q_1}}^{1-\theta} \|\nabla u\|_{L^2}^{\theta} \qquad \text{for } q_1 \leq q_2 \leq a
\end{aligned} \tag{8.10}$$

with $\theta = 1 - \frac{q_1}{q_2} = \left(\frac{1}{q_1} - \frac{1}{q_2}\right) q_1 = \left(\frac{1}{q_1} - \frac{1}{q_2}\right)\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{q_1}\right)^{-1}$.

By collecting (8.5), (8.6), (8.9) and (8.10), the proof of Lemma 8.1 is completed. □

Proof of Lemma 8.2. We prove that Lemma 8.1 holds true even for the case of $0 < q_1 < 1$. To this end, we follow the argument developed in Lemma 3 in Nakao [29]. By the Hölder inequality, it is obtained that

$$\|u\|_{L^1} \leq \|u\|_{L^{q_1}}^{\frac{(q_2-1)q_1}{q_2-q_1}} \|u\|_{L^{q_2}}^{\frac{q_2(1-q_1)}{q_2-q_1}} \qquad \text{for } 0 < q_1 < 1 \leq q_2. \tag{8.11}$$

By Lemma 8.1 with $q_1 = 1$ and (8.11), we see that there exists a positive constant C and $\theta_0 = \left(1 - \frac{1}{q_2}\right)\left(\frac{1}{N} + \frac{1}{2}\right)^{-1}$ such that

$$\begin{aligned} \|u\|_{L^{q_2}} &\leq C\|u\|_{L^1}^{1-\theta_0}\|\nabla u\|_{L^2}^{\theta_0} \\ &\leq C\Big(\|u\|_{L^{q_1}}^{\frac{(q_2-1)q_1}{q_2-q_1}}\|u\|_{L^{q_2}}^{\frac{q_2(1-q_1)}{q_2-q_1}}\Big)^{1-\theta_0}\|\nabla u\|_{L^2}^{\theta_0}. \end{aligned} \tag{8.12}$$

By (8.12), we obtain

$$\|u\|_{L^{q_2}} \leq C^{\frac{1}{\beta}}\|u\|_{L^{q_1}}^{\frac{(q_2-1)q_1}{q_2-q_1}(1-\theta_0)\frac{1}{\beta}}\|\nabla u\|_{L^2}^{\frac{\theta_0}{\beta}} = C^{\frac{1}{\beta}}\|u\|_{L^{q_1}}^{1-\theta}\|\nabla u\|_{L^2}^{\theta},$$

with $\theta = \left(\frac{1}{q_1} - \frac{1}{q_2}\right) \cdot \left(\frac{1}{N} - \frac{1}{2} + \frac{1}{q_1}\right)^{-1}$ and

$$\beta = \frac{q_2 - 1}{q_2 - q_1}\Big(q_1 + (1 - q_1)\frac{2N}{N+2}\Big) \qquad \text{for} \quad 0 < q_1 < 1 \leq q_2.$$

Thus, we complete the proof of Lemma 8.2. □

Proof of Lemma 2.4. We are now in a position to prove Lemma 2.4. By denoting U by $U := u^{\frac{r+m-1}{2}} \in W^{1,2}(\mathbb{R}^N)$ with $r > 0, m \geq 1$,

$$\|u\|_{L^{q_2}} = \Big(\int u^{\frac{r+m-1}{2}\cdot\frac{2q_2}{r+m-1}}\,dx\Big)^{\frac{1}{q_2}} = \|U\|_{L^{\frac{2q_2}{r+m-1}}}^{\frac{2}{r+m-1}}. \tag{8.13}$$

Case (i) Let $1 \leq \frac{2q_1}{r+m-1} \leq 2$ and $1 \leq \frac{2q_2}{r+m-1} \leq a$ and

$$\begin{cases} \frac{r+m-1}{2} \leq q_1 \leq q_2 \leq \infty & \textit{when } N = 1, \\ \frac{r+m-1}{2} \leq q_1 \leq q_2 < \infty & \textit{when } N = 2, \\ \frac{r+m-1}{2} \leq q_1 \leq q_2 \leq \frac{2N}{N-2}\cdot\frac{r+m-1}{2} & \textit{when } N \geq 3. \end{cases}$$

Then, from Lemma 8.1, we have

$$\|U\|_{L^{\frac{2q_2}{r+m-1}}}^{\frac{2}{r+m-1}} \leq \Big(C\|U\|_{L^{\frac{2q_1}{r+m-1}}}^{1-\Theta}\|\nabla U\|_{L^2}^{\Theta}\Big)^{\frac{2}{r+m-1}} \tag{8.14}$$

with $\Theta = \frac{r+m-1}{2} \cdot \left(\frac{1}{q_1} - \frac{1}{q_2}\right) \cdot \left(\frac{1}{N} - \frac{1}{2} + \frac{r+m-1}{2q_1}\right)^{-1}$, where C is a constant depending only on N and a. On the other hand, it is easy to see that

$$\|U\|_{L^{\frac{2q_1}{r+m-1}}}^{\frac{2}{r+m-1}} = \|u\|_{L^{q_1}}. \tag{8.15}$$

Collecting (8.13)–(8.15), we have

$$\|u\|_{L^{q_2}} \leq C^{\frac{2}{r+m-1}}\|u\|_{L^{q_1}}^{1-\Theta}\|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\Theta}{r+m-1}} \tag{8.16}$$

for some constant C depending only on N and a.

Case (ii) Let $0 \leq \frac{2q_1}{r+m-1} \leq 1$ and $1 \leq \frac{2q_2}{r+m-1} \leq a$ and

$$\begin{cases} 1 \leq q_1 \leq q_2 \leq \infty & \textit{when } N = 1, \\ 1 \leq q_1 \leq q_2 < \infty & \textit{when } N = 2, \\ 1 \leq q_1 \leq q_2 \leq \frac{2N}{N-2} \cdot \frac{r+m-1}{2} & \textit{when } N \geq 3. \end{cases} \tag{8.17}$$

From Lemma 8.2, we see that (8.14) holds true even for $0 < \frac{2q_1}{r+m-1} < 1$ by replacing C by $C^{\frac{1}{\beta}}$ with

$$\beta = \frac{q_2 - \frac{r+m-1}{2}}{q_2 - q_1}\Big[\frac{2q_1}{r+m-1} + \Big(1 - \frac{2q_1}{r+m-1}\Big)\frac{2N}{N+2}\Big].$$

Therefore, by an argument similar to that above, we find that (8.16) holds true by replacing C by $C^{\frac{1}{\beta}}$ under the conditions: $0 \leq \frac{2q_1}{r+m-1} \leq 1$ and $1 \leq \frac{2q_2}{r+m-1} \leq a$ and (8). Collecting Case (i) and (ii) above, we prove (8.16) if $q_1 \in [1, r+m-1]$ and $q_2 \in [\frac{r+m-1}{2}, \frac{a(r+m-1)}{2}]$ and q_1, q_2 satisfy (8.17). Thus, we complete the proof of Lemma 2.4. □

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