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ON THE UNIQUENESS OF SOLUTIONS WITH PRESCRIBED NUMBERS OF ZEROS FOR A TWO-POINT BOUNDARY VALUE PROBLEM

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Abstract. The following boundary-value problem

$$\begin{cases} u'' + a(x)f(u) = 0, \quad x_0 < x < x_1, \\ u(x_0) = u(x_1) = 0, \quad u'(x_0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (x_0, x_1), \end{cases} \tag{P_k}$$

is considered under the following conditions: k is a positive integer, $a \in C^2[x_0, x_1]$, $a(x) > 0$ for $x \in [x_0, x_1]$, $f \in C^1(\mathbf{R})$, $f(s) > 0$, $f(-s) = -f(s)$ for $s > 0$. It is shown that if either $(f(s)/s)' > 0$ for $s > 0$ and $([a(x)]^{-\frac{1}{2}})'' \le 0$ for $x \in [x_0, x_1]$ or $(f(s)/s)' < 0$ for $s > 0$ and $([a(x)]^{-\frac{1}{2}})'' \ge 0$ for $x \in [x_0, x_1]$, then (P_k) has at most one solution. To prove the uniqueness of solutions of (P_k), the shooting method is used. The results obtained here are applied to the study of radially symmetric solutions of the Dirichlet problem for semilinear elliptic equations in annular domains.

1. Introduction

We consider the second-order ordinary differential equation

$$u'' + a(x)f(u) = 0, \quad x_0 < x < x_1 \tag{1.1}$$

with the boundary condition

$$u(x_0) = u(x_1) = 0, \tag{1.2}$$

where $a \in C^2[x_0, x_1]$, $a(x) > 0$ for $x \in [x_0, x_1]$, $f \in C^1(\mathbf{R})$, $f(s) > 0$, $f(-s) = -f(s)$ for $s > 0$.

We assume moreover that either the following (F1) or (F2) holds:

$$\Big(\frac{f(s)}{s}\Big)' > 0 \quad \text{for } s > 0; \tag{F1}$$

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$$\left(\frac{f(s)}{s}\right)' < 0 \quad \text{for } s > 0. \tag{F2}$$

The functions

$$f(s) = |s|^{p-1}s \quad (p > 1) \quad \text{and} \quad f(s) = \frac{s}{1+|s|^q} \quad (q > 1)$$

are typical cases satisfying (F1) and (F2), respectively.

Note that if u is a solution of (1.1), so is $-u$, because of $f(-s) = -f(s)$. Therefore we consider solutions u of the problem (1.1) and (1.2) with $u'(x_0) > 0$ only.

In this paper we study the uniqueness of solutions of the problem (1.1) and (1.2) having exactly $k-1$ zeros in (x_0, x_1), and hence consider the following problem:

$$\begin{cases} u'' + a(x)f(u) = 0, \quad x_0 < x < x_1, \\ u(x_0) = u(x_1) = 0, \quad u'(x_0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (x_0, x_1), \end{cases} \tag{P$_k$}$$

where k is a positive integer.

For the case where $f(s) = |s|^{p-1}s$, $p \geq 3$ and $x_0 > 0$, the following two uniqueness results were obtained by Coffman [1].

Theorem A. *Let $k \in \mathbf{N} = \{1, 2, \ldots\}$. Assume that $f(s) = |s|^{p-1}s$, $p \geq 3$, $a(x) = x^\nu$, $\nu \in \mathbf{R}$ and $x_0 > 0$. Then problem* (P$_k$) *has a unique solution.*

Theorem B. *Let $k \in \mathbf{N}$. Assume that $f(s) = |s|^{p-1}s$, $p \geq 3$, $a \in C^1[x_0, x_1]$, $x_0 > 0$ and that*

$$a'(x) < 0, \quad h(x) \equiv x + 2\frac{a(x)}{a'(x)} \leq 0, \quad \text{and} \quad h(x) \text{ is nonincreasing}$$

for $x_0 \leq x \leq x_1$. Then problem (P$_k$) *has a unique solution.*

Ni and Nussbaum [7, Theorem 3.8] later improved Theorem A to the case $p > 1$. Coffman and Marcus [2] generalized Theorem A for equations of the form

$$u'' + x^{-2-\sigma}\varphi(x^\sigma u) = 0, \quad \sigma \in \mathbf{R}.$$

A theorem of a type different from the results quoted above was given by Yanagida [9] as follows:

Theorem C. *Let $k \in \mathbf{N}$. Suppose that $sF(s) \in C^1(\mathbf{R})$, $F(s) > 0$ for $s \neq 0$, $q \in C^1[x_0, x_1]$, and $q(x) > 0$ for $x_0 \leq x \leq x_1$. Assume moreover that one of the following* (i) *or* (ii) *holds*:

(i) *$F'(s) > 0$ for $s > 0$, $F'(s) < 0$ for $s < 0$, and $q'(x)/q(x)$ is nonincreasing in $x \in (x_0, x_1)$;*

(ii) *$F'(s) < 0$ for $s > 0$, $F'(s) > 0$ for $s < 0$, and $q'(x)/q(x)$ is nondecreasing in $x \in (x_0, x_1)$.*

Then the problem

$$\begin{cases} u'' + F(q(x)u)u = 0, \quad x_0 < x < x_1, \\ u(x_0) = u(x_1) = 0, \quad u'(x_0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (x_0, x_1) \end{cases}$$

has at most one solution.

The main result of this paper is as follows.

Theorem 1.1. *Let $k \in \mathbf{N}$. Assume that one of the following* (C1) *or* (C2) *holds*:

(C1) (F1) *holds and $([a(x)]^{-\frac{1}{2}})'' \le 0$ for $x_0 \le x \le x_1$;*

(C2) (F2) *holds and $([a(x)]^{-\frac{1}{2}})'' \ge 0$ for $x_0 \le x \le x_1$.*

Then (P_k) *has at most one solution.*

Since

$$\begin{aligned} ([a(x)]^{-\frac{1}{2}})'' &= \frac{1}{4}[a(x)]^{-\frac{5}{2}}\Big(3[a'(x)]^2 - 2a(x)a''(x)\Big) \\ &= \frac{1}{4}[a(x)]^{-\frac{1}{2}}\Big[\Big(\frac{a'(x)}{a(x)}\Big)^2 - 2\Big(\frac{a'(x)}{a(x)}\Big)'\Big] \end{aligned} \tag{1.3}$$

for $x_0 \le x \le x_1$, by Theorem 1.1, we obtain the next result.

Corollary 1.1. *Let $k \in \mathbf{N}$. Assume that* (F2) *holds, and that either*

$$a''(x) \le 0 \quad \text{or} \quad \Big(\frac{a'(x)}{a(x)}\Big)' \le 0$$

for $x_0 \le x \le x_1$. Then (P_k) *has at most one solution.*

Example 1.1. Consider the problem

$$\begin{cases} u'' + (e^x + \mu)|u|^{p-1}u = 0, \quad 0 < x < 1, \\ u(0) = u(1) = 0, \quad u'(0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (0,1), \end{cases} \tag{1.4}$$

where $p > 1$, $\mu > -1$ and $k \in \mathbf{N}$. From Theorem D below it follows that (1.4) has at least one solution. Theorem C implies that if $-1 < \mu \le 0$, then the solution of (1.4) is unique. Theorem 1.1 shows that if $\mu \ge e/2$, then the solution of (1.4) is unique. Consequently, Theorem 1.1 is a result different from Theorem C.

Remark 1.1. Now we assume that $a \in C^2[x_0, x_1]$, $a(x) > 0$ and $a'(x) \neq 0$ for $x \in [x_0, x_1]$. Then $h(x) \equiv x + 2[a(x)/a'(x)]$ satisfies

$$h'(x) = [a'(x)]^{-2}\Big(3[a'(x)]^2 - 2a(x)a''(x)\Big), \quad x_0 \leq x \leq x_1.$$

In view of (1.3), we see that $([a(x)]^{-\frac{1}{2}})'' \leq 0$ for $x \in [x_0, x_1]$ if and only if $h(x)$ is nonincreasing in $x \in [x_0, x_1]$. The conditions $a'(x) < 0$ and $h(x) \leq 0$, which are assumed in Theorem B, are unnecessary in Theorem 1.1.

For existence of solutions of (P_k), we refer to [1], [2], [3], [5], [6], [7]. In particular we shall describe the result in [6]. Let λ_k be the k-th eigenvalue of

$$\begin{cases} \varphi'' + \lambda a(x)\varphi = 0, \quad x_0 < x < x_1, \\ \varphi(x_0) = \varphi(x_1) = 0. \end{cases}$$

It is known that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

From (F1) and (F2) we see that $f(s)/s$ is a monotone function, and hence there exist limits f_0 and f_∞ such that $0 \leq f_0, f_\infty \leq \infty$,

$$f_0 = \lim_{s\to+0} \frac{f(s)}{s} \quad \text{and} \quad f_\infty = \lim_{s\to\infty} \frac{f(s)}{s}.$$

Applying Theorems 1 and 2 in [6], we have the following Theorem D.

Theorem D. *Let $k \in \mathbf{N}$. Suppose that* (F1) *or* (F2) *holds. Then the following* (i) *and* (ii) *hold*:

(i) *if $f_0 < \lambda_k < f_\infty$ or $f_\infty < \lambda_k < f_0$, then* (P_k) *has at least one solution*;

(ii) *if $f(s)/s < \lambda_k$ for $s > 0$ or $f(s)/s > \lambda_k$ for $s > 0$, then* (P_k) *has no solution.*

Combining Theorem 1.1 with Theorem D and recalling the monotone property of $f(s)/s$, we have the following result.

Corollary 1.2. *Let $k \in \mathbf{N}$. Assume that either* (C1) *or* (C2) *is satisfied. Then the following* (i) *and* (ii) *hold*:

(i) *if $f(s)/s = \lambda_k$ for some $s > 0$, the solution of* (P_k) *exists and is unique*;

(ii) *if $f(s)/s \neq \lambda_k$ for all $s > 0$, then* (P_k) *has no solution.*

In Section 2 we apply Theorem 1.1 to radial solutions of the Dirichlet problem for elliptic equations in annular domains. In Section 3 we give several lemmas. In Section 4 we give a proof of Theorem 1.1 by using the Prüfer transformation.

2. Uniqueness of radial solutions

In this section we consider radial solutions of the Dirichlet problem for elliptic equations in annular domains

$$\begin{cases} \Delta v + K(|x|)f(v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $K \in C^2[R_1, R_2]$, $K(r) > 0$ for $R_1 \le r \le R_2$, $f \in C^1(\mathbf{R})$, $f(s) > 0$, $f(-s) = -f(s)$ for $s > 0$, $\Omega = \{x \in \mathbf{R}^N : R_1 < |x| < R_2\}$, $R_1 > 0$ and $N \ge 2$.

Let $v(r)$ be a radial solution of (2.1), where $r = |x|$. Then $v(r)$ satisfies

$$\begin{cases} \dfrac{d^2v}{dr^2} + \dfrac{N-1}{r}\dfrac{dv}{dr} + K(r)f(v) = 0, \quad R_1 < r < R_2, \\ v(R_1) = v(R_2) = 0. \end{cases} \tag{2.2}$$

Let $u(t) = v(e^t)$ for $N = 2$, and let $u(t) = v(t^{-1/(N-2)})$ for $N \ge 3$. Then it follows that problem (2.2) is transformed into the problem

$$\begin{cases} \dfrac{d^2u}{dt^2} + b(t)f(u) = 0, \quad t_1 < t < t_2, \\ u(t_1) = u(t_2) = 0, \end{cases} \tag{2.3}$$

where $b(t) = e^{2t}K(e^t)$, $t_1 = \log R_1$ and $t_2 = \log R_2$ for $N = 2$, and $b(t) = (N-2)^{-2}t^{-2(N-1)/(N-2)}K(t^{-1/(N-2)})$, $t_1 = R_2^{-(N-2)}$ and $t_2 = R_1^{-(N-2)}$ for $N \ge 3$. An easy computation shows that

$$\frac{d^2}{dt^2}[b(t)]^{-\frac{1}{2}} = \begin{cases} \dfrac{1}{r}\Big[r^2\dfrac{d^2G}{dr^2} - r\dfrac{dG}{dr} + G\Big], & N = 2, \\ \dfrac{r^{N-3}}{N-2}\Big[r^2\dfrac{d^2G}{dr^2} - (N-1)r\dfrac{dG}{dr} + (N-1)G\Big], & N \ge 3, \end{cases}$$

where $G(r) = [K(r)]^{-\frac{1}{2}}$. Thus, by applying Theorem 1.1, we obtain the following result.

Corollary 2.1. *Let* $k \in \mathbf{N}$. *Suppose that one of the following* (i) *or* (ii) *holds*:

(i) (F1) *holds and*

$$r^2G'' - (N-1)rG' + (N-1)G \le 0, \quad R_1 \le r \le R_2;$$

(ii) (F2) *holds and*

$$r^2G'' - (N-1)rG' + (N-1)G \ge 0, \quad R_1 \le r \le R_2,$$

where $G(r) = [K(r)]^{-\frac{1}{2}}$ *and* $' = d/dr$. *Then there exists at most one radial solution* $u(r)$ *of problem* (2.1), *which satisfies* $u'(R_1) > 0$ *and has exactly* $k-1$ *zeros in* (R_1, R_2).

We easily see that, if $G(r) = [K(r)]^{-\frac{1}{2}}$, then

$$\begin{aligned} r^2G'' &- (N-1)rG' + (N-1)G \\ &= \frac{1}{4}K^{-\frac{5}{2}}\left[3r^2(K')^2 - 2r^2KK'' + 2(N-1)rKK' + 4(N-1)K^2\right] \\ &= \frac{1}{4}K^{-\frac{1}{2}}\Big[\Big(\frac{rK'}{K}+2\Big)\Big(\frac{rK'}{K}+2(N-1)\Big) - 2r\Big(\frac{rK'}{K}\Big)'\Big]. \end{aligned}$$

Hence, by Corollary 2.1, we obtain the following results.

Corollary 2.2. *Let* $k \in \mathbf{N}$. *Assume that* (F1) *holds, and that* $-2(N-1) \le rK'(r)/K(r) \le -2$ *and* $[rK'(r)/K(r)]' \ge 0$ *for* $R_1 \le r \le R_2$. *Then there exists at most one radial solution* $u(r)$ *of problem* (2.1), *which satisfies* $u'(R_1) > 0$ *and has exactly* $k-1$ *zeros in* (R_1, R_2).

Corollary 2.3. *Let* $k \in \mathbf{N}$. *Assume that* (F2) *holds and that one of the following* (i)–(iv) *holds*:

(i) $K''(r) \le 0$ *and* $K'(r) \ge 0$ *for* $R_1 \le r \le R_2$;
(ii) $N = 2$ *and* $[rK'(r)/K(r)]' \le 0$ *for* $R_1 \le r \le R_2$;
(iii) $rK'(r)/K(r) \le -2(N-1)$ *and* $[rK'(r)/K(r)]' \le 0$ *for* $R_1 \le r \le R_2$;
(iv) $rK'(r)/K(r) \ge -2$ *and* $[rK'(r)/K(r)]' \le 0$ *for* $R_1 \le r \le R_2$.

Then there exists at most one radial solution $u(r)$ *of problem* (2.1), *which satisfies* $u'(R_1) > 0$ *and has exactly* $k-1$ *zeros in* (R_1, R_2).

3. PRELIMINARIES

In this section we give several lemmas. To prove Theorem 1.1 we use the shooting method. Namely we consider the solution $u(x;\alpha)$ of (1.1) satisfying the initial condition

$$u(x_0) = 0 \quad \text{and} \quad u'(x_0) = \alpha > 0, \tag{3.1}$$

where $\alpha > 0$ is a parameter. Since $a \in C^2[x_0, x_1]$ and $f \in C^1(\mathbf{R})$, by a general theory (see, for example, [8, Chapter III, Section 13]), we see that

$u(x;\alpha)$ exists on $[x_0, x_1]$, it is unique and satisfies $u, u' \in C^1([x_0,x_1]\times(0,\infty))$, and that

$$w(x) \equiv \frac{\partial}{\partial\alpha}u(x;\alpha)$$

satisfies

$$\begin{cases} w'' + a(x)f'(u(x;\alpha))w = 0, \quad x \in (x_0, x_1], \\ w(x_0) = 0, \quad w'(x_0) = 1. \end{cases} \tag{3.2}$$

Let

$$U(x) \equiv u'(x;\alpha). \tag{3.3}$$

Differentiating (1.1) with respect to x and using (3.1), we find that $U(x)$ satisfies

$$\begin{cases} U'' + a(x)f'(u(x;\alpha))U = -a'(x)f(u(x;\alpha)), \quad x \in (x_0, x_1], \\ U(x_0) = \alpha, \quad U'(x_0) = 0. \end{cases} \tag{3.4}$$

The following identity, which plays a crucial part in the proof of Theorem 1.1, has been essentially obtained by Korman and Ouyang [4, Lemma 2.6].

Lemma 3.1. *Let $g \in C^2[x_0, x_1]$. Then solutions w of* (3.2) *and U of* (3.4) *satisfy*

$$\Big[g(w'U - wU') - g'wU\Big]' = -g''wU + (2g'a + ga')f(u(x;\alpha))w.$$

In particular, if $g(x) = [a(x)]^{-\frac{1}{2}}$, then

$$\Big[g(w'U - wU') - g'wU\Big]' = -g''wU. \tag{3.5}$$

Proof. A direct calculation shows that Lemma 3.1 follows immediately, by noting that w and U are solutions of (3.2) and (3.4), respectively, and that $U' = u''(x;\alpha) = -a(x)f(u(x;\alpha))$. □

Hereafter we assume that $u(x;\alpha)$ is a solution of (P_k). Then, by recalling (3.3), there exist sequences $\{z_i\}_{i=0}^k$ and $\{t_i\}_{i=1}^k$ such that

$$x_0 = z_0 < t_1 < z_1 < t_2 < z_2 < \cdots < t_{k-1} < z_{k-1} < t_k < z_k = x_1,$$
$$u(z_i;\alpha) = 0, \quad i = 0, 1, 2, \ldots, k,$$
$$U(t_i) = 0, \quad i = 1, 2, \ldots, k,$$
$$(-1)^{i-1}u(x;\alpha) > 0 \quad \text{for } x \in (z_{i-1}, z_i), \quad i = 1, 2, \ldots, k,$$
$$U(x) > 0 \quad \text{for } x \in [z_0, t_1),$$
$$(-1)^{i-1}U(x) > 0 \quad \text{for } x \in (t_{i-1}, t_i), \quad i = 2, 3, \ldots, k,$$

$$(-1)^k U(x) > 0 \quad \text{for } x \in (t_k, z_k]. \tag{3.6}$$

Lemma 3.2. *Assume that* (F1) *holds. For each* $i \in \{1, 2, \ldots, k\}$*, the solution* w *of* (3.2) *has at least one zero in* (z_{i-1}, z_i).

Proof. From (F1) we easily see that $f'(s) > f(s)/s$ for $s > 0$. Since $f(s)$ is an odd function, we conclude that $f'(s)$ and $f(s)/s$ are even functions. Hence we have

$$f'(s) > \frac{f(s)}{s}, \quad s \neq 0. \tag{3.7}$$

Note that $u \equiv u(x; \alpha)$ is a solution of

$$u'' + a(x)\frac{f(u)}{u}u = 0, \quad x \in (z_{i-1}, z_i)$$

and satisfies $u(z_{i-1}) = u(z_i) = 0$. From (3.7) it follows that

$$a(x)f'(u) > a(x)\frac{f(u)}{u}, \quad x \in (z_{i-1}, z_i).$$

By applying the Sturm–Picone theorem (see, for example, [8, Chapter VI, Section 27]), it follows that w has at least one zero in (z_{i-1}, z_i). □

Lemma 3.3. *Assume that* (F2) *holds. For each* $i \in \{1, 2, \ldots, k\}$*, the solution* w *of* (3.2) *has at most one zero in* $[z_{i-1}, z_i]$.

Proof. Assume to the contrary that there exist numbers r_0 and r_1 such that $z_{i-1} \le r_0 < r_1 \le z_i$ and $w(r_0) = w(r_1) = 0$. By the same argument as in the proof of Lemma 3.2, condition (F2) implies that

$$a(x)f'(u) < a(x)\frac{f(u)}{u}, \quad x \in (r_0, r_1),$$

where $u = u(x; \alpha)$. Then the Sturm–Picone theorem shows that u has at least one zero in (r_0, r_1). This is a contradiction. The proof is complete. □

Lemma 3.4. *Assume that* (C1) *holds. Let* w *be the solution of* (3.2)*. Then* $w(x) > 0$ *for* $x \in (x_0, t_1]$.

Proof. Note that $w(x_0) = 0$ and $w'(x_0) = 1$. Assume that there exists a number $r_1 \in (x_0, t_1]$ such that $w(x) > 0$ for $x \in (x_0, r_1)$ and $w(r_1) = 0$. Then we see that $w'(r_1) < 0$. Put $g(x) = [a(x)]^{-\frac{1}{2}}$. By integrating (3.5) over $[x_0, r_1]$ and using (C1), we have

$$g(r_1)w'(r_1)U(r_1) - \alpha g(x_0) = -\int_{x_0}^{r_1} g''(x)w(x)U(x)dx \ge 0.$$

On the other hand, since $w'(r_1) < 0$ and $U(r_1) \geq 0$, we obtain

$$g(r_1)w'(r_1)U(r_1) - \alpha g(x_0) < 0.$$

This is a contradiction. Consequently we find that $w(x) > 0$ for $x \in (x_0, t_1]$. □

Lemma 3.5. *Assume that* (C1) *holds. Then the solution* w *of* (3.2) *has at most one zero in* $(t_{i-1}, t_i]$ *for each* $i \in \{2, 3, \ldots, k\}$.

Proof. Assume that there exist numbers r_0 and r_1 such that $t_{i-1} < r_0 < r_1 \leq t_i$, $w(r_0) = w(r_1) = 0$ and $w(x) \neq 0$ for $x \in (r_0, r_1)$. Integrating (3.5) over $[r_0, r_1]$, we have

$$g(r_1)w'(r_1)U(r_1) - g(r_0)w'(r_0)U(r_0) = -\int_{r_0}^{r_1} g''(x)w(x)U(x)dx, \qquad (3.8)$$

where $g(x) = [a(x)]^{-\frac{1}{2}}$. Now we suppose that $w(x) > 0$ for $x \in (r_0, r_1)$. Then we find that $w'(r_0) > 0$, $w'(r_1) < 0$, $(-1)^{i-1}U(r_0) > 0$ and $(-1)^{i-1}U(r_1) \geq 0$, so that

$$g(r_1)w'(r_1)(-1)^{i-1}U(r_1) - g(r_0)w'(r_0)(-1)^{i-1}U(r_0) < 0.$$

However (3.8) and (C1) imply that

$$g(r_1)w'(r_1)(-1)^{i-1}U(r_1) - g(r_0)w'(r_0)(-1)^{i-1}U(r_0) \geq 0.$$

This is a contradiction. For the case where $w(x) < 0$ for $x \in (r_0, r_1)$, in exactly the same way, we are led to a contradiction by (3.8) and (C1). Hence w has at most one zero in $(t_{i-1}, t_i]$ for each $i \in \{1, 2, \ldots, k\}$. □

Lemma 3.6. *Assume that* (C1) *holds. Then the solution* w *of* (3.2) *has at most one zero in* $(t_k, x_1]$.

Proof. The proof is similar to that of Lemma 3.5, and hence will be omitted. □

Lemma 3.7. *Assume that* (C2) *holds. Then the solution* w *of* (3.2) *has at least one zero in* $[t_{i-1}, t_i)$ *for each* $i \in \{2, 3, \ldots, k\}$.

Proof. Suppose that $w(x) \neq 0$ for $x \in [t_{i-1}, t_i)$. Integration of (3.5) over $[t_{i-1}, t_i]$ yields

$$-g(t_i)w(t_i)U'(t_i) + g(t_{i-1})w(t_{i-1})U'(t_{i-1}) = -\int_{t_{i-1}}^{t_i} g''(x)w(x)U(x)dx, \qquad (3.9)$$

where $g(x) = [a(x)]^{-\frac{1}{2}}$. We may assume that $w(x) > 0$ for $x \in [t_{i-1}, t_i)$, since the case where $w(x) < 0$ for $x \in [t_{i-1}, t_i)$ can be treated similarly. Then we have $w(t_{i-1}) > 0$, $w(t_i) \geq 0$, $(-1)^{i-1}U'(t_{i-1}) > 0$ and $(-1)^{i-1}U'(t_i) < 0$, which implies

$$-g(t_i)w(t_i)(-1)^{i-1}U'(t_i) + g(t_{i-1})w(t_{i-1})(-1)^{i-1}U'(t_{i-1}) > 0.$$

On the other hand, (C2) and (3.9) show that

$$-g(t_i)w(t_i)(-1)^{i-1}U'(t_i) + g(t_{i-1})w(t_{i-1})(-1)^{i-1}U'(t_{i-1}) \leq 0.$$

This is a contradiction. The proof is complete. □

Proposition 3.1. *If* (C1) *holds, then* $(-1)^k w(z_k) > 0$.

Proposition 3.2. *If* (C2) *holds, then* $(-1)^k w(z_k) < 0$.

Proof of Proposition 3.1. By Lemmas 3.2 and 3.4, there exists a number $c_1 \in (t_1, z_1)$ such that $w(x) > 0$ for $x \in (z_0, c_1)$ and $w(c_1) = 0$. Then Lemma 3.5 implies that $w(x) < 0$ for $x \in (c_1, t_2]$. Hence we have $w(z_1) < 0$.

From Lemma 3.2 it follows that there exists a number $c_2 \in (t_2, z_2)$ such that $w(x) < 0$ for $x \in (t_2, c_2)$ and $w(c_2) = 0$. By Lemma 3.5 we see that $w(x) > 0$ for $x \in (c_2, t_3]$, so that $w(z_2) > 0$.

By continuing this process and using Lemma 3.6, we conclude that

$$(-1)^k w(z_k) > 0.$$

The proof is complete. □

Proof of Proposition 3.2. In view of Lemma 3.3 and the fact that $w(x_0) = 0$ and $w'(x_0) > 0$, we see that $w(x) > 0$ for $x \in (x_0, z_1]$. Then we have $w(z_1) > 0$. By Lemma 3.7, there exists $c_1 \in (z_1, t_2)$ such that $w(x) > 0$ for $x \in (z_1, c_1)$ and $w(c_1) = 0$. From Lemma 3.3 it follows that $w(x) < 0$ for $x \in (c_1, z_2]$, which implies that $w(z_2) < 0$. By continuing this process, we can obtain $(-1)^k w(z_k) < 0$. □

4. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. To this end we employ the Prüfer transformation for the solution $u(x;\alpha)$ of problem (1.1)–(3.1). For the solution $u(x;\alpha)$ with $\alpha > 0$, we define the functions $r(x;\alpha)$ and $\theta(x;\alpha)$ by

$$\begin{cases} u(x;\alpha) = r(x;\alpha)\sin\theta(x;\alpha), \\ u'(x;\alpha) = r(x;\alpha)\cos\theta(x;\alpha), \end{cases}$$

where $' = d/dx$. From the initial condition (3.1) it follows that $r(x_0;\alpha) = \alpha$ and $\theta(x_0;\alpha) \equiv 0 \pmod{2\pi}$. For simplicity we take $\theta(x_0;\alpha) = 0$. Since $u(x;\alpha)$ and $u'(x;\alpha)$ cannot vanish simultaneously, $r(x;\alpha)$ and $\theta(x;\alpha)$ are written in the forms

$$r(x;\alpha) = \Big([u(x;\alpha)]^2 + [u'(x;\alpha)]^2\Big)^{1/2} > 0$$

and

$$\theta(x;\alpha) = \arctan\frac{u(x;\alpha)}{u'(x;\alpha)},$$

respectively. Therefore we find that $r,\,\theta \in C^1([x_0,x_1]\times(0,\infty))$. By a simple calculation we see that

$$\theta'(x;\alpha) = \cos^2\theta(x;\alpha) + a(x)\frac{\sin\theta(x;\alpha) f(r(x;\alpha)\sin\theta(x;\alpha))}{r(x;\alpha)} > 0$$

for $x \in [x_0,x_1]$, which shows that $\theta(x;\alpha)$ is strictly increasing in $x \in [x_0,x_1]$ for each fixed $\alpha > 0$. It is easy to see that $u(x;\alpha)$ is a solution of (P_k) if and only if

$$\theta(x_1;\alpha) = k\pi. \tag{4.1}$$

This means that the number of solutions of (P_k) is equal to the number of roots $\alpha > 0$ of (4.1).

Proposition 4.1. *Let $k \in \mathbf{N}$. Assume that $u(x_1;\alpha_0)$ is a solution of* (P_k) *for some $\alpha_0 > 0$. If* (C1) *holds, then $\frac{\partial}{\partial\alpha}\theta(x_1;\alpha_0) > 0$. If* (C2) *holds, then $\frac{\partial}{\partial\alpha}\theta(x_1;\alpha_0) < 0$.*

Proof. Observe that

$$\frac{\partial}{\partial\alpha}\theta(x;\alpha_0) = \theta_\alpha(x;\alpha_0) = \frac{u_\alpha(x;\alpha_0)u'(x;\alpha_0) - u(x;\alpha_0)u'_\alpha(x;\alpha_0)}{[u(x;\alpha_0)]^2 + [u'(x;\alpha_0)]^2}.$$

Since $u(x_1;\alpha_0) = 0$ and $x_1 = z_k$, we obtain

$$\theta_\alpha(x_1;\alpha_0) = \frac{u_\alpha(x_1;\alpha_0)}{u'(x_1;\alpha_0)} = \frac{w(z_k)}{U(z_k)}.$$

Note that $(-1)^k U(z_k) > 0$, because of (3.6). From Propositions 3.1 and 3.2, it follows that $\theta_\alpha(x_1;\alpha) > 0$ for the case (C1), and $\theta_\alpha(x_1;\alpha) < 0$ for the case (C2). The proof is complete. □

Proof of Theorem 1.1. We give a proof of Theorem 1.1 for the case (C1) only. In exactly the same way, we can show Theorem 1.1 for the case (C2).

Assume to the contrary that there exist numbers $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $u(x;\alpha_1)$ and $u(x;\alpha_2)$ are solutions of (P_k) and $\alpha_1 \neq \alpha_2$. Then

$\theta(x_1;\alpha_1) = \theta(x_1;\alpha_2) = k\pi$. We may assume without loss of generality that $0 < \alpha_1 < \alpha_2$ and $\theta(x_1;\alpha) \neq k\pi$ for $\alpha \in (\alpha_1, \alpha_2)$. In view of Proposition 4.1, we conclude that $\theta_\alpha(x_1;\alpha_1) > 0$ and $\theta_\alpha(x_1;\alpha_2) > 0$. The intermediate value theorem implies that there is a number $\alpha_0 \in (\alpha_1, \alpha_2)$ such that $\theta(x_1;\alpha_0) = k\pi$. This is a contradiction. Consequently, (P_k) has at most one solution. The proof of Theorem 1.1 is complete. □

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