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SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH DAMPING AND SOURCE TERMS

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Abstract. In this article we focus on the global well posedness of the system of nonlinear wave equations

$$u_{tt} - \Delta u + |u_t|^{m-1}u_t = f_1(u,v)$$
$$v_{tt} - \Delta v + |v_t|^{r-1}v_t = f_2(u,v)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, with Dirichlét boundary conditions. Under some restriction on the parameters in the system we obtain several results on the existence of local and global solutions, uniqueness, and the blow up of solutions in finite time.

1. Introduction

1.1. **The model.** Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the C^1-function given by

$$F(u,v) = a\,|u+v|^{p+1} + 2b\,|uv|^{\frac{p+1}{2}}\,,$$

where $p \geq 3$, $a > 1$ and $b > 0$. Let $f_1(u,v) = \frac{\partial F}{\partial u}(u,v)$ and $f_2(u,v) = \frac{\partial F}{\partial v}(u,v)$ for $(u,v) \in \mathbb{R}^2$. Throughout the paper, Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\Gamma = \partial\Omega$, and $n = 1, 2, 3$.

This article is concerned with the global well posedness of the following initial-boundary-value problem:

$$\begin{aligned} u_{tt} - \Delta u + |u_t|^{m-1}u_t &= f_1(u,v), \text{ in } \Omega \times (0,T) \equiv Q_T,\\ v_{tt} - \Delta v + |v_t|^{r-1}v_t &= f_2(u,v), \text{ in } \Omega \times (0,T) \equiv Q_T,\\ u(x,0) = u^0(x) \in H_0^1(\Omega), u_t(x,0) &= u^1(x) \in L_2(\Omega), \end{aligned} \tag{1.1}$$

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$$v(x,0) = v^0(x) \in H_0^1(\Omega), v_t(x,0) = v^1(x) \in L_2(\Omega),$$
$$u = v = 0, \text{on } \Gamma \times (0,T).$$

The nonlinearities $f_1(u,v)$ and $f_2(u,v)$ act as strong source terms in the system (1.1). In addition, the functions f_1, f_2 and F enjoy certain properties. First, it is easy to see that $F(u,v) \leq c_1(|u|^{p+1} + |v|^{p+1})$, for all $(u,v) \in \mathbb{R}^2$, where $c_1 = 2^p a + b$. Moreover, a quick computation will show that for a fixed $a, p > 1$, there exists a constant $c_0 > 0$ such that $F(u,v) \geq c_0(|u|^{p+1} + |v|^{p+1})$, for all $(u,v) \in \mathbb{R}^2$, provided b is chosen large enough. Also, it is easy to see that $uf_1(u,v) + vf_2(u,v) = (p+1)F(u,v)$ for all $(u,v) \in \mathbb{R}^2$. Henceforth, the following conditions are assumed throughout the paper.

Assumption 1.1.

- *$m, r \geq 1$; $p \geq 3$ if $n = 1,2$; $p = 3$ if $n = 3$.*
- *$u^0, v^0 \in H_0^1(\Omega)$, $u^1, v^1 \in L^2(\Omega)$.*
- *There exists constants $c_0, c_1 > 0$ such that*

$$c_0(|u|^{p+1} + |v|^{p+1}) \leq F(u,v) \leq c_1(|u|^{p+1} + |v|^{p+1}) \text{ for all } (u,v) \in \mathbb{R}^2. \tag{1.2}$$

- *In addition,*

$$f_1(u,v) = (p+1)[a|u+v|^{p-1}(u+v) + b|u|^{\frac{p-3}{2}}|v|^{\frac{p+1}{2}}u],$$
$$f_2(u,v) = (p+1)[a|u+v|^{p-1}(u+v) + b|v|^{\frac{p-3}{2}}|u|^{\frac{p+1}{2}}v],$$
$$uf_1(u,v) + vf_2(u,v) = (p+1)F(u,v) \text{ for all } (u,v) \in \mathbb{R}^2. \tag{1.3}$$

Systems of nonlinear wave equations such as (1.1) go back to Reed [28] in 1976 who proposed a similar system in three space dimensions but without the presence of the damping terms $|u_t|^{m-1}u_t$ and $|v_t|^{r-1}v_t$. In this paper, we study the long-time behavior of solutions to the initial-boundary-value problem (1.1). The central interest here is the relationship of the source and damping terms to the behavior of solutions.

As in the case of a single wave equation [9, 17, 24, 36], it is worth noting here that when the damping term $|u_t|^{m-1}u_t$ and $|v_t|^{r-1}v_t$ are absent, then the strong source terms $f_1(u,v)$ and $f_2(u,v)$ should drive the solution of (1.1) to blow-up in finite time. In addition, if the source terms $f_1(u,v)$ and $f_2(u,v)$ are removed from the equations, then damping terms of various forms should yield existence of global solutions (cf. [2, 3, 6, 12]). However, when both damping and source terms are present, then the analysis of their interaction and their influence on the global behavior of solutions becomes more difficult. For work on single wave equations with damping and source terms, we refer the reader to [8, 18, 20, 25, 27, 31, 34] and the references therein.

At this point we remark that the following notation will be used throughout the paper.

$$|u|_{s,\Omega} = \|u\|_{H^s(\Omega)}\,, \ \|u\|_p = \|u\|_{L^p(\Omega)} \ \text{and} \ \langle u, v\rangle = \langle u, v\rangle_{L^2(\Omega)}\,.$$

Also, the following Sobolev imbeddings will be used frequently, and sometimes without mention:

$$\begin{cases} H_0^1(\Omega) \hookrightarrow L^q(\Omega), & \text{for } 1 \le q \le 6, \ n = 3, \\ H_0^1(\Omega) \hookrightarrow L^q(\Omega), & \text{for } 1 \le q < \infty, \ n = 1, 2. \end{cases} \tag{1.4}$$

We finally note that Poincaré's inequality implies that the norms $\|u\|_{H_0^1(\Omega)}$ and $\|\nabla u\|_2$ are equivalent norms on $H_0^1(\Omega)$.

1.2. **Main Results.** In order to state our main results we introduce the definition of a weak solution to (1.1).

Definition 1.2. *A pair of functions (u, v) is said to be a weak solution of (1.1) on $[0, T]$ if $u, v \in C_w([0, T], H_0^1(\Omega))$, $u_t, v_t \in C_w([0, T], L^2(\Omega))$, $u_t \in L^{m+1}(\Omega \times (0, T))$, $v_t \in L^{r+1}(\Omega \times (0, T))$, $(u(0), v(0)) = (u^0, v^0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $(u_t(0), v_t(0)) = (u^1, v^1) \in L^2(\Omega) \times L^2(\Omega)$; and (u, v) satisfies*

$$\begin{aligned} &\left\langle u'(t), \phi\right\rangle_{L^2(\Omega)} - \left\langle u^1, \phi\right\rangle_{L^2(\Omega)} + \int_0^t \langle \nabla u(\tau), \nabla \phi\rangle_{L^2(\Omega)} \\ &+ \int_0^t \left\langle |u'(\tau)|^{m-1} u'(\tau), \phi\right\rangle_{L^2(\Omega)} d\tau = \int_0^t \langle f_1(u(\tau), v(\tau)), \phi\rangle_{L^2(\Omega)}\, d\tau, \end{aligned} \tag{1.5}$$

$$\begin{aligned} &\left\langle v'(t), \psi\right\rangle_{L^2(\Omega)} - \left\langle v^1, \psi\right\rangle_{L^2(\Omega)} + \int_0^t \langle \nabla v(\tau), \nabla \psi\rangle_{L^2(\Omega)} \\ &+ \int_0^t \left\langle |v'(\tau)|^{r-1} v'(\tau), \psi\right\rangle_{L^2(\Omega)} d\tau = \int_0^t \langle f_2(u(\tau), v(\tau)), \psi\rangle_{L^2(\Omega)}\, d\tau, \end{aligned} \tag{1.6}$$

for all test functions $\phi \in H_0^1(\Omega) \cap L^{m+1}(\Omega), \psi \in H_0^1(\Omega) \cap L^{r+1}(\Omega)$, and for almost all $t \in [0, T]$.

We are now in a position to state our main results. Our first theorem establishes the existence and uniqueness of a local weak solution to (1.1) that satisfies an energy identity, without further restrictions on the parameters or the initial data. Specifically, we have the following result.

Theorem 1.3. ***Local weak solutions.*** *Assume the validity of Assumption* 1.1. *Then, there exists a unique local weak solution* (u,v) *to* (1.1) *defined on* $[0,T_0]$ *for some* $T_0>0$. *In addition, the said solution satisfies the energy identity*

$$E(t)+\int_0^t\int_\Omega |u'(\tau)|^{m+1}dxd\tau+\int_0^t\int_\Omega |v'(\tau)|^{r+1}dxd\tau=E(0), \tag{1.7}$$

where

$$E(t):=\frac{1}{2}\left(\left\|u'(t)\right\|_2^2+\left\|v'(t)\right\|_2^2+\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2\right) \\ -\int_\Omega F(u(t),v(t))dx. \tag{1.8}$$

The following theorem asserts that the weak solution furnished by Theorem 1.3 is a global solution provided $p\le\min\{m,r\}$ and without additional restriction on the initial data.

Theorem 1.4. ***Global weak solutions.*** *In addition to Assumption* 1.1, *assume that* $p\le\min\{m,r\}$. *Then, the said solution* (u,v) *in Theorem* 1.3 *is a global solution and* T_0 *may be taken arbitrarily large.*

Our next theorem provides an answer to the existence of a global solution to (1.1) when the condition $p\le\min\{m,r\}$ is violated. In order to state Theorem 1.5, we define

$$J(t):=\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2-4G(t) \text{ where } G(t)=\int_\Omega F(u(t),v(t))dx.$$

Indeed, Theorem 1.5 asserts that the solution constructed in Theorem 1.3 is a global solution, provided $J(0)>0$ and the initial energy $E(0)$ is sufficiently small. More specifically, we have the following result.

Theorem 1.5. ***Global small solutions.*** *In addition to Assumption* 1.1, *assume that* $J(0)>0$ *and* $4^{\frac{p}{2}}c_0\,E(0)^{\frac{p-1}{2}}<1$, *where* c_0 *is a computable positive constant that depends on* p *and* Ω. *Then, the said solution* (u,v) *in Theorem* 1.3 *is a global solution and* T_0 *may be taken arbitrarily large.*

Our next theorem addresses the issue of a *strong source* (large values of p when $p>\max\{m,r\}$) which may lead to a finite-time blow up of solutions. Here, our result in Theorem 1.6 is inspired by the work of Georgiev and Todorova [8] where they proved the blow up of weak solutions to a single wave equation that featured a non-degenerate damping and a source term. In addition, we want to mention that the basic calculus in the proof of Theorem

1.6 draws from the ideas in [8], and also from [4, 25] in their treatment of a single wave equation that contained a degenerate damping term and a source term. We also refer the reader to [18] for general global nonexistence theorems.

Theorem 1.6. ***Blow up of solutions.*** *In addition to Assumption* 1.1, *assume that* $p > \max\{m, r\}$ *and* $E(0) < 0$, *where* $E(0)$ *is the initial energy given by*

$$E(0) := \frac{1}{2}\left(\left\|u^1\right\|_2^2 + \left\|v^1\right\|_2^2 + \left\|\nabla u^0\right\|_2^2 + \left\|\nabla v^0\right\|_2^2\right) - \int_\Omega F(u^0, v^0)dx.$$

Then, the weak solution (u, v) *in Theorem* 1.3 *blows up in finite time.*

We conclude the introduction by reminding the reader of the following elementary inequalities which will be used in various places in the paper. Specifically,

$$\left||a|^k - |b|^k\right| \le C\,|a-b|\,(|a|^{k-1} + |b|^{k-1}), \tag{1.9}$$

for some constant $C > 0$, all $k \ge 1$, and all $a, b \in \mathbb{R}$. Also,

$$||a|^p\ a - |b|^p\ b| \le C\,|a-b|\,(|a|^p + |b|^p), \tag{1.10}$$

for some constant $C > 0$, all $p \ge 0$, and all $a, b \in \mathbb{R}$.

We note here that one can use (1.9)-(1.10) to obtain the following useful inequalities enjoyed by f_1, f_2 and F. Indeed, straightforward computation yields

$$\begin{aligned}|f_1(u,v) - f_1(\tilde{u},\tilde{v})| &\le C_0(|u-\tilde{u}| + |v-\tilde{v}|)(|u|^{p-1} + |v|^{p-1} + |\tilde{u}|^{p-1} + |\tilde{v}|^{p-1})\\ &+ C_1[|v-\tilde{v}||u|^{\frac{p-1}{2}}(|v|^{\frac{p-1}{2}} + |\tilde{v}|^{\frac{p-1}{2}}) + |u-\tilde{u}||\tilde{v}|^{\frac{p+1}{2}}(|u|^{\frac{p-3}{2}} + |\tilde{u}|^{\frac{p-3}{2}})],\end{aligned} \tag{1.11}$$

$$\begin{aligned}|f_2(u,v) - f_2(\tilde{u},\tilde{v})| &\le C_0(|u-\tilde{u}| + |v-\tilde{v}|)(|u|^{p-1} + |v|^{p-1} + |\tilde{u}|^{p-1} + |\tilde{v}|^{p-1})\\ &+ C_1[|u-\tilde{u}||v|^{\frac{p-1}{2}}(|u|^{\frac{p-1}{2}} + |\tilde{u}|^{\frac{p-1}{2}}) + |v-\tilde{v}||\tilde{u}|^{\frac{p+1}{2}}(|v|^{\frac{p-3}{2}} + |\tilde{v}|^{\frac{p-3}{2}})],\end{aligned} \tag{1.12}$$

$$\begin{aligned}|F(u,v) - F(\tilde{u},\tilde{v})| &\le C_0(|u-\tilde{u}| + |v-\tilde{v}|)(|u|^p + |v|^p + |\tilde{u}|^p + |\tilde{v}|^p)\\ &+ C_1(|u-\tilde{u}||v| + |\tilde{u}||v-\tilde{v}|)(|u|^{\frac{p-1}{2}}|v|^{\frac{p-1}{2}} + |\tilde{u}|^{\frac{p-1}{2}}|\tilde{v}|^{\frac{p-1}{2}}),\end{aligned} \tag{1.13}$$

for all $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$, and some positive constants C_0 and C_1.

2. Weak solutions and the proof of Theorem 1.3

This Section is devoted to the proof of Theorem 1.3, which will be carried out in the following four sub-sections.

2.1. **Approximate solutions.** Consider the operator $A = -\Delta$ with its domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is well known that A is positive and self-adjoint, and A is the inverse of a compact operator. Moreover, A has the infinite sequence of positive eigenvalues $\{\lambda_j : j = 1, 2, ...\}$ and a corresponding sequence of eigenfunctions $\{e_j : j = 1, 2, ...\}$ that forms an orthonormal basis for $L^2(\Omega)$. Also, the sequence $\{e_j : j = 1, 2, ...\}$ is an orthogonal basis for $H_0^1(\Omega)$. In addition, one can show that the linear span of $\{e_j : j = 1, 2, ...\}$ is dense in $L^q(\Omega)$ for any $1 \leq q < \infty$.

Let $\mathcal{P}_N$ be the orthogonal projection of $L^2(\Omega)$ onto $V_N :=$ the linear span of $\{e_1, \cdots e_N\}$. In order to establish the existence of a local weak solution to the system (1.1) we shall use a standard Galerkin approximation scheme based on the eigenfunctions $\{e_j\}_{j=1}^{\infty}$ of the operator $A = -\Delta$. That is, let $u_N(t) = \sum_{j=1}^{N} u_{N,j}(t) e_j$ and $v_N(t) = \sum_{j=1}^{N} v_{N,j}(t) e_j$ be the approximate solutions in V_N; i.e., $u_N(t), v_N(t)$ satisfy the following system of ordinary differential equations:

$$\left\langle u_N''(t), e_j \right\rangle_{L^2(\Omega)} + \langle \nabla u_N(t), \nabla e_j \rangle_{L^2(\Omega)} + \left\langle |u_N'(t)|^{m-1} u_N'(t), e_j \right\rangle_{L^2(\Omega)} = \langle f_1(u_N(t), v_N(t)), e_j \rangle_{L^2(\Omega)}, \quad (2.1)$$

$$\left\langle v_N''(t), e_j \right\rangle_{L^2(\Omega)} + \langle \nabla v_N(t), \nabla e_j \rangle_{L^2(\Omega)} + \left\langle |v_N'(t)|^{r-1} v_N'(t), e_j \right\rangle_{L^2(\Omega)} = \langle f_2(u_N(t), v_N(t)), e_j \rangle_{L^2(\Omega)}, \quad (2.2)$$

$$u_N(0) = \mathcal{P}_N u^0, \quad v_N(0) = \mathcal{P}_N v^0, \quad u_N'(0) = \mathcal{P}_N u^1, \quad v_N'(0) = \mathcal{P}_N v^1, \quad (2.3)$$

for $j = 1, 2..., N$. More specifically,

$$u_{N,j}(0) = u_j^0, \quad v_{N,j}(0) = v_j^0, \quad u_{N,j}'(0) = u_j^1, \quad v_{N,j}'(0) = v_j^1, \quad (2.4)$$

where $u_j^0 = \left\langle u^0, e_j \right\rangle_{L^2(\Omega)}$, $v_j^0 = \left\langle v^0, e_j \right\rangle_{L^2(\Omega)}$, $u_j^1 = \left\langle u^1, e_j \right\rangle_{L^2(\Omega)}$ and $v_j^1 = \left\langle v^1, e_j \right\rangle_{L^2(\Omega)}$, for $j = 1, 2..., N$.

Clearly (2.1)-(2.3) is an initial-value problem for a second-order $2N \times 2N$ system of ordinary differential equations with continuous nonlinearities in the unknown functions $u_{N,j}, v_{N,j}$ and their derivatives. It follows from the Cauchy-Peano theorem that for every $N \geq 1$, (2.1)-(2.3) has a solution $u_{N,j}, v_{N,j} \in C^2[0, T_N]$ for some $T_N > 0$.

A priori Estimates. Here, we shall show that T_N can be replaced by some $T > 0$, for all $N \geq 1$.

Lemma 2.1. *There exists a constant $T > 0$ such that the sequence of approximate solutions $\{u_N\}$ and $\{v_N\}$ satisfy the following:*

- *$\{u_N\}$, $\{v_N\}$ are bounded sequences in $L^\infty(0,T;H_0^1(\Omega))$.*
- *$\{u_N'\}$, $\{v_N'\}$ are bounded sequences in $L^\infty(0,T;L^2(\Omega))$.*
- *$\{u_N'\}$ is a bounded sequence in $L^{m+1}(\Omega \times (0,T))$ and $\{v_N'\}$ is a bounded sequence in $L^{r+1}(\Omega \times (0,T))$.*

Proof. Multiply (2.1) by $u_{N,j}'(t)$, (2.2) by $v_{N,j}'(t)$, and sum for $j = 1, \cdots, N$; one has

$$\frac{1}{2}\frac{d}{dt}\left(\left\|u_N'(t)\right\|_2^2 + \|\nabla u_N(t)\|_2^2\right) + \int_\Omega |u_N'(t)|^{m+1}dx$$
$$= \int_\Omega f_1(u_N(t), v_N(t))u_N'(t)dx, \tag{2.5}$$
$$\frac{1}{2}\frac{d}{dt}\left(\left\|v_N'(t)\right\|_2^2 + \|\nabla v_N(t)\|_2^2\right) + \int_\Omega |v_N'(t)|^{r+1}dx$$
$$= \int_\Omega f_2(u_N(t), v_N(t))v_N'(t)dx. \tag{2.6}$$

By summing (2.5) and (2.6) and integrating the resulting identity from 0 to $t \leq T_N$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\left\|u_N'(t)\right\|_2^2 + \left\|v_N'(t)\right\|_2^2 + \|\nabla u_N(t)\|_2^2 + \|\nabla v_N(t)\|_2^2\right) \\
&+ \int_0^t\int_\Omega |u_N'(\tau)|^{m+1}dxd\tau + \int_0^t\int_\Omega |v_N'(\tau)|^{r+1}dxd\tau \\
&= \frac{1}{2}\left(\left\|u_N'(0)\right\|_2^2 + \left\|v_N'(0)\right\|_2^2 + \|\nabla u_N(0)\|_2^2 + \|\nabla v_N(0)\|_2^2\right) \\
&+ \int_0^t\int_\Omega \big(f_1(u_N(\tau), v_N(\tau))u_N'(\tau) + f_2(u_N(\tau), v_N(\tau))v_N'(\tau)\big)\,dxd\tau \\
&\leq C(|u^0|_{1,\Omega}, |v^0|_{1,\Omega}, |u^1|_{0,\Omega}, |v^1|_{0,\Omega}) \\
&+ \int_0^t\int_\Omega \big(f_1(u_N(\tau), v_N(\tau))u_N'(\tau) + f_2(u_N(\tau), v_N(\tau))v_N'(\tau)\big)\,dxd\tau,
\end{aligned} \tag{2.7}$$

where we have used in (2.7) the fact that

$$\begin{aligned}
&u_N(0) \to u^0,\ \ v_N(0) \to v^0 \text{ strongly in } H_0^1(\Omega) \\
&u_N'(0) \to u^1,\ \ v_N'(0) \to v^1 \text{ strongly in } L^2(\Omega).
\end{aligned} \tag{2.8}$$

We estimate the last term in (2.7) as follows. By recalling (1.3) and using Hölder's and Young's inequalities, we have

$$\begin{aligned}
&\Big|\int_\Omega f_1(u_N,v_N)u_N' dx\Big| \leq C\int_\Omega \Big(|u_N+v_N|^p\,|u_N'|+|v_N|^{\frac{p+1}{2}}|u_N|^{\frac{p-1}{2}}|u_N'|\Big)\,dx\\
&\leq C\Big[(\|u_N\|_{2p}^p+\|v_N\|_{2p}^p)\left\|u_N'\right\|_2+\|u_N\|_{3(p-1)}^{\frac{p-1}{2}}\|v_N\|_{\frac{3(p+1)}{2}}^{\frac{p+1}{2}}\left\|u_N'\right\|_2\Big]\\
&\leq C\Big[\,\|u_N\|_{2p}^{2p}+\|v_N\|_{2p}^{2p}+\|u_N\|_{3(p-1)}^{p-1}\|v_N\|_{\frac{3(p+1)}{2}}^{p+1}+\left\|u_N'\right\|_2^2\Big]\\
&\leq C\left[\|\nabla u_N\|_2^{2p}+\|\nabla v_N\|_2^{2p}+\|\nabla u_N\|_2^{p-1}\|\nabla v_N\|_2^{p+1}+\left\|u_N'\right\|_2^2\right], \qquad (2.9)
\end{aligned}$$

where we have used in (2.9) the Sobolev imbeddings in (1.4) and the fact that when $n=3$ then $2p=3(p-1)=\frac{3(p+1)}{2}=6$.

Likewise, we have

$$\begin{aligned}
&\Big|\int_\Omega f_2(u_N,v_N)u_N' dx\Big| \leq C\int_\Omega \Big(|u_N+v_N|^p\,|u_N'|+|u_N|^{\frac{p+1}{2}}|v_N|^{\frac{p-1}{2}}|u_N'|\Big)\,dx\\
&\leq C\left[\|\nabla u_N\|_2^{2p}+\|\nabla v_N\|_2^{2p}+\|\nabla v_N\|_2^{p-1}\|\nabla u_N\|_2^{p+1}+\left\|u_N'\right\|_2^2\right]. \qquad (2.10)
\end{aligned}$$

Let $y_N(t):=1+\|u_N'(t)\|_2^2+\|v_N'(t)\|_2^2+\|\nabla u_N(t)\|_2^2+\|\nabla v_N(t)\|_2^2$. Then, it follows from (2.7)-(2.10) that

$$\begin{aligned}
y_N(t)+2\int_0^t\int_\Omega |u_N'(\tau)|^{m+1}dxd\tau+2\int_0^t\int_\Omega |v_N'(\tau)|^{r+1}dxd\tau&\\
\leq C_0+C\int_0^t y_N(\tau)^p d\tau,& \qquad (2.11)
\end{aligned}$$

where $C_0=C(|u^0|_{1,\Omega},|v^0|_{1,\Omega},|u^1|_{0,\Omega},|v^1|_{0,\Omega})>0$. In particular, $y_N(t)$ satisfies

$$y_N(t)\leq C_0+C\int_0^t y_N(\tau)^p d\tau. \qquad (2.12)$$

Hence, by using a standard comparison theorem, (2.12) yields that $y_N(t)\leq z(t)$, where $z(t)=[C_0^{1-p}-C(p-1)t]^{-\frac{1}{p-1}}$, is the solution of the Volterra integral equation

$$z(t)=C_0+C\int_0^t z(\tau)^p d\tau. \qquad (2.13)$$

Although $z(t)$ blows up in finite time (since $p\geq 3$), nonetheless, there exists a time $0<T<T_N$ such that $y_N(t)\leq z(t)\leq C_1$ for all $t\in[0,T]$, where C_1 is independent of N. Hence, for all $N\geq 1$, one has $y_N(t)\leq C_1$, for all

$t \in [0,T]$, establishing the first two parts of the lemma. The last part of the lemma immediately follows from (2.11). □

It follows from Lemma 2.1 that there exist subsequences of u_N and v_N, which we still denote by u_N and v_N, that satisfy

$$\begin{cases} u_N \to u \text{ and } v_N \to v \text{ weakly}^* \text{ in } L^\infty(0,T;H_0^1(\Omega)), \\ u_N' \to u' \text{ and } v_N' \to v' \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{2.14}$$

The following lemma, however, provides stronger information about the sequences of approximate solutions. More specifically, we have the following result.

Lemma 2.2. *The sequence of approximate solutions $\{u_N\}$ and $\{v_N\}$ satisfy the following:*

- *$\{u_N\}$, $\{v_N\}$ are Cauchy sequences in $L^\infty(0,T;H_0^1(\Omega))$.*
- *$\{u_N'\}$, $\{v_N'\}$ are Cauchy sequences in $L^\infty(0,T;L^2(\Omega))$.*
- *$\{u_N'\}$ is a Cauchy sequence in $L^{m+1}(\Omega\times(0,T))$ and $\{v_N'\}$ is a Cauchy sequence in $L^{r+1}(\Omega\times(0,T))$.*

Proof. Consider two pairs of approximate solutions (u_N,v_N) and (u_L,v_L), and, without loss of generality, we assume $N > L$. Set $u_{NL}(t) := u_N(t) - u_L(t) = \sum_{j=1}^N (u_{N,j}(t) - u_{L,j}(t))e_j$ and $v_{NL} := v_N - v_L = \sum_{j=1}^N (v_{N,j}(t) - v_{L,j}(t))e_j$, with the understanding that $u_{L,j}(t) = v_{L,j}(t) \equiv 0$ when $j > L$. Then, u_{NL} and v_{NL} satisfy

$$\begin{aligned} &\left\langle u_{NL}''(t), e_j \right\rangle + \langle \nabla u_{NL}(t), \nabla e_j\rangle + \left\langle |u_N'(t)|^{m-1}u_N'(t) - |u_L'(t)|^{m-1}u_L'(t), e_j \right\rangle \\ &= \langle f_1(u_N(t),v_N(t)) - f_1(u_L(t),v_L(t)), e_j\rangle, \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\left\langle v_{NL}''(t), e_j \right\rangle + \langle \nabla v_{NL}(t), \nabla e_j\rangle + \left\langle |v_N'(t)|^{r-1}v_N'(t) - |v_L'(t)|^{r-1}v_L'(t), e_j \right\rangle \\ &= \langle f_2(u_N(t),v_N(t)) - f_2(u_L(t),v_L(t)), e_j\rangle, \end{aligned} \tag{2.16}$$

$$\begin{aligned} &u_{NL}(0) = \mathcal{P}_N u^0 - \mathcal{P}_L u^0, \quad v_{NL}(0) = \mathcal{P}_N v^0 - \mathcal{P}_L v^0, \\ &u_{NL}'(0) = \mathcal{P}_N u^1 - \mathcal{P}_L u^1, \quad v_{NL}'(0) = \mathcal{P}_N v^1 - \mathcal{P}_L v^1, \end{aligned} \tag{2.17}$$

for $j = 1, 2..., N$; and where all inner products in (2.15)-(2.16) are $L^2(\Omega)$-inner products.

Multiply (2.15) by $u_{N,j}'(t) - u_{L,j}'(t)$, (2.16) by $v_{N,j}'(t) - v_{L,j}'(t)$, and sum for $j = 1, \cdots, N$; one has

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|u_{NL}'(t)\right\|_2^2 + \|\nabla u_{NL}(t)\|_2^2\right) \\ &+ \left\langle |u_N'(t)|^{m-1}u_N'(t) - |u_L'(t)|^{m-1}u_L'(t), u_{NL}'(t)\right\rangle \end{aligned}$$

$$= \left\langle f_1(u_N(t), v_N(t)) - f_1(u_L(t), v_L(t)), u'_{NL}(t) \right\rangle, \tag{2.18}$$

$$\frac{1}{2}\frac{d}{dt}\left(\left\|v'_{NL}(t)\right\|_2^2 + \|\nabla v_{NL}(t)\|_2^2\right)$$
$$+ \left\langle |v'_N(t)|^{r-1}v'_N(t) - |v'_L(t)|^{r-1}v'_L(t), v'_{NL}(t)\right\rangle$$
$$= \left\langle f_2(u_N(t), v_N(t)) - f_2(u_L(t), v_L(t)), v'_{NL}(t) \right\rangle. \tag{2.19}$$

By recalling (1.11)-(1.12) we can estimate the last terms in (2.18) and (2.19) as follows. To simplify the notation, let us set

$$u = u_N(t), \ \ v = v_N(t), \ \ \tilde{u} = u_L(t), \ \ \tilde{v} = v_L(t), \ \ w = u'_{NL}(t), \ \ \tilde{w} = v'_{NL}(t).$$

Then, with this notation and recalling (1.11), we have

$$\left|\left\langle f_1(u_N(t), v_N(t)) - f_1(u_L(t), v_L(t)), u'_{NL}(t) \right\rangle\right| \leq I_1 + I_2 + I_3, \tag{2.20}$$

where

$$I_1 = C_0 \int_\Omega (|u - \tilde{u}| + |v - \tilde{v}|)\left(|u|^{p-1} + |v|^{p-1} + |\tilde{u}|^{p-1} + |\tilde{v}|^{p-1}\right)|w|dx$$
$$I_2 = C_1 \int_\Omega |v - \tilde{v}|\,|u|^{\frac{p-1}{2}}\left(|v|^{\frac{p-1}{2}} + |\tilde{v}|^{\frac{p-1}{2}}\right)|w|dx$$
$$I_3 = C_1 \int_\Omega |u - \tilde{u}|\,|\tilde{v}|^{\frac{p+1}{2}}\left(|u|^{\frac{p-3}{2}} + |\tilde{u}|^{\frac{p-3}{2}}\right)|w|dx. \tag{2.21}$$

All terms in I_1 are estimated in the same way. In particular, for a typical term in I_1, we have

$$\int_\Omega (|u - \tilde{u}| + |v - \tilde{v}|)\,|u|^{p-1}\,|w|dx$$
$$\leq \|u - \tilde{u}\|_6\,\|u\|_{3(p-1)}^{p-1}\,\|w\|_2 + \|v - \tilde{v}\|_6\,\|u\|_{3(p-1)}^{p-1}\,\|w\|_2$$
$$\leq C\left(\|\nabla(u - \tilde{u})\|_2 + \|\nabla(v - \tilde{v})\|_2\right)\|\nabla u\|_2^{p-1}\,\|w\|_2$$
$$\leq C\left(\|\nabla(u - \tilde{u})\|_2 + \|\nabla(v - \tilde{v})\|_2\right)\|w\|_2, \tag{2.22}$$

where we have used in (2.22) the Sobolev Imbbedding theorem, $3(p-1) = 6$ when $n = 3$, and the boundedness statements in Lemma 2.1. Hence,

$$I_1 \leq C\left(\|\nabla(u - \tilde{u})\|_2^2 + \|\nabla(v - \tilde{v})\|_2^2 + \|w\|_2^2\right). \tag{2.23}$$

A typical term in I_2 is estimated as follows.

$$\int_\Omega |v - \tilde{v}|\,|u|^{\frac{p-1}{2}}\,|v|^{\frac{p-1}{2}}\,|w|dx \leq \|v - \tilde{v}\|_6\,\|u\|_{3(p-1)}^{\frac{p-1}{2}}\,\|v\|_{3(p-1)}^{\frac{p-1}{2}}\,\|w\|_2$$
$$\leq C\,\|\nabla(v - \tilde{v})\|_2\,\|\nabla u\|_2^{\frac{p-1}{2}}\,\|\nabla v\|_2^{\frac{p-1}{2}}\,\|w\|_2 \leq C\,\|\nabla(v - \tilde{v})\|_2\,\|w\|_2. \tag{2.24}$$

Hence,

$$I_2 \le C\left(\|\nabla(v-\tilde{v})\|_2^2 + \|w\|_2^2\right). \tag{2.25}$$

By noting that $p = 3$ when $n = 3$, then a typical term in I_3 is estimated as follows. If $n = 3$ then

$$\begin{aligned}\int_\Omega |u-\tilde{u}|\,|\tilde{v}|^{\frac{p+1}{2}}\,|u|^{\frac{p-3}{2}}\,|w|\,dx &\le \|u-\tilde{u}\|_6\,\|\tilde{v}\|_{\frac{3(p+1)}{2}}^{\frac{p+1}{2}}\,\|w\|_2 \\ &\le C\,\|\nabla(u-\tilde{u})\|_2\left\|\nabla\tilde{v}\right\|_2^2\|w\|_2 \le C\,\|\nabla(u-\tilde{u})\|_2\,\|w\|_2 . \end{aligned}\tag{2.26}$$

The last estimate is also easily obtained when $n = 1, 2$. Hence,

$$I_3 \le C\left(\|\nabla(u-\tilde{u})\|_2^2 + \|w\|_2^2\right). \tag{2.27}$$

It follows from (2.20), (2.23), (2.25) and (2.27) that

$$\begin{aligned}&\left|\left\langle f_1(u_N(t),v_N(t)) - f_1(u_L(t),v_L(t)), u'_{NL}(t)\right\rangle\right| \\ &\le C\left(\|\nabla(u_N(t)-u_L(t))\|_2^2 + \|\nabla(v_N(t)-v_L(t))\|_2^2 + \left\|u'_{NL}(t)\right\|_2^2\right) \\ &= C\left(\|\nabla u_{NL}\|_2^2 + \|\nabla v_{NL}\|_2^2 + \left\|u'_{NL}(t)\right\|_2^2\right).\end{aligned}\tag{2.28}$$

Similarly, by using (1.12) one has

$$\begin{aligned}&\left|\left\langle f_2(u_N(t),v_N(t)) - f_1(u_L(t),v_L(t)), v'_{NL}(t)\right\rangle\right| \\ &\le C\left(\|\nabla u_{NL}\|_2^2 + \|\nabla v_{NL}\|_2^2 + \left\|v'_{NL}(t)\right\|_2^2\right).\end{aligned}\tag{2.29}$$

Now, we note the monotonicity of the damping terms, namely we have elementary inequalities

$$\begin{aligned}(|a|^{m-1}a - |b|^{m-1}b)(a-b) &\ge c_0\,|a-b|^{m+1}, \\ (|a|^{r-1}a - |b|^{r-1}b)(a-b) &\ge c_1\,|a-b|^{r+1},\end{aligned}\tag{2.30}$$

which are valid for all $a, b \in \mathbb{R}$, where c_0, c_1 are some positive constants. By combining this with (2.28)-(2.29), then it follows from (2.18)-(2.19) that

$$\begin{aligned}&\frac{d}{dt}\left(\left\|u'_{NL}(t)\right\|_2^2 + \|\nabla u_{NL}(t)\|_2^2\right) + c_0\int_\Omega \left|u'_{NL}(t)\right|^{m+1}dx \\ &\le C\left(\|\nabla u_{NL}(t)\|_2^2 + \|\nabla v_{NL}(t)\|_2^2 + \left\|u'_{NL}(t)\right\|_2^2\right),\end{aligned}\tag{2.31}$$

$$\begin{aligned}&\frac{d}{dt}\left(\left\|v'_{NL}(t)\right\|_2^2 + \|\nabla v_{NL}(t)\|_2^2\right) + c_1\int_\Omega \left|v'_{NL}(t)\right|^{r+1}dx \\ &\le C\left(\|\nabla u_{NL}(t)\|_2^2 + \|\nabla v_{NL}(t)\|_2^2 + \left\|v'_{NL}(t)\right\|_2^2\right).\end{aligned}\tag{2.32}$$

At this end, we let

$$Y_{NL}(t) := \|\nabla u_{NL}(t)\|_2^2 + \|\nabla v_{NL}(t)\|_2^2 + \left\|u'_{NL}(t)\right\|_2^2 + \left\|v'_{NL}(t)\right\|_2^2.$$

By adding (2.31) and (2.32) and integrating the resulting inequality from 0 to $t \le T$, we obtain

$$Y_{NL}(t) + c_0 \int_0^t \int_\Omega |u'_{NL}(\tau)|^{m+1} dx d\tau + c_1 \int_0^t \int_\Omega |v'_{NL}(\tau)|^{r+1} dx d\tau \le Y_{NL}(0) + C \int_0^t Y_{NL}(\tau) d\tau, \quad (2.33)$$

where $Y_{NL}(0) := \|\nabla u_{NL}(0)\|_2^2 + \|\nabla v_{NL}(0)\|_2^2 + \|u'_{NL}(0)\|_2^2 + \|v'_{NL}(0)\|_2^2$. By recalling the strong convergence of the approximations to the initial data, then $Y_{NL}(0) \to 0$ as $L, N \to \infty$. By Gronwall's inequality, we have

$$Y_{NL}(t) \le C_T Y_{NL}(0) \longrightarrow 0 \text{ as } L, N \longrightarrow \infty. \quad (2.34)$$

Hence, the statements of the lemma follow. □

The following corollary on the convergence of the source terms is an immediate consequence of Lemma 2.2.

Corollary 2.3. *The sequence of approximate solutions $\{u_N\}$ and $\{v_N\}$ satisfy the following:*

$$\begin{cases} f_1(u_N, v_N) \longrightarrow f_1(u, v) \text{ strongly in } L^\infty(0, T; L^2(\Omega)), \\ f_2(u_N, v_N) \longrightarrow f_2(u, v) \text{ strongly in } L^\infty(0, T; L^2(\Omega)). \end{cases} \quad (2.35)$$

Proof. By recalling (1.11) and (1.12), we have

$$\|f_1(u_N(t), v_N(t)) - f_1(u(t), v(t))\|_2^2 \le I_1 + I_2 + I_3 + I_4, \quad (2.36)$$

where

$$\begin{aligned} I_1 &= C_0 \int_\Omega |u_N - u|^2 \left(|u_N|^{2(p-1)} + |v_N|^{2(p-1)} + |u|^{2(p-1)} + |v|^{2(p-1)}\right) dx \\ I_2 &= C_0 \int_\Omega |v_N - v|^2 \left(|u_N|^{2(p-1)} + |v_N|^{2(p-1)} + |u|^{2(p-1)} + |v|^{2(p-1)}\right) dx \\ I_3 &= C_1 \int_\Omega |v_N - v|^2 |u_N|^{p-1} \left(|v_N|^{p-1} + |v|^{p-1}\right) dx \\ I_4 &= C_1 \int_\Omega |u_N - u|^2 |v|^{p+1} \left(|u_N|^{p-3} + |u|^{p-3}\right) dx. \end{aligned} \quad (2.37)$$

A typical term in I_1 and I_2 is estimated as follows.

$$\int_\Omega |u_N - u|^2 \, |u_N|^{2(p-1)} \, dx \le \|u_N - u\|_6^2 \, \|u_N\|_{3(p-1)}^{2(p-1)}$$
$$\le C \, \|\nabla(u_N - u)\|_2^2 \, \|\nabla u_N\|_2^{2(p-1)} \le C \, \|\nabla(u_N - u)\|_2^2 , \qquad (2.38)$$

where we have used in (2.38) the Sobolev Imbedding theorem, the fact that when $n = 3$ then $3(p-1) = 6$, and the bounds furnished by Lemma 2.1. Hence, for all $t \in [0,T]$, we have

$$I_1 + I_2 \le C \left(\|\nabla(u_N(t) - u(t))\|_2^2 + \|\nabla(v_N(t) - v(t))\|_2^2 \right). \qquad (2.39)$$

Similarly, a typical term in I_3 is estimated as follows.

$$\int_\Omega |v_N - v|^2 \, |u_N|^{p-1} \, |v_N|^{p-1} \, dx \le \|v_N - v\|_6^2 \, \|u_N\|_{3(p-1)}^{p-1} \, \|v_N\|_{3(p-1)}^{p-1}$$
$$\le C \, \|\nabla(v_N(t) - v(t))\|_2^2 . \qquad (2.40)$$

Also, a typical term in I_4 is estimated in the same way, and one easily has

$$I_3 + I_4 \le C \left(\|\nabla(u_N(t) - u(t))\|_2^2 + \|\nabla(v_N(t) - v(t))\|_2^2 \right). \qquad (2.41)$$

The strong convergence furnished by Lemma 2.2 combined with (2.36), (2.39) and (2.41) prove the first convergence in (2.35). The second convergence in (2.35) is similar and so it is omitted. □

2.2. **Passage to the limit.** Before passing to the limit, we note that by Lemma 2.2 there exist subsequences of $\{u'_N\}$ and $\{v'_N\}$, which we still denote by $\{u'_N\}$ and $\{v'_N\}$, such that

$$u'_N \to u' \text{ and } v'_N \to v' \text{ almost everywhere in } Q_T = \Omega \times (0,T). \qquad (2.42)$$

Also, by Lemma 2.1, we note that $\{|u'_N|^{m-1} u'_N\}$ is a bounded sequence in $L^{\frac{m+1}{m}}(Q_T)$ and $\{|v'_N|^{r-1} v'_N\}$ is a bounded sequence in $L^{\frac{r+1}{r}}(Q_T)$. Hence, by selecting suitable subsequences, which we still denote by $\{u'_N\}$ and $\{v'_N\}$, we have

$$|u'_N|^{m-1} u'_N \to \xi \text{ weakly in } L^{\frac{m+1}{m}}(Q_T),$$
$$|v'_N|^{r-1} v'_N \to \eta \text{ weakly in } L^{\frac{r+1}{r}}(Q_T), \qquad (2.43)$$

for some functions $\xi \in L^{\frac{m+1}{m}}(Q_T)$, $\eta \in L^{\frac{r+1}{r}}(Q_T)$. By the point-wise almost everywhere convergence in (2.42), we infer that $\xi = |u'|^{m-1} u'$ and $\eta = |v'|^{r-1} v'$. That is,

$$|u'_N|^{m-1} u'_N \to |u'|^{m-1} u' \text{ weakly in } L^{\frac{m+1}{m}}(Q_T),$$

$$|v_N'|^{r-1}v_N' \to |v'|^{r-1}v' \text{ weakly in } L^{\frac{r+1}{r}}(Q_T). \tag{2.44}$$

In addition, by using the results in Lemma 2.2, we can extract subsequences, which we still denote by $\{u_N\}$ and $\{v_N\}$, that satisfy the following:

$$\begin{cases} u_N(t) \to u(t) \text{ and } v_N(t) \to v(t) \text{ strongly in } H_0^1(\Omega), \\ u_N'(t) \to u'(t) \text{ and } v_N'(t) \to v'(t) \text{ strongly in } L^2(\Omega), \end{cases} \tag{2.45}$$

for almost all $t \in [0,T]$.

Now, by integrating (2.1) and (2.2) from 0 to t, we obtain

$$\begin{aligned} &\langle u_N'(t), e_j\rangle - \langle u_N'(0), e_j\rangle + \int_0^t \langle \nabla u_N(\tau), \nabla e_j\rangle \, d\tau \\ &+ \int_0^t \langle |u_N'(\tau)|^{m-1}u_N'(\tau), e_j\rangle \, d\tau = \int_0^t \langle f_1(u_N(\tau), v_N(\tau)), e_j\rangle \, d\tau, \end{aligned} \tag{2.46}$$

$$\begin{aligned} &\langle v_N'(t), e_j\rangle - \langle v_N'(0), e_j\rangle + \int_0^t \langle \nabla v_N(\tau), \nabla e_j\rangle \, d\tau \\ &+ \int_0^t \langle |v_N'(\tau)|^{r-1}v_N'(\tau), e_j\rangle \, d\tau = \int_0^t \langle f_2(u_N(\tau), v_N(\tau)), e_j\rangle \, d\tau, \end{aligned} \tag{2.47}$$

for almost all $t \in [0,T]$, and where all inner products that appear in (2.46) and (2.47) are $L^2(\Omega)$-inner products. By using (2.35), (2.44) and (2.45), and recalling that the linear span of $\{e_j : j = 1, 2, \ldots\}$ is dense in $L^q(\Omega)$ for any $1 \le q < \infty$ we can pass to the limit easily as $N \to \infty$ in (2.46)-(2.47). Indeed, as $N \to \infty$, it follows that (u,v) satisfies all requirements in Definition 1.2. Hence, the proof of the local existence statement of a weak solution in Theorem 1.3 is now complete.

In addition, we have the following result concerning the second-order derivatives $u'', v'', \triangle u, \triangle v$.

Lemma 2.4. *Let (u,v) be a weak solution to* (1.1) *in the sense of Definition* 1.2 *on* $[0,T]$. *Then* $\triangle u, \triangle v \in L^\infty(0,T; H^{-1}(\Omega))$, $u'' \in L^{\frac{m+1}{m}}(0,T; H^{-1}(\Omega) + L^{\frac{m+1}{m}}(\Omega))$ *and* $v'' \in L^{\frac{r+1}{r}}(0,T; H^{-1}(\Omega) + L^{\frac{r+1}{r}}(\Omega))$, *where* $H^{-1}(\Omega) + L^{\frac{m+1}{m}}(\Omega)$ *is the dual space of* $H_0^1(\Omega) \cap L^{m+1}(\Omega)$.

Proof. Throughout the proof, let $< .,. >$ denote the standard duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. It is important to note the inclusions $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$, where each space is dense in the following one and the injections are continuous. In addition,

$$< f, \phi > = \langle f, \phi\rangle_{L^2(\Omega)} \text{ for all } f \in L^2(\Omega) \text{ and all } \phi \in H_0^1(\Omega).$$

We also note that $H^{-1}(\Omega) \subset \mathcal{D}'(\Omega)$ and that every element in $H^{-1}(\Omega)$ can be represented, non-uniquely, as a sum of first-order derivatives of functions in $L^2(\Omega)$ (for more details, see Theorem 12.1, page 71, Lions and Magenes [22]).

Since $\triangle u(t) = \nabla \cdot \nabla u(t)$ and $u(t) \in H_0^1(\Omega)$ then $\triangle u(t) \in H^{-1}(\Omega)$. Moreover,

$$< \triangle u(t), \phi >= - \ll \nabla u(t), \nabla \phi \gg = - \langle \nabla u(t), \nabla \phi \rangle_{L^2(\Omega)}\,, \tag{2.48}$$

for all $\phi \in \mathcal{D}(\Omega)$, and where $\ll .,. \gg$ denotes the duality pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, then $< \triangle u(t), \phi >= - \langle \nabla u(t), \nabla \phi \rangle_{L^2(\Omega)}$ for all $\phi \in H_0^1(\Omega)$.

Now, the fact that $\triangle u \in L^\infty(0,T; H^{-1}(\Omega))$ is trivial, since $u \in L^\infty(0,T; H_0^1(\Omega))$ and

$$\begin{aligned} | < \triangle u(t), \phi > | = | - \langle \nabla u(t), \nabla \phi \rangle_{L^2(\Omega)} | &\le \|u(t)\|_{H_0^1(\Omega)} \|\phi\|_{H_0^1(\Omega)} \\ &\le C \|\phi\|_{H_0^1(\Omega)} \end{aligned} \tag{2.49}$$

for every $\phi \in H_0^1(\Omega))$ and all $t \in [0,T]$. Similarly, $\triangle v \in L^\infty(0,T; H^{-1}(\Omega))$.

By recalling (1.5), for every $\phi \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$, we have

$$\begin{aligned} \left|< u''(t), \phi >\right| = \left| \frac{d}{dt} < u'(t), \phi > \right| &= \left| \frac{d}{dt} \langle u'(t), \phi \rangle_{L^2(\Omega)} \right| \\ &\le \left| \langle \nabla u(t), \nabla \phi) \rangle_{L^2(\Omega)} \right| + \left| \langle |u'(t)|^{m-1} u'(t), \phi \rangle_{L^2(\Omega)} \right| \\ &\qquad + \left| \langle f_1(u(t), v(t)), \phi \rangle_{L^2(\Omega)} \right| \\ &\le \|u(t)\|_{H_0^1(\Omega)} \|\phi\|_{H_0^1(\Omega)} + \left\|u'(t)\right\|^m_{L^{m+1}(\Omega)} \|\phi\|_{L^{m+1}(\Omega)} \\ &\qquad + \left| \langle f_1(u(t), v(t)), \phi \rangle_{L^2(\Omega)} \right|. \end{aligned} \tag{2.50}$$

The last term in (2.50) is estimated as follows.

$$\begin{aligned} &\left| \int_\Omega f_1(u(t), v(t)) \phi dx \right| \le C \int_\Omega \left(|u(t) + v(t)|^p |\phi| + |v(t)|^{\frac{p+1}{2}} |u(t)|^{\frac{p-1}{2}} |\phi| \right) dx \\ &\le C \left[(\|u(t)\|_{2p}^p + \|v(t)\|_{2p}^p) \|\phi\|_2 + \|u(t)\|_{3(p-1)}^{\frac{p-1}{2}} \|v(t)\|_{\frac{3(p+1)}{2}}^{\frac{p+1}{2}} \|\phi\|_2 \right] \\ &\le C \left[\|\nabla u(t)\|_2^p + \|\nabla v(t)\|_2^p + \|\nabla u(t)\|_2^{\frac{p-1}{2}} \|\nabla v(t)\|_2^{\frac{p+1}{2}} \right] \|\phi\|_2 \le C \|\phi\|_{H_0^1(\Omega)}\,, \end{aligned} \tag{2.51}$$

where we have used in (2.51) the Sobolev imbeddings in (1.4) and the fact that when $n = 3$ then $3(p-1) = \frac{3(p+1)}{2} = 6$.

It follows from (2.50)-(2.51) and the fact that $u \in L^\infty(0,T;H_0^1(\Omega))$ and $u' \in L^{m+1}(\Omega \times (0,T))$ that

$$\begin{aligned} &\int_0^T \left| < u''(t), \phi > \right|^{\frac{m+1}{m}} dt \qquad (2.52) \\ &\leq C \left\|\phi\right\|_{H_0^1(\Omega)}^{\frac{m+1}{m}} + \left(\int_0^T \int_\Omega |u'(t)|^{m+1} dxdt \right) \|\phi\|_{L^{m+1}(\Omega)}^{\frac{m+1}{m}} \\ &\leq C \left(\|\phi\|_{H_0^1(\Omega)}^{\frac{m+1}{m}} + \|\phi\|_{L^{m+1}(\Omega)}^{\frac{m+1}{m}} \right), \end{aligned}$$

for all $\phi \in H_0^1(\Omega)) \cap L^{m+1}(\Omega)$. Hence, $u'' \in L^{\frac{m+1}{m}}(0,T;H^{-1}(\Omega) + L^{\frac{m+1}{m}}(\Omega))$. The proof of the statement $v'' \in L^{\frac{r+1}{r}}(0,T;H^{-1}(\Omega) + L^{\frac{r+1}{r}}(\Omega))$ is similar and so it is omitted. □

2.3. **Proof of the energy identity in Theorem 1.3.** We show here that the weak solution (u,v) to (1.1) which we have constructed already satisfies the energy identity (1.7).

Proof. We start by recalling the sequences of the approximate solutions $\{u_N\}$ and $\{v_N\}$ satisfy (2.7), namely

$$\begin{aligned} &\frac{1}{2}\left(\left\|u_N'(t)\right\|_2^2 + \left\|v_N'(t)\right\|_2^2 + \|\nabla u_N(t)\|_2^2 + \|\nabla v_N(t)\|_2^2 \right) \qquad (2.53) \\ &+ \int_0^t \int_\Omega |u_N'(\tau)|^{m+1} dxd\tau + \int_0^t \int_\Omega |v_N'(\tau)|^{r+1} dxd\tau \\ &= \frac{1}{2}\left(\left\|u_N'(0)\right\|_2^2 + \left\|v_N'(0)\right\|_2^2 + \|\nabla u_N(0)\|_2^2 + \|\nabla v_N(0)\|_2^2 \right) \\ &+ \int_0^t \int_\Omega \left(f_1(u_N(\tau), v_N(\tau)) u_N'(\tau) + f_2(u_N(\tau), v_N(\tau)) v_N'(\tau) \right) dxd\tau. \end{aligned}$$

We first note that

$$\begin{aligned} &\int_0^t \int_\Omega \left(f_1(u_N(\tau), v_N(\tau)) u_N'(\tau) + f_2(u_N(\tau), v_N(\tau)) v_N'(\tau) \right) dxd\tau \\ &= \int_0^t \int_\Omega \frac{\partial}{\partial \tau} \left(F(u_N(\tau), v_N(\tau)) \right) dxd\tau \\ &= \int_\Omega F(u_N(t), v_N(t)) dx - \int_\Omega F(u_N(0), v_N(0)) dx. \qquad (2.54) \end{aligned}$$

Let

$$E_N(t) := \frac{1}{2}\left(\left\|u_N'(t)\right\|_2^2 + \left\|v_N'(t)\right\|_2^2 + \|\nabla u_N(t)\|_2^2 + \|\nabla v_N(t)\|_2^2\right) - \int_\Omega F(u_N(t), v_N(t))dx. \tag{2.55}$$

Then, it follows from (2.53)-(2.55) that

$$E_N(t) + \int_0^t\int_\Omega |u_N'(\tau)|^{m+1}dxd\tau + \int_0^t\int_\Omega |v_N'(\tau)|^{r+1}dxd\tau = E_N(0). \tag{2.56}$$

In order to pass to the limit in (2.56) we need to deal with the terms that involve F in (2.56). We first show that

$$\int_\Omega F(u_N(t), v_N(t))dx \longrightarrow \int_\Omega F(u(t), v(t))dx, \tag{2.57}$$

for almost all $t \in [0, T]$.
By recalling 1.13, we have

$$\left|\int_\Omega \left(F(u_N(t), v_N(t)) - F(u(t), v(t))\right) dx\right| \leq I_1 + I_2 + I_3 + I_4, \tag{2.58}$$

where

$$\begin{aligned}
I_1 &= C_0 \int_\Omega |u_N - u| \left(|u_N|^p + |v_N|^p + |u|^p + |v|^p\right) dx \\
I_2 &= C_0 \int_\Omega |v_N - v| \left(|u_N|^p + |v_N|^p + |u|^p + |v|^p\right) dx \\
I_3 &= C_1 \int_\Omega |u_N - u| \, |v_N| \left(|u_N|^{\frac{p-1}{2}} |v_N|^{\frac{p-1}{2}} + |u|^{\frac{p-1}{2}} |v|^{\frac{p-1}{2}}\right) dx \\
I_4 &= C_1 \int_\Omega |v_N - v| \, |u| \left(|u_N|^{\frac{p-1}{2}} |v_N|^{\frac{p-1}{2}} + |u|^{\frac{p-1}{2}} |v|^{\frac{p-1}{2}}\right) dx.
\end{aligned} \tag{2.59}$$

A typical term in I_1 and I_2 is estimated as follows.

$$\begin{aligned}
&\int_\Omega |u_N(t) - u(t)| \, |u_N(t)|^p \, dx \leq \|u_N(t) - u(t)\|_2 \, \|u_N(t)\|_{2p}^p \\
&\leq C \, \|u_N(t) - u(t)\|_2 \, \|\nabla u_N(t)\|_2^p \leq C \, \|u_N(t) - u(t)\|_2 \, ,
\end{aligned} \tag{2.60}$$

where we have used in (2.60) the Sobolev Imbedding theorem, the fact that $2p = 6$ when $n = 3$, and the bounds furnished by Lemma 2.1. Hence, for all $t \in [0, T]$, we have

$$I_1 + I_2 \leq C \left(\|u_N(t) - u(t)\|_2 + \|v_N(t) - v(t)\|_2\right). \tag{2.61}$$

Also, a typical term in I_3 is estimated as follows.

$$\begin{aligned}&\int_\Omega |u_N - u|\,|v_N|\,|u_N|^{\frac{p-1}{2}}\,|v_N|^{\frac{p-1}{2}}\,dx \\ &\le \|u_N(t)-u(t)\|_2\,\|u_N(t)\|_{3(p-1)}^{\frac{p-1}{2}}\,\|v_N(t)\|_{\frac{3(p+1)}{2}}^{\frac{p+1}{2}} \\ &\le \|u_N(t)-u(t)\|_2\,\|\nabla u_N(t)\|_2^{\frac{p-1}{2}}\,\|\nabla v_N(t)\|_2^{\frac{p+1}{2}} \le C\,\|u_N(t)-u(t)\|_2\,,\end{aligned} \tag{2.62}$$

where we have used Lemma 2.1 and the fact that $3(p-1)=\frac{3(p+1)}{2}=6$ when $n=3$.

Similarly, a typical term in I_4 is estimated in the same way, and we have

$$I_3+I_4 \le C\left(\|u_N(t)-u(t)\|_2+\|v_N(t)-v(t)\|_2\right). \tag{2.63}$$

Therefore, (2.57) follows easily from (2.58), (2.61), (2.63) and the strong convergence furnished by Lemma 2.2. Similarly, by the strong convergence given in (2.8), we infer

$$\int_\Omega F(u_N(0),v_N(0))dx \longrightarrow \int_\Omega F(u^0,v^0)dx. \tag{2.64}$$

Finally, by the strong convergence established in Lemma 2.2 and by using (2.57) and (2.64) we can pass to the limit in (2.56). Indeed, by letting $N\to\infty$ in (2.56), we obtain the desired energy identity

$$E(t)+\int_0^t\int_\Omega |u'(\tau)|^{m+1}dxd\tau+\int_0^t\int_\Omega |v'(\tau)|^{r+1}dxd\tau = E(0), \tag{2.65}$$

where

$$\begin{aligned}E(t) := \frac{1}{2}\left(\|u'(t)\|_2^2+\|v'(t)\|_2^2+\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2\right)& \\ -\int_\Omega F(u(t),v(t))dx.&\end{aligned} \tag{2.66}$$

2.4. **Proof of the uniqueness statement in Theorem 1.3.** The proof of Theorem 1.3 will be complete after we prove the following lemma.

Lemma 2.5. *Assume* (u,v) *and* $(\tilde{u},\tilde{v})$ *are two weak solutions to the initial-boundary-value problem* (1.1) *defined on* $[0,T]$ *in the sense of Definition* 1.2. *Then,* $(u,v)=(\tilde{u},\tilde{v})$.

Proof. Let $y = u - \tilde{u}$ and $z = v - \tilde{v}$. Then, y and z satisfy

$$\begin{cases} y_{tt} - \Delta y + |u_t|^{m-1}u_t - |\tilde{u}_t|^{m-1}\tilde{u}_t = f_1(u,v) - f_1(\tilde{u},\tilde{v}), \text{ in } \Omega \times (0,T), \\ z_{tt} - \Delta z + |v_t|^{r-1}v_t - |\tilde{v}_t|^{r-1}\tilde{v}_t = f_2(u,v) - f_2(\tilde{u},\tilde{v}), \text{ in } \Omega \times (0,T), \\ y(x,0) = 0,\ y_t(x,0) = 0,\ z(x,0) = 0,\ z_t(x,0) = 0, \quad \text{in } \Omega \\ y = z = 0, \quad \text{on } \Gamma \times (0,T). \end{cases} \tag{2.67}$$

Since y_t, z_t are not regular enough (we only have $y_t, z_t \in C_w([0,T], L^2(\Omega))$) it is not permissible to directly test the equations in (2.67) with y_t and z_t and apply standard energy estimates to obtain the desired uniqueness of solutions. In order to overcome this difficulty we need to introduce the difference quotients $D_h y$ and $D_h z$ and their well-known properties (see [14] for more details).

For any function $y \in C_w([0,T], L_2(\Omega))$ and $h \neq 0$ we define

$$D_h y(\tau) \equiv \frac{1}{2h}\left(y_e(\tau+h) - y_e(\tau-h)\right) \text{ for } \tau \in (0,T),$$

where $y_e(\tau)$ denotes the extension of $y(\tau)$ to $\mathbb{R}$ given by: $y_e(\tau) = y(\tau)$ for $\tau \in (0,T)$; $y_e(\tau) = y(T)$ for $\tau \geq T$; and $y_e(\tau) = y(0)$ for $\tau \leq 0$.

With the above notation and the fact that $y, z \in C_w([0,T], H_0^1(\Omega))$ and $y_t, z_t \in C_w([0,T], L^2(\Omega))$, for all $t \in [0,T]$, we have (see [14] for more details)

$$\begin{aligned} &\lim_{h\to 0}\int_0^t \langle y_t(\tau), D_h y_t(\tau)\rangle_{L^2(\Omega)}\, d\tau = \frac{1}{2}[\|y_t(t)\|^2_{L^2(\Omega)} - \|y_t(0)\|^2_{L^2(\Omega)}], \\ &\lim_{h\to 0}\int_0^t \langle \nabla y(\tau), D_h \nabla y(\tau)\rangle_{L^2(\Omega)}\, d\tau = \frac{1}{2}[\|\nabla y(t)\|^2_{L^2(\Omega)} - \|\nabla y(0)\|^2_{L^2(\Omega)}], \\ &\lim_{h\to 0}\int_0^t \langle z_t(\tau), D_h z_t(\tau)\rangle_{L^2(\Omega)}\, d\tau = \frac{1}{2}[\|z_t(t)\|^2_{L^2(\Omega)} - \|z_t(0)\|^2_{L^2(\Omega)}], \\ &\lim_{h\to 0}\int_0^t \langle \nabla z(\tau), D_h \nabla z(\tau)\rangle_{L^2(\Omega)}\, d\tau = \frac{1}{2}[\|\nabla z(t)\|^2_{L^2(\Omega)} - \|\nabla z(0)\|^2_{L^2(\Omega)}]. \end{aligned} \tag{2.68}$$

In addition, as $h \to 0$

$$D_h y \to y_t \text{ in } L^{m+1}(\Omega \times (0,T)) \text{ and } D_h z \to z_t \text{ in } L^{r+1}(\Omega \times (0,T)). \tag{2.69}$$

We also note that for any functions $u, v \in H_0^1(\Omega)$, $f_1(u,v) \in L^2(\Omega)$. To see this, we recall (1.3) and by using Hölder's inequality and the Sobolev Imbedding theorem, we have

$$\int_\Omega |f_1(u,v)|^2\, dx \leq C\int_\Omega \left[|u|^{2p} + |v|^{2p} + |u|^{p-1}|v|^{p+1}\right] dx$$

$$\leq C\left[\|u\|_{2p}^{2p}+\|v\|_{2p}^{2p}+\|u\|_{3(p-1)}^{p-1}\|v\|_{\frac{3(p+1)}{2}}^{p+1}\right]\leq C. \quad (2.70)$$

Similarly, $f_2(u,v)\in L^2(\Omega)$ for any functions $u,v\in H_0^1(\Omega)$. In particular, the right-hand side of the equations in (2.67) are in $L^{\frac{m+1}{m}}(\Omega\times(0,T))$ and $L^{\frac{r+1}{r}}(\Omega\times(0,T))$, respectively. Therefore, it follows from (2.67) that

$$\Box y\in L^{\frac{m+1}{m}}(\Omega\times(0,T)) \text{ and } \Box z\in L^{\frac{r+1}{r}}(\Omega\times(0,T)). \quad (2.71)$$

Now, multiply the equations in (2.67) respectively by $D_h y$ and $D_h z$, then integration over $\Omega\times(0,t)$ yields

$$\int_0^t\int_\Omega (y_{tt}-\triangle y)\,D_h y\,dxd\tau+\int_0^t\int_\Omega\left[|u_t|^{m-1}u_t-|\tilde{u}_t|^{m-1}\tilde{u}_t\right]D_h y dxd\tau$$
$$=\int_0^t\int_\Omega [f_1(u,v)-f_1(\tilde{u},\tilde{v})]\,D_h y dxd\tau, \quad (2.72)$$

$$\int_0^t\int_\Omega (z_{tt}-\triangle z)\,D_h z\,dxd\tau+\int_0^t\int_\Omega\left[|v_t|^{r-1}v_t-|\tilde{v}_t|^{r-1}\tilde{v}_t\right]D_h z dxd\tau$$
$$=\int_0^t\int_\Omega [f_2(u,v)-f_2(\tilde{u},\tilde{v})]\,D_h z dxd\tau. \quad (2.73)$$

As in Lemma 2.4, let $<.,.>$ denote the standard duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. In order to pass to the limit in (2.72)-(2.73) we first need to deal with the terms that involve $\Box y$ and $\Box z$. By recalling Lemma 2.4 and (2.48), we have

$$\int_0^t\int_\Omega (y_{tt}-\triangle y)\,D_h y\,dxd\tau=\int_0^t <y_{tt},D_h y>d\tau-\int_0^t<\triangle y,D_h y>d\tau$$
$$=-\int_0^t<y_t,\partial_t D_h y>d\tau+\int_0^t\langle\nabla y,\nabla D_h y\rangle_{L^2(\Omega)}\,d\tau$$
$$+<y_t(t),D_h y(t)>-<y_t(0),D_h y(0)> \quad (2.74)$$
$$=-\int_0^t\langle y_t,D_h y_t\rangle_{L^2(\Omega)}\,d\tau+\int_0^t\langle\nabla y,D_h\nabla y\rangle_{L^2(\Omega)}\,d\tau+\langle y_t(t),D_h y(t)\rangle_{L^2(\Omega)}.$$

In order to pass to the limit in (2.74) as $h\to 0$ it is important to recall (2.69) and (2.71), and also that y satisfies the identities in (2.68). Indeed, by letting $h\to 0$ in (2.74) and noting the identically zero initial conditions in (2.66), we have

$$\lim_{h\to 0}\int_0^t\int_\Omega (y_{tt}-\triangle y)\,D_h y\,dxd\tau=\int_0^t\int_\Omega (y_{tt}-\triangle y)\,y_t\,dxd\tau$$

$$= -\frac{1}{2}\|y_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla y(t)\|^2_{L^2(\Omega)} + \|y_t(t)\|^2_{L^2(\Omega)}$$
$$= \frac{1}{2}\left[\|y_t(t)\|^2_{L^2(\Omega)} + \|\nabla y(t)\|^2_{L^2(\Omega)}\right]. \tag{2.75}$$

By the same exact steps we also have

$$\lim_{h\to 0}\int_0^t\int_\Omega (z_{tt} - \triangle z)\, D_h z\, dxd\tau = \int_0^t\int_\Omega (z_{tt} - \triangle z)\, z_t\, dxd\tau$$
$$= \frac{1}{2}\left[\|z_t(t)\|^2_{L^2(\Omega)} + \|\nabla z(t)\|^2_{L^2(\Omega)}\right]. \tag{2.76}$$

Now, we can pass to the limit (2.72)-(2.72) as $h \to 0$ and obtain

$$\frac{1}{2}\left[\|y_t(t)\|^2_{L^2(\Omega)} + \|\nabla y(t)\|^2_{L^2(\Omega)}\right] + \int_0^t\int_\Omega \left[|u_t|^{m-1}u_t - |\tilde{u}_t|^{m-1}\tilde{u}_t\right] y_t dxd\tau$$
$$= \int_0^t\int_\Omega [f_1(u,v) - f_1(\tilde{u},\tilde{v})]\, y_t dxd\tau, \tag{2.77}$$

$$\frac{1}{2}\left[\|z_t(t)\|^2_{L^2(\Omega)} + \|\nabla z(t)\|^2_{L^2(\Omega)}\right] + \int_0^t\int_\Omega \left[|v_t|^{r-1}v_t - |\tilde{v}_t|^{r-1}\tilde{v}_t\right] z_t dxd\tau$$
$$= \int_0^t\int_\Omega [f_2(u,v) - f_2(\tilde{u},\tilde{v})]\, z_t dxd\tau, \tag{2.78}$$

where we have used (2.69) and (2.70).

At this end, let us set

$$Y(t) := \|y_t(t)\|_2^2 + \|z_t(t)\|_2^2 + \|\nabla y(t)\|_2^2 + \|\nabla z(t)\|_2^2.$$

Then, by adding (2.77)-(2.78) and recalling the monotonicity property in (2.30), one has

$$Y(t) + c_0\int_0^t\int_\Omega \left[|y_t|^{m+1} + |z_t|^{r+1}\right] dxd\tau \leq \int_0^t\int_\Omega [f_1(u,v) - f_1(\tilde{u},\tilde{v})]\, y_t dxd\tau$$
$$+ \int_0^t\int_\Omega [f_2(u,v) - f_2(\tilde{u},\tilde{v})]\, z_t dxd\tau. \tag{2.79}$$

By recalling the estimates we have already obtained in (2.20)-(2.29) but with the replacement: (u_N, v_N) by (u,v), (u_L, v_L) by $(\tilde{u},\tilde{v})$, u'_{NL} by y_t and v'_{NL} by z_t, then it follows from (2.28)-(2.29) that

$$\left|\int_0^t\int_\Omega [f_1(u,v) - f_1(\tilde{u},\tilde{v})]\, y_t dxd\tau\right|$$

$$\leq C\int_0^t \left[\|\nabla y(\tau)\|_2^2 + \|\nabla z(\tau)\|_2^2 + \|y_t(\tau)\|_2^2\right] d\tau, \quad (2.80)$$

$$\left|\int_0^t \int_\Omega [f_2(u,v) - f_2(\tilde{u},\tilde{v})]\, z_t dx d\tau\right|$$
$$\leq C\int_0^t \left[\|\nabla y(\tau)\|_2^2 + \|\nabla z(\tau)\|_2^2 + \|z_t(\tau)\|_2^2\right] d\tau. \quad (2.81)$$

Now, (2.79)-(2.81) yield

$$Y(t) + c_0\int_0^t \int_\Omega \left[|y_t|^{m+1} + |z_t|^{r+1}\right] dx d\tau \leq C\int_0^t Y(\tau) d\tau. \quad (2.82)$$

By Gronwall's inequality we have

$$Y(t) = \|y_t(t)\|_2^2 + \|z_t(t)\|_2^2 + \|\nabla y(t)\|_2^2 + \|\nabla z(t)\|_2^2 = 0$$

for all $t \in [0,T]$, which completes the proof of the lemma. □

3. GLOBAL SOLUTIONS.

This section is devoted to the proofs of Theorems 1.4 and 1.5 which will be complete after proving the estimates in Lemma 3.1 and Lemma 3.2. A global weak solution to (1.1) as stated in Theorems 1.4 and 1.5 then follows from a standard continuation argument.

Lemma 3.1. *Let (u,v) be a weak solution to the initial-boundary-value problem (1.1) defined on $[0,T]$ as furnished by Theorem 1.3 and assume that $p \leq \min\{m,r\}$. Then, for all $t \in [0,T]$, the following inequality holds:*

$$\frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) + \int_\Omega F(u(t),v(t))dx$$
$$+ \int_0^t \int_\Omega \left(|u'(\tau)|^{m+1} + |v'(\tau)|^{r+1}\right) dx d\tau$$
$$\leq C_T(|u^0|_{1,\Omega}, |v^0|_{1,\Omega}, |u^1|_{0,\Omega}, |v^1|_{0,\Omega}), \quad (3.1)$$

where $T > 0$ is arbitrary.

Proof. Define the following energy functions:

$$E_0(t) := \frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right)$$
$$E_1(t) := E_0(t) + \int_\Omega F(u(t),v(t))dx. \quad (3.2)$$

We will show that $E_1(t)$ remains bounded for bounded times and establish (3.1). By recalling (1.2), we have

$$c_0\left(\|u(t)\|_{p+1}^{p+1}+\|v(t)\|_{p+1}^{p+1}\right)\leq\int_\Omega F(u(t),v(t))dx \tag{3.3}$$
$$\leq c_1\left(\|u(t)\|_{p+1}^{p+1}+\|v(t)\|_{p+1}^{p+1}\right)\leq C\left(\|\nabla u(t)\|_2^{p+1}+\|\nabla v(t)\|_2^{p+1}\right).$$

Therefore,

$$E_0(t)\leq E_1(t)\leq C(E_0(t))^{\frac{p+1}{2}} \quad \text{and} \quad \|u(t)\|_{p+1}^{p+1}+\|v(t)\|_{p+1}^{p+1}\leq CE_1(t). \tag{3.4}$$

In addition, the energy identity (1.7) can be written as follows.

$$E_0(t)+\int_0^t\int_\Omega|u'(\tau)|^{m+1}dxd\tau+\int_0^t\int_\Omega|v'(\tau)|^{r+1}dxd\tau$$
$$=E_0(0)+\int_0^t\int_\Omega\frac{\partial}{\partial\tau}\left(F(u(\tau),v(\tau))\right)dxd\tau. \tag{3.5}$$

By adding the term

$$\int_0^t\int_\Omega\frac{\partial}{\partial\tau}\left(F(u(\tau),v(\tau))\right)dxd\tau=\int_\Omega F(u(t),v(t))dx-\int_\Omega F(u^0,v^0)dx$$

to both sides of (3.5), we obtain

$$E_1(t)+\int_0^t\int_\Omega|u'(\tau)|^{m+1}dxd\tau+\int_0^t\int_\Omega|v'(\tau)|^{r+1}dxd\tau$$
$$=E_1(0)+2\int_0^t\int_\Omega\frac{\partial}{\partial\tau}\left(F(u(\tau),v(\tau))\right)dxd\tau. \tag{3.6}$$

Let $Q_t=\Omega\times(0,t)$. In order to obtain a useful estimate for the last term in (3.6), we define

$$\begin{aligned}
Q_{11}&:=\{(x,\tau)\in Q_t:\ |u(x,\tau)|\leq 1,|v(x,\tau)|\leq 1\}\\
Q_{12}&:=\{(x,\tau)\in Q_t:\ |u(x,\tau)|\leq 1,|v(x,\tau)|>1\}\\
Q_{21}&:=\{(x,\tau)\in Q_t:\ |u(x,\tau)|>1,|v(x,\tau)|\leq 1\}\\
Q_{22}&:=\{(x,\tau)\in Q_t:\ |u(x,\tau)|>1,|v(x,\tau)|>1\}.
\end{aligned} \tag{3.7}$$

We first note that

$$2\int_{Q_t}\left|\frac{\partial}{\partial\tau}\left(F(u(\tau),v(\tau))\right)\right|dxd\tau\leq 2\int_{Q_t}\left(|f_1(u,v)|\left|u'\right|+|f_2(u,v)|\left|v'\right|\right)dxd\tau$$
$$\leq I(t)+J(t), \tag{3.8}$$

where

$$I(t) = C \int_{Q_t} \left(|u|^p + |v|^p + |u|^{\frac{p-1}{2}} |v|^{\frac{p+1}{2}} \right) |u'| \, dx d\tau$$
$$J(t) = C \int_{Q_t} \left(|u|^p + |v|^p + |v|^{\frac{p-1}{2}} |u|^{\frac{p+1}{2}} \right) |v'| \, dx d\tau. \tag{3.9}$$

We estimate $I(t)$ and $J(t)$ as follows. We write $I(t) = I_{11} + I_{12} + I_{21} + I_{22}$ and $J(t) = J_{11} + J_{12} + J_{21} + J_{22}$ where

$$I_{ij}(t) = C \int_{Q_{ij}} \left(|u|^p + |v|^p + |u|^{\frac{p-1}{2}} |v|^{\frac{p+1}{2}} \right) |u'| \, dQ_{ij}$$
$$J_{ij}(t) = C \int_{Q_{ij}} \left(|u|^p + |v|^p + |v|^{\frac{p-1}{2}} |u|^{\frac{p+1}{2}} \right) |v'| \, dQ_{ij}, \quad i, j = 1, 2. \tag{3.10}$$

We first note that

$$I_{11}(t) \le C \int_{Q_{11}} |u'| \, dQ_{11} \le \rho |Q_t| + C_\rho \int_{Q_{11}} |u'|^2 \, dQ_{11}$$
$$\le \rho |Q_t| + C_\rho \int_0^t E_1(\tau) d\tau, \tag{3.11}$$

for some constant $\rho > 0$. Also, here and later $|Q_t|$ denotes the Lebesgue measure of Q_t. Similarly,

$$J_{11}(t) \le \rho |Q_t| + C_\rho \int_0^t E_1(\tau) d\tau. \tag{3.12}$$

We also have

$$I_{12}(t) \le C \int_{Q_{12}} \left(1 + |v|^p + |v|^{\frac{p+1}{2}} \right) |u'| \, dQ_{12}$$
$$\le \rho |Q_t| + C_\rho \int_{Q_{12}} |u'|^2 \, dQ_{12} + C \int_{Q_{12}} |v|^p |u'| \, dQ_{12}$$
$$\le \rho |Q_t| + C_\rho \int_0^t E_1(\tau) d\tau + C \int_{Q_{12}} |v|^p |u'| \, dQ_{12}. \tag{3.13}$$

The last term in (3.13) is estimated as follows. Since $p \le m$ then $\alpha := \frac{m-p}{m+1} \ge 0$ and $(p + \alpha)\frac{m+1}{m} = p + 1$. Since $|v| > 1$ on Q_{12}, then by Hölder's and Young's inequalities, and noting (3.4), we have

$$C \int_{Q_{12}} |v|^p |u'| \, dQ_{12} \le C \int_{Q_{12}} |v|^{p+\alpha} |u'| \, dQ_{12}$$

$$\leq C\left(\int_{Q_{12}}\left|u'\right|^{m+1}dQ_{12}\right)^{\frac{1}{m+1}}\left(\int_{Q_{12}}|v|^{(p+\alpha)\frac{m+1}{m}}\,dQ_{12}\right)^{\frac{m}{m+1}}$$
$$\leq \epsilon\int_{Q_{12}}\left|u'\right|^{m+1}dQ_{12}+C_\epsilon\int_{Q_{12}}|v|^{p+1}\,dQ_{12}$$
$$\leq \epsilon\int_0^t\int_\Omega\left|u'(\tau)\right|^{m+1}dxd\tau+C_\epsilon\int_0^t E_1(\tau)d\tau, \tag{3.14}$$

where $\epsilon > 0$ will be chosen later.

It follows from (3.13)-(3.14) that

$$I_{12}(t)\leq \rho\,|Q_t|+(C_\rho+C_\epsilon)\int_0^t E_1(\tau)d\tau+\epsilon\int_0^t\int_\Omega\left|u'(\tau)\right|^{m+1}dxd\tau. \tag{3.15}$$

Similarly, by using the assumption $p \leq r$ we easily obtain

$$J_{12}(t)\leq \rho\,|Q_t|+(C_\rho+C_\epsilon)\int_0^t E_1(\tau)d\tau+\epsilon\int_0^t\int_\Omega\left|v'(\tau)\right|^{r+1}dxd\tau. \tag{3.16}$$

The estimates for I_{21} and J_{21} are similar and they are omitted. Indeed, we have

$$I_{21}(t)\leq \rho\,|Q_t|+(C_\rho+C_\epsilon)\int_0^t E_1(\tau)d\tau+\epsilon\int_0^t\int_\Omega\left|u'(\tau)\right|^{m+1}dxd\tau,$$
$$J_{21}(t)\leq \rho\,|Q_t|+(C_\rho+C_\epsilon)\int_0^t E_1(\tau)d\tau+\epsilon\int_0^t\int_\Omega\left|v'(\tau)\right|^{r+1}dxd\tau. \tag{3.17}$$

We finally estimate I_{22} and J_{22}. For the convenience of the reader we estimate

$$I_{22}=C\int_{Q_{22}}\left(|u|^p+|v|^p+|u|^{\frac{p-1}{2}}\,|v|^{\frac{p+1}{2}}\right)\left|u'\right|dQ_{22}.$$

The first two terms in I_{22} are estimated as in (3.14); that is,

$$C\int_{Q_{22}}(|u|^p+|v|^p)\left|u'\right|dQ_{22}\leq \epsilon\int_0^t\int_\Omega\left|u'(\tau)\right|^{m+1}dxd\tau+C_\epsilon\int_0^t E_1(\tau)d\tau. \tag{3.18}$$

As for the last term in I_{22}, we have

$$C\int_{Q_{22}}|u|^{\frac{p-1}{2}}\,|v|^{\frac{p+1}{2}}\left|u'\right|dQ_{22}$$
$$\leq C\left(\int_{Q_{22}}|u|^{p-1}\left|u'\right|^2dQ_{22}\right)^{\frac{1}{2}}\left(\int_{Q_{22}}|v|^{p+1}\,dQ_{22}\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_0^t\int_\Omega |v|^{p+1}\,dxd\tau + \int_{Q_{22}} |u|^{p-1}\left|u'\right|^2 dQ_{22}\right). \tag{3.19}$$

Since $m \geq p$ then $\beta := \frac{2(m-p)}{m+1} \geq 0$ and $\frac{m+1}{m-1}(p-1+\beta) = p+1$. Therefore, the last term in (3.19) is estimated as follows.

$$\begin{aligned}
C\int_{Q_{22}} |u|^{p-1}\left|u'\right|^2 dQ_{22} &\leq C\int_{Q_{22}} |u|^{p-1+\beta}\left|u'\right|^2 dQ_{22} \\
&\leq C\left(\int_{Q_{22}} |u|^{\frac{m+1}{m-1}(p-1+\beta)}\right)^{\frac{m-1}{m+1}}\left(\int_{Q_{22}}\left|u'\right|^{m+1}\right)^{\frac{2}{m+1}} \\
&\leq \epsilon\int_0^t\int_\Omega\left|u'(\tau)\right|^{m+1} dxd\tau + C_\epsilon\int_0^t\int_\Omega |u|^{p+1}\,dxd\tau.
\end{aligned} \tag{3.20}$$

It follows from (3.18)-(3.20) that

$$I_{22}(t) \leq 2\epsilon\int_0^t\int_\Omega\left|u'(\tau)\right|^{m+1} dxd\tau + C_\epsilon\int_0^t E_1(\tau)d\tau. \tag{3.21}$$

Similarly, one has

$$J_{22}(t) \leq 2\epsilon\int_0^t\int_\Omega\left|v'(\tau)\right|^{r+1} dxd\tau + C_\epsilon\int_0^t E_1(\tau)d\tau. \tag{3.22}$$

By combining the estimates (3.11)-(3.12), (3.15)-(3.17), (3.21)-(3.22), we have

$$\begin{aligned}
I(t)+J(t) &\leq 6\rho|Q_t| + C_{\epsilon,\rho}\int_0^t E_1(\tau)d\tau \\
&\quad + 4\epsilon\int_0^t\int_\Omega |u'(\tau)|^{m+1}dxd\tau + 4\epsilon\int_0^t\int_\Omega |v'(\tau)|^{r+1}dxd\tau.
\end{aligned} \tag{3.23}$$

By choosing $\epsilon > 0$ small enough, then it follows from (3.6), (3.8) and (3.23) that

$$\begin{aligned}
&E_1(t) + c_\epsilon\left(\int_0^t\int_\Omega |u'(\tau)|^{m+1}dxd\tau + \int_0^t\int_\Omega |v'(\tau)|^{r+1}dxd\tau\right) \\
&\qquad \leq E_1(0) + 6\rho|Q_t| + C_{\epsilon,\rho}\int_0^t E_1(\tau)d\tau,
\end{aligned} \tag{3.24}$$

for some constant $c_\epsilon > 0$. By Gronwall's inequality it follows that

$$E_1(t) \leq \left(E_1(0) + 6\rho|Q_t|\right)e^{Ct}, \tag{3.25}$$

for some constant $C > 0$. Finally, (3.25) leads to

$$E_1(t) + \int_0^t \int_\Omega |u'(\tau)|^{m+1} dx d\tau + \int_0^t \int_\Omega |v'(\tau)|^{r+1} dx d\tau \\ \leq C_T \left(E_1(0) + 6\rho |Q_t| \right), \tag{3.26}$$

where the last inequality is valid for all $0 < t \leq T$, and T is being arbitrary. Hence, the proof of the lemma is complete. □

The proof of Theorem 1.5 will be complete after we prove the following lemma. For the convenience of the reader, we recall

$$J(t) := \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 - 4G(t) \text{ where } G(t) := \int_\Omega F(u(t), v(t)) dx.$$

Lemma 3.2. *Let (u, v) be a weak solution to the initial-boundary-value problem (1.1) defined on $[0, T]$ as furnished by Theorem 1.3. Assume that $J(0) > 0$ and $4^{\frac{p}{2}} c_0 E(0)^{\frac{p-1}{2}} < 1$, where c_0 is a computable positive constant. Then, $J(t) > 0$ on $[0, T)$ and, for all $t \in [0, T]$, the following inequality holds:*

$$\frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2 \right) + \frac{1}{4}\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 \right) + G(t) \\ + \int_0^t \int_\Omega \left(|u'(\tau)|^{m+1} + |v'(\tau)|^{r+1} \right) dx d\tau \leq 2E(0), \tag{3.27}$$

where $T > 0$ is arbitrary.

Proof. Recall the energy identity

$$E(t) + \int_0^t \int_\Omega |u'(\tau)|^{m+1} dx d\tau + \int_0^t \int_\Omega |v'(\tau)|^{r+1} dx d\tau = E(0), \tag{3.28}$$

where

$$E(t) := \frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 \right) - G(t). \tag{3.29}$$

We fist note that since $u, v \in C_w([0, T], H_0^1(\Omega))$, $u_t, v_t \in C_w([0, T], L^2(\Omega))$, then after possibly modifying u and v on a set of measure zero in $[0, T]$, one has $u, v \in C([0, T], L^2(\Omega))$. Indeed, this continuity of u and v implies that G is continuous on $[0, T]$. To see this, let $t_N \in [0, T]$ such that $t_N \to t \in [0, T]$ as $N \to \infty$. By recalling the estimates we have obtained in (2.58)-(2.63), and by replacing $(u_N(t), v_N(t))$ by $(u(t_N), v(t_N))$ in these estimates, one has

$$|G(t_N) - G(t)| = \left| \int_\Omega \left[F(u(t_N), v(t_N)) - F(u(t), v(t)) \right] dx \right| \\ \leq C \left(\|u(t_N) - u(t)\|_2 + \|v(t_N) - v(t)\|_2 \right). \tag{3.30}$$

Since $u, v \in C([0,T], L^2(\Omega))$, it follows that G is continuous on $[0,T]$.

We note here that the energy identity (3.28) implies that $E(t)$ is absolutely continuous on $[0,T]$. Therefore, the continuity of G on $[0,T]$ implies that every term in $E(t)$ is continuous on $[0,T]$. Let t_0 be given by $t_0 := \sup\{t \in [0,T] : J(\tau) > 0 \text{ on } [0,t)\}$. By the continuity of $J(t)$ and the assumption $J(0) > 0$, we know that $0 < t_0 \leq T$. We show below that $t_0 = T$.

It follows from the definition of $E(t)$ and $J(t)$ that

$$\begin{aligned} E(0) \geq E(t) &= \tfrac{1}{2}\left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2\right) + \tfrac{1}{4}\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) + \tfrac{1}{4}J(t) \\ &> \frac{1}{4}\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) \end{aligned} \tag{3.31}$$

on $[0,t_0)$. That is,

$$\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 < 4E(0) \text{ on } [0,t_0). \tag{3.32}$$

By recalling (3.3), then, for $t \in [0,t_0)$, we have

$$\begin{aligned} &4G(t) \leq 4c_1\left(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1}\right) \leq c_0\left(\|\nabla u(t)\|_2^{p+1} + \|\nabla v(t)\|_2^{p+1}\right) \\ &\leq 2c_0\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right)^{\frac{p+1}{2}} \leq 2c_0\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right)(4E(0))^{\frac{p-1}{2}} \\ &= 4^{\frac{p}{2}}c_0E(0)^{\frac{p-1}{2}}\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right), \end{aligned} \tag{3.33}$$

where we have used (3.32). Also, we note that the constant c_0 in (3.33) is a positive constant that depends on c_1 and the imbedding constant in (1.4), which itself depends on Ω and p.

By assumption, $4^{\frac{p}{2}}c_0E(0)^{\frac{p-1}{2}} = 1 - \epsilon_0$ for some $0 < \epsilon_0 < 1$. Therefore, it follows from (3.33) and the definition of t_0 that $t_0 = T$ and $J(t) > 0$ on $[0,T)$. Moreover,

$$\epsilon_0\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) \leq J(t) \text{ on } [0,T], \tag{3.34}$$

where T is arbitrary.

Finally, it follows from the energy identity (3.28), (3.31) and (3.33) that

$$\begin{aligned} &\frac{1}{2}\left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2\right) + \frac{1}{4}\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) + G(t) \\ &\qquad + \int_0^t\int_\Omega \left(|u'(\tau)|^{m+1} + |v'(\tau)|^{r+1}\right)dxd\tau \leq 2E(0), \end{aligned} \tag{3.35}$$

for all $t \in [0,T]$, where $T > 0$ is arbitrary. □

4. Proof Of Theorem 1.6–Blow-Up Of Solutions

Proof. Let (u, v) be a solution to (1.1) in the sense of Definition 1.2. Throughout the proof, we assume that $p > \max\{m, r\}$ and $E(0) < 0$. Define

$$N(t) = \|u(t)\|_2^2 + \|v(t)\|_2^2, \quad G(t) = \int_\Omega F(u, v)dx, \tag{4.1}$$

$$H(t) = -\frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) + G(t). \tag{4.2}$$

Note that the assumption $E(0) < 0$ is equivalent to $H(0) > 0$. Also, the energy identity (1.7) implies that

$$H'(t) = \|u'(t)\|_{m+1}^{m+1} + \|v'(t)\|_{r+1}^{r+1} \geq 0. \tag{4.3}$$

Therefore, by recalling (1.2) we have

$$0 < H(0) \leq H(t) \leq G(t) \leq c_1\left(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1}\right) \tag{4.4}$$

For $0 \leq t < T$. Also, (1.2) yields

$$c_0\left(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1}\right) \leq G(t), \ \ 0 \leq t < T. \tag{4.5}$$

Let $0 < \gamma < \min\left\{\frac{1}{m+1} - \frac{1}{p+1}, \ \frac{1}{r+1} - \frac{1}{p+1}, \ \frac{p-1}{2(p+1)}\right\}$. In particular, $\gamma < \frac{1}{2}$. To simplify the notation, we introduce the following constants:

$$K_1 = \frac{1}{c_0^{1/(p+1)}}|\Omega|^{\frac{p-m}{(p+1)(m+1)}}, \quad K_2 = \frac{1}{c_0^{1/(p+1)}}|\Omega|^{\frac{p-r}{(p+1)(r+1)}},$$
$$\delta_1 = \frac{p-1}{4}H(0)^{\frac{1}{m+1}-\frac{1}{p+1}}, \quad \delta_2 = \frac{p-1}{4}H(0)^{\frac{1}{r+1}-\frac{1}{p+1}}, \tag{4.6}$$

where $c_0 > 0$ is the constant that appeared in (4.5) and (1.2).

Since $0 < \gamma < \frac{1}{2}$, we may choose $0 < \epsilon < 1$ small enough such that

$$L := 1 - \gamma - 2\epsilon\left(\delta_1^{-\frac{1}{m}} K_1^{\frac{m+1}{m}} H(0)^{\frac{1}{p+1}-\frac{1}{m+1}+\gamma} + \delta_2^{-\frac{1}{r}} K_2^{\frac{r+1}{r}} H(0)^{\frac{1}{p+1}-\frac{1}{r+1}+\gamma}\right) \geq 0. \tag{4.7}$$

Later, we may need to adjust ϵ again.

In the remainder of the proof, most generic constants will be denoted by C_0, $C, ...$; they may depend on various parameters, but they are totally independent from ϵ and the initial data, and they may change from line to line.

Let us first note that the smallness condition on ϵ in (4.7) implies that

$$1-\gamma \geq \epsilon C \left(H(0)^{\frac{1}{p+1}-\frac{1}{m+1}+\gamma} + H(0)^{\frac{1}{p+1}-\frac{1}{r+1}+\gamma} \right) \geq C\epsilon H(0)^{-\xi}, \tag{4.8}$$

where $\xi > 0$ is given by

$$\xi := \begin{cases} \min\{\frac{1}{m+1}-\frac{1}{p+1}-\gamma,\ \frac{1}{r+1}-\frac{1}{p+1}-\gamma\}, \text{ if } 0 < H(0) < 1 \\ \\ \max\{\frac{1}{m+1}-\frac{1}{p+1}-\gamma,\ \frac{1}{r+1}-\frac{1}{p+1}-\gamma\}, \text{ if } H(0) \geq 1. \end{cases} \tag{4.9}$$

Thus,

$$H(0) \geq C\epsilon^{\theta}, \text{ where } \theta = \frac{1}{\xi} > 0. \tag{4.10}$$

By noting the definition of $N(t)$, we have

$$N'(t) = 2\int_\Omega \big(u'(t)u(t) + v'(t)v(t)\big)\, dx. \tag{4.11}$$

We now recall Lemma 2.4, particularly the fact that

$u'' \in L^{\frac{m+1}{m}}(0,T;H^{-1}(\Omega)+L^{\frac{m+1}{m}}(\Omega))$ and $v'' \in L^{\frac{r+1}{r}}(0,T;H^{-1}(\Omega)+L^{\frac{r+1}{r}}(\Omega))$,

and that (1.5) yields

$$\begin{aligned} < u''(t),\phi >= \frac{d}{dt} < u'(t),\phi >= \frac{d}{dt}\left\langle u'(t),\phi\right\rangle_{L^2(\Omega)} = -\left\langle \nabla u(t),\nabla\phi)\right\rangle_{L^2(\Omega)} \\ -\left\langle |u'(t)|^{m-1}u'(t),\phi\right\rangle_{L^2(\Omega)} + \left\langle f_1(u(t),v(t)),\phi\right\rangle_{L^2(\Omega)}, \end{aligned} \tag{4.12}$$

for all $\phi \in H_0^1(\Omega)\cap L^{m+1}(\Omega)$, and where in (4.12) and later, $< .,. >$ denotes the standard duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Similarly, from (1.6), we have

$$\begin{aligned} < v''(t),\psi >= -\left\langle \nabla v(t),\nabla\psi)\right\rangle_{L^2(\Omega)} - \left\langle |v'(t)|^{r-1}v'(t),\psi\right\rangle_{L^2(\Omega)} \\ + \left\langle f_2(u(t),v(t)),\psi\right\rangle_{L^2(\Omega)}, \end{aligned} \tag{4.13}$$

for all $\psi \in H_0^1(\Omega)\cap L^{r+1}(\Omega)$.

Let us note here that, by assumption, $m, r < p$, and $p = 3$ when $n = 3$. Therefore, $u \in H_0^1(\Omega)\cap L^{m+1}(\Omega)$ and $v \in H_0^1(\Omega)\cap L^{r+1}(\Omega)$. Therefore, it follows from (4.12) that

$$\frac{d}{dt}\int_\Omega u'(t)u(t)dx = \left\|u'(t)\right\|_2^2 + < u''(t),u(t) > \tag{4.14}$$

$$= \|u'(t)\|_2^2 - \|\nabla u(t)\|_2^2 - \int_\Omega |u'(t)|^{m-1}u'(t)u(t)\,dx + \int_\Omega u(t)f_1(u(t),v(t))\,dx.$$

Similarly, (4.13) yields

$$\frac{d}{dt}\int_\Omega v'(t)v(t)dx = \left\|v'(t)\right\|_2^2 - \|\nabla v(t)\|_2^2 - \int_\Omega |v'(t)|^{r-1}v'(t)v(t)\,dx + \int_\Omega v(t)f_2(u(t),v(t))\,dx. \tag{4.15}$$

Therefore, by combining (4.11), (4.14)-(4.15) and recalling (1.3), we obtain

$$\begin{aligned}N''(t) &= 2\left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2\right) - 2\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right)\\ &\quad - 2\int_\Omega \left(\left|u'(t)\right|^{m-1}u'(t)u(t) + \left|v'(t)\right|^{r-1}v'(t)v(t)\right)dx\\ &\quad + 2(p+1)\int_\Omega F(u(t),v(t))dx.\end{aligned} \tag{4.16}$$

As in [4, 8, 25], we let

$$Y(t) = H(t)^{1-\gamma} + \epsilon N'(t), \tag{4.17}$$

where $\epsilon > 0$ is as chosen in (4.7). Now, (4.16)-(4.17) yield

$$\begin{aligned}Y'(t) &= (1-\gamma)H(t)^{-\gamma}H'(t) + 2\epsilon\left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2\right)\\ &\quad - 2\epsilon\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) + 2\epsilon\,(p+1)G(t)\\ &\quad - 2\epsilon\int_\Omega \left(\left|u'(t)\right|^{m-1}u'(t)u(t) + \left|v'(t)\right|^{r-1}v'(t)v(t)\right)dx.\end{aligned} \tag{4.18}$$

Let us note that, from (4.2), we have

$$\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 = -2H(t) - \left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2\right) + 2G(t). \tag{4.19}$$

Therefore, (4.18)-(4.19) yield

$$\begin{aligned}Y'(t) &= (1-\gamma)H(t)^{-\gamma}H'(t) + 4\epsilon\left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2\right) + 4\epsilon H(t) \qquad (4.20)\\ &\quad + 2\epsilon\,(p-1)G(t) - 2\epsilon\int_\Omega \left(\left|u'(t)\right|^{m-1}u'(t)u(t) + \left|v'(t)\right|^{r-1}v'(t)v(t)\right)dx.\end{aligned}$$

The last two terms that involve damping in (4.20) are estimated below. By using Hölder's and Young's inequalities, we have

$$\begin{aligned}&\Big|\int_\Omega |u'(t)^{m-1}u'(t)u(t)dx\Big| \le \int_\Omega |u(t)||u'(t)|^m dx \le \|u(t)\|_{m+1}\left\|u'(t)\right\|_{m+1}^m\\ &\le |\Omega|^{\frac{p-m}{(p+1)(m+1)}}\|u(t)\|_{p+1}\left\|u'(t)\right\|_{m+1}^m \le K_1G(t)^{\frac{1}{p+1}}\left\|u'(t)\right\|_{m+1}^m\end{aligned}$$

$$\leq K_1 H(t)^{\frac{1}{p+1}-\frac{1}{m+1}} G(t)^{\frac{1}{m+1}} \left\|u'(t)\right\|_{m+1}^{m}$$
$$\leq H(t)^{\frac{1}{p+1}-\frac{1}{m+1}} \left[\delta_1 G(t) + \delta_1^{-\frac{1}{m}} K_1^{\frac{m+1}{m}} \left\|u'(t)\right\|_{m+1}^{m+1}\right], \tag{4.21}$$

where we have used (4.4)-(4.6).

By recalling (4.3)-(4.4) and the definition of γ, which implies that $\frac{1}{p+1} - \frac{1}{m+1} + \gamma < 0$, it follows from (4.21) that

$$\left|\int_\Omega \left|u'(t)\right|^{m-1} u'(t)u(t)dx\right| \tag{4.22}$$
$$\leq \delta_1 H(0)^{\frac{1}{p+1}-\frac{1}{m+1}} G(t) + \delta_1^{-\frac{1}{m}} K_1^{\frac{m+1}{m}} H'(t)H(t)^{-\gamma} H(0)^{\frac{1}{p+1}-\frac{1}{m+1}+\gamma}.$$

By repeating the estimates (4.21)-(4.22), replacing $u(t)$ by $v(t)$ and m by r, one obtains

$$\left|\int_\Omega \left|v'(t)\right|^{r-1} v'(t)v(t)dx\right| \tag{4.23}$$
$$\leq \delta_2 H(0)^{\frac{1}{p+1}-\frac{1}{r+1}} G(t) + \delta_2^{-\frac{1}{r}} K_2^{\frac{r+1}{r}} H'(t)H(t)^{-\gamma} H(0)^{\frac{1}{p+1}-\frac{1}{r+1}+\gamma}.$$

By recalling the definition of L in (4.7), it follows from (4.20) and (4.22)-(4.23) that

$$Y'(t) \geq LH(t)^{-\gamma}H'(t) + 4\epsilon H(t) + 4\epsilon\left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2\right)$$
$$+2\epsilon\left[(p-1) - \left(\delta_1 H(0)^{\frac{1}{p+1}-\frac{1}{m+1}} + \delta_2 H(0)^{\frac{1}{p+1}-\frac{1}{r+1}}\right)\right] G(t). \tag{4.24}$$

By the definition of δ_1 and δ_2 in (4.6), we note

$$\left(\delta_1 H(0)^{\frac{1}{p+1}-\frac{1}{m+1}} + \delta_2 H(0)^{\frac{1}{p+1}-\frac{1}{r+1}}\right) = \frac{p-1}{2},$$

and since $L \geq 0$, it follows from (4.24) that

$$Y'(t) \geq 4\epsilon H(t) + 4\epsilon\left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2\right) + \epsilon(p-1)G(t)$$
$$\geq C\epsilon\left[H(t) + \left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2 + \|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1}\right], \tag{4.25}$$

for $t \in [0,T)$ and where $C > 0$ is a constant that does not depend on ϵ. In particular (4.25) shows that $Y(t)$ is increasing on $[0,T)$, with

$$Y(t) = H(t)^{1-\gamma} + \epsilon N'(t) \geq H(0)^{1-\gamma} + \epsilon N'(0). \tag{4.26}$$

If $N'(0) \geq 0$, then no further condition on ϵ is needed. However, if $N'(0) < 0$, then we further adjust ϵ so that $0 < \epsilon \leq -\frac{H(0)^{1-\gamma}}{2N'(0)}$. In any case, one has

$$Y(t) \geq \frac{1}{2}H(0)^{1-\gamma} > 0 \text{ for } t \in [0,T). \tag{4.27}$$

Finally, we show that

$$Y'(t) \geq \epsilon^{1+\sigma} C Y(t)^{\eta} \text{ for } t \in [0,T), \tag{4.28}$$

where

$$\eta = \frac{1}{1-\gamma} > 1 \text{ and } \sigma = \theta\left(1 - \frac{2}{(1-2\gamma)(p+1)}\right) > 0,$$

and $C > 0$ is a generic constant.

If $N'(t) \leq 0$ for some $t \in [0,T)$, then for such values of t we have

$$Y(t)^{\eta} = [H(t)^{1-\gamma} + \epsilon N'(t)]^{\eta} \leq H(t), \tag{4.29}$$

and in this case, (4.25) and (4.29) yield

$$Y'(t) \geq C\epsilon H(t) \geq C\epsilon^{1+\sigma} H(t) \geq C\epsilon^{1+\sigma} Y(t)^{\eta}. \tag{4.30}$$

Hence, (4.28) holds for all $t \in [0,T)$ for which $N'(t) \leq 0$. However, if $t \in [0,T)$ is such that $N'(t) > 0$, then showing the validity of (4.28) requires a bit more effort. First we note that

$$Y(t)^{\eta} \leq 2^{\eta-1}[H(t) + N'(t)^{\eta}]. \tag{4.31}$$

We estimate $N'(t)^{\eta}$ as follows. By using Hölder's and Young's inequalities and noting that $1 < \eta < 2$, we have

$$\begin{aligned}
N'(t)^{\eta} &= 2^{\eta}\left[\int_{\Omega}(u(t)u'(t) + v(t)v'(t))dx\right]^{\eta} \\
&\leq 2^{\eta}\left(\|u(t)\|_2 \left\|u'(t)\right\|_2 + \|v(t)\|_2 \left\|v'(t)\right\|_2\right)^{\eta} \\
&\leq C\left(\|u(t)\|_{p+1} \left\|u'(t)\right\|_2 + \|v(t)\|_{p+1} \left\|v'(t)\right\|_2\right)^{\eta} \\
&\leq C\left(\|u(t)\|_{p+1}^{\eta} \left\|u'(t)\right\|_2^{\eta} + \|v(t)\|_{p+1}^{\eta} \left\|v'(t)\right\|_2^{\eta}\right) \\
&\leq C\left(\|u(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} + \left\|u'(t)\right\|_2^2 + \|v(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} + \left\|v'(t)\right\|_2^2\right).
\end{aligned} \tag{4.32}$$

From the definition of γ, namely, $\gamma < \frac{p-1}{2(p+1)}$, it is easy to see that

$$\frac{2\eta}{(2-\eta)(p+1)} - 1 = \frac{2}{(1-2\gamma)(p+1)} - 1 < 0.$$

Therefore, by recalling (4.4)-(4.5) we have

$$\|u(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} = \left(\|u(t)\|_{p+1}^{p+1}\right)^{\frac{2\eta}{(2-\eta)(p+1)}} \leq CG(t)\,G(t)^{\frac{2\eta}{(2-\eta)(p+1)}-1} \leq CG(t)\,H(0)^{\frac{2\eta}{(2-\eta)(p+1)}-1} \leq C\epsilon^{-\sigma}G(t), \tag{4.33}$$

where we have used (4.10) and the definition of σ. Similarly,

$$\|v(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} \leq C\epsilon^{-\sigma}G(t). \tag{4.34}$$

It follows from (4.32)-(4.34) that

$$N'(t)^{\eta} \leq C\epsilon^{-\sigma}\left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2 + G(t)\right). \tag{4.35}$$

Finally, (4.25), (4.31)-(4.32) and (4.35) allow us to conclude that

$$Y'(t) \geq C\epsilon\left[H(t) + \left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2 + G(t)\right] \geq C\epsilon^{1+\sigma}Y(t)^{\eta}, \tag{4.36}$$

for all values of $t \in [0,T)$ for which $N'(t) > 0$. Hence, (4.28) is valid and therefore, $Y(t)$ blows up in finite time T, where

$$T < C\epsilon^{-1-\sigma}Y(0)^{-\gamma/(1-\gamma)}. \tag{4.37}$$

In addition, (4.27) and (4.37) yield the following upper bound for the life span of the solution:

$$T < C\epsilon^{-1-\sigma}\left[H(0)^{1-\gamma} + \epsilon N'(0)\right]^{-\gamma/(1-\gamma)} \leq C\epsilon^{-1-\sigma}H(0)^{-\gamma}. \tag{4.38}$$

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