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POSITIVE SOLUTION OF LAPLACIAN NONCOOPERATIVE SYSTEM WITH POTENTIAL CONTROL

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Abstract. We are concerned with the uniform positivity preserving property on a domain D of $\mathbb{R}^d$ $(d \geq 3)$, for the noncooperative system

$$\begin{cases} -\Delta u = f(.,u) - \mu a v & \text{in } D, \\ -\Delta v = b u & \text{in } D, \\ \lim_{x \to \partial_\infty D} u(x) = \lim_{x \to \partial_\infty D} v(x) = 0, \end{cases} \tag{0.1}$$

where $\partial_\infty D = \begin{cases} \partial D, & \text{if D is bounded,} \\ \partial D \cup \{+\infty\}, & \text{if not.} \end{cases}$ We give appropriate conditions on a, b and f to get the existence and positivity of the solutions with potential control.

1. Introduction

Consider the systems

$$\begin{cases} -\Delta u = f - \mu a v & \text{in } D, \\ -\Delta v = b u & \text{in } D, \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\Delta u = f(.,u) - \mu a v & \text{in } D, \\ -\Delta v = b u & \text{in } D, \end{cases} \tag{1.2}$$

where a,b and f are Borel functions on D, $\mu \in \mathbb{R}^+$, D is $\mathbb{R}^d$ or a $C^{1,1}$-domain in $\mathbb{R}^d$, $(d \geq 3)$ with compact boundary and $f(.,u)$ is a Borel function on $D \times \mathbb{R}$. Solutions to these problems in D are understood in the distributional sense.

In this paper we will establish the existence and uniqueness of the solutions to the system (1.1) in a certain Banach space and we will show using the same arguments in [4] that the solution to the problem is in $L^1_{loc}(D)$ and that the positivity preserving property also holds for every $\mu \geq 0$ when $a \geq 0$ and $b \geq 0$.

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When a and b change sign we obtain the existence of the solution to the system (1.1) for every nonnegative μ under a critical value μ_c.

Concerning the problem (1.2), we are interested in the case where a and b are nonnegative measurable functions and when $f(.,u)$ satisfies a supplementary condition we will prove the existence of the solutions in a certain space of function for which the positivity preserving property is also satisfied.

Finally, when a, b and $f(.,u)$ satisfy a supplementary hypothesis, we will prove the existence and the continuity of the solutions to the system (0.1) for all $0 \leq \mu \leq \mu_c$.

For $D = \mathbb{R}^d$, $(d \geq 3)$ Stavrakakis and Sweers in [19] proved the existence and uniqueness of the solution to the problem (1.1) for all μ under a critical value $\mu_c > 0$ up to where positivity is preserved which does not depend on f, but they considered the particular case when a and b are in a restrictive space. The same is treated by Belhaj Rhouma, Bezzarga and Mosbah in [4] with more general conditions on a, b and μ. For more references related to this subject one can see [11], [12], [17] and [20].

Notation. As usual, we denote by $\mathcal{B}(D)$ the set of Borel measurable functions in D and by $\mathcal{B}^+(D)$ the set of nonnegative ones. $C(D)$ denotes the set of continuous functions in D. $C_c(D)$ will denote the set of continuous functions in D with compact support and $C_0(D)$ is the closure of $C_c(D)$ with respect to the uniform norm.

If $f \in \mathcal{B}(D)$, we denote $f^+ = \max(f,0)$, $f^- = \max(-f,0)$. For any x in D and $r \geq 0$, put $B(x,r) = \{y \in D : |x-y| < r\}$ and denote by $\delta_D(x)$ the Euclidean distance between x and the boundary ∂D of D and by $\beta_D(x) := \delta_D(x)$ if the domain D is bounded, $\beta_D(x) := 1$ if $D = \mathbb{R}^d$ and $\beta_D(x) := \rho_D(x) := \frac{\delta_D(x)}{\delta_D(x)+1}$ if D is unbounded in $\mathbb{R}^d$ and $\overline{D}^c$ is consisting of finitely many disjoint bounded $C^{1,1}$ domains.

For any real Borel function u on D, we put $\lim_{x\to\partial_\infty D} u(x) = 0$ if for any $x_0 \in \partial D$, $\lim_{\substack{x\to x_o \\ x\in D}} u(x) = 0$ and $\lim_{|x|\to+\infty} u(x) = 0$, if moreover D is unbounded.

For every $p \geq 1$, $L^p(D)$ is the set of all Borel functions f on D such that $|f|^p$ is Lebesgue integrable in D and $\|f\|_p = (\int_D |f(x)|^p dx)^{\frac{1}{p}}$, for $f \in L^p(D)$. G_D denotes the classical Green's function of $(-\Delta)$ on D and if $D = \mathbb{R}^d$, we recall that $G_{\mathbb{R}^d}(x,y) = \frac{1}{|x-y|^{d-2}}$.

Throughout this paper, the letter C will denote a generic positive constant which may vary from line to line. Given a Borel function h on D and $x \in D$, we define

$$G^h f(x) = \int_D G_D(x,y)h(y)f(y)dy$$

whenever the integral makes sense. When $h \equiv 1$, we simply denote $G^h f$ by Gf. We recall also the classes of functions $K^d_{loc}(\mathbb{R}^d), K_d(\mathbb{R}^d)$ and $K^\infty_d(\mathbb{R}^d)$ introduced respectively by Aizenmann, Simon in [1] and Zhao in [21] as follows:

Definition 1. *A Borel function f is said to be in the local Kato class $K^d_{loc}(\mathbb{R}^d)$ if and only if, for every $R \geq 0$,*

$$\lim_{r\to 0}(\sup_{|x|\leq R}\int_{B(x,r)}\frac{|f(y)|}{|x-y|^{d-2}}dy) = 0.$$

A Borel function f is said to be in the Kato class $K_d(\mathbb{R}^d)$ if and only if

$$\lim_{r\to 0}(\sup_{x\in\mathbb{R}^d}\int_{B(x,r)}\frac{|f(y)|}{|x-y|^{d-2}}dy) = 0.$$

$$K^\infty_d(\mathbb{R}^d) = \{f \in K^d_{loc}(\mathbb{R}^d) : \lim_{M\to\infty}(\sup_{x\in\mathbb{R}^d}\int_{|y|\geq M}\frac{|f(y)|}{|x-y|^{d-2}}dy) = 0\}.$$

For the remainder D is assumed to be $\mathbb{R}^d$ $(d \geq 3)$, a bounded domain with $C^{1,1}$ boundary or an unbounded domain with compact boundary such that $\overline{D}^c$ is consisting of finitely many disjoints bounded $C^{1,1}$ domains.

2. The Class $S_\infty(X^D)$

It is well known that the Green's function G_D is a symmetric function on $D\times D$ which is positive on $D\times D$, continuous on $D\times D$ except along the diagonal, satisfying for all $y \in D, -\triangle_x G_D(.,y) = \varepsilon_y$ in the distributional sense, where ε_y is the Dirac measure at y and $G_D(.,y)$ vanishes on ∂D.

Lemma 1. (*see* [14], [2] *and* [10]) *There exists a constant $C > 0$ such that for each $x, y, z \in D$*

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leq C\Big(\frac{\beta_D(y)}{\beta_D(x)}G_D(x,y) + \frac{\beta_D(y)}{\beta_D(z)}G_D(y,z)\Big). \qquad (2.1)$$

In the sequel we will recall the class of function $K(D)$ ($S_\infty(X^D)$ respectively) introduced by Maagli and Zribi in [15] and Bachar, Maagli and Zeddini in [2] (Chen and Song in [7] respectively) as follows:

Definition 2. *A Borel measurable function f in D belongs to the Kato class $K(D)$ if f satisfies the following conditions*

$$\lim_{r\to 0}(\sup_{x\in D}\int_{D\cap B(x,r)}\frac{\beta_D(y)}{\beta_D(x)}G_D(x,y)|f(y)|dy) = 0 \qquad (2.2)$$

and

$$\lim_{M\to+\infty}(\sup_{x\in D}\int_{D\cap(|y|\geq M)}\frac{\beta_D(y)}{\beta_D(x)}G_D(x,y)|f(y)|dy)=0. \tag{2.3}$$

Definition 3. *A Borel function f is said to be in the class $S_\infty(X^D)$ if, for every $\varepsilon>0$, there exists a constant $\delta=\delta(\varepsilon)>0$ such that for all measurable sets $B\subset D$ with Lebesgue measure $|B|<\delta$,*

$$\sup_{(x,z)\in D\times D}\int_B\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}|f(y)|dy\leq\varepsilon \tag{2.4}$$

and there is a Borel subset $K=K(\varepsilon)$ of finite Lebesgue measure such that

$$\sup_{(x,z)\in D\times D}\int_{D\setminus K}\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}|f(y)|dy\leq\varepsilon. \tag{2.5}$$

Remark 1. (i) If D is a bounded domain the condition (2.2) ((2.4) respectively) is enough to obtain $f\in K(D)$ ($S_\infty(X^D)$ respectively).

(ii) Due to (2.1), we have $K(D)\subset S_\infty(X^D)$.

(iii) f is in $S_\infty(X^D)$ if and only if f^+ and f^- are also in $S_\infty(X^D)$.

In the sequel, we will recall and prove some properties of functions belonging to the class $S_\infty(X^D)$ which will be needed later.

Proposition 1. *Let $f\in S_\infty(X^D)$. Then* (*see* [7], *Proposition* 4.1)

$$\|f\|_D:=\sup_{x,z\in D}\int_D\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}|f(y)|dy<\infty. \tag{2.6}$$

In addition, if h is a nonnegative superharmonic function in D, then for all $x\in D$ we have

$$\int_D G_D(x,y)|f(y)|h(y)dy\leq\|\varphi\|_Dh(x). \tag{2.7}$$

Moreover, if $x_0\in\overline{D}$, from Proposition 3.1 in [7], *$f\in S_\infty(X^D)$ if and only if for any superharmonic function h in D we have:*

$$\lim_{\varepsilon\to0}(\sup_{x\in D}\frac{1}{h(x)}\int_{D\cap B(x_0,\varepsilon)}G_D(x,y)h(y)|f(y)|dy)=0, \tag{2.8}$$

and

$$\lim_{M\to+\infty}(\sup_{x\in D}\frac{1}{h(x)}\int_{D\cap(|y|\geq M)}G_D(x,y)h(y)|f(y)|dy)=0. \tag{2.9}$$

Proof. The assertion (2.7) follows by integration for potential superharmonic functions $h=Gf$ and then by monotone approximation with potentials for general positive superharmonic functions. $\square$

Remark 2. If $D=\mathbb{R}^d$, then

(i) Due to the last proposition $S_\infty(X^{\mathbb{R}^d})\subset K_d^\infty(\mathbb{R}^d)\cap K^d(\mathbb{R}^d)$.

(ii) Due to (2.1), (2.2), (2.3), (2.8) and (2.9), $f\in S_\infty(X^{\mathbb{R}^d})$ if and only if f is Green-tight on $\mathbb{R}^d$; i.e.,

$$\lim_{r\to 0}(\sup_{x\in\mathbb{R}^d}\int_{B(x,r)}\frac{|f(y)|}{|x-y|^{d-2}}dy)=0,\quad \lim_{M\to+\infty}(\sup_{x\in\mathbb{R}^d}\int_{|y|\geq M}\frac{|f(y)|}{|x-y|^{d-2}}dy)=0,$$

see definition 2.1 in [9].

Corollary 1. *For any $f\in S_\infty(X^D)$, we have*

$$\sup_{x\in D}G|f|(x)<\infty. \tag{2.10}$$

Proof. (2.10) follows from (2.7) with $h\equiv 1$. □

Remark 3. Using a superharmonicity argument, we deduce that for any $f\in B^+(D)$ such that $Gf<+\infty$, we have $Gf\in L^1_{loc}(D)$.

Proposition 2. *Assume that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ or D is an unbounded domain with compact boundary; then for any $f\in S_\infty(X^D)$ we have $Gf\in C_0(D)$. In particular $G|f|\in L^1_{loc}(D)$.*

For the proof see [3] and [6].

Proposition 3. (*See* [13]*, Lemma* 6.24*, page* 121) *Let D be a connected open set having a Green's function G, finite Lebesgue measure, if $d\geq 3$, and bounded if $d=2$. If f is a bounded measurable function on D and $\lim_{x\to x_0\in\partial D}G(x,z_0)=0$ for some $z_0\in D$, then $\lim_{x\to x_0\in\partial D}Gf(x)=0$.*

Proposition 4. (*See* [13]*, Theorem* 6.22) *If D is an open set having a Green's function G and f is a bounded Lebesgue integrable function on D, then Gf is continuous on D.*

Proposition 5. *For any $f\in S_\infty(X^{\mathbb{R}^d})$, we have $Gf\in C(\mathbb{R}^d)$. Moreover, if $f\in L^p(\mathbb{R}^d)$ for $1<p<\frac{d}{2}$, then Gf is in $C_0(\mathbb{R}^d)$.*

Proof. Due to (i) in remark 2 and using the fact that for any function f in $K_d^\infty(\mathbb{R}^d)$, Gf is in $C(\mathbb{R}^d)$ (see [21]), the proof of the first part of the proposition is achieved. The second part is also a consequence of (i) in remark 2 and Theorem 1.4 in [5]. □

Proposition 6. (*see* [16]) *For any $f\in L^p(\mathbb{R}^d)$, $1<p<\frac{d}{2}$, we have $G_{\mathbb{R}^d}f$ in $L^{P^*}(\mathbb{R}^d)$, with $p^*=\frac{dp}{d-2p}$. Moreover, there exists a constant C=C(d,p) such that $\|g\|_{p^*}\leq C\|f\|_p$.*

Example 1. (i) (See Proposition 4.8 in [3]) Let D be a bounded $C^{1,1}$ domain, assume that D contains 0 and consider the function g defined on D by $g(x) = \frac{1}{\delta(x)^\lambda |x|^\mu}$ with $\lambda < 2$ and $\mu < 2$. Then $g \in S_\infty(X^D)$.

(ii) (See Proposition 4.3 in [6]) Let D be an unbounded domain with compact boundary such that $\overline{D}^c$ is consisting of finitely many disjoint bounded $C^{1,1}$ domains and consider the function g defined on D by $g(x) = \frac{1}{|x|^\mu(\rho_D(x))^\lambda}$, then the function g is in $S_\infty(X^D)$ if and only if $\lambda < 2 < \mu$.

(iii) (See Proposition 1 in [22]) If $D = \mathbb{R}^d$, a sufficient condition for a real-valued Borel measurable function f to be in $S_\infty(X^{\mathbb{R}^d})$ is that $f \in K_d^{loc}(\mathbb{R}^d)$ and there is $L > 0$ such that $|f(x)| \leq \frac{\psi(|x|)}{|x|^2}$ for all $|x| \geq L$, where ψ is a positive function defined on the interval $[L, \infty)$ such that $\int_L^\infty \frac{\psi(r)}{r} dr < \infty$.

For the remainder of this paper, let f be in $\mathcal{B}(D)$ with $G|f| < +\infty$, so that $G|f| \in L^1_{loc}(D)$ (to get an example see corollary 1).

3. Existence and uniqueness of solutions to the system (1.1)

Since, for all $y \in D$, $-\triangle_x G_D(.,y) = \varepsilon_y$ in the distributional sense, it follows that for appropriate p (see Proposition 5 and see both propositions 3 and 4), the function $u = Gf$ with $f \in L^p(D)$ solves

$$\begin{cases} -\Delta u & = f \quad \text{on } D, \\ \lim\limits_{x \to \partial_\infty D} u(x) & = 0; \end{cases} \tag{3.1}$$

i.e., $u \in L^1_{loc}(D)$ and satisfies

$$-\int u \triangle \varphi = \int f \varphi, \ \forall \varphi \in C_c^\infty(D).$$

In the sequel, let a and b be two measurable functions. Using the operator G we can replace the system (1.1) by

$$\begin{cases} u = & Gf - \mu G(av), \\ v = & G(bu), \end{cases}$$

which gives $(I + \mu G^a G^b)u = Gf$.

3.1. Nonnegative weights.

Let a_i, b_i $(i = 1, 2)$ be nonnegative measurable functions. In the sequel, we denote

$$c_{1,2} = \sup_{x,t \in D} \frac{1}{G_D(x,t)} \Big[\sum_{1 \leq i,j \leq 2} \iint_{D \times D} G_D(x,y) a_i(y) G_D(y,z) b_j(z) G_D(z,t) dz dy \Big]$$

and

$$\widetilde{K} = \sum_{1\le i,j\le 2} G^{a_i} G^{b_j}. \tag{3.2}$$

Hence,

$$c_{1,2} = \sup_{x,t\in D} \frac{1}{G_D(x,t)} \widetilde{K} G_D(.,t)(x). \tag{3.3}$$

Proposition 7. *If a_i, $b_i \in S_\infty(X^D)$,$1 \le i \le 2$, then $c_{1,2} < +\infty$.*

Proof. As

$$\iint_{y,z\in D} \frac{G_D(x,y)a_i(y)G_D(y,z)b_j(z)G_D(z,t)}{G_D(x,t)} dydz$$
$$= \int_{z\in D} \left(\int_{y\in D} \frac{G_D(x,y)a_i(y)G_D(y,z)}{G_D(x,z)} dy \right) \frac{G_D(x,z)b_j(z)G_D(z,t)}{G_D(x,t)} dz,$$

the proof is achieved by using (2.6). □

Lemma 2. *For each $n \in \mathbb{N}^*$, we have*

$$\widetilde{K}^n(Gf) \le c_{1,2}^n Gf \quad \textit{for every } f \in \mathcal{B}^+(D). \tag{3.4}$$

Proof. From (3.3), we obtain $\widetilde{K}(G_D(.,t)) \le c_{1,2} G_D(.,t)$, for all $t \in D$. Hence, $\widetilde{K}^2(G_D(.,t)) \le c_{1,2}\widetilde{K}(G_D(.,t)) \le c_{1,2}^2 G_D(.,t)$ and inductively, we obtain

$$\widetilde{K}^n(G_D(.,t)) \le c_{1,2}^n G_D(.,t). \tag{3.5}$$

Therefore,

$$\int_D \widetilde{K}^n(G_D(.,t))(x).f(t)dt \le c_{1,2}^n \int_D G_D(x,t)f(t)dt = c_{1,2}^n Gf(x) \tag{3.6}$$

for every $n \in \mathbb{N}^*$ and $f \in \mathcal{B}^+(D)$. Set

$$I_n(x) := \int_D \widetilde{K}^n(G_D(.,t))(x)f(t)dt.$$

Next, we claim that

$$I_n(x) = \widetilde{K}^n(Gf)(x). \tag{3.7}$$

Indeed, for $n = 1$,

$$\begin{aligned} I_1 &= \int_D \widetilde{K} G_D((.,t))(x).f(t)dt \\ &= \sum_{1\le i,j\le 2} \int_{t\in D} \Big[\int_{y\in D} G_D(x,y)a_i(y) \Big(\int_{z\in D} G_D(y,z)b_j(z)G_D(z,t)dz \Big) dy \Big] f(t)dt \\ &= \sum_{1\le i,j\le 2} \int_{y\in D} G_D(x,y)a_i(y) \Big(\int_{z\in D} G_D(y,z)b_j(z) \Big(\int_{t\in D} G_D(z,t)f(t)dt \Big) dz \Big) dy \end{aligned}$$

$$= \sum_{1\le i,j\le 2} \int_{y\in D} G_D(x,y)a_i(y)\Big(\int_{z\in D} G_D(y,z)b_j(z)Gf(z)dz\Big)dy = \widetilde{K}(Gf)(x).$$

Assume that $I_n(x) = \widetilde{K}^n(Gf)(x)$ for all $x \in D$. It follows that

$$\begin{aligned}
I_{n+1} &= \int_{t\in D} \widetilde{K}\left(\widetilde{K}^n G_D(.,t)\right)(x)f(t)dt \\
&= \sum_{1\le i,j\le 2} \int_{t\in D}\Big[\int_{y\in D} G_D(x,y)a_i(y)\Big(\int_{z\in D} G_D(y,z)b_j(z)\widetilde{K}^n G_D(z,t)dz\Big)dy\Big] f(t)dt \\
&= \sum_{1\le i,j\le 2} \int_{y\in D} G_D(x,y)a_i(y)\Big(\int_{z\in D} G_D(y,z)b_j(z)\Big(\int_{t\in D}\widetilde{K}^n(G_D(.,t))(z).f(t)dt\Big)dz\Big)dy \\
&= \sum_{1\le i,j\le 2} \int_{y\in D} G_D(x,y)a_i(y)\Big(\int_{z\in D} G_D(y,z)b_j(z)\widetilde{K}^n(Gf)(z)dz\Big)dy \\
&= \widetilde{K}^{n+1}(Gf)(x).
\end{aligned}$$

Hence, (3.4) follows. □

Denote by E_{s_0} the real ordered Banach space

$$E_{s_0} := \{f : D \to \mathbb{R} \text{ measurable, } |f| \le cs_0 \text{ for some } c > 0\}$$

equipped with the norm $\|f\|_{E_{s_0}} = \inf\{c > 0 : |f| \le cs_0\}$.

For the remainder of this paper, let f_0 be a positive function in $\mathcal{B}(D)$ such that the function $s_0 := Gf_0$ satisfies $1 \le \|s_0\|_\infty < \infty$. Let us observe that an operator L on E_{s_0} is bounded if and only if the operator $M_{s_0}^{-1}LM_{s_0}$ is a bounded operator on E_1. ($M_{s_0} : f \to s_0 f$ is a multiplication operator). Moreover, its norm $\|L\|_{E_{s_0}}$ is the norm of $M_{s_0}^{-1}LM_{s_0}$ on E_1.

In particular

$$\|L\|_{E_{s_0}} = \Big\|\frac{Ls_0}{s_0}\Big\|_{E_1} = \sup_{x\in D}\Big|\frac{Ls_0(x)}{s_0(x)}\Big|.$$

Proposition 8. *Suppose that* $\mu \in [0, \frac{1}{2c_{1,2}}]$, *then the operator* $I + \mu\widetilde{K}$ *is invertible on* E_{s_0}. *Moreover,*

$$0 \le \big(I + \mu\widetilde{K}\big)^{-1} Gf \le Gf \tag{3.8}$$

for every nonnegative function f *satisfying* Gf *in* E_{s_0}.

Proof. Note that $\|\mu\widetilde{K}\|_{E_{s_0}} = \mu\sup_{x\in D}|\frac{\widetilde{K}s_0(x)}{s_0(x)}|$. By (3.4) we have $\mu\widetilde{K}s_0 \le \mu c_{1,2}s_0$. So, $\|\mu\widetilde{K}\|_{E_{s_0}} \le \mu c_{1,2} \le \frac{1}{2}$. Hence, the operator $I + \mu\widetilde{K}$ is invertible on E_{s_0}. Let $f \in \mathcal{B}^+(D)$ such that $Gf \in E_{s_0}$. Using (3.4), we get

$$\begin{aligned}\big(I+\mu\widetilde{K}\big)^{-1}Gf &= Gf + \sum_{n=1}^{+\infty}(-1)^n(\mu\widetilde{K})^n Gf \geq Gf - \sum_{n=1}^{+\infty}(\mu\widetilde{K})^n Gf \\ &\geq Gf - \sum_{n=1}^{+\infty}(\mu c_{1,2})^n Gf = (1-\frac{\mu c_{1,2}}{1-\mu c_{1,2}})Gf \geq 0. \qquad \square\end{aligned}$$

Proposition 9. *Let $\mu \in [0, \frac{1}{2c_{1,2}}]$ and $f \in \mathcal{B}^+(D)$ such that $Gf \in E_{s_0}$. Then, for any $n \in \mathbb{N}$ we have*

$$0 \leq \mu^n((I+\mu\widetilde{K})^{-1}\widetilde{K})^n Gf \leq Gf. \tag{3.9}$$

Proof. We have

$$\begin{aligned}\mu\big(I+\mu\widetilde{K}\big)^{-1}\widetilde{K}Gf &= \big(I+\mu\widetilde{K}\big)^{-1}\big(\big(I+\mu\widetilde{K}\big)-I\big)Gf \\ &= Gf - \big(I+\mu\widetilde{K}\big)^{-1}Gf \leq Gf.\end{aligned}$$

The result follows by using (3.8). $\square$

We similarly obtain the following result:

Proposition 10. *For any $\beta \geq 0$, the operator $I+\beta\widetilde{K}$ is invertible on E_{s_0}. Moreover, for every $f \in \mathcal{B}^+(D)$ such that $Gf \in E_{s_0}$, we have*

$$0 \leq (I+\beta\widetilde{K})^{-1}Gf \leq Gf. \tag{3.10}$$

Proof. Let $\mu \in \mathbb{R}^+$ such that $\mu c_{1,2} \leq \frac{1}{2}$ and let $\beta \in [\mu, \mu(1+\frac{1}{2\|s_0\|_\infty})]$. Using the fact that $\widetilde{K}s_0 = G(\sum_{1\leq i,j\leq 2} a_i G^{b_j})Gf_0$ lies in E_{s_0} and Proposition 8, we get

$$\big(I+\mu\widetilde{K}\big)^{-1}\widetilde{K}s_0 \leq \widetilde{K}s_0 \leq c_{1,2}s_0 \leq \frac{1}{\mu}s_0.$$

So

$$\big\|\big(I+\mu\widetilde{K}\big)^{-1}\widetilde{K}\big\|_{E_{s_0}} \leq \frac{1}{\mu} \tag{3.11}$$

and

$$\big\|(\beta-\mu)\big(I+\mu\widetilde{K}\big)^{-1}\widetilde{K}\big\|_{E_{s_0}} \leq \frac{\beta-\mu}{\mu} \leq \frac{1}{2\,\|s_0\|_\infty} \leq \frac{1}{2}.$$

Therefore, the operator $I+(\beta-\mu)(I+\mu\widetilde{K})^{-1}\widetilde{K}$ is invertible on E_{s_0} and hence $I+\beta\widetilde{K} = (I+\mu\widetilde{K})(I+(\beta-\mu)(I+\mu\widetilde{K})^{-1}\widetilde{K})$ is invertible on E_{s_0}. From (3.9), we get for every $n \in \mathbb{N}^*$

$$\Big((I+\mu\widetilde{K})^{-1}\widetilde{K}\Big)^n Gf \leq \frac{1}{\mu^n}Gf. \tag{3.12}$$

In what follows let $\beta \in [\mu, \mu(1+\frac{1}{2\|s_0\|_\infty})]$. Since $(I+\beta\widetilde{K}) = (I+(\beta-\mu)(I+\mu\widetilde{K})^{-1}\widetilde{K})(I+\mu\widetilde{K})$, we get from Proposition 8 and (3.12):

$$\begin{aligned}(I+\beta\widetilde{K})^{-1}Gf &= (I+\mu\widetilde{K})^{-1}(I+(\beta-\mu)(I+\mu\widetilde{K})^{-1}\widetilde{K})^{-1}Gf\\ &= (I+\mu\widetilde{K})^{-1}\sum_{n=0}^{+\infty}(-1)^n\left((\beta-\mu)^n((I+\mu\widetilde{K})^{-1}\widetilde{K})^nGf\right)\\ &= (I+\mu\widetilde{K})^{-1}\Big(Gf+\sum_{n=1}^{+\infty}(-1)^n(\beta-\mu)^n\left((I+\mu\widetilde{K})^{-1}\widetilde{K}\right)^nGf\Big)\\ &\geq (I+\mu\widetilde{K})^{-1}Gf-(I+\mu\widetilde{K})^{-1}\sum_{n=1}^{+\infty}(\beta-\mu)^n\left((I+\mu\widetilde{K})^{-1}\widetilde{K}\right)^nGf\\ &\geq (I+\mu\widetilde{K})^{-1}Gf-(I+\mu\widetilde{K})^{-1}\sum_{n=1}^{+\infty}(\frac{\beta-\mu}{\mu})^nGf\\ &\geq \left(\frac{1-2(\frac{\beta-\mu}{\mu})}{1-(\frac{\beta-\mu}{\mu})}\right)\left(I+\mu\widetilde{K}\right)^{-1}Gf \geq 0. \qquad \square\end{aligned}$$

Corollary 2. *Let $f_1, f_2 \in \mathcal{B}^+(D)$ such that $f_1 \geq f_2$ and $Gf_1, Gf_2 \in E_{s_0}$, then*

$$(I+\beta\widetilde{K})^{-1}Gf_1 \geq (I+\beta\widetilde{K})^{-1}Gf_2.$$

Proof. It is clear under these conditions that $f_1 - f_2 \in B^+(D)$, moreover $G(f_1-f_2) = Gf_1-Gf_2 \leq Gf_1$, thus $G(f_1-f_2)$ is in E_{s_0} and so by proposition 10 we get: $0 \leq (I+\beta\widetilde{K})^{-1}G(f_1-f_2) = (I+\beta\widetilde{K})^{-1}Gf_1-(I+\beta\widetilde{K})^{-1}Gf_2$. □

Theorem 1. *Let a, b in $\mathcal{B}^+(D)$ be such that*

$$\sup_{x,t\in D}\left[\iint_{D\times D}\frac{G_D(x,y)a(y)G_D(y,z)b(z)G_D(z,t)}{G_D(x,t)}dzdy\right] < +\infty. \qquad (3.13)$$

Then, for every $\mu \geq 0$ and $f \in \mathcal{B}^+(D)$ such that $Gf < \infty$, the system (1.1) *has a solution u. Moreover, $f \geq 0$ implies $0 \leq u \leq Gf$.*

Proof. Take $a_1 = a, b_1 = b$ and $a_2 = b_2 = 0$; we get $\widetilde{K} = K$. Let $f \in \mathcal{B}^+(D)$ such that $Gf < \infty$. For each $n \in \mathbb{N}$, set $f_n := f \wedge nf_0$. It is obvious that $Gf_n \in E_{s_0}$ and that $(f_n)_n$ is increasing and converges to f on D.

By Proposition 10, for each n, there exists $u_n = (I+\mu K)^{-1}Gf_n$, a solution of (1.1) such that $0 \leq u_n \leq Gf_n \leq Gf$. The sequence $(u_n)_n$ is then increasing and converges to a nonnegative function u such that $u \leq Gf$. From the relation $u_n + \mu Ku_n = Gf_n$ we get $u + \mu Ku = Gf$; i.e., u is a solution of (1.1). □

Corollary 3. *Let $a \in \mathcal{B}^+(D)$ such that*

$$\sup_{x,t\in D} \frac{1}{G_D(x,t)} \left[\iint_{D\times D} G_D(x,y)a(y)G_D(y,z)G_D(z,t)dzdy \right] < +\infty.$$

Then, for every $\mu \geq 0$ the biharmonic equation

$$(-\Delta)^2 v + \mu a v = f$$

has a solution which is nonnegative whenever $f \geq 0$.

3.2. **General Case.** Let a and b be two measurable functions satisfying the condition $c_{a,b} < +\infty$, where $c_{a,b} = c_{1,2}$ with $a_1 = a^+, a_2 = a^-$, $b_1 = b^+$, $b_2 = b^-$. Next, we set:

$$K = G^a G^b, \quad K^+ = G^{a^+}G^{b^+} + G^{a^-}G^{b^-}, \quad K^- = G^{a^+}G^{b^-} + G^{a^-}G^{b^+}.$$

Proposition 11. *For every $\mu \in [0, \frac{1}{c_{1,2}}]$, the operator $I + \mu K$ is invertible on E_{s_0} and for every $f \geq 0$ such that $Gf \in E_{s_0}$, we have*

$$0 \leq (I + \mu K)^{-1} Gf \leq \frac{1}{1 - \mu c_{1,2}} Gf. \tag{3.14}$$

Proof. Since $(I + \mu K^+)$ is invertible for every $\mu \in \mathbb{R}^+$, we denote by

$$L := \left(I + \mu K^+\right)^{-1} \mu K^-.$$

So, $I + \mu K = (I + \mu K^+)(I - L)$. Now, we have

$$\left\|\mu K^-\right\|_{E_{s_0}} = \sup_{x\in D} \left| \frac{\mu K^- s_0(x)}{s_0(x)} \right| \leq \mu c_{1,2} < 1.$$

Hence, $I - L$ is invertible on E_{s_0} and the solution u of system (1.1) satisfies

$$u = (I + \mu K)^{-1} Gf = (I - L)^{-1} \left(I + \mu K^+\right)^{-1} Gf$$

which yields that

$$u = (I - L)^{-1} \left(I + \mu K^+\right)^{-1} Gf = \sum_{n=0}^{+\infty} L^n \left(\left(I + \mu K^+\right)^{-1} Gf \right) \geq 0 \text{ if } f \geq 0.$$

On the other hand, using (3.4) and (3.10) we have for every nonnegative function f such that $Gf \in E_{s_0}$

$$(I + \mu K)^{-1} Gf = \sum_{n=0}^{+\infty} L^n \left(\left(I + \mu K^+\right)^{-1} Gf \right) \leq \sum_{n=0}^{+\infty} (\mu c_{1,2})^n Gf$$
$$= \left(\frac{1}{1 - \mu c_{1,2}}\right) Gf.$$

□

Theorem 2. *Let a and b in $\mathcal{B}(D)$ be such that $c_{a,b} < +\infty$. Then, for every $\mu \in [0, \frac{1}{2c_{a,b}}]$ the system* (1.1) *has a solution u which is nonnegative whenever $f \geq 0$. Moreover, there exists a positive constant c such that*

$$0 \leq u \leq cGf. \tag{3.15}$$

Proof. Let $f \in \mathcal{B}^+(D)$ be such that $Gf < \infty$. For each $n \in \mathbb{N}$, set $f_n := f \wedge nf_0$. It is obvious that $Gf_n \in E_{s_0}$ and that $(f_n)_n$ is increasing and converges to f on D. By Proposition 11, for each n, there exists $u_n = (I + \mu K)^{-1} Gf_n$, a solution of (1.1) such that

$$0 \leq u_n \leq \frac{1}{1 - \mu c_{1,2}} Gf_n \leq \frac{1}{1 - \mu c_{1,2}} Gf.$$

The sequence $(u_n)_n$ is then increasing and converges to a function u satisfying

$$0 \leq u \leq \frac{1}{1 - \mu c_{1,2}} Gf.$$

From the relation $u_n + \mu K u_n = Gf_n$ we get $u + \mu K u = Gf$; i.e., u is a solution of (1.1). □

Corollary 4. *Let $a \in \mathcal{B}(D)$ and suppose that $c_{a,1} < +\infty$. Then, for every $\mu \in [0, \frac{1}{2c_{a,1}}]$, the biharmonic equation*

$$(-\Delta)^2 v + \mu a v = f$$

has a solution which is nonnegative whenever $f \geq 0$.

In the sequel, let

$$\mathrm{S} = \{s \in \mathcal{B}(D) : s = Gf \text{ for some } f \in \mathcal{B}^+(D) \text{ such that } 1 \leq \|Gf\|_\infty < \infty\}$$

and let $\mathrm{E} = \bigcup_{s \in \mathrm{S}} E_s$. It is obvious that the family $(E_s)_{s \in \mathrm{S}}$ is upper directed and if s, $s' \in \mathrm{E}$, we have E_s continuously embedded in $E_{s+s'}$. Hence, the following result holds.

Proposition 12. *The system* (1.1) *has a unique solution $u \in \mathrm{E}$ whenever it exists.*

4. Existence of the solutions to the system (1.2)

In this section, we are interested in proving under which conditions we get the existence of the solutions to the system (1.2), where $D = \mathbb{R}^d$ ($d \geq 3$), a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ or unbounded such that $\overline{D}^c$ is consisting of finitely many disjoint $C^{1,1}$ domains, f is assumed to be a measurable nonnegative function in $D \times \mathbb{R}$ and a, b are two measurable nonnegative functions. Solutions to this problem are understood as in the distributional sense in D. Using the operator G we can replace the system (1.2) by

$$\begin{cases} u = & Gf(.,u) - \mu G(av), \\ v = & G(bu), \end{cases}$$

which gives $u = Gf(.,u) - \mu G^a G^b u$, or equally

$$u = PTu, \tag{4.1}$$

where $P := (I + \mu G^a G^b)^{-1}$ and $Tu := Gf(.,u)$.

Consider now the closed subset H of E_{s_0}, given by $H = \{u \in \mathcal{B}(D) : 0 \leq u \leq s_0\}$. We get the following lemmas:

Lemma 3. *Assume that $c_{1,2} < +\infty$, $\mu \in [0, \frac{1}{2c_{1,2}}]$, $\|s_0\|_\infty < +\infty$ and $|f(x,u(x))| \leq f_0(x)$, for all $u \in H, x \in D$, then $PTH \subset H$.*

Proof. Put $u \in H$, then $0 \leq Tu \leq s_0$, consequently using Proposition 8, we get $0 \leq PTu \leq Ps_0 \leq s_0$, which ends the proof. □

Lemma 4. *Assume that $a, b \in \mathcal{B}^+(D)$, $c_{1,2} < +\infty$ and $\|s_0\|_\infty < +\infty$. Let $(u_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$, then one has pointwise: if $\lim_{n\to\infty} u_n = u$, then $\lim_{n\to\infty} \tilde{K}u_n = \tilde{K}u$.*

Proof. Using Tonelli's theorem we can write:

$$\begin{aligned} \tilde{K}u_n(x) = Ku_n(x) &= G^a G^b(u_n)(x) \\ &= \int_{z\in D}\int_{y\in D} G_D(x,y)a(z)G_D(z,y)u_n(y)b(y)dydz \\ &\leq \int_{z\in D}\int_{y\in D} G_D(x,y)a(z)G_D(z,y)Gf_0(y)b(y)dydz \\ &\leq \int_{z\in D}\int_{y\in D}\int_{t\in D} G_D(x,y)a(z)G_D(z,y)G_D(y,t)b(y)dydzf_0(t)dt. \end{aligned}$$

Now put $h(y,z) = G_D(x,y)a(z)G_D(z,y)G_D(y,t)b(y)$, since $c_{1,2} < +\infty$, then there exists $C > 0$ such that

$$\int_{z\in D}\int_{y\in D} h(y,z)dydz < CG_D(x,t)$$

and consequently,

$$\int_{z\in D}\int_{y\in D} G_D(x,y)a(z)G_D(z,y)u_n(y)b(y)dydz \leq Cs_0(x).$$

The proof is then achieved using Lebesgue's dominated convergence theorem. □

Lemma 5. *Assume that $c_{1,2} < +\infty$, $1 \leq \|s_0\|_\infty < +\infty$ and $f(x, u(x)) \leq f_0(x)$, for all $u \in H, x \in D$, then for every $\mu \geq 0$, we get $PTH \subset H$.*

Proof. Put $u \in H$, then $0 \leq Tu \leq s_0$, consequently, using Proposition 10 and corollary 2, we get $0 \leq PTu \leq Ps_0 \leq s_0$. □

Our existence results are the following:

Theorem 3. *Assume the conditions of Lemmas* 3 *and* 4 *hold. Moreover, we assume that the mapping $t \to f(x, t)$ is increasing and continuous, then the system* (1.2) *has a solution u in H.*

Proof. Put $(u_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ such that $u_0 = 0$ and $u_{n+1} = PTu_n \in H \subset E_{s_0}$. Since f is increasing with respect to the second variable, then $Tu_n = Gf(., u_n) \in E_{s_0}$ for all $n \in \mathbb{N}$ is increasing, and using Proposition 8, we get $PTu_n \leq PTu_{n+1}$, for all $n \in \mathbb{N}$ and so $(u_n)_{n\in\mathbb{N}}$ is increasing in H for all $n \in \mathbb{N}$; moreover, due to the same proposition, u_n is dominated by s_0 for all $n \in \mathbb{N}$, then there exists $u \in H$ such that $\lim_{n\to+\infty} u_n = u$ in H. On the other hand, using the continuity of f with respect to the second variable and Lebesgue's dominated convergence theorem, we found that $\lim_{n\to+\infty} Tu_n = Tu$. Finally using (4.1) and lemma 4, we conclude that $u = PTu$, which ends the proof.

Theorem 4. *Assume that a, b and f_0 are nonnegative functions in $S_\infty(X^D)$ ($S_\infty(X^{\mathbb{R}^d}) \cap L^p(\mathbb{R}^d)$, $1 < p < \frac{d}{2}$ respectively) and $\mu \in [0, \frac{1}{2c_{1,2}}]$. Moreover, we suppose that $f(x, u(x)) \leq f_0(x)$, for all $u \in H, x \in D$ and the mapping $t \to f(x, t)$ is increasing and continuous; then the system* (1.2) *has a solution u in H, moreover $u \in C_0(D)$.*

Proof. It is clear under the hypothesis on a, b and f_0 that the conditions of lemmas 3 and 4 hold which proves the existence of the solution to the system (1.2) in H, thus

$$u = (I + \mu K)^{-1} Gf(., u) = \sum_{n=0}^{\infty} (-1)^n (\mu K)^n Gf(., u).$$

Now, we notice that under the condition $\|s_0\|_\infty < +\infty$, we deduce that $\|.\|_\infty \leq \|.\|_{E_{s_0}}$, which implies that the series

$$\sum_{n=0}^{\infty} (-1)^n (\mu K)^n Gf(., u)$$

converges normally and consequently uniformly on D. Besides we have $K^n(Gf(., u))$ is in $C_0(D)$ in fact, for $n = 0$, using that $f(x, u(x)) \leq f_0(x)$, for all $u \in H, x \in D$ and Proposition 2 (Proposition 5 respectively), we conclude that $K^n(Gf(.,u)) = Gf(.,u) \in C_0(D)$. Now suppose that $K^n(Gf(.,u)) \in C_0(D)$ and prove that $K^{n+1}(Gf(.,u)) \in C_0(D)$,

$$K^{n+1}(Gf(.,u)) = K(K^n(Gf(.,u))) = G^aG^bK^nGf(.,u) = G(aG^bK^nGf(.,u)).$$

Since $K^n(Gf(.,u)) \le c_{1,2}^n Gf(.,u) \le c_{1,2}^n Gf_0$ and $Gf_0 \in C_0(D)$, then there exists $C > 0$ such that

$$aG^bK^nGf(.,u) \le CaG^bGf_0 = CaG(bGf_0) \le CaGb.$$

Finally, using the fact that a and b are in $S_\infty(X^D)$ $(S_\infty(X^{\mathbb{R}^d}) \cap L^p(\mathbb{R}^d)$, $1 < p < \frac{d}{2}$ respectively) we can conclude that $aG^bK^nGf(.,u) \le Ca \in S_\infty(X^D)$ $(S_\infty(X^{\mathbb{R}^d}) \cap L^p(\mathbb{R}^d), 1 < p < \frac{d}{2}$ respectively) and so $aG^bK^nGf(.,u) \in S_\infty(X^D)$ $(S_\infty(X^{\mathbb{R}^d}) \cap L^p(\mathbb{R}^d), 1 < p < \frac{d}{2}$ respectively) which implies that $K^{n+1}(Gf(.,u)) = G(aG^bK^nGf(.,u))$ is in $C_0(D)$. □

Corollary 5. *Assume that a, b and f_0 are nonnegative functions in $S_\infty(X^D)$ $(S_\infty(X^{\mathbb{R}^d}) \cap L^p(\mathbb{R}^d), 1 < p < \frac{d}{2}$ respectively), $\mu \in [0, \frac{1}{2c_{1,2}}]$. Moreover, we suppose that $f(x, u(x)) \le f_0(x)$, for all $u \in H, x \in D$ and the mapping $t \to f(x,t)$ is increasing and continuous, then the system* (0.1) *has a solution u in H.*

Proof. It is clear that $0 \le u \le Gf_0$. Since f_0 is in $S_\infty(X^D)$ $(S_\infty(X^{\mathbb{R}^d}) \cap L^p(\mathbb{R}^d), 1 < p < \frac{d}{2}$ respectively), then $\lim_{x\to\partial_\infty D} u(x) = 0$. Consequently, using the fact that u is bounded, $0 \le v = G(bu)$ and Proposition 2 (Proposition 5 respectively) we conclude that $\lim_{x\to\partial_\infty D} v(x) = 0$. □

Theorem 5. *Assume that the conditions of Lemmas* 4 *and* 5 *hold, moreover we assume that the mapping $t \to f(x,t)$ is increasing and continuous; then for all $\mu \ge 0$ the system* (1.2) *has a solution u in H.*

Proof. As in the proof of Theorem 3, let $(u)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ such that $u_0 = 0$ and $u_{n+1} = PTu_n \in H \subset E_{s_0}$. Using the same steps of the proof of Theorem 3, Proposition 10 and corollary 2, we conclude that there exists u in H such that $u = PTu$, which ends the proof. □

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