Differential and Integral Equations Volume 20, Number 1 (2007), 77–92

ON LOCAL COMPACTNESS IN QUASILINEAR ELLIPTIC PROBLEMS

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(Submitted by: Ildefonso Diaz)

Abstract. One of the major difficulties in nonlinear elliptic problems involving critical nonlinearities is the compactness of Palais-Smale sequences. In their celebrated work [7], Brézis and Nirenberg introduced the notion of critical level for these sequences in the case of a critical perturbation of the Laplacian homogeneous eigenvalue problem. In this paper, we give a natural and general formula of the critical level for a large class of nonlinear elliptic critical problems. The sharpness of our formula is established by the construction of suitable Palais-Smale sequences which are not relatively compact.

1. Introduction

In nonlinear elliptic variational problems involving critical nonlinearities, one of the major difficulties is to recover the compactness of Palais-Smale sequences of the associated Euler-Lagrange functional. Such questions were first studied, to our knowledge, by Brézis and Nirenberg in their well-known work [7]. The concentration-compactness principle due to Lions [12] is widely used to overcome these difficulties. Other methods, based on the convergence almost everywhere of the gradients of Palais-Smale sequences, can be also used to recover the compactness. We refer the reader to the papers by Boccardo and Murat [5] and by J. M. Rakotoson [14] for bounded domains. For arbitrary domains, we refer to the recent work by A. El Hamidi and J. M. Rakotoson [9].

In [7], the authors studied the critical perturbation of the eigenvalue problem:

$$\begin{cases} -\Delta u &= \lambda u + u^{2^*-1} \text{ in } \Omega, \\ u &> 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

Accepted for publication: August 2006.
AMS Subject Classifications: 35J70, 46E35, 35B65.

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary, $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent of the embedding $W^{1,2}(\Omega) \subset L^p(\Omega)$, and λ is a positive parameter. The authors introduced an important condition on the level corresponding to the energy of Palais-Smale sequences which guarantees their relative compactness. Indeed, let (u_n) be a Palais-Smale sequence for the Euler-Lagrange functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}.$$

More precisely, the authors showed that if

$$\lim_{n\to+\infty} I_\lambda(u_n) < \frac{1}{N} S^{\frac{N}{2}}, \tag{1.2}$$

then (u_n) is relatively compact, which implies the existence of nontrivial critical points of I_λ. Here, S denotes the best Sobolev constant in the embedding $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$. In this work, we begin by giving the generalization of condition (1.2) for the quasilinear equation

$$\begin{aligned} &-\Delta_p u = \lambda f(x,u) + |u|^{p^*-2}u \ \text{ in } \ \Omega, \\ &u|_\Gamma = 0 \ \text{ and } \ \tfrac{\partial u}{\partial \nu}|_\Sigma = 0, \end{aligned} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Sigma}$, where Γ and Σ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$. Δ_p is the p-Laplacian and $\frac{\partial}{\partial\nu}$ is the outer normal derivative. Here, f is a subcritical perturbation of $|u|^{p^*-1}$.

The sharpness of our result is established by the construction of suitable Palais-Smale sequences (corresponding to the critical level) which are not relatively compact.

Then we give the condition analogous to (1.2) for a general system with critical exponents

$$\begin{cases} -\Delta_p u &= \lambda f(x,u) + u|u|^{\alpha-1}|v|^{\beta+1} \ \text{ in } \ \Omega \\ -\Delta_q v &= \mu g(x,v) + |u|^{\alpha+1}|v|^{\beta-1}v \ \text{ in } \ \Omega \end{cases}$$

together with Dirichlet or mixed boundary conditions, where f and g are subcritical perturbations of $|u|^{p^*-1}$ and $|v|^{q^*-1}$ respectively, $p^* = \frac{Np}{N-p}$ (respectively $q^* = \frac{Nq}{N-q}$) is the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \subset L^r(\Omega)$ (respectively $W^{1,q}(\Omega) \subset L^r(\Omega)$). Our approach provides a general condition based on the Nehari manifold, which can be extended to a large class of critical nonlinear problems. In this work, we confine ourselves to systems involving $(p,q)-$Laplacian operators and critical nonlinearities.

The sharpness of our result is established, in the special case $p = q$, by the construction of suitable Palais-Smale sequences which are not relatively compact. The question of sharpness corresponding to the case $p \neq q$ is still open.

For a more complete description of nonlinear elliptic systems, we refer the reader to the papers by De Figueiredo [10] and by De Figueiredo & Felmer [11] and the references therein.

2. A general local compactness result

For the reader's convenience, we start with the scalar case and to render the paper self contained we will recall or show some well-known facts.

2.1. **The scalar case.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with smooth boundary $\partial\Omega$. Let $f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function which is measurable in x, continuous in u and satisfies the growth condition at infinity

$$|f(x,u)| = o(u^{p^*-1}) \text{ as } u \to +\infty, \text{ uniformly in } x. \tag{2.1}$$

This situation occurs, for example, in the special cases $f(x,u) = u$ or $f(x,u) = u^{q-1}$, $1 < q < p^*$.

Consider the problem

$$\begin{aligned} &-\Delta_p u = \lambda f(x,u) + |u|^{p^*-2}u \text{ in } \Omega, \\ &u|_\Gamma = 0 \text{ and } \tfrac{\partial u}{\partial \nu}|_\Sigma = 0, \end{aligned} \tag{2.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Sigma}$, where Γ and Σ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$. Problem (2.2) is posed in the framework of the Sobolev space $W^{1,p}_\Gamma(\Omega) = \{u \in W^{1,p}(\Omega) : u|_\Gamma = 0\}$, which is the closure of $C^1_0(\Omega \cap \Gamma, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$. Notice that $meas(\Gamma) > 0$ implies that the Poincaré inequality is still available in $W^{1,p}_\Gamma(\Omega)$, so it can be endowed with the norm

$$||u|| = ||\nabla u||_p$$

and $(W^{1,p}_\Gamma(\Omega), ||\,.\,||)$ is a reflexive and separable Banach space. The associated Euler-Lagrange functional is given by

$$J_\lambda(u) := \frac{1}{p}||\nabla u||_p^p - \frac{1}{p^*}||u||_{p^*}^{p^*} - \lambda \int_\Omega F(x,u(x))\,dx,$$

the corresponding Euler-Lagrange functional, where $F(x,u) := \int_0^u f(x,s)\,ds$.

We recall here that the Nahari manifold associated to the functional J_λ is given by:

$$\mathcal{N}_{J_\lambda} = \{u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\} \;:\; J'_\lambda(u)(u) = 0\},$$

and it is clear that $\mathcal{N}_{J_\lambda}$ contains all nontrivial critical points of J_λ. This manifold can be characterized more explicitly by the following

$$\mathcal{N}_{J_\lambda} = \left\{ tu,\ (t,u) \in (\mathbb{R} \setminus \{0\}) \times (W^{1,p}_\Gamma(\Omega) \setminus \{0\}) \;:\; \frac{d}{dt} J_\lambda(tu) = 0 \right\},$$

where $t \mapsto J_\lambda(tu)$ is a function defined from $\mathbb{R}$ to itself, for every u given in $W^{1,p}_\Gamma(\Omega) \setminus \{0\}$. We define the critical level associated to Problem (2.2) by:

$$c^*(\lambda) := \inf_{w \in \mathcal{N}_{J_0}} J_0(w) + \inf_{w \in \mathcal{N}_{J_\lambda} \cup \{0\}} J_\lambda(w). \tag{2.3}$$

At this stage, we can state and show our first result.

Theorem 2.1. *Let $\lambda \in \mathbb{R}$ and (u_n) be a Palais-Smale sequence of J_λ such that*

$$\lim_{n \to +\infty} J_\lambda(u_n) < c^*(\lambda). \tag{2.4}$$

Then (u_n) is relatively compact.

Proof. Let $\lambda \in \mathbb{R}$ and (u_n) be a Palais-Smale sequence for J_λ of level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ for short) satisfying the condition (2.4). We claim that (u_n) is bounded in $W^{1,p}_\Gamma(\Omega)$. Indeed, one has on the one hand

$$\frac{1}{p}||\nabla u_n||_p^p - \frac{1}{p^*}||u_n||_{p^*}^{p^*} - \lambda \int_\Omega F(x,u_n)\,dx = c + o_n(1), \tag{2.5}$$

and

$$||\nabla u_n||_p^p - ||u_n||_{p^*}^{p^*} - \lambda \int_\Omega f(x,u_n)u_n\,dx = o_n(||\nabla u_n||_p). \tag{2.6}$$

Then

$$\begin{aligned}&\Big(\frac{1}{p} - \frac{1}{p^*}\Big)||u_n||_{p^*}^{p^*} + \frac{\lambda}{p}\int_\Omega f(x,u_n)u_n\,dx - \lambda\int_\Omega F(x,u_n)\,dx\\ &= c + o_n(1) + o_n(||\nabla u_n||_p).\end{aligned}$$

Now, letting $\varepsilon > 0$, using the growth condition (2.1), there exists $c_1(\varepsilon) > 0$ such that

$$|f(x,u)| \le \varepsilon |u|^{p^*-1} + c_1 \ \text{ and } \ |F(x,u)| \le \frac{\varepsilon}{p^*}|u|^{p^*} + c_1, \ \text{ a.e. } x \in \Omega$$

and for every $u \in \mathbb{R}$. Applying the Hölder and the Young inequalities to the last relations, it follows that

$$||u_n||_{p^*}^{p^*} \leq \varepsilon ||\nabla u_n||_p + c_2(|\Omega|, \lambda, \varepsilon). \tag{2.7}$$

Combining (2.7) and (2.5), we deduce that (u_n) is in fact bounded in $W^{1,p}_\Gamma(\Omega)$. So passing, if necessary to a subsequence, we can consider that

$$u_n \rightharpoonup u \text{ in } W^{1,p}_\Gamma(\Omega), \qquad u_n \to u \text{ a.e. in } \Omega.$$

On the other hand, the growth condition (2.1) implies also that, for almost every $x \in \Omega$, the functions $s \mapsto F(x,s)$ and $s \mapsto sf(x,s)$ satisfy the conditions of the Brézis-Lieb lemma (see Theorem 2 in [6]). Thus, we get the identities

$$\begin{aligned}\int_\Omega F(x,v_n)\,dx &= \int_\Omega F(x,u_n) - \int_\Omega F(x,u) + o_n(1),\\ \int_\Omega f(x,v_n)v_n\,dx &= \int_\Omega f(x,u_n)u_n - \int_\Omega f(x,u)u + o_n(1).\end{aligned}$$

Moreover, letting $\varepsilon > 0$, there is $c_1(\varepsilon) > 0$ such that

$$\left|\int_\Omega f(x,v_n)v_n\,dx\right| \leq \varepsilon ||v_n||_{p^*}^{p^*} + c_1 ||v_n||_1.$$

Let $C > 0$ (which is independent of n and ε), such that $||v_n||_{p^*}^{p^*} \leq C$. Since (v_n) converges strongly to 0 in $L^1(\Omega)$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that $||v_n||_1 \leq \varepsilon/c_1$, for every $n \geq n_0(\varepsilon)$, and consequently

$$|\int_\Omega f(x,v_n)v_n\,dx| \leq \varepsilon(1+C), \quad \forall n \geq n_0(\varepsilon).$$

In the same way, rewriting $F(x,v_n) = \int_0^{v_n} f(x,s)\,ds$ and using the same arguments as above, we deduce that

$$\int_\Omega F(x,v_n)\,dx = o_n(1) \tag{2.8}$$

$$\int_\Omega f(x,v_n)v_n\,dx = o_n(1). \tag{2.9}$$

Applying once again the Brézis-Lieb lemma, we conclude that $u \in \mathcal{N}_{J_\lambda} \cup \{0\}$ and

$$||v_n||^p - ||v_n||_{p^*}^{p^*} = o_n(1), \tag{2.10}$$

$$J_0(v_n) := \frac{1}{p}||v_n||^p - \frac{1}{p^*}||v_n||_{p^*}^{p^*} = c - J_\lambda(u) + o_n(1). \tag{2.11}$$

A direct computation gives $\mathcal{N}_{J_0} = \left\{ t_0(u)u \ : \ u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\} \right\}$, where $t_0(u) := \left(\frac{||u||^p}{||u||^{p^*}_{p^*}} \right)^{\frac{1}{p^*-p}}$. Now, let b be the common limit of $||v_n||^p$ and $||v_n||^{p^*}_{p^*}$. Suppose that $b \neq 0$. On the one hand we have

$$J_0(t_0(v_n)v_n) = \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{||v_n||^p}{||v_n||^p_{p^*}}\right)^{\frac{p^*}{p^*-p}} \geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w).$$

Then

$$\lim_{n \to +\infty} J_0(t_0(v_n)v_n) = \frac{b}{N} \geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w).$$

On the other hand, the identity (2.11) leads to $\frac{b}{N} = c - J_\lambda(u)$. It follows then

$$c \geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w) + J_\lambda(u) \geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w) + \inf_{w \in \mathcal{N}_{J_\lambda} \cup \{0\}} J_\lambda(w),$$

which contradicts the condition (2.4). This achieves the proof. □

2.2. **Sharpness of the critical level formula in the scalar case.** To show the sharpness of the critical level formula (2.4), it suffices to carry out a Palais-Smale sequence for J_λ of level $c^*(\lambda)$ which contains no convergent subsequence.

Consider, for a given $\varepsilon > 0$, the extremal function

$$\Phi_\varepsilon(x) = C_N \varepsilon^{\frac{N-p}{p^2}} \left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{\frac{p-N}{p}} \ \text{ with } \ C_N := \left(N\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{(N-p)/p^2}$$

which attains the best constant S of the Sobolev embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$.

Without loss of generality, we can consider that $0 \in \Sigma$. Moreover, the set $\partial\Omega$ satisfies the following property (see more details in Adimurthi, Pacella and Yadava [1]):

There exist $\delta > 0$, an open neighborhood $\mathcal{V}$ of 0 and a diffeomorphism $\Psi : B_\delta(0) \longrightarrow \mathcal{V}$ which has a Jacobian determinant equal to one at 0, with $\Psi(B^+_\delta) = \mathcal{V} \cap \Omega$, where $B^+_\delta = B_\delta(0) \cap \{x \in \mathbb{R}^N \ : \ x_N > 0\}$.

Let $\varphi \in C^\infty_0(\mathbb{R}^N)$ such that $\varphi \equiv 1$ in a neighborhood of the origin.

We define the sequence defined by

$$\psi_n(x) := \varphi(x)\Phi_{1/n}(x), \quad \text{for} \quad n \in \mathbb{N}^*. \tag{2.12}$$

It is well known that the sequence $(\psi_n) \subset W^{1,p}_\Gamma(\Omega)$ is a Palais-Smale sequence for J_0 of level $\inf_{w\in\mathcal{N}_{J_0}} J_0(w)$, which satisfies

$$\begin{aligned}
\psi_n &\to 0 \text{ a.e. in } \Omega,\\
\nabla\psi_n &\to 0 \text{ a.e. in } \Omega,\\
||\psi_n||_{p^*}^{p^*} &\longrightarrow \Big[N \inf_{w\in\mathcal{N}_{J_0}} J_0(w)\Big]^{p/N} := \ell \text{ as } n\longrightarrow+\infty,\\
||\nabla\psi_n||_p^p &\longrightarrow \Big[N \inf_{w\in\mathcal{N}_{J_0}} J_0(w)\Big]^{p/N} := \ell \text{ as } n\longrightarrow+\infty.
\end{aligned}$$

Now, let (u_n) be a Palais-Smale sequence of J_λ of level $\inf_{w\in\mathcal{N}_{J_\lambda}\cup\{0\}} J_\lambda(w)$. We will not go into further details concerning which subcritical terms $f(u)$ allow the existence of such sequences, but in the literature, this occurs for various classes of subcritical terms. Applying Theorem 2.1, there exists a subsequence, still denoted by (u_n), which converges to some $u\in W^{1,p}_\Gamma(\Omega)$. Then

$$\begin{aligned}
||u_n+\psi_n||_{p^*} &\le C,\\
u_n+\psi_n &\to u \text{ a.e. in } \Omega,\\
||\nabla u_n+\nabla\psi_n||_p &\le C,\\
\nabla u_n+\nabla\psi_n &\to \nabla u \text{ a.e. in } \Omega,
\end{aligned}$$

where C is a positive constant independent of n. We apply the Brézis-Lieb lemma to the sequence $(u_n+\psi_n)$ and get

$$||u_n+\psi_n||_{p^*}^{p^*} = ||(u_n-u)+\psi_n||_{p^*}^{p^*} + ||u||_{p^*}^{p^*} + \mathrm{o}_n(1).$$

Moreover, one has

$$\begin{aligned}
-||u_n-u||_{p^*} + ||\psi_n||_{p^*} - \ell^{1/p^*} &\le ||(u_n-u)+\psi_n||_{p^*} - \ell^{1/p^*}\\
&\le ||u_n-u||_{p^*} + ||\psi_n||_{p^*} - \ell^{1/p^*}
\end{aligned}$$

which implies that

$$||(u_n-u)+\psi_n||_{p^*} - \ell^{1/p^*} = \mathrm{o}_n(1).$$

Therefore, we conclude that

$$||u_n+\psi_n||_{p^*}^{p^*} = ||u||_{p^*}^{p^*} + \ell + \mathrm{o}_n(1).$$

The same arguments applied to the sequence $(\nabla u_n+\nabla\psi_n)$ give

$$||\nabla u_n+\nabla\psi_n||_p^p = ||\nabla u||_p^p + \ell + \mathrm{o}_n(1).$$

Finally, using the fact that

$$|\psi_n|^{p^*} \overset{*}{\rightharpoonup} \ell\,\delta_0 \quad \text{weakly} * \text{in } \mathcal{M}^+(\Omega) \tag{2.13}$$

$$|\nabla\psi_n|^{p} \overset{*}{\rightharpoonup} \ell\,\delta_0 \quad \text{weakly} * \text{in } \mathcal{M}^+(\Omega) \tag{2.14}$$

where δ_0 is the Dirac measure concentrated at the origin and $\mathcal{M}^+(\Omega)$ is the space of positive finite measures [20]), we get that the sequence $(u_n + \psi_n)$ is a Palais-Smale sequence of J_λ of level $c^*(\lambda)$.

We hence have constructed a Palais-Smale sequence $(u_n + \psi_n)$ of J_λ of level $c^*(\lambda)$ which can not be relatively compact in $W^{1,p}_\Gamma(\Omega)$. This justifies the sharpness of the critical level formula (2.4).

Remark 2.1. If we are interested in the homogeneous Dirichlet conditions; *i.e.,* if $\Sigma = \emptyset$, the same arguments developed above are still valid. It suffices to assume that the origin $0 \in \Omega$ and consider $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ in a neighborhood of the origin.

2.3. **The system case.** Now, consider the system

$$\begin{cases} -\Delta_p u &= \lambda f(x,u) + u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v &= \mu g(x,v) + |u|^{\alpha+1}|v|^{\beta-1}v, \end{cases} \tag{2.15}$$

together with Dirichlet or mixed boundary conditions

$$\begin{cases} u|_{\Gamma_1} = 0 \quad \text{and} \quad \frac{\partial u}{\partial\nu}|_{\Sigma_1} = 0, \\ v|_{\Gamma_2} = 0 \quad \text{and} \quad \frac{\partial v}{\partial\nu}|_{\Sigma_2} = 0, \end{cases} \tag{2.16}$$

where, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma}_i \cup \overline{\Sigma}_i$, where Γ_i and Σ_i are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma_i \cap \Sigma_i = \emptyset$, $i \in \{1,2\}$. Δ_p is the p-Laplacian and $\frac{\partial}{\partial\nu}$ is the outer normal derivative. Also, it is clear that when $\Gamma_1 = \Gamma_2 = \partial\Omega$, one deals with homogeneous Dirichlet boundary conditions. We assume here that

$$1 < p < N, \;\; 1 < q < N, \tag{2.17}$$

and the critical condition

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1. \tag{2.18}$$

Indeed, this condition represents the maximal growth such that the integrability of the product term $|u|^{\alpha+1}|v|^{\beta+1}$ (which will appear in the Euler-Lagrange functional) can be guaranteed by suitable Hölder estimates.

The functions f and g are two Caratheodory functions which satisfy the growth conditions

$$|f(x,u)| = o(u^{p^*-1}) \text{ as } u \to +\infty, \text{ uniformly in } x, \qquad (2.19)$$

$$|g(x,v)| = o(v^{q^*-1}) \text{ as } v \to +\infty, \text{ uniformly in } x. \qquad (2.20)$$

Problem (2.15), together with (2.16), is posed in the framework of the Sobolev space $W = W^{1,p}_{\Gamma_1}(\Omega) \times W^{1,q}_{\Gamma_2}(\Omega)$, where

$$W^{1,p}_{\Gamma_1}(\Omega) = \{u \in W^{1,p}(\Omega) : u|_{\Gamma_1} = 0\}, \;\; W^{1,q}_{\Gamma_2}(\Omega) = \{u \in W^{1,q}(\Omega) : u|_{\Gamma_2} = 0\},$$

which are respectively the closure of $C^1_0(\Omega \cap \Gamma_1, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$ and $C^1_0(\Omega \cap \Gamma_2, \mathbb{R})$ with respect to the norm of $W^{1,q}(\Omega)$. Notice that $meas(\Gamma_i) > 0$, $i = 1, 2$, imply that the Poincaré inequality is still available in $W^{1,p}_{\Gamma_1}(\Omega)$ and $W^{1,q}_{\Gamma_2}(\Omega)$, so W can be endowed with the norm

$$||(u,v)|| = ||\nabla u||_p + ||\nabla v||_q$$

and $(W, ||\,.\,||)$ is a reflexive and separable Banach space. The associated Euler-Lagrange functional $I_{\lambda,\mu} \in C^1(W, \mathbb{R})$ is given by

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= (\alpha+1)\Big(\frac{P(u)}{p} - \lambda \int_\Omega F(x,u)\Big) \\ &\quad + (\beta+1)\Big(\frac{Q(v)}{q} - \mu \int_\Omega G(x,v)\Big) - R(u,v), \end{aligned}$$

where

$$P(u) = ||\nabla u||_p^p, \quad Q(v) = ||\nabla v||_q^q, F(x,u) = \int_0^u f(x,s)\,ds,$$

$$G(x,v) = \int_0^v g(x,t)\,dt, \quad R(u,v) = \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}dx.$$

Notice that $R(u,v) \le ||u||_{p^*}^{\alpha+1}||v||_{q^*}^{\beta+1} < +\infty$.

Consider the Nehari manifold associated to Problem (2.15) given by

$$\mathcal{N}_{\lambda,\mu} = \{(u,v) \in W \setminus \{(0,0)\} : D_1 I_{\lambda,\mu}(u,v)(u) = D_2 I_{\lambda,\mu}(u,v)(v) = 0\},$$

where $D_1 I_{\lambda,\mu}$ and $D_2 I_{\lambda,\mu}$ are the derivative of $I_{\lambda,\mu}$ with respect to the first variable and the second variable respectively.

An interesting and useful characterization of $\mathcal{N}_{\lambda,\mu}$ is the following

$$\mathcal{N}_{\lambda,\mu} = \{(su,tv) : (s,u,t,v) \in \mathcal{Z}^* \text{ and } \partial_s I_{\lambda,\mu}(su,tv) = \partial_t I_{\lambda,\mu}(su,tv) = 0\},$$

where

$$\mathcal{Z}^* = \{(s,u,t,v) : (s,t) \in \mathbb{R}^2, \; (u,v) \in W^{1,p}_{\Gamma_1}(\Omega) \times W^{1,q}_{\Gamma_2}(\Omega), (su,tv) \ne (0,0)\}$$

and $I_{\lambda,\mu}$ is considered as a functional of four variables (s,u,t,v) in $\mathcal{Z} := \mathbb{R}\times W^{1,p}_{\Gamma_1}(\Omega)\times\mathbb{R}\times W^{1,q}_{\Gamma_2}(\Omega)$.

Definition 2.1. *Let λ and μ be two real parameters. A sequence $(u_n,v_n)\in W$ is a Palais-Smale sequence of the functional $I_{\lambda,\mu}$ if*

- *there exists $c\in\mathbb{R}$ such that* $\lim_{n\to+\infty} I_{\lambda,\mu}(u_n,v_n)=c$ (2.21)
- *$DI_{\lambda,\mu}(u_n,v_n)$ converges strongly in the dual W' of W* (2.22)

where $DI_{\lambda,\mu}(u_n,v_n)$ denotes the Gâteaux derivative of $I_{\lambda,\mu}$.

The last condition (2.22) implies that

$$D_1I_{\lambda,\mu}(u_n,v_n)(u_n) = \mathrm{o}\left(||u_n||_{p^*}\right) \tag{2.23}$$
$$D_2I_{\lambda,\mu}(u_n,v_n)(v_n) = \mathrm{o}\left(||v_n||_{q^*}\right), \tag{2.24}$$

where $D_1I_{\lambda,\mu}(u_n,v_n)$ (respectively $D_2I_{\lambda,\mu}(u_n,v_n)$) denotes the Gâteaux derivative of $I_{\lambda,\mu}$ with respect to its first (respectively second) variable.

We introduce the critical level corresponding to Problem (2.15) by

$$c^*(\lambda,\mu) := \inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w) + \inf_{w\in\mathcal{N}_{\lambda,\mu}\cup\{(0,0)\}} I_{\lambda,\mu}(w). \tag{2.25}$$

Then we have the following.

Theorem 2.2. *Let λ and μ be two real parameters and (u_n,v_n) be a Palais-Smale sequence of $I_{\lambda,\mu}$ such that*

$$c := \lim_{n\to+\infty} I_{\lambda,\mu}(u_n,v_n) < c^*(\lambda,\mu). \tag{2.26}$$

Then (u_n,v_n) is relatively compact.

Proof. Let λ and μ be two real parameters and (u_n,v_n) be a Palais-Smale sequence of $I_{\lambda,\mu}$ satisfying the condition (2.26). We claim that (u_n,v_n) is bounded in W. Indeed, on the one hand conditions (2.21), (2.23) and (2.24) can be rewritten as the following.

$$I_{\lambda,\mu}(u_n,v_n) = c + o_n(1) \tag{2.27}$$
$$P(u_n) - \lambda\int_\Omega f(x,u_n)u_n\,dx = R(u_n,v_n) + \mathrm{o}\left(||u_n||_{p^*}\right) \tag{2.28}$$
$$Q(v_n) - \mu\int_\Omega f(x,v_n)v_n\,dx = R(u_n,v_n) + \mathrm{o}\left(||v_n||_{q^*}\right). \tag{2.29}$$

Using (2.18), one gets

$$R(u_n,v_n) = \frac{\alpha+1}{p^*}\Big(P(u_n) - \lambda\int_\Omega f(x,u_n)u_n\Big) + \mathrm{o}\left(||u_n||_{p^*}\right)$$

$$+ \quad \frac{\beta+1}{q^*}\Big(Q(v_n) - \mu \int_\Omega g(x,v_n)v_n\Big) + \mathrm{o}\,(||v_n||_{q^*}). \quad (2.30)$$

Suppose that there is a subsequence, still denoted by (u_n, v_n), in W which is unbounded; *i.e.,* $||\nabla u_n||_p + ||\nabla v_n||_q$ tends to $+\infty$ as n goes to $+\infty$. If

$$\lim_{n\to+\infty} ||\nabla u_n||_p = +\infty,$$

then using (2.19) one has

$$\int_\Omega |f(x,u_n)u_n| = \mathrm{o}\,(P(u_n)), \quad \int_\Omega |F(x,u_n)| = \mathrm{o}\,(P(u_n)),$$

since (2.19) implies that, for every $\varepsilon > 0$, there exists $c_1(\varepsilon) > 0$ such that

$$|f(x,s)| \le \varepsilon |s|^{p^*-1} + c_1 \;\text{ and }\; |F(x,s)| \le \frac{\varepsilon}{p^*}|s|^{p^*} + c_1, \;\text{ a.e. } x \in \Omega, \;\forall\, s \in \mathbb{R}.$$

Similarly, if

$$\lim_{n\to+\infty} ||\nabla v_n||_q = +\infty,$$

then using (2.20) it follows that

$$\int_\Omega |g(x,v_n)v_n| = \mathrm{o}\,(Q(v_n)), \quad \int_\Omega |G(x,v_n)| = \mathrm{o}\,(Q(v_n)).$$

On the one hand, suppose that

$$\lim_{n\to+\infty} ||\nabla u_n||_p = \lim_{n\to+\infty} ||\nabla v_n||_q = +\infty.$$

Substituting (2.30) in (2.27), we obtain

$$\begin{aligned} c + \mathrm{o}_n(1) \quad &= \quad (\alpha+1)\left(\frac{1}{p} - \frac{1}{p^*} + \mathrm{o}\,(P(u_n))^{\frac{p^*-p}{p}}\right) P(u_n) \\ &+ \quad (\beta+1)\left(\frac{1}{q} - \frac{1}{q^*} + \mathrm{o}\,(Q(v_n))^{\frac{q^*-q}{q}}\right) Q(v_n) \;\longrightarrow_{n\to+\infty} +\infty, \end{aligned}$$

which can not hold true. On the other hand, suppose that

$$\lim_{n\to+\infty} ||\nabla u_n||_p = +\infty \;\text{ and the sequence }\; ||\nabla v_n||_q \;\text{ is bounded,}$$

then (2.28) implies that $R(u_n, v_n)$ is unbounded while (2.29) implies, on the contrary, that $R(u_n, v_n)$ is bounded. The case

$$\lim_{n\to+\infty} ||\nabla v_n||_q = +\infty \;\text{ and the sequence }\; ||\nabla u_n||_p \;\text{ is bounded,}$$

leads to a contradiction with the same argument, which achieves the claim. At this stage, we can assume, up to a subsequence, that

$$u_n \;\rightharpoonup\; u \;\text{ in }\; W^{1,p}_{\Gamma_1}(\Omega), \quad v_n \rightharpoonup v \;\text{ in }\; W^{1,q}_{\Gamma_2}(\Omega),$$

$$u_n \quad \to \quad u \;\text{ a.e. in }\; \Omega, \quad v_n \to v \;\text{ a.e. in }\; \Omega.$$

It is clear that $(u, v) \in \mathcal{N}_{\lambda,\mu} \cup \{(0, 0)\}$. Let us set

$$X_n = u_n - u \;\text{ and }\; Y_n = v_n - v.$$

Using again the growth conditions (2.19) and (2.20), we show easily that the functions, which are defined on $\Omega \times \mathbb{R}$: $(x, s) \mapsto sf(x, s)$, $(x, s) \mapsto sg(x, s)$, $(x, s) \mapsto F(x, s)$ and $(x, s) \mapsto G(x, s)$ satisfy the conditions of the Brézis-Lieb lemma [6]. Then, we have the decompositions

$$\begin{aligned}
\int_\Omega F(x, X_n) &= \int_\Omega F(x, u_n) - \int_\Omega F(x, u) + o_n(1), \\
\int_\Omega f(x, X_n)X_n &= \int_\Omega f(x, u_n)u_n - \int_\Omega f(x, u)u + o_n(1), \\
\int_\Omega G(x, Y_n) &= \int_\Omega G(x, v_n) - \int_\Omega G(x, v) + o_n(1), \\
\int_\Omega g(x, Y_n)Y_n &= \int_\Omega g(x, v_n)v_n - \int_\Omega g(x, v)v + o_n(1).
\end{aligned}$$

Moreover, let $\varepsilon > 0$; then there is $c_1(\varepsilon) > 0$ such that

$$\left| \int_\Omega f(x, X_n)X_n \, dx \right| \leq \varepsilon ||X_n||_{p^*}^{p^*} + c_1 ||X_n||_1.$$

Let C be a positive constant such that $||X_n||_{p^*}^{p^*} \leq C$. Since X_n converges to 0 in $L^1(\Omega)$, there exists $n_0(\varepsilon) \in \mathbb{N}$ satisfying $||X_n||_1 \leq \varepsilon / c_1$, for every $n \geq n_0(\varepsilon)$, thus

$$\left| \int_\Omega f(x, X_n)X_n \, dx \right| \leq \varepsilon(1 + C), \quad \forall n \geq n_0(\varepsilon).$$

In the same manner, writing $F(x, X_n) = \int_0^{X_n} f(x, s) \, ds$ and using the same arguments as above, we get

$$\int_\Omega F(x, X_n) = o_n(1) \;\text{ and }\; \int_\Omega f(x, X_n)X_n = o_n(1).$$

Similarly, it follows that

$$\int_\Omega G(x, Y_n) = o_n(1) \;\text{ and }\; \int_\Omega g(x, Y_n)Y_n = o_n(1).$$

Applying a slightly modified version of the Brézis-Lieb lemma [13], one has

$$R(X_n, Y_n) = R(u_n, v_n) - R(u, v) + o_n(1).$$

It follows that

$$P(X_n) - R(X_n, Y_n) = o_n(1), \quad Q(Y_n) - R(X_n, Y_n) = o_n(1),$$
$$I_{0,0}(X_n, Y_n) = c - I_{\lambda,\mu}(u, v) + o_n(1).$$

Notice that the Nehari manifold associated to $I_{0,0}$ is given by

$$\mathcal{N}_{0,0} = \left\{(s_0(u,v)u, t_0(u,v)v) : (u,v) \in W^{1,p}_{\Gamma_1}(\Omega) \times W^{1,q}_{\Gamma_2}(\Omega),\ u \not\equiv 0,\ v \not\equiv 0\right\},$$

where

$$s_0(u,v) = \Big[\frac{P(u)Q(v)^{\frac{r(\beta+1)}{q(\alpha+1)}}}{R(u,v)^{\frac{r}{\alpha+1}}}\Big]^{\frac{1}{r-p}}, \ \ t_0(u,v) = t(s_0(u,v)),$$

and

$$r = \frac{(\alpha+1)q}{q-(\beta+1)} > p, \quad t(s) = \left[\frac{R(u,v)}{Q(v)}\right]^{\frac{r}{q(\alpha+1)}} s^{\frac{r}{q}}.$$

Let ℓ be the common limit of $P(X_n)$, $Q(Y_n)$ and $R(X_n, Y_n)$. We claim that $\ell = 0$. By contradiction, suppose that $\ell \neq 0$, then on the one hand we get

$$\begin{aligned} I_{0,0}(s_0(X_n,Y_n)X_n, t_0(X_n,Y_n)Y_n) &= (\alpha+1)\Big(\frac{1}{p} - \frac{1}{r}\Big)K(X_n, Y_n) \\ &\geq \inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w), \end{aligned} \tag{2.31}$$

where

$$K(X_n, Y_n) = \Big[\frac{P(X_n)^{(\alpha+1)}Q(Y_n)^{(\beta+1)\frac{p}{q}}}{R(X_n, Y_n)^p}\Big]^{\frac{r}{(\alpha+1)(r-p)}}.$$

A direct computation shows that

$$\lim_{n\to+\infty} K(X_n, Y_n) = \ell,$$

therefore

$$\lim_{n\to+\infty} I_{0,0}(s_0(X_n,Y_n)X_n, t_0(X_n,Y_n)Y_n) = \ell(\alpha+1)\left(\frac{1}{p} - \frac{1}{r}\right).$$

On the other hand,

$$\lim_{n\to+\infty} I_{0,0}(X_n, Y_n) = \ell\Big(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\Big) = \ell(\alpha+1)\Big(\frac{1}{p} - \frac{1}{r}\Big).$$

Hence, we obtain

$$\ell(\alpha+1)\left(\frac{1}{p} - \frac{1}{r}\right) = c - I_{\lambda,\mu}(u, v),$$

and consequently

$$\begin{aligned} c &\geq \inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w) + I_{\lambda,\mu}(u,v) \\ &\geq \inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w) + \inf_{w\in\mathcal{N}_{\lambda,\mu}\cup\{(0,0)\}} I_{\lambda,\mu}(w). \end{aligned}$$

This leads to a contradiction with (2.26), thus $\ell = 0$, which achieves the proof. □

Remark 2.2. 1) In the scalar case, we obtain the analog of Theorem 2.2; the proof follows easily with the same arguments. We note here that if we consider the special case (1.1), direct computations show that

$$\inf_{w\in\mathcal{N}_0} I_0(w) = \frac{1}{N} S^{\frac{N}{2}} \quad \text{and} \quad \inf_{w\in\mathcal{N}_\lambda\cup\{0\}} I_\lambda(w) = 0,$$

which recovers the famous Brézis-Nirenberg condition (1.2).

2) It is clear that our condition (2.4) or (2.26) can be extended to a large class of quasilinear or semilinear differential operators: Leray-Lions type operators, fourth-order operators.

3) Using the Hölder inequality in the denominator $R(u,v)$, we get

$$\inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) \geq (\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right)\left[S_p S_q^{\frac{p(\beta+1)}{q(\alpha+1)}}\right]^{\frac{r}{r-p}}, \tag{2.32}$$

where S_p (respectively S_q) denotes the best Sobolev constant in the embedding $W^{1,p}_{\Gamma_1}(\Omega) \subset L^{p^*}(\Omega)$ (respectively $W^{1,q}_{\Gamma_2}(\Omega) \subset L^{q^*}(\Omega)$).

We end this note by the following interesting relation arising in the special case $p = q$ and $\Gamma_1 = \Gamma_2$.

Proposition 2.1. *Assume that $p = q > 1$. Then,*

$$\inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) = \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

Proof. In the special case $p = q$, direct computations give

$$p^* = \alpha+\beta+2 \quad \text{and} \quad (\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right) = \frac{p}{N-p}.$$

Then, using (2.32), we conclude that

$$\inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) \geq \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

On the other hand, let $(u_n) \subset W^{1,p}_{\Gamma_1}(\Omega)$ be a minimizing sequence of S_p. Then using the identity (2.31), we get

$$\inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w) \leq I_{0,0}(s_0(u_n,u_n)u_n, t_0(u_n,u_n)u_n)$$
$$= \frac{p}{N-p}\Big[\frac{||\nabla u_n||_p^p}{||u_n||_{p^*}^p}\Big]^{\frac{rp^*}{(\alpha+1)(r-p)}} = \frac{p}{N-p}\Big[\frac{||\nabla u_n||_p^p}{||u_n||_{p^*}^p}\Big]^{\frac{N}{p}}.$$

It is clear that the last quantity goes to $\frac{p}{N-p}S_p^{\frac{N}{p}}$ as $n+\infty$, which achieves the proof. □

Remark 2.3. For the sharpness of the critical level (2.26), we define the sequence $\psi_n(x) := \varphi(x)\Phi_{1/n}(x)$ as in (2.12). We consider then a Palais-Smale sequence (u_n, v_n) for $J_{\lambda,\mu}$ of level $\inf_{w\in\mathcal{N}_{\lambda,\mu}\cup\{(0,0)\}} I_{\lambda,\mu}(w)$. Following the same arguments developed in the scalar case and using Proposition 2.1, we prove that the sequence $(u_n+\psi_n, v_n+\psi_n)$ is a Palais-Smale sequence for $J_{\lambda,\mu}$ of level $c^*(\lambda,\mu)$ which can not be relatively compact in W. This implies the sharpness of the critical level formula (2.26).

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