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# ON A NEUMANN PROBLEM WITH $p$-LAPLACIAN AND NON-SMOOTH POTENTIAL

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**Abstract.** Three existence results for a homogeneous Neumann problem with partial $p$-Laplacian and non-smooth potential (i.e., hemivariational inequality) are established through a locally Lipschitz continuous version of the classical mountain pass theorem.

## Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$, and let $2 \leq p < +\infty$. This paper treats the existence of weak solutions $u \in W^{1,p}(\Omega)$ to the hemivariational inequality problem

$$\begin{cases} -\Delta_p u \in \partial J(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, the potential $(x,\xi) \mapsto J(x,\xi)$, $(x,\xi) \in \Omega \times \mathbb{R}$, is measurable with respect to $x$ and locally Lipschitz continuous in $\xi$, $\partial J(x,\xi)$ stands for the generalized gradient of $J(x,\xi)$ with respect to $\xi$, while $\frac{\partial u}{\partial n_p} := |\nabla u|^{p-2}\nabla u \cdot n$, where $n(x)$ denotes the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$.

The $p$-Laplacian operator $\Delta_p$ arises from a variety of physical phenomena. For instance, it is employed in the mathematical modelings of non-Newtonian fluids, some reaction-diffusion problems, as well as flows through porous media. The $p$-Laplacian also appears in nonlinear elasticity, glaceology, and petroleum extraction.

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Hemivariational inequalities have been introduced by Panagiotopoulos [18,17] for mathematically investigating several complicated mechanical and engineering questions, where the relevant energy functionals are neither convex nor smooth (the so-called superpotentials). As an example, this is the case for non-monotone multivalued interface laws or constitutive relations that occur in certain contact and friction processes, as well as for phenomena related to large displacements and deformations expressed by nonlinear strain-displacement laws. The theoretical formulation of such concrete questions basically relies on the notion of Clarke's generalized gradient, which replaces the subdifferential in the sense of convex analysis.

Let $\theta(x) := \limsup_{|\xi|\to+\infty} \frac{pJ(x,\xi)}{|\xi|^p}$ uniformly with respect to $x \in \Omega$ and let $\lambda_2$ be the first nonzero eigenvalue of the Neumann problem

$$-\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n_p} = 0 \quad \text{on } \partial\Omega. \tag{0.2}$$

In this paper, three existence results (Theorems 2.1–2.3) for Problem (0.1) are established according to whether, roughly speaking,

(i) $\int_\Omega \theta(x)dx < 0$, *or*
(ii) $\theta(x) \le \lambda_2$ *for almost all* $x \in \Omega$ *while* $N < p$, *or else*
(iii) $\theta(x) < \lambda_2$ *almost everywhere in* $\Omega$.

From a technical point of view, the proofs of Theorems 2.1–2.3 chiefly exploit a locally Lipschitz continuous version, due to Motreanu-Varga [16, Theorem 2.1] (cf. also [12, Theorem 5]), of Ambrosetti-Rabinowitz's mountain pass theorem [21, Theorem 2.2].

Neumann problems with $p$-Laplacian and non-smooth potential have recently been studied in several papers [20,13,2,11,19,9] under various hypotheses. For instance, [9] requires that $\xi \mapsto J(x,\xi)$ be bounded, while a Landesman-Lazer type condition is assumed in [19]. The approaches of [13,2,11] do not employ mountain pass type arguments. Finally, the interesting work [20] treats the case when

$$J(x,\xi) := \frac{\lambda}{p}|\xi|^p + \int_0^\xi j(x,t)dt, \quad (x,\xi) \in \Omega \times \mathbb{R},$$

where $0 < \lambda < \lambda_2$ and $t \mapsto |j(x,t)|$ behaves as $|t|^s$ with $0 \le s < p-1$; see [20, Theorem 6]. Obviously, such a $J$ fulfils (iii).

Hemivariational inequalities have by now been widely investigated. Besides [18,17,14] we cite the very recent monographs [15,10] as general references on this subject.

## 1. Basic definitions and auxiliary results

Let $(X, \|\cdot\|)$ be a real Banach space. The symbol $X^*$ indicates the dual space of $X$, while $\langle \cdot, \cdot \rangle_X$ stands for the duality pairing between $X$ and $X^*$. A function $\Phi : X \to \mathbb{R}$ is called coercive when

$$\lim_{\|x\| \to +\infty} \Phi(x) = +\infty.$$

If to every $x \in X$ there correspond a neighbourhood $V_x$ of $x$ and a constant $L_x \geq 0$ such that

$$|\Phi(z) - \Phi(w)| \leq L_x \|z - w\| \quad \forall\, z, w \in V_x,$$

then we say that $\Phi$ is locally Lipschitz continuous. In this case $\Phi^0(x; z)$, $x, z \in X$, denotes the generalized directional derivative of $\Phi$ at the point $x$ along the direction $z$, namely

$$\Phi^0(x; z) := \limsup_{w \to x,\, t \to 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t}.$$

It is known [6, Proposition 2.1.1] that the function $\Phi^0$ is upper semicontinuous on $X \times X$. The generalized gradient of the function $\Phi$ at $x$, indicated by $\partial \Phi(x)$, is the set

$$\partial \Phi(x) := \left\{ x^* \in X^* : \langle x^*, z \rangle_X \leq \Phi^0(x; z)\; \forall\, z \in X \right\}.$$

Clearly, $\partial \Phi(x) = \{\Phi'(x)\}$ whenever $\Phi \in C^1(X)$ and if $\lambda$ is any real number then $\partial \Phi(\lambda x) = \lambda \partial \Phi(x)$. Proposition 2.1.2 of [6] ensures that $\partial \Phi(x)$ is nonempty and convex, in addition to being weak* compact, and that

$$\Phi^0(x; z) = \max\{\langle x^*, z \rangle_X : x^* \in \partial \Phi(x)\}, \quad z \in X.$$

Hence, it makes sense to write

$$m_\Phi(x) := \min\{\|x^*\|_{X^*} : x^* \in \partial \Phi(x)\}.$$

For locally Lipschitz continuous $\Phi_1, \Phi_2 : X \to \mathbb{R}$ one evidently has

$$(\Phi_1 + \Phi_2)^0(x; z) \leq \Phi_1^0(x; z) + \Phi_2^0(x; z) \quad \forall\, x, z \in X,$$

which implies

$$\partial(\Phi_1 + \Phi_2)(x) \subseteq \partial \Phi_1(x) + \partial \Phi_2(x), \quad x \in X.$$

In this setting, the classical Palais-Smale compactness condition for $C^1$ functions becomes (see [5, Definition 2]):

$(\mathrm{PS})_\Phi$ *Every sequence* $\{x_n\} \subseteq X$ *such that* $\{\Phi(x_n)\}$ *is bounded and* $m_\Phi(x_n) \to 0$ *as* $n \to +\infty$ *possesses a convergent subsequence.*

We say that $x \in X$ is a critical point of $\Phi$ when $0 \in \partial\Phi(x)$, namely $\Phi^0(x;z) \geq 0$ for all $z \in X$.

The next non-smooth version of Theorem 2.1 in [8] is due to Motreanu-Varga [16, Theorem 2.1] (cf. also [12, Theorem 5]) and will be exploited in Section 2.

Let $X$ be reflexive and let $Q$, $Q_0$, $S$ be nonempty closed subsets of $X$ such that $Q_0 \subseteq Q$. Write

$$\Gamma := \left\{ \gamma \in C^0(Q,X) : \gamma|_{Q_0} = \mathrm{id}|_{Q_0} \right\}.$$

The pair $(Q,Q_0)$ is said to link with $S$ provided $Q_0 \cap S = \emptyset$ and $\gamma(Q) \cap S \neq \emptyset$ for every $\gamma \in \Gamma$.

**Theorem 1.1.** *Suppose $(Q,Q_0)$ links with $S$ while $\Phi : X \to \mathbb{R}$ satisfies the following assumptions in addition to $(\mathrm{PS})_\Phi$.*

($a_1$) $\sup_{x\in Q} \Phi(x) < +\infty$.
($a_2$) $\sup_{x\in Q_0} \Phi(x) \leq \inf_{x\in S} \Phi(x)$.

*Then the function $\Phi$ has a critical point $x \in X \setminus Q_0$.*

**Remark 1.1.** This result remains true if $(\mathrm{PS})_\Phi$ is replaced by condition $(\mathrm{C})_\Phi$ below, which is more general. For the proof we refer to [12, Theorem 5].

$(\mathrm{C})_\Phi$ *Every sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ is bounded and*

$$\lim_{n\to+\infty} (1 + \|x_n\|) m_\Phi(x_n) = 0$$

*possesses a convergent subsequence.*

Throughout the paper $\Omega$ denotes a bounded domain of the real Euclidean $N$-space $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$, with a smooth boundary $\partial\Omega$, $p \in [2,+\infty[$, $p' := p/(p-1)$, and $\|\cdot\|_p$ is the usual norm of $W^{1,p}(\Omega)$; i.e.,

$$\|u\|_p := \left( \int_\Omega (|\nabla u(x)|^p + |u(x)|^p) dx \right)^{1/p}.$$

It is known (see for instance [7, Theorem 3.9.34]) that there exists a unique linear continuous operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ satisfying $\tau(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$. Given $u \in W^{1,p}(\Omega)$, the function $\tau(u)$ is usually called the trace of $u$ on $\partial\Omega$, while $W^{1/p',p}(\partial\Omega) := \tau(W^{1,p}(\Omega))$. Indicate by $p^*$ the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \subseteq L^q(\Omega)$. Recall that if $p < N$ then $p^* = \frac{Np}{N-p}$, for every $q \in [1,p^*]$ there exists a constant $c_q > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq c_q \|u\|_p, \quad u \in W^{1,p}(\Omega),$$

and the embedding is compact whenever $q \in [1,p^*[$.

The symbol $\lambda_2$ denotes the first nonzero eigenvalue of Problem (0.2). One clearly has $\lambda_2 > 0$. Further properties of $\lambda_2$ can be found in [20, Section 3].

Finally, to shorten notation we define, if $u : \Omega \to \mathbb{R}$ and $M \in \mathbb{R}$,

$$\Omega(u(x) < M) := \{x \in \Omega : \, u(x) < M\}.$$

The meaning of $\Omega(u(x) \leq M)$, $\Omega(u(x) > M)$, and $\Omega(u(x) \geq M)$ is analogous. From now on, 'measurable' always signifies Lebesgue measurable while $m(E)$ indicates the measure of $E$.

## 2. Existence results

The following hypotheses on the non-smooth potential $J : \Omega \times \mathbb{R} \to \mathbb{R}$ will be posited in the sequel.

($j_1$) *$x \mapsto J(x, \xi)$ is measurable for all $\xi \in \mathbb{R}$ and $J(\cdot, 0) \in L^\infty(\Omega)$.*

($j_2$) *$\xi \mapsto J(x, \xi)$ is locally Lipschitz continuous for almost every $x \in \Omega$.*

On account of ($j_2$) we may consider both the generalized directional derivative $J^0(x, \xi; \cdot)$ and the generalized gradient $\partial J(x, \xi)$ of $J(x, \xi)$ in the variable $\xi$.

A function $u \in W^{1,p}(\Omega)$ is called a weak solution to Problem (0.1) provided there exists $v \in L^{p'}(\Omega)$ such that $v(x) \in \partial J(x, u(x))$ almost everywhere in $\Omega$ and

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx = \int_\Omega v(x) w(x) dx \quad \forall\, w \in W^{1,p}(\Omega).$$

In this case,

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx \leq \int_\Omega J^0(x, u(x); w(x)) dx, \quad w \in W^{1,p}(\Omega),$$

namely $u$ turns out to be a solution of the hemivariational inequality stemming from (0.1).

($j_3$) *There is a constant $a_1 > 0$ such that for almost all $x \in \Omega$ and every $\xi \in \mathbb{R}$ one has*

$$|y| \leq a_1 \left(1 + |\xi|^{p-1}\right) \quad \forall\, y \in \partial J(x, \xi).$$

Since $J(\cdot, 0) \in L^\infty(\Omega)$, thanks to [14, Theorem 1.1] the above inequality immediately forces

$$|J(x, \xi)| \leq a_2 \left(1 + |\xi|^p\right) \quad \text{a.e. in } \Omega \text{ and for any } \xi \in \mathbb{R}, \tag{2.1}$$

where $a_2 > 0$.

($j_4$) *$J(x, \xi) \leq \frac{\lambda_2}{p} |\xi|^p$ almost everywhere in $\Omega$ and for all $\xi \in \mathbb{R}$.*

($j_5$) *There exists a function* $\theta \in L^\infty(\Omega)$ *such that*

$$\limsup_{|\xi|\to+\infty} \frac{pJ(x,\xi)}{|\xi|^p} \leq \theta(x)$$

*uniformly for almost all* $x \in \Omega$ *and* $\int_\Omega \theta(x)dx < 0$.

($j_6$) *There are constants* $\mu > p$, $M > 0$ *such that* $\mu J(x,\xi) \leq -J^0(x,\xi;-\xi)$ *almost everywhere in* $\Omega$ *and for any* $|\xi| \geq M$.

Obviously, ($j_6$) looks like a non-smooth version of the classical Ambrosetti-Rabinowitz condition [21, page 9]. However, we do not require here that $J(x,\xi) > 0$ in $\Omega \times \mathbb{R}$.

($j_7$) *There exists an* $\eta > 0$ *such that* $\int_\Omega J(x,\pm\eta)dx \geq 0$.

**Example 2.1.** A simple verification shows that the function $J : \Omega \times \mathbb{R} \to \mathbb{R}$ given by setting, for every $(x,\xi) \in \Omega \times \mathbb{R}$,

$$J(x,\xi) := \begin{cases} \frac{\lambda_2}{p}|\xi|^p & \text{if } |\xi| \leq 1, \\ -\frac{1}{p}(|\xi|^p - 1) + \frac{\lambda_2}{p} & \text{otherwise,} \end{cases}$$

fulfils ($j_1$)–($j_5$), ($j_6$) with any $\mu > p$, and ($j_7$).

Now, write $X := W^{1,p}(\Omega)$ and define

$$\Phi(u) := \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega J(x,u(x))dx \quad \forall\, u \in X.$$

Due to ($j_1$)–($j_3$) the function $\Phi$ is locally Lipschitz continuous; see for instance [7, Theorem 5.6.39].

**Proposition 2.1.** *If hypotheses* ($j_1$)–($j_3$), ($j_5$), *and* ($j_6$) *hold true then* $\Phi$ *satisfies condition* $(\mathrm{PS})_\Phi$.

**Proof.** Pick a sequence $\{u_n\} \subseteq X$ such that $\{\Phi(u_n)\}$ is bounded and $m_\Phi(u_n) \to 0$ as $n \to +\infty$. One clearly has $m_\Phi(u_n) = \|u_n^*\|_{X^*}$ for some point $u_n^* \in \partial\Phi(u_n)$. Exploiting Theorem 5.6.39 of [7] we then get

$$u_n^* = A(u_n) - v_n,$$

where $A : X \to X^*$ is the nonlinear operator defined by

$$\langle A(u), v\rangle_X := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx \quad \forall\, u,v \in X, \tag{2.2}$$

while $v_n \in L^{p'}(\Omega)$ and $v_n(x) \in \partial J(x,u_n(x))$ for almost every $x \in \Omega$. In particular, it means that

$$v_n(x)\xi \leq J^0(x,u_n(x);\xi), \quad \xi \in \mathbb{R}.$$

Put $\varepsilon_n := \|u_n^*\|_{X^*}$ and observe that, thanks to the above inequality,

$$-\|\nabla u_n\|^p_{L^p(\Omega,\mathbb{R}^N)} - \int_\Omega J^0(x,u_n(x);-u_n(x))dx$$
$$\leq -\|\nabla u_n\|^p_{L^p(\Omega,\mathbb{R}^N)} + \int_\Omega u_n(x)v_n(x)dx$$
$$= -\langle A(u_n)-v_n,u_n\rangle_X = -\langle u_n^*,u_n\rangle_X \leq \varepsilon_n\|u_n\|_p \quad \forall\, n\in\mathbb{N}. \tag{2.3}$$

Since $|\Phi(u_n)| \leq M_1$ for some $M_1>0$, we also have

$$\frac{\mu}{p}\|\nabla u_n\|^p_{L^p(\Omega,\mathbb{R}^N)} - \int_\Omega \mu J(x,u_n(x))dx \leq \mu M_1\,, \quad n\in\mathbb{N}. \tag{2.4}$$

Gathering (2.3) and (2.4) together yields

$$\Big(\frac{\mu}{p}-1\Big)\|\nabla u_n\|^p_{L^p(\Omega,\mathbb{R}^N)} - \int_\Omega \big[\mu J(x,u_n(x)) + J^0(x,u_n(x);-u_n(x))\big]\,dx$$
$$\leq \mu M_1 + \varepsilon_n\|u_n\|_p \quad \forall\, n\in\mathbb{N}. \tag{2.5}$$

On account of ($j_3$), besides (2.1), we obtain

$$\int_{\Omega(|u_n(x)|<M)} \big[\mu J(x,u_n(x)) + J^0(x,u_n(x);-u_n(x))\big]\,dx \leq M_2$$

for some $M_2>0$. Since, owing to ($j_6$),

$$\int_{\Omega(|u_n(x)|\geq M)} \big[\mu J(x,u_n(x)) + J^0(x,u_n(x);-u_n(x))\big]\,dx \leq 0,$$

from (2.5) it immediately follows

$$\Big(\frac{\mu}{p}-1\Big)\|\nabla u_n\|^p_{L^p(\Omega,\mathbb{R}^N)} \leq \varepsilon_n\|u_n\|_p + M_3 \quad \forall\, n\in\mathbb{N}, \tag{2.6}$$

with $M_3 := \mu M_1 + M_2$. Inequality (2.6) implies that $\{u_n\}$ is bounded. Indeed, if the assertion were false then, passing to a subsequence when necessary,

$$\lim_{n\to+\infty}\|u_n\|_p = +\infty. \tag{2.7}$$

Define $w_n := \frac{u_n}{\|u_n\|_p}$, $n\in\mathbb{N}$. Obviously, we may suppose

$$w_n \rightharpoonup w \text{ in } X \text{ and } w_n \to w \text{ in } L^p(\Omega), \tag{2.8}$$

because $\{w_n\}\subseteq X$ turns out bounded. Moreover, by (2.6),

$$\Big(\frac{\mu}{p}-1\Big)\|\nabla w_n\|^p_{L^p(\Omega,\mathbb{R}^N)} \leq \frac{\varepsilon_n}{\|u_n\|_p^{p-1}} + \frac{M_3}{\|u_n\|_p^p}, \quad n\in\mathbb{N}, \tag{2.9}$$

which, on account of (2.7)–(2.8), leads to $\|\nabla w\|^p_{L^p(\Omega,\mathbb{R}^N)} = 0$; i.e., $w(x) \equiv \bar{\xi}$ in $\Omega$ for some $\bar{\xi} \in \mathbb{R}$. Now, let

$$V := \Big\{ v \in X : \int_\Omega v(x)dx = 0 \Big\}. \tag{2.10}$$

Since $X = \mathbb{R} \oplus V$, each $w_n$ can be written as $w_n = \bar{w}_n + \hat{w}_n$, with $\bar{w}_n \in \mathbb{R}$, $\hat{w}_n \in V$. Exploiting the Poincaré-Wirtinger inequality [1, page 194], besides (2.9), provides $\lim_{n\to+\infty} \|\hat{w}_n\|_p = 0$. Consequently, thanks to (2.8),

$$w_n \to \bar{\xi} \quad \text{in } X \tag{2.11}$$

and $\bar{\xi} \neq 0$ because $\|w_n\|_p = 1$ for all $n \in \mathbb{N}$. Pick $\varepsilon > 0$. Combining ($\mathrm{j}_5$) with (2.1) we can find a $c_\varepsilon > 0$ such that

$$J(x,\xi) < \frac{1}{p}\left(\theta(x) + \varepsilon\right)|\xi|^p + c_\varepsilon \quad \text{a.e. in } \Omega \text{ and for all } \xi \in \mathbb{R}.$$

On account of (2.7), the choice of $w_n$, and (2.11) one has

$$\limsup_{n\to+\infty} \int_\Omega \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx \le \frac{|\bar{\xi}|^p}{p} \int_\Omega \left(\theta(x) + \varepsilon\right) dx.$$

As $\varepsilon$ was arbitrary, we actually get

$$\limsup_{n\to+\infty} \int_\Omega \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx \le \frac{|\bar{\xi}|^p}{p} \int_\Omega \theta(x) dx.$$

Thus, due to ($\mathrm{j}_5$),

$$\limsup_{n\to+\infty} \int_\Omega \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx < 0. \tag{2.12}$$

On the other hand, the inequality $|\Phi(u_n)| \le M_1$, $n \in \mathbb{N}$, forces

$$\frac{1}{p}\|\nabla w_n\|^p_{L^p(\Omega,\mathbb{R}^N)} \le \frac{M_1}{\|u_n\|_p^p} + \int_\Omega \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx \quad \forall\, n \in \mathbb{N}.$$

By (2.7) and (2.11) this yields

$$\liminf_{n\to+\infty} \int_\Omega \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx \ge 0,$$

which contradicts (2.12). Therefore, the sequence $\{u_n\} \subseteq X$ is bounded. Along a subsequence when necessary, we may thus suppose

$$u_n \rightharpoonup u \text{ in } X, \quad u_n \to u \text{ in } L^p(\Omega), \tag{2.13}$$

and so $u_n(x) \to u(x)$, $|u_n(x)| \le w(x)$, $n \in \mathbb{N}$, for almost every $x \in \Omega$, where $w \in L^p(\Omega)$. Now, bearing in mind ($j_3$), one has

$$|\langle A(u_n), u_n - u\rangle_X| \le \|u_n^*\|_{X^*}\|u_n - u\|_p + \left|\int_\Omega v_n(x)(u_n(x) - u(x))dx\right|$$
$$\le \varepsilon_n\|u_n - u\|_p + a_1\int_\Omega (1 + w(x)^{p-1})|u_n(x) - u(x)|dx \quad \forall\, n \in \mathbb{N},$$

which evidently implies

$$\lim_{n\to\infty}\langle A(u_n), u_n - u\rangle_X = 0.$$

Since $p \ge 2$, a standard argument (see e.g. [4, page 3]) leads to

$$\langle A(u_n) - A(u), u_n - u\rangle_X \ge C_p\|\nabla u_n - \nabla u\|^p_{L^p(\Omega,\mathbb{R}^N)}, \quad n \in \mathbb{N},$$

where $C_p > 0$. Hence, $\nabla u_n \to \nabla u$ in $L^p(\Omega, \mathbb{R}^N)$ as $n \to +\infty$. Via (2.13) we finally achieve $u_n \to u$ in $X$. □

Define

$$S := \left\{u \in X : \int_\Omega |u(x)|^{p-2}u(x)dx = 0\right\}. \tag{2.14}$$

A simple computation ensures that $S$ is a closed set such that $\lambda u \in S$ for all $\lambda \ge 0$, $u \in S$. Moreover, the following Poincaré-Wirtinger's type inequality holds:

$$\lambda_2\|u\|^p_{L^p(\Omega)} \le \|\nabla u\|^p_{L^p(\Omega,\mathbb{R}^N)} \quad \forall\, u \in S. \tag{2.15}$$

This can be easily verified through the same arguments exploited to prove Corollary 4.2.1 in [10].

Combining Theorem 1.1 with Proposition 2.1 provides the next existence result.

**Theorem 2.1.** *Let* ($j_1$)–($j_7$) *be satisfied. Then Problem* (0.1) *possesses at least one weak solution* $u \in W^{1,p}(\Omega)$.

**Proof.** Write $Q_0 := \{-\eta, \eta\}$, where $\eta > 0$ is given by ($j_7$), and

$$Q := \{u \in X : \|u\|_p \le \eta m(\Omega)^{1/p},\ \ |u(x)| \le \eta \text{ for almost every } x \in \Omega\}.$$

One clearly has $Q_0 \cap S = \emptyset$, where $S$ is as in (2.14), $Q_0 \subseteq Q$, while $Q_0$ and $Q$ are closed subsets of $X$. We claim that $(Q, Q_0)$ links with $S$. To see this, pick $\gamma \in C^0(Q, X)$ fulfilling $\gamma|_{Q_0} = \mathrm{id}|_{Q_0}$. If

$$g(u) := \int_\Omega |u(x)|^{p-2}u(x)dx \quad \forall\, u \in X,$$

then $g(\gamma(-\eta)) < 0 < g(\gamma(\eta))$. Since $g \circ \gamma \in C^0(Q)$ and $Q$ is convex, there exists a $\tilde{u} \in Q$ such that $g(\gamma(\tilde{u})) = 0$, which implies $\gamma(\tilde{u}) \in S$. Hence, $\gamma(Q) \cap S \neq \emptyset$, and the assertion follows.

Let us next note that $\sup_{u\in Q} \Phi(u) < +\infty$ because, on account of (2.1),

$$\Phi(u) \le \frac{1}{p}\eta^p m(\Omega) + a_2(1+\eta^p)m(\Omega)$$

for any $u \in Q$. Through ($j_7$) we then get

$$\Phi(\pm\eta) = -\int_\Omega J(x, \pm\eta)dx \le 0.$$

Moreover, by ($j_4$) and (2.15),

$$\Phi(u) \ge \frac{1}{p}\left(\|\nabla u\|^p_{L^p(\Omega,\mathbb{R}^N)} - \lambda_2\|u\|^p_{L^p(\Omega)}\right) \ge 0 \quad \forall\, u \in S.$$

Consequently,

$$\sup_{u\in Q_0} \Phi(u) \le 0 \le \inf_{u\in S} \Phi(u).$$

Finally, thanks to Proposition 2.1, the function $\Phi$ satisfies $(\mathrm{PS})_\Phi$. So, Theorem 1.1 can be applied, and we obtain a point $u \in X \setminus Q_0$ such that $0 \in \partial\Phi(u)$. In view of [7, Theorem 5.6.39] from this it follows that $A(u) = v$ for some $v \in L^{p'}(\Omega)$, with $A$ given by (2.2), and

$$v(x) \in \partial J(x, u(x)) \quad \text{for almost every } x \in \Omega. \tag{2.16}$$

Now, write $Y := W^{1,p}_0(\Omega)$. Due to [7, page 362] we have

$$\begin{aligned}\langle -\Delta_p u, w\rangle_Y &= \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w(x)dx \\ &= \langle A(u), w\rangle_Y = \int_\Omega v(x)w(x)dx, \quad w \in Y,\end{aligned}$$

which evidently implies

$$-\Delta_p u = v. \tag{2.17}$$

Exploiting [3, Corollary 2] provides

$$\begin{aligned}\int_\Omega \Delta_p u(x)w(x)dx + \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w(x)dx \\ = \langle \frac{\partial u}{\partial n_p}, \tau(w)\rangle_Z \quad \forall\, w \in X,\end{aligned} \tag{2.18}$$

where $Z := W^{1/p',p}(\partial\Omega)$. Owing to (2.17)–(2.18) we get

$$\langle \frac{\partial u}{\partial n_p}, \tau(w)\rangle_Z = -\int_\Omega v(x)w(x)dx + \langle A(u), w\rangle_X = 0, \quad w \in X.$$

Since $\tau(X) = Z$, this forces $\frac{\partial u}{\partial n_p} = 0$ in $Z^*$. Therefore, by (2.16)–(2.18), the function $u \in X$ turns out to be a weak solution of (0.1). □

The next existence result concerns "low dimensional" problems, in the sense that $N < p$. Let the non-smooth potential $J : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy $(j_1)$–$(j_3)$. Suppose also

- $(j_8)$ *there is a function* $\beta \in L^1(\Omega)$ *such that for almost every* $x \in \Omega$ *and any* $\xi \in \mathbb{R}$ *one has* $\sup \partial J(x,\xi) \leq \beta(x)$*,*
- $(j_9)$ $\limsup_{|\xi|\to+\infty} \dfrac{pJ(x,\xi)}{|\xi|^p} \leq \lambda_2$ *uniformly for almost all* $x \in \Omega$*, and*
- $(j_{10})$ $\lim_{|\eta|\to+\infty} \int_\Omega J(x,\eta)dx = +\infty$.

**Example 2.2.** An easy verification ensures that the function $J : \Omega \times \mathbb{R} \to \mathbb{R}$ given by setting, for every $(x,\xi) \in \Omega \times \mathbb{R}$,

$$J(x,\xi) := \begin{cases} \frac{\lambda_2}{p}|\xi|^p + \frac{p-\lambda_2}{p} & \text{if } \xi < -1, \\ |\xi| & \text{if } |\xi| \leq 1, \\ \sqrt{\xi} & \text{otherwise,} \end{cases}$$

where $p > N$, fulfils $(j_1)$–$(j_3)$ and $(j_8)$–$(j_{10})$.

**Proposition 2.2.** *If* $p > N$ *and assumptions* $(j_1)$–$(j_3)$*,* $(j_8)$*,* $(j_{10})$ *hold true then* $\Phi$ *satisfies condition* $(\mathrm{PS})_\Phi$*.*

**Proof.** Pick a sequence $\{u_n\} \subseteq X$ such that $\{\Phi(u_n)\}$ is bounded and $m_\Phi(u_n) \to 0$ as $n \to +\infty$. As in the proof of Proposition 2.1 one has $m_\Phi(u_n) = \|u_n^*\|_{X^*}$ for some point $u_n^* \in \partial\Phi(u_n)$. Moreover, $u_n^* = A(u_n) - v_n$, where $A$ is given by (2.2) while $v_n \in L^{p'}(\Omega)$ and $v_n(x) \in \partial J(x, u_n(x))$ for almost every $x \in \Omega$. To shorten notation, write $\varepsilon_n := \|u_n^*\|_{X^*}$. Observe that

$$\Big|\int_\Omega v_n(x)dx\Big| = |\langle u_n^*, 1\rangle_X| \leq \varepsilon_n m(\Omega)^{1/p} \quad \forall\, n \in \mathbb{N}.$$

Consequently, on account of $(j_8)$,

$$\begin{aligned}\Big|\int_{\Omega(v_n(x)<0)} v_n(x)dx\Big| &\leq \Big|\int_\Omega v_n(x)dx\Big| + \int_{\Omega(v_n(x)\geq 0)} v_n(x)dx \\ &\leq \varepsilon_n m(\Omega)^{1/p} + \|\beta\|_{L^1(\Omega)}, \quad n \in \mathbb{N}.\end{aligned}$$

From this it follows that

$$\int_\Omega |v_n(x)|dx = \int_{\Omega(v_n(x)\geq 0)} v_n(x)dx - \int_{\Omega(v_n(x)<0)} v_n(x)dx$$

$$\leq \|\beta\|_{L^1(\Omega)} + \Big| \int_{\Omega(v_n(x)<0)} v_n(x)dx \Big| \leq \varepsilon_n m(\Omega)^{1/p} + 2\|\beta\|_{L^1(\Omega)},$$

namely

$$\|v_n\|_{L^1(\Omega)} \leq M_2 \quad \forall\, n \in \mathbb{N}, \tag{2.19}$$

where $M_2 > 0$.

Next, since $X = \mathbb{R} \oplus V$, with $V$ given by (2.10), each $u_n$ can be written as

$$u_n = \bar{u}_n + \hat{u}_n \quad \text{for some } \bar{u}_n \in \mathbb{R},\ \hat{u}_n \in V.$$

Due to the properties of $u_n^*$ we get

$$\Big| \langle A(u_n), \hat{u}_n\rangle_X - \int_\Omega v_n(x)\hat{u}_n(x)dx \Big| = |\langle u_n^*, \hat{u}_n\rangle_X| \leq \varepsilon_n \|\hat{u}_n\|_p, \quad n \in \mathbb{N}.$$

Recalling that $\hat{u}_n \in L^\infty(\Omega)$, because $p > N$, and exploiting the above inequality, besides (2.19), yields

$$\begin{aligned} \|\nabla \hat{u}_n\|^p_{L^p(\Omega,\mathbb{R}^N)} &\leq \Big| \int_\Omega v_n(x)\hat{u}_n(x)dx \Big| + \varepsilon_n \|\hat{u}_n\|_p \\ &\leq M_2 \|\hat{u}_n\|_{L^\infty(\Omega)} + \varepsilon_n \|\hat{u}_n\|_p \quad \forall\, n \in \mathbb{N}. \end{aligned} \tag{2.20}$$

Thanks to the Poincaré-Wirtinger inequality [1, page 194] one has

$$\|\hat{u}_n\|_p^p \leq (M_3 + 1)\|\nabla \hat{u}_n\|^p_{L^p(\Omega,\mathbb{R}^N)}, \quad n \in \mathbb{N}, \tag{2.21}$$

where $M_3 > 0$. Once we bear in mind that $W^{1,p}(\Omega)$ continuously embeds in $L^\infty(\Omega)$, while $\varepsilon_n \to 0^+$, from (2.20) and (2.21) it immediately follows that

$$\|\hat{u}_n\|_p^p \leq M_4 \|\hat{u}_n\|_p, \quad n \in \mathbb{N},$$

for some $M_4 > 0$. Hence, $\{\hat{u}_n\}$ is bounded in $X$. If $\{u_n\}$ does not enjoy the same property then, passing to a subsequence when necessary,

$$\lim_{n\to+\infty} \|u_n\|_p = +\infty. \tag{2.22}$$

Define $w_n := \frac{u_n}{\|u_n\|_p}$, $n \in \mathbb{N}$. Obviously, we may suppose

$$w_n \rightharpoonup w \quad \text{in } X \quad \text{and} \quad w_n \to w \quad \text{in } C^0(\overline{\Omega}). \tag{2.23}$$

Since $X = \mathbb{R} \oplus V$, each $w_n$ can be written as $w_n = \bar{w}_n + \hat{w}_n$, with $\bar{w}_n \in \mathbb{R}$, $\hat{w}_n \in V$. Using (2.22) provides

$$\lim_{n\to+\infty} \|\hat{w}_n\|_p = \lim_{n\to+\infty} \frac{\|\hat{u}_n\|_p}{\|u_n\|_p} = 0.$$

Consequently, $w(x) \equiv \bar\xi$ for some $\bar\xi \in \mathbb{R}$, $w_n \to \bar\xi$ in $X$ by (2.23), and $\bar\xi \neq 0$ because $\|w_n\|_p = 1$ for all $n \in \mathbb{N}$. On account of (2.22)–(2.23) this implies $\lim_{n\to+\infty} |u_n(x)| = +\infty$ in $\overline{\Omega}$. Let us next verify that

$$\lim_{n\to+\infty} |\max_{x\in\overline{\Omega}} u_n(x)| = +\infty. \tag{2.24}$$

Thanks to (2.23) there exists a positive integer $\nu$ fulfilling

$$|w_n(x)| > \frac{|\bar\xi|}{2} \quad \forall\, x \in \overline{\Omega},\ n > \nu,$$

which forces

$$|u_n(x)| = |w_n(x)|\|u_n\|_p > \frac{|\bar\xi|}{2}\|u_n\|_p, \quad x \in \overline{\Omega},\ n > \nu.$$

Hence,

$$|\max_{x\in\overline{\Omega}} u_n(x)| > \frac{|\bar\xi|}{2}\|u_n\|_p \quad \forall\, n > \nu,$$

and (2.24) follows. Let $\eta_n := \max_{x\in\overline{\Omega}} u_n(x)$, $n \in \mathbb{N}$. Bearing in mind assumption ($\mathrm{j}_{10}$) we thus have

$$\lim_{n\to+\infty} \int_\Omega J(x,\eta_n)dx = +\infty. \tag{2.25}$$

By [6, Theorem 7.1.1] and a straightforward measurable selection argument there exist two measurable functions $v_n : \Omega \to \mathbb{R}$, $\lambda_n : \Omega \to [0,1]$ such that

$$J(x,u_n(x)) - J(x,\eta_n) = v_n(x)\left(u_n(x) - \eta_n\right),$$
$$v_n(x) \in \partial J(x, \lambda_n(x)u_n(x) + (1-\lambda_n(x))\eta_n)$$

for almost every $x \in \Omega$. Since the sequence $\{\Phi(u_n)\}$ is bounded, while, on account of ($\mathrm{j}_8$),

$$v_n(x)\left(u_n(x) - \eta_n\right) \geq \beta(x)\left(u_n(x) - \eta_n\right) \quad \text{a.e. in } \Omega,$$

we achieve

$$\begin{aligned} M_5 &\geq -\frac{1}{p}\|\nabla u_n\|^p_{L^p(\Omega,\mathbb{R}^N)} + \int_\Omega J(x,u_n(x))dx - \int_\Omega J(x,\eta_n)dx + \int_\Omega J(x,\eta_n)dx \\ &\geq -\frac{1}{p}\|\nabla u_n\|^p_{L^p(\Omega,\mathbb{R}^N)} + \int_\Omega \beta(x)(u_n(x)-\eta_n)dx + \int_\Omega J(x,\eta_n)dx \qquad (2.26) \\ &\geq -\frac{1}{p}\|\nabla u_n\|^p_{L^p(\Omega,\mathbb{R}^N)} - \|\beta\|_{L^1(\Omega)}\|u_n - \eta_n\|_{L^\infty(\Omega)} + \int_\Omega J(x,\eta_n)dx, \end{aligned}$$

for some $M_5 > 0$. Define $\hat\eta_n := \max_{x\in\overline{\Omega}} \hat u_n(x)$, $n \in \mathbb{N}$. One clearly has

$$\|u_n - \eta_n\|_{L^\infty(\Omega)} = \max_{x\in\overline{\Omega}} |u_n(x) - \eta_n| = \hat\eta_n - \min_{x\in\overline{\Omega}} \hat u_n(x) \leq M_6 \quad \forall\, n \in \mathbb{N}, \tag{2.27}$$

where $M_6 > 0$, because $\{\hat{u}_n\}$ is bounded in $X$ and so in $C^0(\overline{\Omega})$. Likewise,

$$\|\nabla u_n\|_{L^p(\Omega,\mathbb{R}^N)} = \|\nabla \hat{u}_n\|_{L^p(\Omega,\mathbb{R}^N)} \leq M_7, \quad n \in \mathbb{N}, \tag{2.28}$$

for some $M_7 > 0$. Exploiting (2.26)–(2.28) we realize that

$$\int_\Omega J(x, \eta_n)dx \leq M_8 \quad \forall\, n \in \mathbb{N},$$

where $M_8 > 0$, which contradicts (2.25). Therefore, the sequence $\{u_n\} \subseteq X$ turns out bounded too. Now, the proof goes on exactly as that of Proposition 2.1. □

**Proposition 2.3.** *Let* ($j_1$)–($j_3$), ($j_8$), *and* ($j_9$) *be satisfied. Then the function* $\Phi|_S$, *where* $S$ *is given by* (2.14), *is coercive.*

**Proof.** If the conclusion were false then there would exist a sequence $\{u_n\} \subseteq S$, besides a constant $M_9 > 0$, such that

$$\lim_{n\to+\infty} \|u_n\|_p = +\infty \quad \text{but} \quad \Phi(u_n) \leq M_9 \quad \forall\, n \in \mathbb{N}. \tag{2.29}$$

Define $w_n := \frac{u_n}{\|u_n\|_p}$, $n \in \mathbb{N}$. Obviously, we may suppose (2.23) holds. Moreover, $w \in S$ because $\{w_n\} \subseteq S$ and $w_n \to w$ in $C^0(\overline{\Omega})$. Using (2.1) gives

$$\frac{|J(x, u_n(x))|}{\|u_n\|_p^p} \leq a_2 \left( \frac{1}{\|u_n\|_p^p} + |w_n(x)|^p \right), \quad n \in \mathbb{N}, \tag{2.30}$$

for almost every $x \in \Omega$. Hence, on account of (2.23) and the Dunford-Pettis Theorem (see e.g. [1, Theorem IV.30]), we can find a function $\tilde{w} \in L^1(\Omega)$ fulfilling

$$\frac{J(\cdot, u_n(\cdot))}{\|u_n\|_p^p} \rightharpoonup \tilde{w} \quad \text{in } L^1(\Omega), \tag{2.31}$$

where a subsequence is considered when necessary. Let $\Omega_0 := \{x \in \Omega : w(x) = 0\}$. Inequality (2.30) implies

$$\tilde{w}(x) = 0 \quad \text{for almost every } x \in \Omega_0. \tag{2.32}$$

Furthermore, by (2.23), $|u_n(x)| \to +\infty$ whenever $x \in \Omega \setminus \Omega_0$. Consequently, if $\varepsilon > 0$, $n \in \mathbb{N}$,

$$\Omega_{\varepsilon,n} := \left\{ x \in \Omega :\, u_n(x) \neq 0,\ \frac{J(x, u_n(x))}{|u_n(x)|^p} \leq \frac{1}{p}(\lambda_2 + \varepsilon) \right\},$$

and $\chi_{\varepsilon,n}$ indicates the characteristic function of $\Omega_{\varepsilon,n}$, then, thanks to ($j_9$) we have

$$\lim_{n\to+\infty} \chi_{\varepsilon,n}(x) = 1 \quad \text{a.e. in } \Omega \setminus \Omega_0. \tag{2.33}$$

Combining (2.31) with (2.33) easily yields

$$\chi_{\varepsilon,n}\frac{J(\cdot,u_n(\cdot))}{\|u_n\|_p^p} \rightharpoonup \tilde{w} \quad \text{in } L^1(\Omega\setminus\Omega_0).$$

Now, since

$$\chi_{\varepsilon,n}(x)\frac{J(x,u_n(x))}{\|u_n\|_p^p} \le \frac{1}{p}(\lambda_2+\varepsilon)|w_n(x)|^p \tag{2.34}$$

for almost every $x \in \Omega\setminus\Omega_0$, by the Mazur theorem (see e.g. [7, Corollary 3.4.17]) we get

$$\tilde{w}(x) \le \frac{1}{p}(\lambda_2+\varepsilon)|w(x)|^p \quad \text{a.e. in } \Omega\setminus\Omega_0.$$

On account of (2.32) this inequality can be written with $\Omega\setminus\Omega_0$ replaced by $\Omega$. Thus,

$$\lim_{n\to+\infty}\int_\Omega \frac{J(x,u_n(x))}{\|u_n\|_p^p}dx = \int_\Omega \tilde{w}(x)dx \le \frac{1}{p}(\lambda_2+\varepsilon)\|w\|^p_{L^p(\Omega)}. \tag{2.35}$$

Observe next that, due to (2.29),

$$\frac{1}{p}\|\nabla w_n\|^p_{L^p(\Omega,\mathbb{R}^N)} - \int_\Omega \frac{J(x,u_n(x))}{\|u_n\|_p^p}dx \le \frac{M_9}{\|u_n\|_p^p} \quad \forall\, n\in\mathbb{N}, \tag{2.36}$$

which implies, in view of (2.35),

$$\|\nabla w\|^p_{L^p(\Omega,\mathbb{R}^N)} \le (\lambda_2+\varepsilon)\|w\|^p_{L^p(\Omega)}.$$

Letting $\varepsilon \to 0^+$ and using (2.15) we obtain

$$\|\nabla w\|^p_{L^p(\Omega,\mathbb{R}^N)} \le \lambda_2\|w\|^p_{L^p(\Omega)} \le \|\nabla w\|^p_{L^p(\Omega,\mathbb{R}^N)};$$

i.e., either $w=0$ or $w$ is an eigenfunction of Problem (0.2) corresponding to $\lambda_2$. The second assertion derives from arguments analogous to those made in the proof of [20, Proposition 4]. If $w=0$ then, by (2.35), (2.36), besides (2.29), one should have $\lim_{n\to+\infty}\|\nabla w_n\|_{L^p(\Omega,\mathbb{R}^N)}=0$. Thanks to (2.23) this forces $w_n \to 0$ in $X$, which is impossible because $\|w_n\|_p = 1$ for all $n\in\mathbb{N}$. So,

$$w \in S\setminus\{0\} \quad \text{and} \quad \|\nabla w\|^p_{L^p(\Omega,\mathbb{R}^N)} = \lambda_2\|w\|^p_{L^p(\Omega)}. \tag{2.37}$$

Through [6, Theorem 7.1.1] as well as a straightforward measurable selection argument there exist two measurable functions $v_n : \Omega \to \mathbb{R}$, $\lambda_n : \Omega \to [0,1]$ such that

$$J(x,u_n(x)) - J(x,0) = v_n(x)u_n(x), \quad v_n(x) \in \partial J(x,\lambda_n(x)u_n(x))$$

for almost all $x \in \Omega$. Consequently,

$$\frac{J(x,u_n(x))}{\|u_n\|_p^p} = \frac{J(x,0)}{\|u_n\|_p^p} + \frac{v_n(x)}{\|u_n\|_p^p} u_n(x).$$

Let $\Omega_1 := \Omega(w(x) > 0)$ and let $\Omega_2 := \Omega(w(x) \le 0)$. Since $w_n \to w$ in $C^0(\overline{\Omega})$, by means of ($j_8$) we get, provided $n$ is large enough,

$$\frac{J(x,u_n(x))}{\|u_n\|_p^p} \le \frac{J(x,0)}{\|u_n\|_p^p} + \frac{\beta(x)}{\|u_n\|_p^{p-1}} w_n(x) \quad \text{a.e. in } \Omega_1.$$

Hence, due to (2.33) and (2.29),

$$\limsup_{n\to+\infty} \int_{\Omega_1} \chi_{\varepsilon,n}(x) \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx \le 0. \tag{2.38}$$

Using (2.31)–(2.32) produces

$$\limsup_{n\to+\infty} \int_{\Omega_2} \chi_{\varepsilon,n}(x) \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx = \limsup_{n\to+\infty} \int_{\Omega_2\setminus\Omega_0} \chi_{\varepsilon,n}(x) \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx.$$

Because of (2.30), Fatou's Lemma can be applied, and one has

$$\limsup_{n\to+\infty} \int_{\Omega_2} \chi_{\varepsilon,n}(x) \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx \le \int_{\Omega_2\setminus\Omega_0} \limsup_{n\to+\infty} \chi_{\varepsilon,n}(x) \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx.$$

Owing to (2.34) we thus achieve

$$\limsup_{n\to+\infty} \int_{\Omega_2} \chi_{\varepsilon,n}(x) \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx \le \frac{\lambda_2+\varepsilon}{p} \int_{\Omega_2\setminus\Omega_0} |w(x)|^p dx. \tag{2.39}$$

Since, by (2.31)–(2.33),

$$\lim_{n\to+\infty} \int_\Omega \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx = \lim_{n\to+\infty} \int_\Omega \chi_{\varepsilon,n}(x) \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx$$

for any $\varepsilon > 0$, gathering (2.38)–(2.39) together and letting $\varepsilon \to 0^+$ yields

$$\lim_{n\to+\infty} \int_\Omega \frac{J(x,u_n(x))}{\|u_n\|_p^p} dx \le \frac{\lambda_2}{p} \int_{\Omega_2\setminus\Omega_0} |w(x)|^p dx < \frac{\lambda_2}{p} \|w\|_{L^p(\Omega)}^p, \tag{2.40}$$

where the strict inequality stems from the fact that $w \in S \setminus \{0\}$. Finally, from (2.36) and (2.40) it follows that

$$\|\nabla w\|_{L^p(\Omega,\mathbb{R}^n)}^p < \lambda_2 \|w\|_{L^p(\Omega)}^p,$$

which contradicts (2.37). This completes the proof. □

Combining Theorem 1.1 with Propositions 2.2–2.3 provides the next existence result.

**Theorem 2.2.** *Let $p > N$ and let* $(j_1)$–$(j_3)$, $(j_8)$–$(j_{10})$ *be satisfied. Then Problem* (0.1) *possesses at least one weak solution $u \in W^{1,p}(\Omega)$.*

**Proof.** Thanks to Proposition 2.3 one easily has $\inf_{u\in S}\Phi(u) > -\infty$. Through $(j_{10})$ we thus obtain a constant $\eta > 0$ such that

$$-\int_\Omega J(x, \pm\eta)dx < \inf_{u\in S}\Phi(u). \tag{2.41}$$

Write $Q_0 := \{-\eta, \eta\}$ and

$$Q := \{u \in X : \|u\|_p \le \eta m(\Omega)^{1/p}, \ \ |u(x)| \le \eta \text{ a.e. in } \Omega\}.$$

Evidently, $Q_0 \cap S = \emptyset$, $Q_0 \subseteq Q$, while $Q_0$ and $Q$ are closed subsets of $X$. Arguing as in the proof of Theorem 2.1 then shows that $(Q, Q_0)$ links with $S$, besides $\sup_{u\in Q}\Phi(u) < +\infty$. Since, on account of (2.41),

$$\sup_{u\in Q_0}\Phi(u) < \inf_{u\in S}\Phi(u)$$

and, by Proposition 2.2, condition $(\mathrm{PS})_\Phi$ holds true, Theorem 1.1 can be applied. Hence, there exists a point $u \in X \setminus Q_0$ such that $0 \in \partial\Phi(u)$. Now, the proof goes on exactly as that of Theorem 2.1. □

The last existence result of this section does not require that $N < p$, but a strict inequality must appear in $(j_9)$. Let the non-smooth potential $J : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfil $(j_1)$–$(j_3)$ and $(j_{10})$. Suppose also

- $(j_9')$ $\limsup_{|\xi|\to+\infty} \dfrac{pJ(x,\xi)}{|\xi|^p} < \lambda_2$ *uniformly for almost all $x \in \Omega$, and*
- $(j_{11})$ $\lim_{|\xi|\to+\infty} \sup_{y\in\partial J(x,\xi)} (\xi y - pJ(x,\xi)) = -\infty$ *uniformly almost everywhere in $\Omega$.*

**Example 2.3.** An easy verification ensures that the function $J : \Omega\times\mathbb{R} \to \mathbb{R}$ given by setting, for every $(x,\xi) \in \Omega \times \mathbb{R}$,

$$J(x,\xi) := \begin{cases} \xi^2 \log|\xi| & \text{if } |\xi| \le 1, \\ \frac{\bar\theta}{p}|\xi|^p + |\xi| - \frac{\bar\theta}{p} - 1 & \text{otherwise,} \end{cases}$$

where $0 < \bar\theta < \lambda_2$, satisfies $(j_1)$–$(j_3)$, $(j_9')$, $(j_{10})$–$(j_{11})$.

**Proposition 2.4.** *If hypotheses* $(j_1)$–$(j_3)$, $(j_9')$, *and* $(j_{11})$ *hold true then $\Phi$ fulfils condition* $(\mathrm{C})_\Phi$.

**Proof.** Pick a sequence $\{u_n\} \subseteq X$ such that $\{\Phi(u_n)\}$ is bounded and

$$\lim_{n\to+\infty}(1 + \|u_n\|_p)m_\Phi(u_n) = 0. \tag{2.42}$$

As in the proof of Proposition 2.1 one has $m_\Phi(u_n) = \|u_n^*\|_{X^*}$ for some point $u_n^* \in \partial\Phi(u_n)$. Moreover, $u_n^* = A(u_n) - v_n$, where $A$ is given by (2.2) while $v_n \in L^{p'}(\Omega)$ and $v_n(x) \in \partial J(x, u_n(x))$ almost everywhere in $\Omega$. We may assume that $\{\Phi(u_n)\}$ converges to a real number, say $l$, which leads to

$$\lim_{n\to+\infty} \int_\Omega (v_n(x)u_n(x) - pJ(x, u_n(x)))\, dx = \lim_{n\to+\infty} (p\Phi(u_n) - \langle u_n^*, u_n\rangle) = pl \tag{2.43}$$

because of (2.42) besides the obvious inequality

$$|\langle u_n^*, u_n\rangle_X| \le (1 + \|u_n\|_p) m_\Phi(u_n) \quad \forall\, n \in \mathbb{N}.$$

If $\{u_n\}$ were unbounded then, passing to a subsequence when necessary, (2.22) holds. Define $w_n := \frac{u_n}{\|u_n\|_p}$, $n \in \mathbb{N}$. It is not restrictive to suppose

$$w_n \rightharpoonup w \quad \text{in } X \quad \text{and} \quad w_n \to w \quad \text{in } L^p(\Omega). \tag{2.44}$$

Arguing as in the proof of Proposition 2.3 provides a function $\tilde{w} \in L^1(\Omega)$ such that

$$\frac{J(\cdot, u_n(\cdot))}{\|u_n\|_p^p} \rightharpoonup \tilde{w} \quad \text{in } L^1(\Omega) \quad \text{and} \quad \tilde{w}(x) \le \frac{\bar{\theta}}{p}|w(x)|^p \quad \text{a.e. in } \Omega \tag{2.45}$$

for some $\bar{\theta} \in ]0, \lambda_2[$. Since $\{\Phi(u_n)\}$ is bounded, one has

$$\frac{1}{p}\|\nabla w_n\|^p_{L^p(\Omega,\mathbb{R}^N)} \le \int_\Omega \frac{J(x, u_n(x))}{\|u_n\|_p^p} dx + \frac{M_{10}}{\|u_n\|_p^p} \quad \forall\, n \in \mathbb{N}, \tag{2.46}$$

where $M_{10} > 0$. In view of (2.44)–(2.46), letting $n \to +\infty$ we obtain

$$\frac{1}{p}\|\nabla w\|^p_{L^p(\Omega,\mathbb{R}^N)} \le \int_\Omega \tilde{w}(x) dx \le \frac{\bar{\theta}}{p}\|w\|^p_{L^p(\Omega)}. \tag{2.47}$$

If $w = 0$ then, by (2.46)–(2.47), besides (2.44), $\|w_n\|_p \to 0$, against $\|w_n\|_p = 1$ for all $n \in \mathbb{N}$. So, $w \ne 0$ and, consequently,

$$\lim_{n\to+\infty} |u_n(x)| = +\infty \quad \text{a.e. in } \Omega \setminus \Omega_0,$$

with $\Omega_0 := \{x \in \Omega :\, w(x) = 0\}$. Gathering ($j_3$) and ($j_{11}$) together yields a constant $M_{11} > 0$ such that, for almost every $x \in \Omega$,

$$\sup_{y \in \partial J(x,\xi)} (\xi y - pJ(x, \xi)) \le M_{11}, \quad \xi \in \mathbb{R}.$$

Since $m(\Omega \setminus \Omega_0) > 0$ while, due to the above inequality,

$$\int_\Omega (v_n(x)u_n(x) - pJ(x, u_n(x)))\, dx \le M_{11} m(\Omega_0)$$

$$+\int_{\Omega\setminus\Omega_0}\sup_{y\in\partial J(x,u_n(x))}(yu_n(x)-pJ(x,u_n(x)))dx \quad \forall\, n\in\mathbb{N},$$

from ($j_{11}$) again it follows that

$$\lim_{n\to+\infty}\int_\Omega (v_n(x)u_n(x)-pJ(x,u_n(x)))\,dx=-\infty,$$

which contradicts (2.43). Therefore, the sequence $\{u_n\}$ turns out bounded. Now, the proof goes on exactly as that of Proposition 2.1. □

**Proposition 2.5.** *Let* ($j_1$)–($j_3$) *and* ($j'_9$) *be satisfied. Then the function* $\Phi|_S$*, where* $S$ *is given by* (2.14)*, is coercive.*

**Proof.** Combining (2.1) with ($j'_9$) we get constants $\bar\theta\in]0,\lambda_2[$, $M_{12}>0$ such that

$$J(x,\xi)\le\frac{\bar\theta}{p}|\xi|^p+M_{12}\quad \text{a.e. in } \Omega \text{ and for all } \xi\in\mathbb{R}.$$

On account of (2.15) this implies, whenever $u\in S$,

$$\begin{aligned}\Phi(u)\ge\frac{1}{p}\|\nabla u\|^p_{L^p(\Omega,\mathbb{R}^N)}-\frac{\bar\theta}{p}\|u\|^p_{L^p(\Omega)}-M_{12}m(\Omega)\\ \ge\frac{1}{p}\left(1-\frac{\bar\theta}{\lambda_2}\right)\|\nabla u\|^p_{L^p(\Omega,\mathbb{R}^N)}-M_{12}m(\Omega).\end{aligned}$$

Since $\bar\theta<\lambda_2$, from preceding inequality we immediately infer the conclusion. □

Gathering Propositions 2.4–2.5 and Theorem 1.1 together yields the next existence result, whose proof is similar to that of Theorem 2.2.

**Theorem 2.3.** *Assume* ($j_1$)–($j_3$), ($j'_9$), ($j_{10}$), *and* ($j_{11}$) *hold true. Then Problem* (0.1) *possesses at least one weak solution* $u\in W^{1,p}(\Omega)$.

**Remark 2.1.** Through standard results from nonlinear regularity theory (see for instance [10, Section 1.5.3]) we realize that the solutions $u$ to Problem (0.1) given by Theorems 2.1–2.3 lie in $C^1(\overline{\Omega})$ and one has $\frac{\partial u(x)}{\partial n}=0$ for all $x\in\partial\Omega$.

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## References

[1] H. Brézis, "Analyse Fonctionnelle - Théorie et Applications," Masson, Paris, 1983.

[2] P. Candito, *Infinitely many solutions to the Neumann problem for elliptic equations involving the p-Laplacian and with discontinuous nonlinearities,* Proc. Edinburgh Math. Soc. **45** (2002), 397–409.

[3] E. Casas and L.A. Fernández, *A Green's formula for quasilinear elliptic operators,* J. Math. Anal. Appl. **142** (1989), 62–73.

[4] J. Chabrowski, "Variational Methods for Potential Operator Equations," de Gruyter Ser. Nonlinear Anal. Appl. **24**, de Gruyter, Berlin, 1997.

[5] K.-C. Chang, *Variational methods for non-differentiable functionals and their applications to partial differential equations,* J. Math. Anal. Appl. **80** (1981), 102–129.

[6] F.H. Clarke, "Optimization and Nonsmooth Analysis," Classics Appl. Math. **5**, SIAM, Philadelphia, 1990.

[7] Z. Denkowski, S. Migorski, and N.S. Papageorgiou, "An Introduction to Nonlinear Analysis: Theory," Kluwer/Plenum, New York, 2003.

[8] Y. Du, *A deformation lemma and some critical points theorems,* Bull. Austral. Math. Soc. **43** (1991), 161–168.

[9] M. Filippakis, L. Gasiński, and N.S. Papageorgiou, *Multiplicity results for nonlinear Neumann problems,* to appear.

[10] L. Gasiński and N.S. Papageorgiou, "Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems," Ser. Math. Anal. Appl. **8**, Chapman and Hall/CRC Press, Boca Raton, 2005.

[11] N. Halidias, *Neumann boundary value problems with discontinuities,* Appl. Math. Lett. **16** (2003), 729–732.

[12] N.C. Kourogenis and N.S. Papageorgiou, *Nonsmooth critical point theory and nonlinear elliptic equations at resonance,* J. Austral. Math. Soc. (Series A) **69** (2000), 245–271.

[13] S.A. Marano and D. Motreanu, *Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian,* J. Differential Equations **182** (2002), 108–120.

[14] D. Motreanu and P.D. Panagiotopoulos, "Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities," Nonconvex Optim. Appl. **29**, Kluwer, Dordrecht, 1998.

[15] D. Motreanu and V. Radulescu, "Variational and Non-variational Methods in Nonlinear Analysis and Boundary Value Problems," Nonconvex Optim. Appl. **67**, Kluwer, Dordrecht, 2003.

[16] D. Motreanu and C. Varga, *Some critical point results for locally Lipschitz functionals,* Comm. Appl. Nonlinear Anal. **4** (1997), 17–33.

[17] Z. Naniewicz and P.D. Panagiotopoulos, "Mathematical Theory of Hemivariational Inequalities and Applications," Marcel Dekker, New York, 1995.

[18] P.D. Panagiotopoulos, "Hemivariational Inequalites. Applications to Mechanics and Engineering," Springer-Verlag, New York, 1993.

[19] N.S. Papageorgiou and G. Smyrlis, "On nonlinear hemivariational inequalities," Dissertationes Math. (Rozprawy Math.) **419** (2003).

[20] F. Papalini, *Nonlinear eigenvalue Neumann problems with discontinuities,* J. Math. Anal Appl. **273** (2002), 137–152.

[21] P. H. Rabinowitz, "'Minimax methods in critical point theory with applications to differential equations," CBMS Reg. Conf. Ser. in Math. **65**, Amer. Math. Soc., Providence, 1986.