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ON SINGULAR p-LAPLACIAN PROBLEMS

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Abstract. In this work we combine perturbation arguments and variational methods to prove existence and uniqueness results for singular p-Laplacian problems.

1. Introduction

Consider the boundary-value problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\, \nabla u\right)$ is the p-Laplacian of u, $1 < p < \infty$, and f is a Carathéodory function on $\Omega \times (0,\infty)$ satisfying

(f_1) given $0 < t_1 \leq t_2 < \infty$, there are $h_1 \in L^1_{loc}(\Omega)$ and $h_2 \in L^1(\Omega)$ such that

$$-h_1(x) \leq f(x,t) \leq h_2(x) \text{ for all } 0 < t_1 \leq t \leq t_2, \text{ a.e. in } \Omega,$$

(f_2) there are a nontrivial function $a \geq 0$ in $L^1(\Omega)$ and $t_0 > 0$ such that

$$f(x,t) \geq a(x) \text{ for all } 0 < t \leq t_0, \text{ a.e. in } \Omega,$$

(f_3) there is a nontrivial function $\alpha \geq 0$ such that

$$\liminf_{t\to\infty} \frac{\lambda_1 t^{p-1} - f(x,t)}{t^{p-2}} \geq \alpha(x), \text{ unif. a.e. in } \Omega,$$

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where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$. Note that, under the conditions (f_1) and (f_2), the function f may be singular at $t = 0$ and it may change sign. The condition (f_3) allows Problem (1.1) to be resonant at the first eigenvalue λ_1. Note also that the conditions $(f_1) - (f_3)$ do not imply any growth restriction from below on f with respect to the variable t.

In a pioneering paper Crandall, Rabinowitz, and Tartar [6] considered the semilinear case $p = 2$, with a more general second-order uniformly elliptic operator satisfying a maximum principle in place of the Laplacian and under the stronger assumptions $\partial\Omega \in C^3$, $f \in C(\overline{\Omega} \times (0,\infty))$,

$$\lim_{t\to 0^+} f(x,t) = \infty \quad \text{uniformly in } \overline{\Omega}, \qquad \sup_{[1,\infty)\times\overline{\Omega}} f < \infty,$$

and obtained a generalized solution in $W^{2,q}_{loc}(\Omega) \cap C_0(\overline{\Omega})$ for some $q > n$, satisfying (1.1) almost everywhere. They further showed that this solution is a classical $C^2(\Omega)$ solution if $f \in C^1(\overline{\Omega}\times(0,\infty))$ and is unique if in addition f is nonincreasing in t.

Since then singular equations of the form

$$-\Delta\, u = a(x)\, u^{-\gamma} + \mu\, g(x,u), \tag{1.2}$$

where $a \geq 0$ is a nontrivial function in $L^2(\Omega)$, $\gamma > 0$ is a constant, $\mu \geq 0$ is a parameter, and $g \geq 0$ is a Carathéodory function on $\Omega \times [0,\infty)$ have been studied extensively (see, e.g., [3 - 5, 7, 10, 11, 13, 17, 19, 21] and their references). Similar equations for the p-Laplacian and small γ were studied by Agarwal, Perera, and O'Regan [1], Perera and Silva [14, 15], and Perera and Zhang [16].

In the present paper we consider the problem

$$\begin{cases} -\Delta_p u = f(x,u) + \mu\, g(x,u) & \text{in } \Omega \\ \quad u > 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where f satisfies (f_1) - (f_3), $\mu \geq 0$ is a parameter, and g is a Carathéodory function on $\Omega \times [0,\infty)$ satisfying

(g_1) given $t_1 > 0$, there are $h_3 \in L^1_{loc}(\Omega)$ and $h_4 \in L^{q_1}(\Omega)$, $q_1 = 1$ if $p > n$, and $q_1 > n/p$ if $p \leq n$, such that

$$-h_3(x) \leq g(x,t) \leq h_4(x) \text{ for all } 0 \leq t \leq t_1, \text{ a.e. in } \Omega,$$

(g_2) there is $c_1 > 0$ such that

$$g(x,t) \geq -c_1 a(x) \text{ for all } 0 \leq t \leq t_0, \text{ a.e. in } \Omega.$$

Note that conditions $(g_1) - (g_2)$ do not impose any growth restriction on g with respect to the variable t.

In view of conditions (f_1), (f_2) and (g_1), we look for a solution of Problem (1.3) that solves the quasilinear equation in the sense of distributions and satisfies the boundary condition in a more general sense. More specifically, we define:

Definition 1.1. *By a solution of* (1.3) *we mean a function* $u \in W^{1,p}_{loc}(\Omega)$ *such that*

(i) $\operatorname{ess\ inf}_{\Omega'} u > 0$ *for all* $\Omega' \subset\subset \Omega$,

(ii) $\displaystyle\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - (f(x,u) + \mu\, g(x,u))\, \varphi = 0$ *for all* $\varphi \in C_0^\infty(\Omega)$,

(iii) $(u-\varepsilon)^+ \in W_0^{1,p}(\Omega)$ *for all* $\varepsilon > 0$.

Actually we will show that $u = 0$ on $\partial\Omega$ whenever $p > n$ or h_2 and h_4 are in $L^\infty(\Omega)$ and $\partial\Omega$ is regular (see [20]) in the sense that

$$\liminf_{\rho \to 0^+} \frac{|K_x(\rho) \setminus \Omega|}{\rho^n} > 0 \text{ for all } x \in \partial\Omega,$$

where $K_x(\rho)$ denotes the cube with center x and sides of length ρ parallel to the coordinate axes.

Now we may state:

Theorem 1.2. *If* (f_1) - (f_3), (g_1), *and* (g_2) *hold, then there is a* $\mu_0 > 0$ *such that Problem* (1.3) *has a solution* $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$ *for all* $0 \leq \mu < \mu_0$. *If* $p > n$ *or* $\partial\Omega$ *is regular and* (f_1), (g_1) *hold with* h_2, $h_4 \in L^\infty(\Omega)$, *respectively, then* $u = 0$ *on* $\partial\Omega$.

Supposing the following stronger versions of (f_1) and (g_1),

$(\hat{f}_1)$ given $0 < t_1 \leq t_2 < \infty$, there are $h_1 \in L^{q_2}_{loc}(\Omega)$ $h_2 \in L^{q_2}(\Omega)$ and $q_2 > pn(p-1)^{-1}$, such that

$$-h_1(x) \leq f(x,t) \leq h_2(x) \text{ for all } 0 < t_1 \leq t \leq t_2, \text{ a.e. in } \Omega,$$

$(\hat{g}_1)$ given $t_1 > 0$, there are $h_3 \in L^{q_3}_{loc}(\Omega)$ and $h_4 \in L^{q_3}(\Omega)$, $q_3 > pn(p-1)^{-1}$, such that

$$-h_3(x) \leq g(x,t) \leq h_4(x) \text{ for all } 0 \leq t \leq t_1, \text{ a.e. in } \Omega,$$

we may combine Theorem 1.2 and the local regularity theory results of DiBenedetto [8] to obtain

Theorem 1.3. *If $(\hat{f}_1)$, (f_2), (f_3), $(\hat{g}_1)$, and (g_2) hold, then there is a $\mu_0 > 0$ such that Problem* (1.3) *has a solution $u \in C^{1,\alpha}_{loc}(\Omega) \cap L^\infty(\Omega)$ for all $0 \le \mu < \mu_0$. If $p > n$ or $\partial\Omega$ is regular and $(\hat{f}_1)$, $(\hat{g}_1)$ hold with h_2, $h_4 \in L^\infty(\Omega)$, respectively, then $u \in C_0(\overline{\Omega})$.*

Finally, we present a uniqueness result for solutions of Problem (1.1).

Theorem 1.4. *If (f_1) holds and f is nonincreasing in t, then Problem* (1.1) *has at most one solution $u \in W^{1,p}_{loc}(\Omega)$ satisfying $u = 0$ on $\partial\Omega$.*

Example 1.5. *There is a $\mu_0 > 0$ such that the problem*

$$\begin{cases} -\Delta_p u = e^{1/u} + \mu\, e^u & \text{in } \Omega \\ \quad u > 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

has a solution $u \in C^{1,\alpha}_{loc}(\Omega) \cap L^\infty(\Omega)$ for all $\mu < \mu_0$. If $p > n$ or $\partial\Omega$ is regular, then $u \in C_0(\overline{\Omega})$ and is unique when $\mu \le 0$.

2. Preliminaries

In order to solve Problem (1.3), we approximate it by the sequence of problems

$$\begin{cases} -\Delta_p u = f_j(x,u) + \mu\, g_j(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where

$$\begin{aligned} f_j(x,t) &= \min\{\max\{f(x,(t-\varepsilon_j)^+ + \varepsilon_j), -\varepsilon_j^{-1}\}\}, \\ g_j(x,t) &= \max\{g(x,t), -c_1\varepsilon_j^{-1}\}, \end{aligned} \tag{2.2}$$

with $c_1 > 0$ given by (g_2) and (ε_j) a sequence on $(0,\infty)$ such that $\varepsilon_j \to 0^+$ as $j \to \infty$. Note that $f_j(x,t)$ and $g_j(x,t)$ are Carathéodory function on $\Omega \times \mathbb{R}$ which satisfy $f_j(x,t) \to f(x,t)$ and $g_j(x,t) \to g(x,t)$, as $j \to \infty$, for every $(x,t) \in \Omega \times (0,\infty)$.

Lemma 2.1. *If (f_1) and (g_1) hold, $\mu \ge 0$, and $(u_j) \subset W^{1,p}_0(\Omega)$ is a sequence of weak solutions of Problems* (2.1) *satisfying*

$$\delta_{\Omega'} := \inf_j \operatorname{ess\ inf}_{\Omega'} u_j > 0 \textit{ for all } \Omega' \subset\subset \Omega, \tag{2.3}$$

$$M := \sup_j \|u_j\|_\infty < \infty, \tag{2.4}$$

then a subsequence of (u_j) *converges almost everywhere to a solution* $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$ *of* (1.3). *If* $p > n$ *or* $\partial\Omega$ *is regular and* (f_1), (g_1) *hold with* h_2, $h_4 \in L^\infty(\Omega)$, *respectively, then* $u = 0$ *on* $\partial\Omega$.

Proof. Let (Ω_k) be a sequence of open subsets of Ω such that $\Omega_k \subset\subset \Omega_{k+1}$ for each k and $\bigcup_k \Omega_k = \Omega$. Set $\delta_k = \delta_{\Omega_k}$, δ_{Ω_k} given by (2.3). Taking $\varphi = (u_j - \delta_1)^+$ in

$$\int_\Omega |\nabla u_j|^{p-2}\, \nabla u_j \cdot \nabla\varphi - (f_j(x,u_j) + \mu\, g_j(x,u_j))\, \varphi = 0 \tag{2.5}$$

gives

$$\int_{\Omega_1} |\nabla u_j|^p \le \int_{u_j > \delta_1} |\nabla u_j|^p = \int_{u_j > \delta_1} (f_j(x,u_j) + \mu\, g(x,u_j))\,(u_j - \delta_1). \tag{2.6}$$

The last integral is bounded by (2.2), (2.4), (f_1), and (g_1), so (u_j) is bounded in $W^{1,p}(\Omega_1)$ and, consequently, it has a subsequence $(u_{j^1_k})$ which converges weakly in $W^{1,p}(\Omega_1)$, strongly in $L^p(\Omega_1)$, and almost everywhere in Ω_1. Denote by u_{Ω_1} the corresponding limit in $W^{1,p}(\Omega_1)$.

Arguing by induction, for each k we obtain a subsequence $(u_{j^k_l})$ of (u_j) and $u_{\Omega_k} \in W^{1,p}(\Omega_k)$ such that $(u_{j^k_l})$ converges to u_{Ω_k} weakly in $W^{1,p}(\Omega_k)$, strongly in $L^p(\Omega_k)$, and almost everywhere in Ω_k. Moreover we may assume that $(u_{j^{k+1}_l})$ is a subsequence of $(u_{j^k_l})$, for every k, and that $j^k_k \to \infty$ as $k \to \infty$.

By construction $u_{\Omega_{k+1}}|_{\Omega_k} = u_{\Omega_k}$, so setting $u = u_{\Omega_1}$ and $u = u_{\Omega_{k+1}}$ on $\Omega_{k+1} \setminus \Omega_k$ for each k, we get a well-defined function $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$, to which the diagonal subsequence $(u_{j_k}) := (u_{j^k_k})$ converges weakly in $W^{1,p}_{loc}(\Omega_k)$, strongly in $L^p_{loc}(\Omega_k)$, and almost everywhere in Ω. We claim that, actually, the diagonal sequence (u_{j_k}) converges strongly to u in $W^{1,p}_{loc}(\Omega_k)$.

Assuming the claim, we may verify that the function $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.3); i.e., u satisfies the conditions (i)-(iii) given in Definition 1.1. First note that u satisfies the condition (i) since (u_j) satisfies (2.3) and (u_{j_k}) converges to u almost everywhere in Ω. Next, given $\varphi \in C^\infty_0(\Omega)$, fix $k_1 \ge 1$ such that $\operatorname{supp}\varphi \subset \Omega_{k_1}$. Hence the relation (2.4) with j_k in place of j reduces to

$$\int_{\Omega_{k_1}} |\nabla u_{j_k}|^{p-2}\, \nabla u_{j_k} \cdot \nabla\varphi - (f_{j_k}(x,u_{j_k}) + \mu\, g_{j_k}(x,u_{j_k}))\, \varphi = 0. \tag{2.7}$$

As $|(f_{j_k}(x,u_{j_k}) + \mu\, g_{j_k}(x,u_{j_k}))|\,\varphi$ is bounded by an L^1 function in Ω, from (f_1), (g_1), (2.2)-(2.4), and the above claim, we may pass to the limit in (2.7), obtaining

$$\int_\Omega |\nabla u|^{p-2}\,\nabla u\cdot\nabla\varphi - (f(x,u)+\mu\, g(x,u))\,\varphi = 0 \tag{2.8}$$

since $supp\varphi \subset \Omega_{k_1}$. Therefore the condition (ii) holds. Now, given $\varepsilon > 0$, we may argue as in (2.5) to show that $((u_{j_k}-\varepsilon)^+)$ is bounded in $W_0^{1,p}(\Omega)$, and hence it has a subsequence which converges to some $v \in W_0^{1,p}(\Omega)$ weakly in $W_0^{1,p}(\Omega)$ and almost everywhere in Ω. Then $v = (u-\varepsilon)^+$ since $(u_{j_k}-\varepsilon)^+ \to (u-\varepsilon)^+$ almost everywhere in Ω. This implies that the condition (iii) holds.

Now we prove the claim. Let $\Omega' \subset\subset \Omega$. We must show that $(u_{j_k}|_{\Omega'})$ converges strongly to $u|_{\Omega'}$ in $W^{1,p}(\Omega')$. Take $\eta \in C_0^\infty(\Omega,[0,1])$ such that $\eta = 1$ on Ω', and consider $k_1 \geq 1$ such that $\operatorname{supp}\eta \subset \Omega_{k_1}$. For every $k, m \geq 1$, we have

$$\begin{aligned}
&\int_{\Omega'} \left(|\nabla u_{j_k}|^{p-2}\,\nabla u_{j_k} - |\nabla u_{j_m}|^{p-2}\,\nabla u_{j_m}\right)\cdot\nabla(u_{j_k}-u_{j_m})\\
&\leq \int_{\Omega} \left(|\nabla u_{j_k}|^{p-2}\,\nabla u_{j_k} - |\nabla u_{j_m}|^{p-2}\,\nabla u_{j_m}\right)\cdot\nabla(\eta(u_{j_k}-u_{j_m})) \qquad (2.9)\\
&- \int_{\Omega_{k_1}} \left[\left(|\nabla u_{j_k}|^{p-2}\,\nabla u_{j_k} - |\nabla u_{j_m}|^{p-2}\,\nabla u_{j_m}\right)\cdot\nabla\eta\right](u_{j_k}-u_{j_m}).
\end{aligned}$$

Using the fact that (u_{j_k}) is bounded in $W^{1,p}(\Omega_{k_1})$ and converges strongly in $L^p(\Omega_{k_1})$, we obtain

$$\int_{\Omega_{k_1}} \left[\left(|\nabla u_{j_k}|^{p-2}\,\nabla u_{j_k} - |\nabla u_{j_m}|^{p-2}\,\nabla u_{j_m}\right)\cdot\nabla\eta\right](u_{j_k}-u_{j_m}) \to 0, \tag{2.10}$$

as $k, m \to \infty$. On the other hand, taking $\varphi = \eta(u_{j_k}-u_{j_m})$ and either $j = j_k$ or $j = j_m$ in (2.5), gives us

$$\Big|\int_\Omega |\nabla u_{j_l}|^{p-2}\nabla u_{j_l}\cdot\nabla(\eta(u_{j_k}-u_{j_m}))\Big| \leq \int_{\Omega_{k_1}}\!|(f_{j_l}(x,u_{j_l})+\mu\, g_{j_l}(x,u_{j_l}))(u_{j_k}-u_{j_m})|,$$

for $l = k, m$. Arguing as in the verification of (2.8), we pass to the limit in the above inequality to obtain

$$\int_\Omega |\nabla u_{j_l}|^{p-2}\nabla u_{j_l}\cdot\nabla(\eta(u_{j_k}-u_{j_m})) \to 0, \text{ as } k, m \to \infty, \tag{2.11}$$

for $l = k, m$. The relations (2.9)-(2.11) and a standard argument (see e. g. [14] and the references therein) enable us to show that

$$\int_{\Omega'} |\nabla u_{j_k} - \nabla u_{j_m}|^p \to 0, \text{ as } k, m \to \infty.$$

This implies that (u_{j_k}) is a Cauchy sequence in $W^{1,p}(\Omega')$ since we already know that (u_{j_k}) converges strongly in $L^p(\Omega')$. The claim is proved.

Our final task is to verify that $u = 0$ on $\partial\Omega$ if $p > n$ or $\partial\Omega$ is regular and (f_1), (g_1) hold with $h_2, h_4 \in L^\infty(\Omega)$. Given $0 < \varepsilon < M := \|u\|_\infty$, we find $h_\varepsilon > 0 \in L^q(\Omega)$, $q = 1$ if $p > n$ and $q = \infty$ if $p \leq n$, such that

$$f(x,t) + \mu g(x,t) \leq h_\varepsilon(x), \text{ for every } 0 < \varepsilon \leq t \leq M, \text{ a.e. in } \Omega. \tag{2.12}$$

Now let $\varphi_\varepsilon > 0$ in $W_0^{1,p}(\Omega) \cap C_{loc}^{1,\alpha}(\Omega)$ be the unique solution of

$$\begin{cases} -\Delta_p \varphi_\varepsilon = h_\varepsilon & \text{in } \Omega \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.13}$$

Then $\varphi_\varepsilon \in C_0(\overline{\Omega})$ by the Sobolev imbedding theorem if $p > n$ and by Trudinger [20] if $\partial\Omega$ is regular. Taking $j = j_k$ and $\varphi = (u_{j_k} - \varepsilon - \varphi_\varepsilon)^+$ in (2.5), and considering (2.2), (2.4) and (2.12), we obtain

$$\int_\Omega |\nabla u_{j_k}|^{p-2} \nabla u_{j_k} \cdot \nabla (u_{j_k} - \varepsilon - \varphi_\varepsilon)^+ \leq \int_\Omega h_\varepsilon(x)(u_{j_k} - \varepsilon - \varphi_\varepsilon)^+.$$

Therefore, since φ_ε is a solution of (2.12),

$$\int_{u_{j_k} > \varepsilon + \varphi_\varepsilon} \left(|\nabla u_{j_k}|^{p-2} \nabla u_{j_k} - |\nabla(\varepsilon + \varphi_\varepsilon)|^{p-2} \nabla(\varepsilon + \varphi_\varepsilon) \right) \cdot \nabla (u_{j_k} - \varepsilon - \varphi_\varepsilon) \leq 0,$$

and, consequently, $u_{j_k} \leq \varepsilon + \varphi_\varepsilon$. Thus $0 < u \leq \varepsilon + \varphi_\varepsilon$ almost everywhere in Ω. Since $\varphi_\varepsilon \in C_0(\bar{\Omega})$, there is a neighborhood U of $\partial\Omega$ such that $0 < u(x) < 2\varepsilon$ for almost every $x \in U \cap \Omega$. The fact that $\varepsilon > 0$ can be chosen arbitrarily small implies that $u = 0$ on $\partial\Omega$. □

Since $a \geq 0$ is nontrivial, the function $a_0(x) = \min\{a(x), t_0^{-1}\} \geq 0$ is also nontrivial, furthermore, it belongs to $L^\infty(\Omega)$. Consequently the problem

$$\begin{cases} -\Delta_p u_0 = a_0(x) & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{2.14}$$

has a unique solution $u_0 > 0$ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Fixing $0 < \lambda_0 < 1$ so small that $\underline{u} := \lambda_0^{\frac{1}{p-1}} u_0 \leq t_0$ almost everywhere in Ω, for all j so large that

$\varepsilon_j \le t_0$, we may invoke (f_2), (g_2) and (2.2) to obtain

$$-\Delta_p \underline{u} - f_j(x,\underline{u}) - \mu\, g_j(x,\underline{u}) \le -((1-\lambda_0) - c_1\mu)\, a_0(x).$$

Hence there is $\mu_1 > 0$ so that $\underline{u}$ is a subsolution of Problems (2.1) whenever $0 \le \mu < \mu_1$.

Now consider the problem

$$\begin{cases} -\Delta_p u = f_j(x,u) + h_4(x)\eta(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where h_4 is given by (g_1), and $\eta : \Omega \to \mathbb{R}$ is a continuous function satisfying

(η_1) $0 < \eta(t) \le 1$ for all $t \in \mathbb{R}$, and

$$\frac{\eta(t)}{t^{p-2}} \to 0, \text{ as } t \to \infty.$$

Here, we note that, without loss of generality, we suppose that $h_4 \ge 1$ in Ω.

Lemma 2.2. *If (f_1)-(f_3), (g_1), and (g_2) hold and the Problems (2.15) have a sequence of weak solutions $(\overline{u}_j) \subset W_0^{1,p}(\Omega)$ such that*

$$\sup_j \|(\overline{u}_j - \varepsilon)^+\| < \infty \quad \text{for some } \varepsilon > 0, \tag{2.16}$$

then there is a $\mu_0 > 0$ such that Problem (1.3) has a solution $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$ for all $0 \le \mu < \mu_0$. If $p > n$ or $\partial\Omega$ is regular and (f_1), (g_1) hold with h_2, $h_4 \in L^\infty(\Omega)$, respectively, then $u = 0$ on $\partial\Omega$.

Proof. For all j so large that $\varepsilon_j \le t_0$,

$$f_j(x,t) \ge a(x) \ge \varepsilon_0\, a(x), \text{ for all } -\infty < t \le t_0, \text{ a.e. in } \Omega,$$

by (f_2). Hence $\overline{u}_j \ge \underline{u}$. We claim that $(\overline{u}_j)$ is bounded in $L^\infty(\Omega)$. Effectively, if $p > n$, this assertion is a consequence of (2.16) and the Sobolev imbedding theorem. On the other hand, if $p \le n$, we invoke (f_3), (g_1), (η_1), and (2.2) to find $b \ge \varepsilon > 0$ and $a_1 \in L^r(\Omega)$, with $r > n/p$, such that

$$|f_j(x,t) + h_4(x)\eta(t)| \le a_1(x)(1 + t^{p-1}), \text{ for all } t \ge b, \text{ a.e. in } \Omega.$$

Hence, by (2.16) and Lemma A.1 of the appendix, $(\overline{u}_j)$ is bounded in $L^\infty(\Omega)$. The claim is proved.

In view of the above claim, (g_1), (2.2), and (η_1), we may find $0 < \mu_0 < \mu_1$ such that, for $0 \le \mu < \mu_0$,

$$-\Delta_p \overline{u}_j - f_j(x,\overline{u}_j) - \mu\, g_j(x,\overline{u}_j) \ge (\eta(\overline{u}_j) - \mu)\, h_4(x) \ge 0,$$

so $\overline{u}_j$ is a supersolution of (2.1). Thus (2.1) has a weak solution u_j in the order interval $[\underline{u}, \overline{u}_j]$ by a standard argument (see e.g. [16]), and the conclusion follows from Lemma 2.1. $\square$

3. PROOFS

Proof of Theorem 1.2. We apply Lemma 2.2. Weak solutions of (2.15) are the critical points of the C^1 functional

$$\Phi_j(u) = \int_\Omega \frac{1}{p} |\nabla u|^p - \hat{F}_j(x, u), \text{ for all } u \in W_0^{1,p}(\Omega),$$

where $\hat{F}_j(x, t) = \int_0^t \hat{f}_j(x, s)\, ds$, $\hat{f}_j(x, t) = f_j(x, t) + h_4(x)\eta(t)$. By (f_1), (g_1), (2.2), (η_1), and the Sobolev imbedding theorem,

$$\Phi_j(u) \geq \frac{1}{p}\|u\|^p - C_j \|u\|$$

for some $C_j > 0$, so Φ_j is bounded from below and coercive, and hence it has a global minimizer $\overline{u}_j$ since it is weakly lower semicontinuous.

Now we fix $0 < \varepsilon < t_0$ such that

$$0 < \lambda_1 \varepsilon \int_\Omega \varphi_1^{p-1}(x) < \int_\Omega \alpha(x) \varphi_1^{p-1}(x)\, dx, \tag{3.1}$$

where α is given by (f_3) and $\varphi_1 > 0$ is the first normalized eigenvalue of $-\Delta_p$ on $W^{1,p}(\Omega)$. Note that, replacing α by $\min\{\alpha, 1\}$ if necessary, we may always suppose that $\alpha \in L^\infty(\Omega)$.

In the following our objective is to prove that (2.16) holds for the value of ε given above. Arguing by contradiction, and taking a subsequence if necessary, we suppose

$$\|(\overline{u}_j - \varepsilon)^+\| \to \infty, \text{ as } j \to \infty. \tag{3.2}$$

Taking $\gamma > 0$, to be chosen posteriorly, by (f_3) there is $t_\infty > t_0$ such that

$$f(x, t) \leq -(\alpha(x) - \gamma) t^{p-2} + \lambda_1 t^{p-1}, \text{ for all } t \geq t_\infty, \text{ a.e. in } \Omega.$$

Hence, by (2.2) and (η_1), and taking t_∞ larger if necessary, for every $j \geq 1$, we have

$$\hat{f}_j(x, t) \leq -(\alpha(x) - \gamma h_5(x)) t^{p-2} + \lambda_1 t^{p-1}, \text{ for all } t \geq t_\infty, \text{ a.e. in } \Omega., \tag{3.3}$$

where $h_5 := h_4 + 1 \in L^q(\Omega)$, $q = 1$ for $p > n$, and $q > n/p$ for $p \leq n$. Next, we invoke (f_1), (g_1), (2.2), and (η_1) to find $h = h(\varepsilon, \gamma) \in L^1(\Omega)$ such that

$$\hat{f}_j(x, t) \leq h(x), \text{ for all } \varepsilon \leq t \leq t_\infty, \text{ a.e. in } \Omega. \tag{3.4}$$

Considering (3.3)-(3.4) and taking $\varphi = v_j := (\overline{u}_j - \varepsilon)^+$ in

$$\int_\Omega |\nabla \overline{u}_j|^{p-2} \nabla \overline{u}_j \cdot \nabla \varphi - (\hat{f}_j(x, \overline{u}_j) + 1)\, \varphi = 0$$

gives

$$\|v_j\|^p \leq \lambda_1 \int_{\overline{u}_j > t_\infty} (v_j + \varepsilon)^{p-1} v_j - \int_{\overline{u}_j > t_\infty} (\alpha(x) - \gamma h_5(x))(v_j + \varepsilon)^{p-2}\, v_j + C, \tag{3.5}$$

where C denotes a generic positive constant that depends on ε and γ, but not on j.

Now, defining $z_j := \|v_j\|^{-1} v_j$, for $j \geq 1$, we may suppose there is $z \in W_0^{1,p}(\Omega)$ such that $\|z\| \leq 1$ and

$$\begin{cases} z_j \rightharpoonup z \text{ weakly in } W_0^{1,p}(\Omega), \\ z_j \to z \text{ strongly in } L^p(\Omega), \\ z_j(x) \to z(x) \text{ a.e in } \Omega, \\ |z_j| \leq \psi_q(x) \in L^q(\Omega) \text{ a.e in } \Omega, \end{cases} \tag{3.6}$$

where $1 \leq q \leq \infty$ if $p > n$, $1 \leq q < \infty$ if $p = n$, $1 \leq q < p^* = pn/(n-p)$ if $p < n$.

We claim that actually $z = \varphi_1$. Indeed, from (3.5)

$$1 \leq \lambda_1 \int_{\overline{u}_j > t_\infty} \frac{(v_j + \varepsilon)^{p-1} z_j}{\|v_j\|^{p-1}} + \gamma \int_{\overline{u}_j > t_\infty} \frac{h_5(x))(v_j + \varepsilon)^{p-2}\, z_j}{\|v_j\|^{p-1}} + \frac{C}{\|v_j\|^p}.$$

Hence, from (3.2), (3.6), and the variational characterization of λ_1,

$$\lambda_1 \|z\|_p^p \leq \|z\|^p \leq 1 \leq \lambda_1 \|z\|_p^p.$$

Consequently, $z = \varphi_1$ since $z \geq 0$ almost everywhere in Ω. The claim is proved.

Invoking (3.5) and the variational characterization of λ_1 one more time, we find a constant $\hat{C} > 0$, not depending on j, such that

$$\int_\Omega \alpha(x) \frac{(v_j + \varepsilon)^{p-2} z_j}{\|v_j\|^{p-1}} \leq \lambda_1 \int_\Omega \frac{(v_j + \varepsilon)^{p-1} - v_j^{p-1}) v_j}{\|v_j\|^{p-1}}$$

$$+ \gamma \int_\Omega \frac{h_5(x))(v_j + \varepsilon)^{p-2}\, v_j}{\|v_j\|^{p-1}} + \frac{\hat{C}}{\|v_j\|^{p-1}}.$$

Therefore, using (3.6), the above claim, and passing to the limit in the above inequality,

$$0 < \int_\Omega \alpha(x)\varphi_1^{p-1}(x) \le \lambda_1 \varepsilon \int_\Omega \varphi_1^{p-1}(x) + \gamma \int_\Omega h_5(x)\varphi_1^{p-1}(x). \tag{3.7}$$

Hence, if $\gamma > 0$ is sufficiently small, (3.1) and (3.7) imply that

$$0 < \int_\Omega \alpha(x)\varphi_1^{p-1}(x) < \int_\Omega \alpha(x)\varphi_1^{p-1}(x).$$

This shows that $\|(\overline{u}_j - \varepsilon)^+\|$ is bounded. □

Proof of Theorem 1.3. Consider $\mu_0 > 0$ given by Theorem 1.3. For every $0 \le \mu < \mu_0$, Problem (1.3) has a solution $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$. Then $f(\cdot, u) + \mu\, g(\cdot, u) \in L^q_{loc}(\Omega)$, $q > np/(p-1)$, by (i) of Definition 1.1, $(\hat{f}_1)$, and $(\hat{g}_1)$, and hence $u \in C^{1,\alpha}_{loc}(\Omega)$ by the local regularity results of DiBenedetto [8]. The last part of Theorem 1.3 implies that $u \in C_0(\bar{\Omega})$ if $p > n$ or $\partial\Omega$ is regular and $(\hat{f}_1)$, $(\hat{g}_1)$ hold with h_2, $h_4 \in L^\infty(\Omega)$. □

Proof of Theorem 1.4. Let u_1 and u_2 be solutions of (1.1) in $W^{1,p}_{loc}(\Omega)$ such that $u = 0$ on $\partial\Omega$. Then, given $\varepsilon > 0$, by compactness we find an open neighborhood U of $\partial\Omega$ such that

$$0 \le u_1 < \varepsilon, \ i = 1, 2, \ \text{a.e. in } \Omega,$$

so $\varphi = (u_1 - u_2 - \varepsilon)^+ \in W^{1,p}_{loc}(\Omega)$ has compact support in Ω. Taking $u = u_1, u_2$ with this φ in (ii) of Definition 1.1 and subtracting gives

$$\begin{aligned} &\int_{u_1 > u_2 + \varepsilon} \left(|\nabla u_1|^{p-2}\, \nabla u_1 - |\nabla u_2|^{p-2}\, \nabla u_2\right) \cdot \nabla(u_1 - u_2) \\ &\qquad = \int_{u_1 > u_2 + \varepsilon} \left(f(x, u_1) - f(x, u_2)\right)(u_1 - u_2 - \varepsilon) \le 0 \end{aligned} \tag{3.8}$$

since f is nonincreasing in t, which implies that $u_1 \le u_2 + \varepsilon$. Since ε is arbitrary, $u_1 \le u_2$, and the reverse inequality follows similarly. □

Appendix

Consider the boundary-value problem

$$\begin{cases} -\Delta_p u = h(x, u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n > 1$, $1 < p \leq n$, and h is a Carathéodory function on $\Omega \times (0, \infty)$ satisfying

(h_1) there are $a \in L^r(\Omega)$, $r \geq n/p$ ($r > 1$ if $n = p$) and $b > 0$ such that

$$h(x,t) \leq a(x)(1 + t^{p-1}), \text{ for all } b \leq t < \infty, \text{ a.e. in } \Omega.$$

Our main result in this appendix provides an L^∞ bound for weak solutions of Problem (A.1) as a function of its norm on $L^p(\Omega)$.

Lemma A.1. *If (h_1) holds and $u \in W_0^{1,p}(\Omega)$ is a weak solution of Problem* (A.1)*, then $u \in L^\infty(\Omega)$. Furthermore $\|u\|_\infty \leq M < \infty$, with the constant M depending only on b, Ω, a, and $\|u\|_p$.*

We observe that the proof of Lemma A.1 will be established through the verification of two main steps. First we show that $u \in L^q(\Omega)$ for every $q \in [1, \infty)$. Moreover,

$$\|u\|_q \leq M_1, \tag{A.2}$$

with the constant $M_1 > 0$, depending only on b, Ω, q, a, and $\|u\|_p$.

Our proof of (A.2) is based on an estimate due to Brezis and Kato [2]. Here we adapt the argument presented by Struwe [18] for the Laplacian operator (see also Guedda and Veron [9] where a corresponding argument for the p-Laplacian operator is employed).

In our second and final step, we obtain an L^∞ bound for the weak solutions of (A.1) by combining (A.2) and an L^∞ estimate for functions in $W_0^{1,p}(\Omega)$ provided by the following lemma (see Ladyzhenskaya and Uraltseva [12]).

Lemma A.2. *If $u \in W_0^{1,p}(\Omega)$, $p \leq n$ and there is $c_0 \geq 0$ such that for $c \geq c_0$,*

$$\int_{A_c} |\nabla u|^p \leq \gamma c^\alpha |A_c|^{(1 - \frac{p}{n} + \varepsilon)},$$

where $A_c = \{x \in \Omega : u(x) > b\}$, $\varepsilon > 0$, $0 \leq \alpha \leq \varepsilon + p$, then $u^+ = \max\{u, 0\} \in L^\infty(\Omega)$. Furthermore

$$\|u^+\|_\infty \leq M < \infty,$$

with the constant M depending only on γ, α, ε, c_0, Ω and $\|u\|_{L^1(A_{c_0})}$.

Proof of Lemma A.1. We start by verifying the estimate (A.2). We note that in order to prove (A.2) it suffices to show that, given $q \in [1, \infty)$, then $(u - b)^+ \in L^q(\Omega)$ and

$$\|(u - b)^+\|_q \leq M_2 < \infty, \tag{A.3}$$

with the constant M_2 depending only on b, Ω, q, a, and $\|(u - b)^+\|_p$.

Taking $v = (u-b)^+$ and $L > b$, we define $w \in W_0^{1,p}(\Omega)$ by

$$w(x) = v(x)\min\{v^{ps}(x), L^p\}, \text{ for every } x \in \Omega. \tag{A.4}$$

Using the above definition, we have

$$\nabla w(x) = \nabla v(x)\min\{v^{ps}(x), L^p\} + z(x), \text{ for a.e. } x \in \Omega, \tag{A.5}$$

where

$$z(x) = \begin{cases} psv^{ps}\nabla v(x) & \text{if } 0 \le v(x) < L, \\ 0 & \text{if } v(x) \ge L. \end{cases} \tag{A.6}$$

From (A.4)-(A.6), (h_1) and the fact that u is a weak solution of (A.1), we obtain

$$\begin{aligned} \int_\Omega |\nabla v|^p \min\{v^{ps}, L^p\} + ps\int_{0\le v<L} v^{ps}|\nabla v|^p \le \\ \int_\Omega a(x)(1+u^{p-1})v\min\{v^{ps}, L^p\}. \end{aligned} \tag{A.7}$$

Hence, we may find $a_1 = a_1(s,b)$ such that

$$\begin{aligned} \int_\Omega |\nabla(v\min\{v^s, L\})|^p &\le a_1\int_\Omega a(x)(1+v^{p-1})v\min\{v^{ps}, L^p\} \\ &\le 2a_1\int_\Omega a(x) + 2a_1\int_\Omega a(x)v^p\min\{v^{ps}, L^p\}. \end{aligned} \tag{A.8}$$

Consequently, there is $a_2 = a_2(s,b,|\Omega|,a)$ such that

$$\int_\Omega |\nabla(v\min\{v^s, L\})|^p \le a_2 + 2a_1\int_\Omega a(x)v^p\min\{v^{ps}, L^p\}. \tag{A.9}$$

Now we suppose that $v \in L^{p(s+1)}(\Omega)$. Then, for $p < n$, given $\varepsilon > 0$, we find $c = c(a) > 0$ such that

$$2a_1\Big(\int_{\{a(x)\ge c\}} a(x)^{\frac{n}{p}}\Big)^{\frac{n}{p}} < \varepsilon.$$

Hence, invoking (A.9), Hölder's inequality, the Sobolev imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, and taking $\varepsilon > 0$ sufficiently small, we obtain

$$\Big(\int_\Omega |(v\min\{v^s, L\})|^{p^*}\Big)^{\frac{p}{p^*}} \le a_3(1 + \|v\|_{p(s+1)}^{p(s+1)}), \tag{A.10}$$

where $a_3 = a_3(s, b, \Omega, a)$ (note that a similar argument provides A.10 when $p = n$). Then, letting $L \to \infty$, we conclude that $v^{s+1} \in L^{p^*}(\Omega)$ whenever $v \in L^{p(s+1)}(\Omega)$. Moreover,

$$\|v^{(s+1)}\|_{p^*}^p \le a_3(1 + \|v\|_{p(s+1)}^{p(s+1)}). \tag{A.11}$$

Finally, given $q \in [1, \infty)$, we take $s_0 = 0$ and $s_i + 1 = n(s_{i-1} + 1)/(n - p)$, $1 \le i \le j$, such that $s_j > q$. Applying (A.11) j times, we conclude that (A.3) must hold with M_2 depending only on b, Ω, q a, and $\|v\|_p$. This completes the first part of the proof of Lemma A.1.

Now we shall establish the L^∞ bound for the weak solutions of (A.1). As above we take $v = (u - b)^+ \in W_0^{1,p}(\Omega)$. Moreover, we set $c_0 = 1$ and we denote by A_c the set $\{x \in \Omega : v(x) > c\}$ for every $c \ge c_0$.

Invoking the fact that u is a weak solution of (A.1), (h_1) and our choice of c_0, we obtain

$$\begin{aligned}\int_{A_c} |\nabla v|^p &= \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla((v - c)^+) \\ &\le \int_{A_c} a(x)(1 + u^{p-1})u \le 2 \int_{A_c} a(x) u^p.\end{aligned} \tag{A.12}$$

Now, considering $q_1 > 1$ such that $q_1 < n(r - 1)[r(n - p)]^{-1}$, from (A.12) and Hölder's inequality, we get

$$\int_{A_c} |\nabla v|^p \le 2\|a\|_r \|u\|_q^p |A_c|^{\frac{r-1}{rq_1}}, \tag{A.13}$$

where $q = rpq_1[(r - 1)(q_1 - 1)]^{-1} > 1$. Setting $\varepsilon = [(r - 1)n - (n - p)rq_1](rnq_1)^{-1} > 0$ and $\alpha = 0$, we complete the proof of Lemma A.1 by invoking (A.2) and Lemma A.2. $\square$

Remark A.3. If (h_1) is satisfied with $r > n/p$, then Lemma A.1 holds with the constant M depending only on b, Ω, $\|a\|_r$, and $\|u\|_p$.

Considering

(h_2) there are constants c, $b_1 > 0$ and $p < s < \infty$ ($s \le p^*$ if $p < n$) such that $h(x, t) \le c(1 + t^{s-1})$, $\quad t \in [b_1, \infty)$.

As a direct consequence of Lemma A.1 and the above remark, we have:

Lemma A.4. *If* (h_2) *holds and* $u \in W_0^{1,p}(\Omega)$ *is a weak solution of Problem* (A.1)*, then* $u \in L^\infty(\Omega)$*. Furthermore, if* $p = n$ *or* $s < p^*$*, then*

$$\|u\|_\infty \le M < \infty,$$

with the constant M *depending only on* b_1, Ω *and* $\|(u - b_1)^+\|$.

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