Differential and Integral Equations Volume 19, Number 10 (2006), 1081–1102

NON-UNIFORM DEPENDENCE ON INITIAL DATA FOR A FAMILY OF NON-LINEAR EVOLUTION EQUATIONS

Erika A. Olson
Department of Mathematics, University of Notre Dame
Notre Dame, IN 46556

(Submitted by: Gustavo Ponce)

Abstract. We show that solutions to the periodic Cauchy problem for a family of non-linear evolution equations, which contains the Camassa-Holm equation, do not depend uniformly continuously on initial data in the Sobolev space $H^s(\mathbb{T})$, when $s = 1$ or $s \geq 2$.

1. Introduction

For nonzero γ, we consider the following periodic Cauchy problem

$$u_t - u_{txx} + 3uu_x = \gamma(2u_xu_{xx} + uu_{xxx}),\ x \in \mathbb{T}, \quad t \in \mathbb{R}, \tag{1.1}$$

$$u(x,0) = u_0(x),\ \ u_0 \in H^s(\mathbb{T}). \tag{1.2}$$

Equation (1.1) is referred to as the hyperelastic rod equation. It was first derived by Dai [11] and subsequently studied by Constantin and Strauss [10], Dai [12], Dai and Huo [13], Holden and Raynaud [19], Lenells [21], and Zhou [25]. The periodic initial-value problem for equation (1.1) is locally well posed in $H^s(\mathbb{T})$ for $s > 3/2$ with the solutions depending continuously on the initial data (see [25]). The problem is well posed for initial data in $H^1(\mathbb{R})$ if one includes a Radon measure that corresponds to the energy of the system with the initial data (see [19]).

When $\gamma = 1$, equation (1.1) is an integrable equation commonly called the Camassa-Holm (CH) equation. The CH equation was discovered independently by Fokas and Fuchssteiner [16] and by Camassa and Holm [5]. In Himonas and Misiołek [18], in Misiołek [24], and in Danchin [14], it was shown that the periodic initial-value problem for the CH equation is locally well posed in $H^s(\mathbb{T})$ for $s > 3/2$ with the solutions depending continuously on the initial data; De Lellis, Kappeler, and Topalov [15] have shown local well posedness in $H^s(\mathbb{T}) \cap \mathrm{Lip}(\mathbb{T})$ for $1 \leq s \leq 3/2$, also with the solutions depending continuously on the initial data. (For results in different directions

Accepted for publication: July 2006.
AMS Subject Classifications: 35Q53; 35B65.

including in the non-periodic case we refer the reader to Constantin and Escher [7], McKean [23], Beals, Sattinger and Szmigielski [2], Constantin and Strauss [9], Li and Olver [22], Alber, Camassa, Holm, and Marsden [1], and the references therein.)

Himonas and Misiołek [17] have shown, using appropriate sequences of smooth traveling wave solutions, that dependence on initial data for the periodic initial-value problem for the Camassa-Holm equation cannot be better than continuous. More precisely, for $s \geq 2$ they have shown that the solution map $u_0 \to u$ is not uniformly continuous from any bounded set in $H^s(\mathbb{T})$ into $C([0,T], H^s(\mathbb{T}))$. Using sequences of periodic cuspons, Byers [4] proved this result for $s = 1$. In this work, we arrive at the same results for the initial-value problem (1.1)-(1.2) using constructions analogous to those in [17] and [4]. For initial data in $H^s(\mathbb{T}), s \geq 2$, we require $\gamma \neq 3$. Specifically, we show

Theorem 1.1. *Let γ be a nonzero real number.*

(i) *If $s = 1$ and $T > 0$, then there exist two sequences of non-smooth (cuspon) solutions u_n and v_n in $H^s(-\pi, \pi)$ of the Cauchy problem (1.1)–(1.2) and a positive constant c such that, for any $t \in (0, T]$,*

$$\sup_n ||u_n(\cdot, t)||_{H^s} + \sup_n ||v_n(\cdot, t)||_{H^s} \leq c, \tag{1.3}$$

$$\lim_{n\to\infty} ||v_n(\cdot, 0) - u_n(\cdot, 0)||_{H^s} = 0, \tag{1.4}$$

$$\liminf_n ||v_n(\cdot, T) - u_n(\cdot, T)||_{H^s} > 0. \tag{1.5}$$

(ii) *Assume $\gamma \neq 3$. If $s \geq 2$ and $T \in (0, 1]$, then there exist two sequences of smooth solutions u_n and v_n in $H^s(-\pi, \pi)$ of the Cauchy problem (1.1)–(1.2) satisfying (1.3)–(1.5).*

We note that $\gamma = 1$ is the only choice of γ for which equation (1.1) is known to be integrable. We also note that Theorem 1.1 provides a contrast between equation (1.1) and the KdV equation. While for equation (1.1) dependence on initial data is not uniform, for the KdV equation this dependence is better than uniform. More precisely, the solution map of the KdV equation is Lipschitz continuous on bounded sets of $H^s(\mathbb{T})$ for any $s \geq 1/2$ (see Bourgain [3], Kenig, Ponce and Vega [20] and Colliander, Keel, Staffilani, Takaoka, and Tao [6]).

Finally, we remark that for given k_1 and k_2 nonzero real numbers, the equation

$$v_t - v_{txx} + 3k_1 v v_x = k_2(2v_x v_{xx} + v v_{xxx}), \ x \in \mathbb{T}, \quad t \in \mathbb{R} \tag{1.6}$$

may be transformed into equation (1.1) by setting $\gamma = \frac{k_2}{k_1}$ and $u(x,t) = k_1 v(x,t)$.

In the next section we derive the differential equation of the unit speed traveling wave solutions of (1.1). In Section 3, we choose the parameters of this equation appropriately to construct non-smooth periodic solutions (cuspons) and show that the period and the L^2-norm of the derivative of each such solution can be estimated by its amplitude. Using two sequences of these solutions, we prove Theorem 1.1(i) in Section 4. In Section 5, with a different choice of the parameters of the equation derived in Section 2, we construct smooth traveling wave solutions and estimate the period of these solutions. In Section 6, we prove the H^s-estimates needed for the proof of Theorem 1.1(ii), which is given in the final section.

2. Construction of traveling wave solutions

Scaling. Notice that equation (1.1) has the following scaling property: if $u(x,t)$ is a solution of equation (1.1), then, for any constant c, so is the function

$$u_c(x,t) = cu(x,ct). \tag{2.1}$$

ODE of traveling wave solutions. Let $u(x,t) = \varphi(x-t)$ be a unit speed traveling wave solution of (1.1). Then the function φ must satisfy

$$-\varphi' + \varphi''' + 3\varphi\varphi' = \gamma(2\varphi'\varphi'' + \varphi\varphi''').$$

So, since $\frac{d}{dx}(\varphi\varphi'') = \varphi'\varphi'' + \varphi\varphi'''$, integrating this equation yields

$$-\varphi + \varphi'' + \frac{3}{2}\varphi^2 = \gamma\left(\frac{1}{2}(\varphi')^2 + \varphi\varphi''\right) + \frac{M_1}{2} \tag{2.2}$$

which for smooth φ is equivalent to

$$\frac{\gamma}{2}(\varphi')^2 - \varphi + \frac{3}{2}\varphi^2 = \left(\frac{\gamma}{2}\varphi^2 - \varphi\right)'' + \frac{M_1}{2}. \tag{2.3}$$

Note that for equation (2.3) to make sense for distributions it is sufficient that $\varphi \in H^1(\mathbb{T})$. We will use (2.3) as our weak formulation for unit speed traveling wave solutions of (1.1). In Section 5 we construct classical smooth solutions of (1.1), while in Section 3 we construct distribution solutions of (2.3).

Multiplying (2.2) by $2\varphi'$ yields

$$-2\varphi\varphi' + 2\varphi''\varphi' + 3\varphi^2\varphi' = \gamma\left((\varphi')^3 + 2\varphi\varphi''\varphi'\right) + M_1\varphi',$$

from which, since $\frac{d}{dx}(\varphi(\varphi')^2) = (\varphi')^3 + 2\varphi\varphi'\varphi''$, integration yields

$$-\varphi^2 + (\varphi')^2 + \varphi^3 = \gamma\varphi(\varphi')^2 + M_1\varphi + L_1,$$

which we can rewrite as

$$(\gamma\varphi - 1)\,(\varphi')^2 = \varphi^3 - \varphi^2 + M\varphi + L. \tag{2.4}$$

Using the identity $(\varphi-a)(\varphi-b)(\varphi-c) = \varphi^3-(a+b+c)\varphi^2+(ab+ac+bc)\varphi-abc$, we find that (2.4) holds for certain constants M and L if

$$(\gamma\varphi - 1)(\varphi')^2 = (\varphi - a)(\varphi - b)(\varphi - c) \tag{2.5}$$

for constants a,b, and c satisfying

$$a + b + c = 1. \tag{2.6}$$

On the largest open set on which φ satisfies (2.5)–(2.6), is smooth, and has nonzero derivative, φ also satisfies (2.3). It follows that if $\varphi \in H^1(\mathbb{T})$ and if, except on a discrete set, φ satisfies (2.5)–(2.6), is smooth, and has nonzero derivative, then φ satisfies our notion of weak solution of (1.1). The periodic cuspon solutions we construct in Section 3 will be of this type. In Section 5, we construct smooth solutions of (2.5), which are, in turn, smooth solutions of (1.1).

To prove Theorem 1.1, we use a sequence of the constructed solutions and a second sequence of solutions obtained by scaling the first sequence by a factor which goes to one as n gets large. Thus the two sequences will be close at time zero. We then show that the sequences are no longer close at time $T > 0$. In the case of the cuspons, this is due to the fact that the scaling factor is chosen so that at time $T > 0$ the sign of the derivative of the function in the first sequence is the opposite of the sign of the derivative of the corresponding function in the second sequence. In the case of the smooth solutions, we will need to take extra care when constructing the first sequence of solutions to ensure that the two sequences are apart at time $T > 0$.

3. Periodic cuspons

To construct periodic cuspons solutions, we will choose a, b, and c in (2.5) so that $|\varphi'| \to \infty$ as $\varphi \to \frac{1}{\gamma}$.

For $\gamma > 0$, we begin with the stipulation that

$$\tfrac{1}{\gamma}(1 - \epsilon) < \varphi < \tfrac{1}{\gamma} \tag{3.1}$$

for some $\epsilon > 0$ and choose $a = \frac{1}{\gamma}(1+\beta)$ and $b = \frac{1}{\gamma}(1-\epsilon)$ for some $\beta > 0$. Then, by (2.6), we must choose $c = \frac{1}{\gamma}(\epsilon - 2 - \beta + \gamma)$. For convenience, let

$$y = 1 - \gamma\varphi. \tag{3.2}$$

Note that (3.1) then says that $0 < y < \epsilon$. With the above choices for a, b, and c, equation (2.5) can be written as

$$(y')^2 = \frac{\frac{1}{\gamma}(y+\beta)(\epsilon - y)(3 - \gamma + \beta - \epsilon - y)}{y}. \tag{3.3}$$

We want to choose $\epsilon > 0$ and $\beta > 0$ to ensure that the right-hand side of (3.3) is positive for $y \in (0, \epsilon)$. It is easy to check that if we fix any $\beta > \max\{0, \gamma - 3\}$ and $\epsilon_0 \in (0, \frac{1}{2}(\beta + 3 - \gamma))$, then for any $\epsilon \in (0, \epsilon_0)$ the right-hand side of (3.3) will be positive for $y \in (0, \epsilon)$.

Remark. If $\gamma < 0$, we begin with, instead of (3.1), the stipulation that $\frac{1}{\gamma}(1+\epsilon) < \varphi < \frac{1}{\gamma}$ and choose $a = \frac{1}{\gamma}(1-\beta)$ and $b = \frac{1}{\gamma}(1+\epsilon)$. Letting $y = \gamma\varphi - 1$, we can then proceed in an almost identical manner as we do below for the case $\gamma > 0$.

Taking $y' > 0$, the initial-value problem for (3.3) with initial data $y(x_0) = y_0$, where $x_0 \in \mathbb{R}$ and $y_0 \in (0, \epsilon)$ will have an increasing local solution with

$$y' = \sqrt{\frac{\frac{1}{\gamma}(y+\beta)(\epsilon - y)(3 - \gamma + \beta - \epsilon - y)}{y}}. \tag{3.4}$$

Notice that since $y' > 0$ and $y' \to \infty$ as $y \to 0^+$, there must be an $\tilde{x_0} < x_0$ such that $y(\tilde{x_0}) = 0$. Without loss of generality, we can assume $y(0) = 0$. (Define $\tilde{y}$ by $\tilde{y}(x) = y(x + \tilde{x_0})$. Then $\tilde{y}$ solves (3.3) and $\tilde{y}(0) = 0$.)

Since y is increasing (and differentiable) when $y \in (0, \epsilon)$, it is invertible and its inverse $x = x(y)$ is defined and differentiable in the interval $(0, \epsilon)$ with

$$x'(y) = \frac{1}{y'(x)} = \sqrt{\frac{y}{\frac{1}{\gamma}(y+\beta)(\epsilon - y)(3 - \gamma + \beta - \epsilon - y)}}.$$

So, for $y \in (0, \epsilon)$,

$$\begin{aligned} x(y) &= \int_0^y \sqrt{\frac{\tau}{\frac{1}{\gamma}(\tau+\beta)(\epsilon-\tau)(3-\gamma+\beta-\epsilon-\tau)}}\,d\tau \\ &\le \sqrt{\frac{1}{\frac{1}{\gamma}\beta(3-\gamma+\beta-2\epsilon_0)}} \int_0^y \frac{\sqrt{\epsilon}}{\sqrt{\epsilon-\tau}}\,d\tau = c\left(-2\sqrt{\epsilon}\sqrt{\epsilon-\tau}\right)\Big|_0^y. \end{aligned}$$

Thus,

$$x(y) \le c \cdot \epsilon \quad \text{for all} \quad y \in (0, \epsilon), \tag{3.5}$$

where c depends only on ϵ_0. Let

$$\ell := \lim_{y \to \epsilon^-} x(y). \tag{3.6}$$

By the estimate (3.5), ℓ is finite. We now have a function $y : [0, \ell] \to [0, \epsilon]$ that satisfies (3.3) on $(0, \ell)$. Extend this to $y : [-\ell, \ell] \to [0, \epsilon]$ by defining $y(-x) = y(x)$. Then extend the domain of y to all of $\mathbb{R}$ by defining $y(x+2n\ell) = y(x)$ for every nonzero integer n and every $x \in [-\ell, \ell]$. Note that y is 2ℓ-periodic; y is C^1 at $x = \ell$, with $y'(\ell) = 0$; and $\lim_{x\to 0} |y'(x)| = \infty$.

In the next section, we will show that $\varphi(x) = \frac{1}{\gamma}(1 - y(x))$ is in $H^1(\mathbb{T})$. Then, fixing β so that we can choose $\epsilon_0 = 1$, we have the following lemma.

Lemma 3.1. *Fix $\beta > \max\{0, \gamma - 1\}$. Then for every $\epsilon \in (0, 1)$ there exists a positive number $\ell = \ell(\epsilon)$ and an even, 2ℓ-periodic continuous function $y = y(x)$ which, on the set $\mathbb{R} - \{n\ell : n \in \mathbb{Z}\}$, is smooth and satisfies (3.3). The function y (see Figure 1) satisfies the bounds*

$$0 \le y(x) \le \epsilon < 1, \tag{3.7}$$

its derivative satisfies

$$y' > 0 \quad \text{on} \quad (0, \ell) \quad \text{and} \quad y' < 0 \quad \text{on} \quad (\ell, 2\ell), \tag{3.8}$$

and the function $u(x,t) = \varphi(x - t)$, where $\varphi(x) = \frac{1}{\gamma}(1 - y(x))$, is a solution of (1.1) *in the weak sense.*

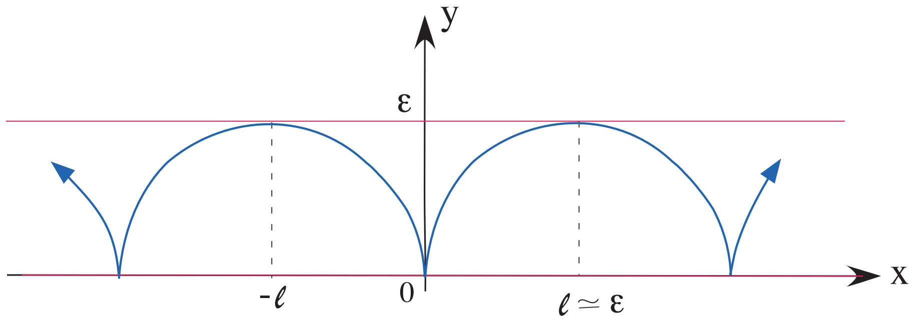

FIGURE 1

In the next lemma we show that the period of each solution constructed above can be estimated by the corresponding value of ϵ.

Lemma 3.2. *The period 2ℓ of the solution described above satisfies*

$$\ell \simeq \epsilon. \tag{3.9}$$

Proof. By (3.6) and (3.5), it remains only to show that $\ell \gtrsim \epsilon$. Estimating we have

$$\begin{aligned} \ell &= \int_0^\epsilon \sqrt{\frac{\tau}{\frac{1}{\gamma}(\tau+\beta)(\epsilon-\tau)(3-\gamma+\beta-\epsilon-\tau)}}d\tau \\ &\geq \sqrt{\frac{1}{\frac{1}{\gamma}(1+\beta)(3-\gamma+\beta)}}\int_0^\epsilon \frac{\sqrt{\tau}}{\sqrt{\epsilon}}d\tau = c\frac{2\tau^{3/2}}{3\sqrt{\epsilon}}\bigg|_0^\epsilon \simeq \epsilon, \end{aligned}$$

as desired. □

The function $\ell = \ell(\epsilon)$ is continuous and increasing on the interval $(0,1)$ and $\lim_{\epsilon\to 0^+}\ell(\epsilon) = 0$. Thus, the image of $(0,1)$ under the map ℓ is an interval $(0,\ell_1)$, for some $\ell_1 > 0$. So, for any sufficiently large n (i.e., $n > \pi/\ell_1$), there is a corresponding $\epsilon \in (0,1)$ such that

$$\ell = \ell(\epsilon) = \frac{\pi}{n}. \tag{3.10}$$

To obtain the desired Sobolev estimates on the solutions constructed above, we will use the following L^2-estimate.

Proposition 3.3. *The 2ℓ-periodic function y constructed above satisfies*

$$||y'||^2_{L^2(-\ell,\ell)} \simeq \epsilon. \tag{3.11}$$

Proof. Estimating from above,

$$\begin{aligned} ||y'||^2_{L^2(-\ell,\ell)} &= 2\int_0^\ell y'(x)\cdot y'(x)dx = 2\int_0^\epsilon y'dy \\ &= 2\int_0^\epsilon \sqrt{\frac{\frac{1}{\gamma}(y+\beta)(\epsilon-y)(3-\gamma+\beta-\epsilon-y)}{y}}dy \\ &\leq 2\sqrt{\frac{1}{\gamma}(1+\beta)(3-\gamma+\beta)}\sqrt{\epsilon}\int_0^\epsilon \frac{1}{\sqrt{y}}dy \\ &= 4\sqrt{\frac{1}{\gamma}(1+\beta)(3-\gamma+\beta)}\cdot\epsilon. \end{aligned}$$

Estimating from below,

$$||y'||^2_{L^2(-\ell,\ell)} = 2\int_0^\epsilon \sqrt{\frac{\frac{1}{\gamma}(y+\beta)(\epsilon-y)(3-\gamma+\beta-\epsilon-y)}{y}}dy$$

$$\geq \quad 2\sqrt{\frac{\beta}{\gamma}(3-\gamma+\beta-2)}\int_0^\epsilon \frac{\sqrt{\epsilon-y}}{\sqrt{y}}dy,$$

so the desired estimate follows since

$$\begin{aligned}\int_0^\epsilon \frac{\sqrt{\epsilon-y}}{\sqrt{y}}dy &= \int_0^{\epsilon/2} \frac{\sqrt{\epsilon-y}}{\sqrt{y}}dy + \int_{\epsilon/2}^\epsilon \frac{\sqrt{\epsilon-y}}{\sqrt{y}}dy \\ &\geq \sqrt{\frac{\epsilon}{2}}\int_0^{\epsilon/2} \frac{1}{\sqrt{y}}dy + \frac{1}{\sqrt{\epsilon}}\int_{\epsilon/2}^\epsilon \sqrt{\epsilon-y}dy \\ &= 2\sqrt{\frac{\epsilon}{2}}\sqrt{\frac{\epsilon}{2}} + \frac{2}{3}\frac{1}{\sqrt{\epsilon}}\left(\frac{\epsilon}{2}\right)^{3/2} = \left(1+\frac{1}{3\sqrt{2}}\right)\epsilon.\end{aligned}$$

4. Proof of Theorem 1.1(i)

By the results in the previous section, for all sufficiently large n we can construct a $\frac{2\pi}{n}$-periodic function φ_n such that $u_n(x,t) = \varphi_n(x,t)$ is a solution of (1.1) in the weak sense and such that φ_n satisfies

$$0 < \varphi_n \leq \tfrac{1}{\gamma}, \tag{4.1}$$

$$\varphi_n' > 0 \quad \text{on} \quad \big(-\frac{\pi}{n}, 0\big) \quad \text{and} \quad \varphi_n' < 0 \quad \text{on} \quad \big(0, \frac{\pi}{n}\big), \tag{4.2}$$

and

$$\frac{1}{n} \simeq \epsilon \simeq \|\varphi_n'\|^2_{L^2(-\frac{\pi}{n},\frac{\pi}{n})}. \tag{4.3}$$

By the scaling property (2.1), we can define a second sequence of traveling wave solutions

$$v_n(x,t) = c_n\varphi_n(x - c_n t) \tag{4.4}$$

where

$$c_n = 1 + \frac{\pi}{nT}. \tag{4.5}$$

To show the sequences are bounded, that is, to show (1.3), it suffices to estimate the H^1-norm of v_n since it is larger than the H^1-norm of u_n. By (4.4), (4.1), and (4.3),

$$\begin{aligned}||v_n(\cdot,t)||^2_{H^1(-\pi,\pi)} = c_n^2||\varphi_n||^2_{H^1(-\pi,\pi)} &\simeq c_n^2\Big(||\varphi_n||^2_{L^2(-\pi,\pi)} + ||\varphi_n'||^2_{L^2(-\pi,\pi)}\Big) \\ &\lesssim c_n^2\Big(\frac{2\pi}{\gamma^2} + n\cdot\frac{1}{n}\Big) \leq c,\end{aligned}$$

as desired.

At time $t = 0$,

$$||v_n(\cdot,0) - u_n(\cdot,0)||^2_{H^1(-\pi,\pi)} = (c_n - 1)^2 ||\varphi_n||^2_{H^1(-\pi,\pi)} \lesssim \frac{1}{(nT)^2},$$

which goes to zero as $n \to \infty$, while at time T, by (4.2) and (4.3),

$$\begin{aligned} &||v_n(\cdot,T) - u_n(\cdot,T)||^2_{H^1(-\pi,\pi)} \\ &= \left\| \left(1 + \frac{\pi}{nT}\right) \varphi_n \left(\cdot - T - \frac{\pi}{n}\right) - \varphi_n(\cdot - T) \right\|^2_{H^1(-\pi,\pi)} \\ &\geq \left\| \left(1 + \frac{\pi}{nT}\right) \varphi_n' \left(\cdot - \frac{\pi}{n}\right) - \varphi_n'(\cdot) \right\|^2_{L^2(-\pi,\pi)} \geq ||\varphi_n'||^2_{L^2(-\pi,\pi)} \simeq n \cdot \frac{1}{n} = 1. \end{aligned}$$

5. Smooth periodic solutions

To construct a smooth solution f of (2.5) with global maximum M and global minimum m satisfying $\frac{1}{\gamma} \notin [m, M]$, we write (2.5) as

$$(\varphi')^2 = \frac{(\varphi - m)(M - \varphi)(1 - m - M - \varphi)}{\gamma\varphi - 1}. \tag{5.1}$$

We want the right-hand side of (5.1) to be nonnegative for $m \leq \varphi \leq M$. If $0 < \gamma < 3$, we can achieve this by stipulating that

$$m = \frac{1}{\gamma}(1 - (\delta + \epsilon)) \text{ and } M = \frac{1}{\gamma}(1 - \delta)$$

where $\epsilon > 0$ and $\delta > 0$ satisfy $3\delta + 2\epsilon < 3 - \gamma$. The other cases will require different choices of m and M, as summarized in the table below.

	$m \doteq f_{\min}$	$M \doteq f_{\max}$	Constraint on $\delta > 0$ and $\epsilon > 0$
$0 < \gamma < 3$	$\frac{1}{\gamma}(1 - (\delta + \epsilon))$	$\frac{1}{\gamma}(1 - \delta)$	$3\delta + 2\epsilon < 3 - \gamma$
$\gamma > 3$	$\frac{1}{\gamma}(1 + \delta)$	$\frac{1}{\gamma}(1 + \delta + \epsilon)$	$3\delta + 2\epsilon < \gamma - 3$
$\gamma < 0$	$\frac{1}{\gamma}(1 + \delta + \epsilon)$	$\frac{1}{\gamma}(1 + \delta)$	–

We will continue in the case $0 < \gamma < 3$. One can follow along similar lines for the other cases to establish that Theorem 1.1(ii) holds for all nonzero $\gamma \neq 3$.

Letting $y = 1 - \gamma\varphi$, (5.1) becomes

$$(y')^2 = \frac{\frac{1}{\gamma}(y - \delta)(\delta + \epsilon - y)((3 - \gamma) - 2\delta - \epsilon - y)}{y} \tag{5.2}$$

the right-hand side of which is nonnegative for $y \in [\delta, \delta + \epsilon]$. Taking $y' \geq 0$, the initial-value problem for (5.2) with initial data $y(0) = \delta$, will have an increasing local solution with

$$y' = \sqrt{\frac{\frac{1}{\gamma}(y-\delta)(\delta+\epsilon-y)((3-\gamma)-2\delta-\epsilon-y)}{y}}. \tag{5.3}$$

Since this solution is increasing (and differentiable), it is invertible and its inverse $x = x(y)$ is defined and differentiable on the interval $[\delta, \delta+\epsilon]$. Let

$$\ell = x(\delta+\epsilon) - x(\delta). \tag{5.4}$$

At this point we have a function $y : [0, \ell] \to [0, \epsilon]$ that satisfies (5.2) on $[0, \ell]$. Extend this to $y : [-\ell, \ell] \to [0, \epsilon]$ by defining $y(-x) = y(x)$. Then extend the domain of y to all of $\mathbb{R}$ by defining $y(x + 2n\ell) = y(x)$ for every nonzero integer n and every $x \in [-\ell, \ell]$. Note that y is smooth. We have established the following lemma.

Lemma 5.1. *Suppose* $0 < \gamma < 3$. *Let* $c = \frac{1}{2} \cdot \frac{1}{5}(3-\gamma)$. *For any* $0 < \epsilon, \delta < c$, *there exists a positive number* $l = l(\epsilon, \delta)$ *and an even* 2ℓ*-periodic smooth function* $y = y(x)$ *which solves equation* (5.2). *The function* y *satisfies the bounds* $\delta \le y(x) \le \delta + \epsilon$ *and the function* $u(x,t) = \varphi(x-t)$, *where* $\varphi(x) = \frac{1}{\gamma}(1 - y(x))$ *is a solution of* (1.1).

Next we estimate, in terms of the parameters δ and ϵ, the period of each solution constructed above.

Lemma 5.2. *The period* 2ℓ *of the solution* y *described in Lemma* 5.1 *depends continuously on the parameters* ϵ *and* δ *and satisfies*

$$\frac{1}{3}\frac{1}{\sqrt{\frac{3}{\gamma}-1}}\sqrt{\delta+\epsilon} \le \ell \le \frac{4\sqrt{2}}{\sqrt{\frac{3}{\gamma}-1}}\sqrt{\delta+\epsilon}. \tag{5.5}$$

Proof. As noted above, $y = y(x)$ is increasing (and differentiable) on $[0, \ell]$, so it is invertible with differentiable inverse $x = x(y)$ defined on $[\delta, \delta+\epsilon]$, so we have

$$\begin{aligned} \ell &= \int_\delta^{\delta+\epsilon} x'(y)dy = \int_\delta^{\delta+\epsilon} \frac{1}{y'(x)}dy \\ &= \int_\delta^{\delta+\epsilon} \sqrt{\frac{y}{\frac{1}{\gamma}(\delta+\epsilon-y)(y-\delta)((3-\gamma)-2\delta-\epsilon-y)}}dy, \end{aligned}$$

which shows in particular that the period 2ℓ is a continuous function of the parameters ϵ and δ. Since $\delta < y < \delta + \epsilon$, we have the estimate

$$\begin{aligned} \ell &\le \frac{\sqrt{\delta+\epsilon}}{\sqrt{(3-\gamma)-2\delta-\epsilon-(\delta+\epsilon)}}\sqrt{\gamma}\int_{\delta}^{\delta+\epsilon}(y-\delta)^{-\frac{1}{2}}(\delta+\epsilon-y)^{-\frac{1}{2}}dy \\ &= \frac{\sqrt{\delta+\epsilon}}{\sqrt{3-\gamma-3\delta-2\epsilon}}\sqrt{\gamma}\cdot I, \end{aligned}$$

where $I \doteq \int_{\delta}^{\delta+\epsilon}(y-\delta)^{-\frac{1}{2}}(\delta+\epsilon-y)^{-\frac{1}{2}}dy$. Then, since

$$\begin{aligned} I &= \int_{\delta}^{\delta+\frac{\epsilon}{2}}(y-\delta)^{-\frac{1}{2}}(\delta+\epsilon-y)^{-\frac{1}{2}}dy + \int_{\delta+\frac{\epsilon}{2}}^{\delta+\epsilon}(y-\delta)^{-\frac{1}{2}}(\delta+\epsilon-y)^{-\frac{1}{2}}dy \\ &\le \frac{1}{\sqrt{\epsilon/2}}\int_{\delta}^{\delta+\frac{\epsilon}{2}}(y-\delta)^{-\frac{1}{2}}dy + \frac{1}{\sqrt{\epsilon/2}}\int_{\delta+\frac{\epsilon}{2}}^{\delta+\epsilon}(\delta+\epsilon-y)^{-\frac{1}{2}}dy \\ &= \frac{2}{\sqrt{\epsilon/2}}\cdot\sqrt{\frac{\epsilon}{2}} + \frac{2}{\sqrt{\epsilon/2}}\cdot\sqrt{\frac{\epsilon}{2}} = 4, \end{aligned}$$

we have

$$\ell \le 4\sqrt{\gamma}\frac{\sqrt{\delta+\epsilon}}{\sqrt{3-\gamma-3\delta-2\epsilon}} < 4\sqrt{\gamma}\frac{\sqrt{\delta+\epsilon}}{\sqrt{\frac{1}{2}(3-\gamma)}} = \frac{4\sqrt{2}}{\sqrt{\frac{3}{\gamma}-1}}\sqrt{\delta+\epsilon}.$$

On the other hand,

$$\begin{aligned} \ell &\ge \int_{\delta+\frac{\epsilon}{4}}^{\delta+3\frac{\epsilon}{4}}\sqrt{\frac{y}{\frac{1}{\gamma}(\delta+\epsilon-y)(y-\delta)((3-\gamma)-2\delta-\epsilon-y)}}dy \\ &\ge \frac{1}{\sqrt{\frac{3}{\gamma}-1}}\int_{\delta+\frac{\epsilon}{4}}^{\delta+3\frac{\epsilon}{4}}\frac{\sqrt{\delta+\epsilon/4}}{\sqrt{(\delta+\epsilon-(\delta+\epsilon/4))(\delta+3\epsilon/4-\delta)}}dy \\ &= \frac{1}{\sqrt{\frac{3}{\gamma}-1}}\cdot\frac{1}{3\epsilon/4}\cdot\frac{\epsilon}{2}\sqrt{\delta+\frac{\epsilon}{4}} \ge \frac{1}{3}\frac{1}{\sqrt{\frac{3}{\gamma}-1}}\sqrt{\delta+\epsilon}. \qquad \square \end{aligned}$$

To ensure that the solutions constructed satisfy desired Sobolev bounds, we require

$$\delta = \epsilon^{2/s}, \quad \text{where } s \ge 2. \tag{5.6}$$

In the light of this relationship between δ and ϵ, we can think of ℓ as a function of the parameter ϵ alone. The function $\ell = \ell(\epsilon)$ is continuous and increasing on the interval $(0, c)$ (where c is as in Lemma 5.1) and satisfies

$\lim_{\epsilon\to 0^+} \ell(\epsilon) = 0$. So, for any sufficiently large n, there is a corresponding $\epsilon \in (0, c)$ such that

$$\ell = \ell(\epsilon) = \frac{\pi}{n}, \tag{5.7}$$

and, by Lemma 5.2,

$$n \simeq \frac{1}{\sqrt{\delta+\epsilon}}. \tag{5.8}$$

We close this section by deriving an expression for y'' which will be of use for us in the next section.

Lemma 5.3. *We have*

$$y'' = \frac{1}{\gamma}\Big(y - \frac{1}{2}(3-\gamma) + \frac{B}{2y^2}\Big) \tag{5.9}$$

where

$$B = B(\epsilon, \delta) = \delta(\delta+\epsilon)\left((3-\gamma) - 2\delta - \epsilon\right). \tag{5.10}$$

Proof. Write (5.2) as

$$y(y')^2 = \frac{1}{\gamma}(y-\delta)(\delta+\epsilon-y)\left((3-\gamma)-2\delta-\epsilon-y\right). \tag{5.11}$$

Differentiating (5.11) gives

$$\begin{aligned} y'^3 + 2yy'y'' &= \frac{1}{\gamma}y'\Big\{(\delta+\epsilon-y)\left((3-\gamma)-2\delta-\epsilon-y\right) \\ &- (y-\delta)\left((3-\gamma)-2\delta-\epsilon-y\right) - (y-\delta)(\delta+\epsilon-y)\Big\}, \end{aligned}$$

which implies that, where $y' \neq 0$, we have

$$\begin{aligned} 2y^2y'' = -y(y')^2 &+ \frac{1}{\gamma}y\Big\{(\delta+\epsilon-y)\left((3-\gamma)-2\delta-\epsilon-y\right) \\ &- (y-\delta)\left((3-\gamma)-2\delta-\epsilon-y\right) - (y-\delta)(\delta+\epsilon-y)\Big\}, \end{aligned}$$

so, by (5.11),

$$\begin{aligned} 2y^2y'' = -\frac{1}{\gamma}\Big\{&(y-\delta)(\delta+\epsilon-y)\left((3-\gamma)-2\delta-\epsilon-y\right) \\ &- y(\delta+\epsilon-y)\left((3-\gamma)-2\delta-\epsilon-y\right) \\ &+ y(y-\delta)\left((3-\gamma)-2\delta-\epsilon-y\right) + y(y-\delta)(\delta+\epsilon-y)\Big\}. \end{aligned}$$

Then using the identity $(y-a)(y-b)(y-c) = y^3 - (a+b+c)y^2 + (ab + ac + bc)y - abc$ yields

$$\begin{aligned} 2y^2y'' &= -\frac{1}{\gamma}\Big\{ -2y^3 - \Big[(3-\gamma) - (3-\gamma-\delta) \\ &\quad - (3-\gamma-\delta-\epsilon) - (2\delta+\epsilon)\Big]y^2 \\ &\quad + \Big[\delta(\delta+\epsilon) + \delta\,(3-\gamma-2\delta-\epsilon) + (\delta+\epsilon)\,(3-\gamma-2\delta-\epsilon) \\ &\quad -(\delta+\epsilon)\,(3-\gamma-2\delta-\epsilon) - \delta\,(3-\gamma-2\delta-\epsilon) - \delta(\delta+\epsilon)\Big]y \\ &\quad -\delta(\delta+\epsilon)\,((3-\gamma)-2\delta-\epsilon)\Big\}; \end{aligned}$$

that is,

$$2y^2y'' = \frac{1}{\gamma}\left(2y^3 - (3-\gamma)\,y^2 + \delta(\delta+\epsilon)\,((3-\gamma)-2\delta-\epsilon)\right).$$

Dividing the above equation by $2y^2$ gives (5.9). □

6. Sobolev estimates for smooth solutions

The next lemma gives L^∞-estimates on the derivatives of the 2ℓ-periodic function y constructed in the previous section. The proof follows along the lines of the proof of the corresponding estimates in [17], but we include it here for completeness.

Lemma 6.1. *If $\delta \geq \epsilon$, then for any integer $k \geq 0$, there is a constant $c_k > 0$ depending only on k such that*

$$||y^{(k)}||_{L^\infty} \leq \frac{c_k}{(\sqrt{\delta})^{k-2}}. \tag{6.1}$$

For $k = 1$, we also have the estimate

$$\|y'\|_{L^\infty} \leq \left(\sqrt{\frac{3}{\gamma}-1}\right)\frac{\epsilon}{\sqrt{\delta}}. \tag{6.2}$$

Proof. For $k = 0$, we have

$$||y||_{L^\infty} = \delta + \epsilon \leq 2\delta = \frac{2}{(\sqrt{\delta})^{-2}}. \tag{6.3}$$

For $k = 1$,

$$||y'||_{L^\infty} \;=\; \sup_{\delta\leq y\leq\delta+\epsilon} \sqrt{\frac{\frac{1}{\gamma}(\delta+\epsilon-y)(y-\delta)((3-\gamma)-2\delta-\epsilon-y)}{y}}$$

$$\begin{aligned}
&\leq \sqrt{\frac{\frac{1}{\gamma}\cdot\epsilon\cdot\epsilon\cdot(3-\gamma)}{\delta}} = \left(\sqrt{\frac{3}{\gamma}-1}\right)\frac{\epsilon}{\sqrt{\delta}}, \\
&\leq \left(\sqrt{\frac{3}{\gamma}-1}\right)\frac{1}{(\sqrt{\delta})^{-1}}. \qquad (6.4)
\end{aligned}$$

For $k=2$, by (5.9) we have

$$\begin{aligned}
||y''||_{L^\infty} &= \sup_{\delta\leq y\leq\delta+\epsilon}\Big|\frac{1}{\gamma}\Big(y-\frac{1}{2}(3-\gamma)+\frac{B}{2y^2}\Big)\Big| \\
&\leq \frac{1}{\gamma}\Big[\frac{1}{5}(3-\gamma)+\frac{1}{2}(3-\gamma)+(3-\gamma)\Big] \leq 2\Big(\frac{3}{\gamma}-1\Big).
\end{aligned}$$

Now we will establish (6.1) for $k=m+2$, $m\geq 0$, by induction. We will assume (6.1) is true for $k=2,3,...,m+2$, and then show (6.1) for $k=m+3$. Multiplying both sides of (5.9) by y^2 we have

$$y^2y'' = \frac{1}{\gamma}\left(y^3-\frac{1}{2}(3-\gamma)y^2+\frac{B}{2}\right).$$

Differentiating both sides of the above equation yields

$$2yy'y''+y^2y''' = \frac{1}{\gamma}\left(3y^2y'-(3-\gamma)yy'\right),$$

which gives

$$yy''' = -2y'y''+\frac{3}{\gamma}yy'-\left(\frac{3}{\gamma}-1\right)y'.$$

Taking m derivatives of both sides of the last equation and solving for $y^{(m+3)}$ gives

$$\begin{aligned}
y^{(m+3)} = \frac{1}{y}\Bigg(&- \sum_{j=0}^{m-1}\binom{m}{j}y^{(m-j)}y^{(j+3)} - 2\sum_{j=0}^{m}\binom{m}{j}y^{(m-j+1)}y^{(j+2)} \\
&+ \frac{3}{\gamma}\sum_{j=0}^{m}\binom{m}{j}y^{(m-j)}y^{(j+1)} - \left(\frac{3}{\gamma}-1\right)y^{(m+1)}\Bigg). \qquad (6.5)
\end{aligned}$$

Equation (6.5) together with (6.3), (6.4), and the induction assumption gives

$$||y^{(m+3)}||_{L^\infty} \leq \frac{1}{\delta}\sum_{j=0}^{m-1}\binom{m}{j}\frac{c_{m-j}}{(\sqrt{\delta})^{m-j-2}}\cdot\frac{c_{j+3}}{(\sqrt{\delta})^{j+3-2}}$$

$$
\begin{aligned}
&+\frac{2}{\delta}\sum_{j=0}^{m}\binom{m}{j}\frac{c_{m-j+1}}{(\sqrt{\delta})^{m-j+1-2}}\cdot\frac{c_{j+2}}{(\sqrt{\delta})^{j+2-2}}\\
&+\frac{3}{\delta\gamma}\sum_{j=0}^{m}\binom{m}{j}\frac{c_{m-j}}{(\sqrt{\delta})^{m-j-2}}\cdot\frac{c_{j+1}}{(\sqrt{\delta})^{j+1-2}}+\frac{(\frac{3}{\gamma}-1)c_{m+1}}{\delta(\sqrt{\delta})^{m+1-2}}\\
= \;& c_{m+3}\left[\frac{1}{(\sqrt{\delta})^{m+1}}+\frac{1}{(\sqrt{\delta})^{m+1}}+\frac{1}{(\sqrt{\delta})^{m-1}}+\frac{1}{(\sqrt{\delta})^{m+1}}\right]\\
\leq \;& \frac{c_{m+3}}{(\sqrt{\delta})^{(m+3)-2}}.
\end{aligned}
$$

□

We now use the L^∞- estimates of Lemma 6.1 to establish L^2-estimates in the following proposition. Again, the proof follows along the lines of the proof of the corresponding estimates in [17], but we include it for completeness.

Proposition 6.2. *For any integer $k \geq 1$, there is a constant $c_k > 0$ depending only on k such that*

$$||y^{(k)}||^2_{L^2(-\ell,\ell)} \leq \frac{c_k}{\delta^{k-1}}||y'||^2_{L^2(-\ell,\ell)}. \tag{6.6}$$

For $k = 1$, we also have

$$||y'||^2_{L^2(-\ell,\ell)} \leq 2\Big(\sqrt{\frac{3}{\gamma}-1}\Big)\frac{\epsilon^2}{\sqrt{\delta}}. \tag{6.7}$$

Proof. For $k = 1$, we have, by (6.2),

$$
\begin{aligned}
||y'||^2_{L^2(-\ell,\ell)} &= 2\int_0^\ell y'(x)\cdot y'(x)dx = 2\int_\delta^{\delta+\epsilon} y'dy\\
&\leq 2\epsilon||y'||_{L^\infty} \leq 2\Big(\sqrt{\frac{3}{\gamma}-1}\Big)\frac{\epsilon^2}{\sqrt{\delta}}.
\end{aligned}
$$

For $k = 2$, we have, using integration by parts,

$$||y''||^2_{L^2(-\ell,\ell)} = 2\int_0^\ell y''(x)\cdot y''(x)dx = -2\int_0^\ell y'''(x)\cdot y'(x)dx.$$

Differentiating (5.9) once gives

$$y''' = \frac{1}{\gamma}\Big(1-\frac{B}{y^3}\Big)y',$$

so we have

$$||y''||^2_{L^2(-\ell,\ell)} = \frac{2}{\gamma}\int_0^\ell \Big(\frac{B}{(y(x))^3} - 1\Big)(y'(x))^2 dx \le \frac{1}{\gamma}\Big(\frac{B}{\delta^3} - 1\Big)||y'||^2_{L^2(-\ell,\ell)}$$
$$\le \frac{1}{\gamma}\frac{2(3-\gamma)}{\delta}||y'||^2_{L^2(-\ell,\ell)} = \frac{2(\frac{3}{\gamma} - 1)}{\delta}||y'||^2_{L^2(-\ell,\ell)},$$

as desired. To complete the proof, we proceed by induction as in the proof of Lemma 6.1. From (6.5) we have

$$\|y^{(m+3)}\|_{L^2(-\ell,\ell)}$$
$$\le \frac{1}{\delta}\Big(\sum_{j=0}^{m-1}\binom{m}{j}\|y^{(m-j)}y^{(j+3)}\|_{L^2(-\ell,\ell)} + 2\sum_{j=0}^{m}\binom{m}{j}\|y^{(m-j+1)}y^{(j+2)}\|_{L^2(-\ell,\ell)}$$
$$+ \frac{3}{\gamma}\sum_{j=0}^{m}\binom{m}{j}\|y^{(m-j)}y^{(j+1)}\|_{L^2(-\ell,\ell)} + \Big(\frac{3}{\gamma} - 1\Big)\|y^{(m+1)}\|_{L^2(-\ell,\ell)}\Big).$$

Thus,

$$\|y^{(m+3)}\|_{L^2(-\ell,\ell)} \le \frac{1}{\delta}\Big(\sum_{j=0}^{m-1}\binom{m}{j}\|y^{(m-j)}\|_{L^\infty}\|y^{(j+3)}\|_{L^2(-\ell,\ell)}$$
$$+ 2\sum_{j=0}^{m}\binom{m}{j}\|y^{(m-j+1)}\|_{L^\infty}\|y^{(j+2)}\|_{L^2(-\ell,\ell)}$$
$$+ \frac{3}{\gamma}\sum_{j=0}^{m}\binom{m}{j}\|y^{(m-j)}\|_{L^\infty}\|y^{(j+1)}\|_{L^2(-\ell,\ell)} + \Big(\frac{3}{\gamma} - 1\Big)\|y^{(m+1)}\|_{L^2(-\ell,\ell)}\Big).$$

The L^2-estimates of Proposition 6.2 and the induction hypothesis then give

$$\|y^{(m+3)}\|_{L^2(-\ell,\ell)} \le \frac{1}{\delta}\Big(\sum_{j=0}^{m-1}\binom{m}{j}\frac{c_{m-j}}{\big(\sqrt{\delta}\big)^{m-j-2}}\cdot\frac{c_{j+3}}{\big(\sqrt{\delta}\big)^{j+2}}\|y'\|_{L^2(-\ell,\ell)}$$
$$+ 2\sum_{j=0}^{m}\binom{m}{j}\frac{c_{m-j+1}}{(\sqrt{\delta})^{m-j-1}}\cdot\frac{c_{j+2}}{(\sqrt{\delta})^{j+1}}\|y'\|_{L^2(-\ell,\ell)}$$
$$+ \frac{3}{\gamma}\sum_{j=0}^{m}\binom{m}{j}\frac{c_{m-j}}{(\sqrt{\delta})^{m-j-2}}\cdot\frac{c_{j+1}}{(\sqrt{\delta})^{j}}\|y'\|_{L^2(-\ell,\ell)} + (\frac{3}{\gamma} - 1)\frac{c_{m+1}}{(\sqrt{\delta})^m}\|y'\|_{L^2(-\ell,\ell)}\Big)$$
$$= c_{m+3}\Big[\frac{1}{(\sqrt{\delta})^{m+2}} + \frac{1}{(\sqrt{\delta})^{m+2}} + \frac{1}{(\sqrt{\delta})^{m}} + \frac{1}{(\sqrt{\delta})^{m+2}}\Big]||y'||_{L^2(-\ell,\ell)}$$

$$\overset{\delta<1}{\leq} \frac{c_{m+3}}{(\sqrt{\delta})^{(m+3)-1}} ||y'||_{L^2(-\ell,\ell)}. \qquad \square$$

From the above proposition, we immediately obtain estimates on the H^k-norm for integer values of $k \geq 1$, which is the content of the following corollary.

Corollary 6.3. *Let y be the $\frac{2\pi}{n}$-periodic smooth solution constructed above. For any integer $k \geq 1$ there is a positive constant c_k depending only on k such that*

$$\|y\|^2_{H^k(-\pi,\pi)} \leq c_k\Big(\frac{1}{\delta^{k-1}}\|y'\|^2_{L^2(-\pi,\pi)} + 1\Big). \tag{6.8}$$

Using Corollary 6.3 and an interpolation argument (see [17] for details), we obtain the following estimates on the H^s-norm for all real numbers $s \geq 1$.

Proposition 6.4. *Let y be the $\frac{2\pi}{n}$-periodic smooth solution constructed above. For any $s \geq 1$ there is a positive constant c_s depending only on s such that*

$$\|y\|^2_{H^s(-\pi,\pi)} \leq c_s\Big(\frac{1}{\delta^{s-1}}\|y'\|^2_{L^2(-\pi,\pi)} + 1\Big). \tag{6.9}$$

7. PROOF OF THEOREM 1.1(ii)

The proof of Theorem 1.1(ii) given below follows along the lines of the proof of the main result in [17].

For n sufficiently large (see the discussion following Lemma 5.2), let $u_n(x,t) = \varphi_n(x-t)$ be the $\frac{2\pi}{n}$-periodic smooth traveling wave solution constructed above of the Cauchy problem (1.1)–(1.2). By the scaling property (2.1), we can define a second sequence of traveling wave solutions

$$v_n(x,t) = c_n\varphi_n(x - c_n t), \tag{7.1}$$

where

$$c_n = 1 + \frac{1}{n}. \tag{7.2}$$

As in the proof of part (i) of the theorem, to show boundedness it suffices to estimate the H^s-norm of v_n. By the H^s- and L^2-estimates established above we have

$$\begin{aligned}\|v_n(\cdot,t)\|^2_{H^s(-\pi,\pi)} &= c_n^2\|\varphi_n\|^2_{H^s(-\pi,\pi)} = c_n^2\Big(\frac{1}{\gamma}\Big)^2\|(1-y_n)\|^2_{H^s(-\pi,\pi)} \\ &\lesssim 1 + \|y_n\|^2_{H^s(-\pi,\pi)}.\end{aligned}$$

So, by (6.9),

$$\begin{aligned}\|v_n(\cdot,t)\|^2_{H^s(-\pi,\pi)} &\lesssim 1+\Big(\frac{1}{\delta^{s-1}}\|y_n'\|^2_{L^2(-\pi,\pi)}+1\Big)\\ &= 1+\Big(\frac{1}{\delta^{s-1}}\cdot n\|y_n'\|^2_{L^2(-\frac{\pi}{n},\frac{\pi}{n})}+1\Big).\end{aligned}$$

Thus, by (6.7),

$$\|v_n(\cdot,t)\|^2_{H^s(-\pi,\pi)} \lesssim 1+\Big(\frac{1}{\delta^{s-1}}\cdot\frac{1}{\sqrt{\delta+\epsilon}}\cdot\frac{\epsilon^2}{\sqrt{\delta}}+1\Big)\le 1+\frac{\epsilon^2}{\delta^s}+1=3,$$

by the stipulation that $\delta=\epsilon^{2/s}$.

At time $t=0$, we have

$$||v_n(\cdot,0)-u_n(\cdot,0)||^2_{H^s(-\pi,\pi)}=(c_n-1)^2||\varphi_n||^2_{H^s(-\pi,\pi)}\simeq\frac{1}{n^2},$$

which goes to zero as $n\to\infty$.

On the other hand, at time $T>0$, we have

$$\begin{aligned}\|v_n(\cdot,T)-u_n(\cdot,T)\|^2_{H^s(-\pi,\pi)} &= \|c_n\varphi_n(\cdot-c_nT)-\varphi_n(\cdot-T)\|^2_{H^s(-\pi,\pi)}\\ &= \sum_{\xi\in\mathbb{Z}}(1+\xi^2)^s\left|c_n\widehat{\varphi_n}(\cdot-c_nT)(\xi)-\widehat{\varphi_n}(\cdot-T)(\xi)\right|^2,\end{aligned}\tag{7.3}$$

where $\widehat{\varphi_n}(\cdot-cT)(\xi)$ denotes the Fourier transform of the function $\varphi_n(x-cT)$ with respect to x. So, since $\widehat{\varphi_n}(\cdot-cT)(\xi)=e^{-icT\xi}\widehat{\varphi_n}(\xi)$, we have

$$\|v_n(\cdot,T)-u_n(\cdot,T)\|^2_{H^s(-\pi,\pi)}=\sum_{\xi\in\mathbb{Z}}(1+\xi^2)^s|c_ne^{-ic_nT\xi}-e^{-iT\xi}|^2|\widehat{\varphi_n}(\xi)|^2.$$

Using only the term corresponding to $\xi=n$ yields

$$\begin{aligned}&\|v_n(\cdot,T)-u_n(\cdot,T)\|^2_{H^s(-\pi,\pi)}\\ &\ge(1+n^2)^s\Big|\Big(1+\frac{1}{n}\Big)e^{-i(1+\frac{1}{n})Tn}-e^{-iTn}\Big|^2|\widehat{\varphi_n}(n)|^2\\ &=(1+n^2)^s\Big|(e^{-iT}-1)+\frac{1}{n}e^{-iT}\Big|^2|\widehat{\varphi_n}(n)|^2.\end{aligned}$$

Since φ_n is $\frac{2\pi}{n}$-periodic and even,

$$\widehat{\varphi_n}(n)=\frac{n}{\sqrt{2\pi}}\int_{-\pi/n}^{\pi/n}e^{-ixn}\varphi_n(x)dx=\frac{2}{\sqrt{2\pi}}\int_0^{\pi/n}n\cos(nx)\varphi_n(x)dx.$$

Then, integrating by parts, we obtain

$$\widehat{\varphi_n}(n)=-\frac{2}{\sqrt{2\pi}}\int_0^{\pi/n}\sin(nx)y_n'(x)dx.$$

Thus,

$$\|v_n(\cdot,T)-u_n(\cdot,T)\|^2_{H^s(-\pi,\pi)} \geq \frac{2}{\pi}(1+n^2)^s\left|(e^{-iT}-1)+\frac{1}{n}e^{-iT}\right|^2|B_n|^2, \quad (7.4)$$

where

$$B_n = \int_0^{\pi/n} \sin(nx)y_n'(x)dx.$$

To complete the proof, we use the following lemma whose proof is given below.

Lemma 7.1. *There exists a constant $c_0 > 0$ independent of n such that $B_n \geq c_0 \cdot \epsilon$.*

By the lemma, (7.4) gives

$$\|v_n(\cdot,T)-u_n(\cdot,T)\|^2_{H^s(-\pi,\pi)} \gtrsim n^{2s}\epsilon^2\left|(e^{-iT}-1)+\frac{1}{n}e^{-iT}\right|^2.$$

By (5.8) and (5.6),

$$n^{2s}\epsilon^2 \simeq \frac{\epsilon^2}{(\delta+\epsilon)^s} = \frac{\epsilon^2}{(\epsilon^{2/s}+\epsilon)^s} \geq \frac{\epsilon^2}{(\epsilon^{2/s}+\epsilon^{2/s})^s} = \frac{\epsilon^2}{2^s\epsilon^2} = \frac{1}{2^s}.$$

Thus

$$\|v_n(\cdot,T)-u_n(\cdot,T)\|^2_{H^s(-\pi,\pi)} \gtrsim \left|(e^{-iT}-1)+\frac{1}{n}e^{-iT}\right|^2.$$

Therefore,

$$\begin{aligned}\liminf_n \|v_n(\cdot,T)-u_n(\cdot,T)\|^2_{H^s(-\pi,\pi)} &\gtrsim |e^{-iT}-1|^2 = (e^{-iT}-1)(e^{iT}-1)\\ &= 2-(e^{iT}+e^{-iT}) = 2(1-\cos T) > 0. \qquad \square\end{aligned}$$

We now prove Lemma 7.1.

Proof. By a change of variables, we have

$$B_n = \int_0^{\pi/n} \sin(nx)y_n'(x)dx = \int_\delta^{\delta+\epsilon} \sin(nx(y))dy.$$

Notice that

$$\sin(nx(\cdot)) : [\delta, \delta+\epsilon] \to [0,1],$$

so, for any $0 < \alpha < 1$, we have

$$B_n \geq \int_{\delta+\frac{\alpha}{2}\epsilon}^{\delta+\alpha\epsilon} \sin(nx(y))dy.$$

Thus, if we choose α so that $nx(\delta+\alpha\epsilon) \le \frac{\pi}{2}$, then the map $\sin(nx(\cdot))$ will be increasing on the interval $[\delta+\frac{\alpha}{2}\epsilon, \delta+\alpha\epsilon]$, and we will have

$$B_n \ge \frac{\alpha}{2}\sin\left(nx\left(\delta+\frac{\alpha}{2}\epsilon\right)\right)\cdot\epsilon.$$

So, to finish the proof of the lemma, we will derive bounds on $nx(\delta+\alpha\epsilon)$. By Lemma 5.2, we have

$$\frac{1}{3}\frac{1}{\sqrt{\frac{3}{\gamma}-1}}\sqrt{\delta+\epsilon} \le \frac{\pi}{n} \le \frac{4\sqrt{2}}{\sqrt{\frac{3}{\gamma}-1}}\sqrt{\delta+\epsilon};$$

that is,

$$\frac{\pi}{4\sqrt{2}}\Big(\sqrt{\frac{3}{\gamma}-1}\Big)\frac{1}{\sqrt{\delta+\epsilon}} \le n \le 3\pi\Big(\sqrt{\frac{3}{\gamma}-1}\Big)\frac{1}{\sqrt{\delta+\epsilon}}. \tag{7.5}$$

Let $0<\alpha<1$. We have

$$x(\delta+\alpha\epsilon) = \int_{\delta}^{\delta+\alpha\epsilon}\sqrt{\frac{y}{\frac{1}{\gamma}(\delta+\epsilon-y)(y-\delta)((3-\gamma)-2\delta-\epsilon-y)}}dy.$$

Then, since $0<\alpha<1$ and $0<\delta,\epsilon<\frac{1}{2}\cdot\frac{1}{5}(3-\frac{1}{\gamma})$,

$$\begin{aligned} x(\delta+\alpha\epsilon) &\le \frac{\sqrt{\delta+\epsilon}}{\sqrt{\frac{1}{\gamma}\cdot\epsilon(1-\alpha)\cdot\frac{1}{2}(3-\gamma)}}\int_{\delta}^{\delta+\alpha\epsilon}(y-\delta)^{-1/2}dy \\ &= \frac{\sqrt{2}\sqrt{\delta+\epsilon}}{\sqrt{\epsilon(1-\alpha)(\frac{3}{\gamma}-1)}}\cdot 2\sqrt{\alpha}\sqrt{\epsilon} = \frac{2\sqrt{2}}{\sqrt{\frac{3}{\gamma}-1}}\cdot\frac{\sqrt{\alpha}}{\sqrt{1-\alpha}}\cdot\sqrt{\delta+\epsilon} \end{aligned}$$

and

$$\begin{aligned} x(\delta+\alpha\epsilon) &\ge \frac{\sqrt{\delta}}{\sqrt{\epsilon(\frac{3}{\gamma}-1)}}\int_{\delta}^{\delta+\alpha\epsilon}(y-\delta)^{-1/2}dy = \frac{\sqrt{\delta}}{\sqrt{\epsilon(\frac{3}{\gamma}-1)}}\cdot 2\sqrt{\alpha}\sqrt{\epsilon} \\ &= \frac{2}{\sqrt{\frac{3}{\gamma}-1}}\sqrt{\alpha}\sqrt{\frac{\delta}{2}+\frac{\delta}{2}} \ge \frac{\sqrt{2}}{\sqrt{\frac{3}{\gamma}-1}}\sqrt{\alpha}\sqrt{\delta+\epsilon}. \end{aligned}$$

Combining the above results, we obtain

$$\frac{\pi}{4}\sqrt{\alpha} \le nx(\delta+\alpha\epsilon) \le 6\pi\sqrt{2}\frac{\sqrt{\alpha}}{\sqrt{1-\alpha}}.$$

Solving

$$6\pi\sqrt{2}\frac{\sqrt{\alpha}}{\sqrt{1-\alpha}} = \frac{\pi}{2}$$

yields $\alpha = \frac{1}{289}$, so we have

$$B_n \geq \frac{1/289}{2} \sin\left(\frac{\pi}{4}\sqrt{\frac{1/289}{2}}\right)\epsilon. \qquad \square$$

Acknowledgements. The author would like to thank Alex Himonas for useful conversations during the preparation of this work and the referee for valuable suggestions which led to improvements in its presentation.

References

[1] M. Alber, R. Camassa, D. Holm, and J. Marsden, *The geometry of peaked solitons and billiard solutions of a class of integrable PDEs*, Lett. Math. Phys. **32** (1994), 137–151.
[2] R. Beals, D. Sattinger and J. Szmigielski, *Multipeakons and the classical moment problem*, Adv. Math. **154** (2000), 229–257.
[3] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part 2: KdV equation*, Geom. Funct. Anal. **3** (1993), 209–262.
[4] P. Byers, *Spiked traveling waves and ill-posedness for the Camassa-Holm equation on the circle*, Illinois J. Math. **48** (2004), 1031–1040.
[5] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661–1664.
[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc. **16** (2003), 705–749.
[7] A. Constantin and J. Escher, *Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. **51** (1998), 475–504.
[8] A. Constantin and H. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math. **52** (1999), 949–982.
[9] A. Constantin and W. Strauss, *Stability of the Camassa-Holm solitons*, J. Nonlinear Sci. **12** (2002), 415–422.
[10] A. Constantin and W. Strauss, *Stability of a class of solitary waves in compressible elastic rods*, Phys. Lett. A **270** (2000), 140–148.
[11] H.-H. Dai, *Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod*, Acta Mech. **127** (1998), 193–207.
[12] H.-H. Dai, *Exact travelling-wave solutions of an integrable equation arising in hyperelastic rods*, Wave Motion **28** (1998), 367–381.
[13] H.-H. Dai and Y. Huo, *Solitary shock waves and other travelling waves in a general compressible hyperelastic rod*, Proc. R. Soc. Lond. A **456** (2000), 331–363.
[14] R. Danchin, *A few remarks on the Camassa-Holm equation*, Differential Integral Equations **14** (2001), 953–988.
[15] C. De Lellis, T. Kappeler and P. Topalov, *Low-regularity solutions of the periodic Camassa-Holm equation*, preprint (2005).
[16] A. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäklund transformations and hereditary symmetries*, Phys. D **4** (1981/82), 47–66.

[17] A. A. Himonas and G. Misiołek, *High frequency smooth soltuions and well-posedness of the Camassa-Holm equation*, Int. Math. Res. Not. (2005), 3135–3151.
[18] A. Himonas and G. Misiołek, *The Cauchy problem for an integrable shallow water equation*, Differential Integral Equations **14** (2001), 821–831.
[19] H. Holden and X. Raynaud, *Global conservative solutions for the generalized hyperelastic-rod equation*, submitted.
[20] C. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), 573–603.
[21] J. Lenells, *Traveling waves in compressible elastic rods*, Disc. Cont. Dyn. Sys. B **6** (2006), 151–168.
[22] Y. Li and P. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*, J. Differential Equations, **162** (2000), 27–63.
[23] H. McKean, *Breakdown of the Camassa-Holm equation*, Comm. Pure Appl. Math. **57** (2004), 416–418.
[24] G. Misiołek, *Classical solutions of the periodic Camassa-Holm equation*, Geom. Funct. Anal. **12** (2002), 1080–1104.
[25] Y. Zhou, *Local well-posedness and blow-up criteria of solutions for a rod equation*, Math. Nachr. **278** (2005), 1726–1739.