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EXTENSIONS OF FIRST-ORDER PARTIAL DIFFERENTIAL EXPRESSIONS AND VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

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1. Introduction.

In this paper we introduce a new function space which is a complete metric space, a conditionally complete lattice, a vector space and a commutative ring. This space is specifically intended for extensions of nonlinear differential expressions. In particular, the problem of such extensionss appears if one needs to consider nondifferentiable or even noncontinuous functions as solutions of partial differential equations. The classical function space are not often well adapted for such extensions. For example, the continuous solutions obtained by the vanishing viscosity method may be also obtained by monotone extensions of the corresponding operators. However, there is not a common essential domain for the extensions by closure even for very natural families of operators. In the case of discontinuous viscosity solutions there is not an appropriate extension not only for families of operators but even for individual operators.

The space introduced in this paper and denoted by S possesses the following remarkable property: the subset C of continuous functions is dense in it both in the sense of its metric and in the sense of its order. In other words S is the completion of C in metric and order senses simultaneously. At the same time the metric induces uniform convergence on C. The simplest example of such a two-fold completion gives the couple $\mathbb{Q}, \mathbb{R}$ of rational and real numbers. But in the known function spaces containing continuous functions this property is not true. The discussed property of the two-fold density for the couple (C, S) implies some links between extensions by monotonicity and by closure. It allows us to obtain the existence and the unicity

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of viscosity solutions in S. Given the great number of existence and unicity results for viscosity solutions of Hamilton-Jacobi equations [1-3] we are not presented with this kind of problem but rather with the description of essential domains for the discussed extensions.

In Section 3 we obtain a common essential domain in S for Hamilton-Jacobi equations corresponding to the Lagrange problem of minimization of functionals that is not possible in classical function spaces. This domain is defined by a principal algebraic structural property of solutions of the Lagrange problem. In Section 4 we apply for these equations an approach based on the compactness (in S) which is usual in functional analysis.

Notation. Let $X \subseteq \mathbb{R}^n, X'$ be a dense subset in X and f be a bounded function from X' to $\mathbb{R}$. By $(x \to f(x) : x \in X' \subset X)^*$ we denote the upper semicontinuous (u.s.c.) hull of f on X, i.e. the function $x \to \limsup f(y)$, where $y \to x \in X, y \in X'$.

Respectively, $(x \to f(x) : x \in X' \subset X)_*$ is the lower semicontinuous (l.s.c.) hull of f on X. If f is defined on X then we use the simplified notation f^* for the u.s.c. hull, respectively f_* for the l.s.c. hull.

$\partial_+ f(x)$ (respectively $\partial_- f(x)$) denotes the super (respectively sub)-derivative of a function $f : X \to \mathbb{R}$ in $x \in X$.

In $\mathbb{R}^n < \cdot, \cdot >$ and $| \cdot |$ mean the usual Euclidean scalar product and the norm, $B(x, r)$ is the open ball in $\mathbb{R}^n$ centered in $x \in \mathbb{R}^n$. Let A be a metric space, $A' \subseteq A$ and $F : A' \to A$ be a map. We say that F is a preclosed operator in A A', if $\{y_i \to y, z_i \to y, Ay_i \to a, Az_i \to b\} \Rightarrow \{a = b\}$. The extension of a preclosed operator F in A from A' by closure is the operator with the domain $\Delta = \{y \in A : \exists (y_i)_i \subset A'$ such that $y_i \to y$ and the sequence Ay_i converges$\}$ which associates the element $Ay = \lim Ay_i$ to $y = \lim y_i, y \in \Delta$. It is a closed operator in A; i.e., $\{y_i \in \Delta, y_i \to y, Ay_i \to a\} \Rightarrow \{y \in \Delta$ and $Ay = a\}$.

2. THE SPACE S

In this paragraph we present a metric such that the completeness of the lattice of continuous functions by this metric becomes at the same time the order completeness. The corresponding abstract construction for inductive limits of lattices that are complete ones in both senses is given in [5].

Suppose that X is a compact subset in $\mathbb{R}^n$ that is the closure of its interior, $Lip_k(X)$ is the set of real-valued Lipschitzian functions on X with Lipschitz constant $k, Lip(X) = \bigcup_{k \in \mathbb{N}} Lip_k(X)$. We endow the set $Lip(X)$ with the metric ρ defined as follows : $\rho(f, g)$ equals to the Hausdorff distance in

$X \times \mathbb{R}$ between the graphs of f and g. (Recall that the Hausdorff distance between two compact sets A and B is

$$Inf\{\varepsilon > 0 : A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\},$$

where ϕ_ε is the ε-neighborhood of ϕ).

Definition 2.1. Consider the family of bounded real functions on X for any of which the set of continuity points is of the second Baire category in X. We say that two such functions φ and ψ are equivalent ($\varphi \sim \psi$), if φ coincides with ψ on some set of the second category in X. We denote the set of equivalence classes by $S(X)$.

In the set $S(X)$ introduce the following partial order : $f \leq g$, if $\varphi(x) \leq \psi(x)$ for some (hence for every) $\varphi \in f, \psi \in g$ on a subset of the second category in X.

Evidently the set $C(X)$ of continuous functions on X is a subset of $S(X)$ and $\leq$ induces the usual partial order on $C(X))(f \leq g$, if $f(x) \leq g(x)$ for any $x \in X)$.

For every $k \in \mathbb{N}$ the sublattice $Lip_k(X)$ is a conditionally complete one. This makes it possible to define projectors

$$T_k^- : S(X) \to Lip_k(X), T_k^+ : S(X) \to Lip_k(X)$$

by the rules

$$\begin{aligned} T_k^- f &= \sup\{\varphi \in Lip_k(X) : \varphi \leq f) \\ T_k^+ f &= \inf\{\varphi \in Lip_k(X) : \varphi \geq f). \end{aligned}$$

Let $f, g \in S(X)$. We define

$$s(f,g) = \sup_{k \in \mathbb{N}} \max(\rho(T_k^- f, T_k^- g), \rho(T_k^+ f, T_k^+ g)).$$

Theorem [5]. *The set $S(X)$ endowed with the metric $s(\cdot,\cdot)$ and with the introduced partial order is a complete metric space and a conditionally complete lattice. Moreover $(S(X), \leq)$ is the order completion of $(C(X), \leq)$ and $(S(X), s)$ is the metric completion of $(C(X), s)$. The convergence generated on $C(X)$ by the metric s is uniform convergence.*

Recall that a function $\varphi : X \to \mathbb{R}$ is said to be quasicontinuous, if for any open subsets $U \subset X$ and $V \subset \mathbb{R}$ the intersection $f^{-1}(V) \cap \mathcal{U}$ is either empty or contains a nonempty open subset (see [4], where many equivalent definitions are given).

Proposition 2.1. [6]. *Let $f \in S(X)$. Then the class f contains a unique lower semicontinuous quasicontinuous representative f_* and a unique upper* *semicontinuous quasicontinuous representative f^*. They are linked by the relations*

$$(f_*)^* = f^* \text{ and } (f^*)_* = f_* \tag{2.1}$$

(here $(\cdot)^*((\cdot)_*)$ denotes the upper (lower) semicontinuous hull). Conversely, any pair (f_*, f^*) of a lower semicontinuous and an upper semicontinuous function that satisfy relations (2.1) determines an element f in $S(X)$ for which f_* and f^* are representatives.

There are natural structures of a real vector space and a commutative ring in $S(X)$: if $f \in S, g \in S, \lambda \in \mathbb{R}$ we denote by $\lambda f + g$, respectively $f \cdot g$, the equivalence classes of $\lambda\varphi + \psi$, respectively of $\varphi \cdot \psi$, where $\varphi \in f, \psi \in g$ are some representatives. If $f \in S(X), x \in X$, then we denote the interval

$$[f_*(x), f^*(x)]$$

by $f(x)$ and we call it the value of f in x. $f(x)$ may be also defined as

$$[\lim_{\to x,} \inf_{y \in C_\varphi} \varphi(y), \lim_{y \to x,} \sup_{y \in C_\varphi} \varphi(y)],$$

where $\varphi \in f$ and C_φ is the subset of the second category of continuity points of φ.

The values of elements from $S(X)$ appear in the following auxiliary propositions.

Proposition 2.2. *Let f, g be two elements from $S(X), f \neq g$. Then there exist an open subset $\mathcal{U}$ in X and $\alpha > 0$ such that for all $x \in \mathcal{U}$, for all $y \in \mathcal{U}$, for all $a \in f(x)$, for all $b \in g(y)$, we have $\mid a - b \mid > \alpha$.*

Proof. Let $h \in S(X)$. By grh denote the following subset in $X \times \mathbb{R} : grf = \{(x, y) : x \in X, y \in h(x)\}$. Whenever $f \neq g$ then $clgrf \neq clgrg$ and the Hausdorff distance between $clgrf$ and $clgrg$ is positive. Hence there exists an open ball in $X \times \mathbb{R}$ with center $(x, f_*(x))$ which does not contain any point of $clgrg$. Since f_* is a quasicontinuous function, there exists an open ball $\mathcal{U}$ in X such that every $a \in f(x')$ for $x' \in \mathcal{U}$ is sufficiently close to $f_*(x)$. □

Proposition 2.3. *Let $\{f_i\}$ be a sequence of elements from $S(X)$ that converges to $f \in S(X)$. Then for every open subset $\mathcal{U} \subset X$ and for every $\varepsilon > 0$ there exist an open subset $V \subset \mathcal{U}$ and a natural number N such that for any $y \in V, n \geq N, a_n \in f_n(y)$ and $a \in f(y)$ the inequality $|a_n - a| < \varepsilon$ is valid.*

Proof. Suppose by contradiction that V does not exist. This means that for every open subset $\mathcal{U} \in X$ there exists $\varepsilon > 0$ for which the subset $\Omega_N = \{y \in \mathcal{U} : \exists n > N, \exists a_n \in f_n(y), \exists a \in f(y)$ such that $|a_n - a| > \varepsilon\}$ is dense in $\mathcal{U}$ for every $N \in \mathbb{N}$.

Let $x \in \mathcal{U}$ be a continuity point of f; i.e., $f_*(x) = f^*(x)$. Let $B(x, \delta)$ be an open ball in $\mathcal{U}$ such that $|a - a'| < \frac{\varepsilon}{2}$ for any $v \in B(x, \delta), w \in B(x, \delta), a \in f(v), a' \in f(w)$.

Let $\eta = \min(\frac{\varepsilon}{2}, \frac{\delta}{2})$ and $N_1 \in \mathbb{N}$ be a number such that $s(f_n, f) < \eta$ for any $n > N_1$. The density of the subset Ω_{N_1} implies that there exists y belonging to $B(x, \frac{\delta}{2}) \cap \Omega_{N_1}$. But it contradicts the inequality

$$\max(h(T_k^- f_n, T_k^- f), h(T_k^+ f_n, T_k^+ f)) < \eta$$

for a sufficiently large $k \in \mathbb{N}$ and any $n \geq N_1$. □

Proposition 2.4. *Let $(f_i)_i$ be a sequence in $S(X), (x_i)_i$ be a sequence in $X, \xi_i \in f_i(x_i)$. If $f_i \to f$ in $S(X), x_i \to x, \xi_i \to \xi$ as $i \to \infty$, then $\xi \in f(x)$.*

Proof. This follows from the inequalities

$$\liminf f_{i*}(x_i) \geq f_*(x), \quad \lim_{i \to \infty} \sup f_i^*(x_i) \leq f^*(x),$$

and from the equality $f(x) = [f_*(x), f^*(x)]$. □

3. The Lagrange problem

Numerous problems may be reduced to the study of the following evolutionary process

$$(U_t f)(x) = Inf(\int_0^t L(\xi(\tau), \dot{\xi}(\tau)) d\tau + f(\xi(t))), \tag{3.1}$$

where Inf is taken over the set of absolutely continuous trajectories $\xi(\cdot)$ in X satisfying $\xi(0) = x$. Let H be the Legendre transform of the function L in (3.1). The following partial differential equations

$$\frac{\partial y}{\partial t} = H(x, \text{ grad } y) \text{ or } H(x, \text{ grad } y) = h(x) \tag{3.2}$$

are associated with the process (3.1).

Recall [1.3] that the viscosity solution (which is discontinuous in general) of the equation

$$F(\xi, y(\xi), \text{ grad } y(\xi)) = h(\xi) \tag{3.3}$$

is a function $y : \Omega \to \mathbb{R}$ such that y_* is a subsolution of (3.3) (i.e., for all $\xi \in \Omega$ such that $\partial_- y_*(\xi) \neq \phi$, for all $a \in \partial_- y_*(\xi)$ the inequality

$$F_*(\xi, y_*(\xi), a) \leq h_*(\xi)$$

holds) and y^* is a supersolution of (3.3) (i.e., for all $\xi \in X$ such that $\partial_+ y^*(\xi) \neq \phi$, for all $f \in \partial_+ y^*(\xi)$ the inequality

$$F^*(\xi, y^*(\xi), b) \geq h^*(\xi)$$

holds).

Let us recall here that the viscosity solutions of the left equation (3.2), where $y = (t, x)$, give the functions defined by formula (3.1).

In view of Proposition 2.1 any $f \in S(\Omega)$ corresponds to the couple (f_*, f^*) of semi-continuous functions on Ω. Let us replace couples (y_*, y^*) in the cited definition of viscosity solutions with $y = (y_*, y^*) \in S(\Omega)$. Then it is quite correct to say that $y \in S(\Omega)$ is (or is not) a viscosity solution of one of equations in (3.2).

Note that the Hamiltonian H, which is the Legendre transform of the function L, is convex in the second argument. The form (3.1) of desired solutions of equations (3.2) implies the following structural property of the set of solutions : if y_1 and y_2 are two solutions, then $\min(y_1, y_2)$ is also a solution of (3.2). We shall consider in the space $S(X)$ the partial differential operator defined by

$$(\mathcal{H}f)(x) = H(x, \text{ grad } f(x)).$$

This operator is defined, of course, on the set $C^1(X)$ of continuously differentiable functions on X. Moreover if f is a piecewise continuously differentiable function (even discontinuous) then $\mathcal{H}f$, being a piecewise continuous function, defines an element from $S(X)$.

The mentioned structural property of the set of solutions of (2.2) gives rise to the next condition : the domain of a desired extension must be stable with respect to the operation min. The minimal set possessing this property and containing $C^1(X)$ is the set of functions of the type

$$min(\varphi_1, \ldots, \varphi_k), \tag{3.4}$$

where $k \in \mathbb{N}$ and all functions $\varphi_1, \ldots, \varphi_k$ are from $C^1(X)$.

Definition 3.1. We denote by $C^1_{min}(X)$ the set of functions of the type (3.4).

Theorem 3.1. *Let $H : X \times \mathbb{R}^n \to \mathbb{R}$ be a uniformly continuous function convex in the second argument. Then the partial differential operator*

$$f \to H(\cdot, \text{ grad } f(\cdot))$$

from $S(X)$ to $S(X)$ with the domain $C^1_{min}(X)$ is preclosed. Let $\mathcal{H}$ be its extension by closure from $C^1_{min}(X)$ and Δ be the domain of this extension. Then every solution $f \in \Delta$ of the equation $\mathcal{H}y = h, h \in S(X)$ is a viscosity solution of the equation

$$H(x, \text{ grad } y(x)) = h(x). \tag{3.5}$$

Conversely, every viscosity solution of (3.5) *from* $S(X)$ *belongs to* Δ *and satisfies* $\mathcal{H}y = h$.

Before the proof of Theorem 3.1, we prove some auxiliary propositions. Let us introduce the following two metrics in $S(X)$: $m_-(f,g)$ equals the Hausdorff distance between the epigraphs of f_* and g_*; $m_+(f,g)$ equals the Hausdorff distance between the hypographs of f^* and g^*.

Proposition 3.1. *Let* $(f_i)_i$ *be a sequence of elements of* $S(X)$ *converging to* $f \in S(X)$ *in both the metrics* m_+ *and* m_-. *Then* $(f_i)_i$ *converges to* f *in the metric* s *on* $S(X)$.

Proof. Suppose by contradiction that the sequence $(f_i)_i$ from Proposition 3.1 does not converge in the metric s. Then there exist a sequence $(n_i)_i, n_i \in \mathbb{N}$, a point $x \in X, \delta > 0$ and a number $b \in \mathbb{R}$ such that in the δ-neighborhood $\mathcal{U}$ of the point x we have either $T^-_{n_i} f < b - \delta$ and simultaneously $T^-_{n_i} f_i > b + \delta$ or $T^+_{n_i} f > b + \delta$ and simultaneously $T^-_{n_i} f_i < b + \delta$ or these two cases at the same time.

We consider the first case; the second one may be treated similarly and the third one is included in one of the preceding cases.

From the convergence of $(f_i)_i$ to f in the metric m_- it follows that for every n the sequence $(T^-_n f_i)_i$ converges to $T^*_n f$ in the metric m_-. Then our supposition by contradiction implies that $n_i \to \infty$ as $i \to \infty$.

According to Proposition 2.3, $f \in S'(X)$ implies that there are a number $N \in \mathbb{N}$ and an open subset V in the δ-neighborhood of x such that for all $y \in V$, for all $i > N$,

$$|f_{i*}(y) - f_*(x)| < \delta$$

is valid The inequality $T^-_{n_i} f_i > b + \delta$ in the open subset $\mathcal{U}$ implies that in some open neighborhood V_1 of the point x the inequality $f_{i*} > b + \delta$ takes place for some sufficiently large i.

Since the sequence $(f_{i*})_i$ converges to f_* in the metric m_-, there is a point $x' \in V_1$ for which the inequality

$$f_*(x') > b + \frac{\delta}{2}$$

holds. For sufficiently large i and consequently sufficiently large n_i, this inequality contradicts the suppositions $T^-_{n_i} f < b - \delta$ in the δ-neighborhood of the point x. Hence $(f_i)_i$ converges to $f \in S(X)$. □

We omit the prove of the following proposition because it is a particular case of Proposition 8.1 from [3].

Proposition 3.2. *Suppose that* $f = (f_*, f^*) \in S(X)$, $\varepsilon > 0$, $x \in X$, $\partial_- f_*(x) \neq \phi$, $a \in \partial_- f_*(x)$. *Then there exists a* $\delta > 0$ *such that for every* $g = (g_*, g^*) \in S(X)$ *satisfying* $m_-(f, g) < \delta$, *there exists* $x' \in X, \partial_- g_*(x') \neq \phi, a' \in \partial_- g_*(x')$ *for which the inequalities*

$$||x - x'|| < \varepsilon, | \ f_*(x) - g_*(x') \ | < \varepsilon, ||a - a'|| < \varepsilon$$

hold. The corresponding proposition is true, if one change f_*, g_* *to* f^*, g^*, $\partial_- f_*$, $\partial_- g_*$ *to* $\partial_+ f^*, \partial_+ g^*$ *and* m_- *to* m_+.

The following proposition concerns Hamiltonians that are more general than in Theorem 3.1.

Proposition 3.3. *Let* $H : X \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *be a continuous function,* $f \in S(X, \mathbb{R})$. *Define*

$$\mathcal{D}_- f \quad = (x \to \inf_{a \in \partial_- f_*(x)} H(x, f_*(x), a) : x \in \{x : \partial_- f_*(x) \neq \phi\} \subset X)_*$$
$$\mathcal{D}_+ f \quad = (x \to \sup_{b \in \partial_+ f^*(x)} H(x, f^*(x), b) : x \in \{x : \partial_+ f^*(x) \neq \phi\} \subset X)^*.$$

Let Δ *be the subset of* $S(X)$ *consisting of all* $f = (f_*, f^*)$ *such that* $\mathcal{D}_\pm f$ *are bounded functions and the couple* $(\mathcal{D}_- f, \mathcal{D}_+ f)$ *satisfies relation (2.1) (hence the couple* $(\mathcal{D}_- f, \mathcal{D}_+ f)$ *defines an element from* $S(X)$*). Then the operator*

$$f \to \mathcal{D} f = (\mathcal{D}_- f, \mathcal{D}_+ f)$$

in $S(X)$ *with the domain* Δ *is closed.*

(We say that this operator $\mathcal{D}$ is the *viscose extension* of the differential expression

$$y \to H(x, y(x), \ \text{grad} \ y(x)))$$

in $S(X)$.

Proof. Suppose that there is a sequence $f_i \to f$ such that $f_i \in \Delta, \mathcal{D} f_i \to F$.

From Proposition 3.2 and the continuity of H it follows that

i) for all $\varepsilon > 0, (x, a) \in X \times \mathbb{R}^n$ such that $\partial_- f_*(x) \neq \phi$ and $a \in \partial_- f_*(x)$ there exists $N \in \mathbb{N}$ and an open subset $\mathcal{U}$ with the following properties :

$$x \in \mathcal{U}, \forall n > N, \exists (y, b) \in \mathcal{U} \times \mathbb{R}^n$$

such that

$$\partial_- f_{n*}(y) \neq \phi \ \ \exists b \in \partial f_{n*}(y)$$

for which the inequality

$$|H(y, f_{n*}(y), b) - H(x, f_*(x), a| < \varepsilon$$

holds.

We argue by reductio ad absurdum to prove that $f \in \delta$ and $F = \mathcal{D} f$. Suppose that $F_* \neq \mathcal{D}_- f$. Let x' be a point where $F_*(x') \neq \mathcal{D}_- f(x')$. Two cases are possible. In the first one we have the inequality $F_*(x') < \mathcal{D}_f - f(x')$. Both functions F_* and $\mathcal{D}_- f$ are lower semicontinuous, moreover F_* is quasicontinuous (see Proposition 2.1). Then there exist an open subset V in $X, x' \in V$ and $\alpha > 0$ such that

$$(\mathcal{D}_- f)(x) > F_*(y) + \alpha$$

for every x and y from V.

From the equality $F = \lim \mathcal{D} f_n$ and from the definition of $\mathcal{D}_- f$ it follows that there exist $(x, a) \in V \times \mathbb{R}^n, \partial_- f_*(x) \neq \phi$ and $a \in \partial_- f_*(x)$ such that for all $N \in \mathbb{N}$, there exists $n > N$ with the following property:

for every $(y, b) \in V \times \mathbb{R}^n$, with $\partial_- f_{n*}(y) \neq \phi, b \in \partial_- f_{n*}(y)$, we have the inequality

$$|H(y, f_{n*}(y), b) - H(x, f_*(x), a)| > \frac{\alpha}{2}.$$

The last statement is the negation of the statement i). Consider the second case when

$$F_*(x') > \mathcal{D}_- f(x').$$

The lower semicontinuity of F_* and of $\mathcal{D}_- f$ and the quasicontinuity of F_* imply the existence of an open subset V in $X (x' \in ClV)$ and of $\alpha > 0$ such that

$$F_*(y) > \mathcal{D}_- f(x) + \alpha$$

for every $y \in V$ and for some $x \in V$, where $(\partial_- f_*)(x) \neq \phi$.

From the equality $F = \lim \mathcal{D} f_n$ and the definition of $\mathcal{D}_- f$ it follows that for every $N \in \mathbb{N}$ and $(x, a) \in V \times \mathbb{R}^n$, where $a \in \partial_- f_*(x)$ there exists $n > N$ with the following property:

for every $(y, b) \in V \times \mathbb{R}^n$ such that $\partial_- f_{n*}(y) \neq \phi$ and $b \in \partial_- f_{n*}(y)$ we have the inequality

$$|H(x, f_*(x), a) - H(y, f_{n*}(y), b)| > \frac{\alpha}{2}.$$

This contradicts the statement i).

Then $F_* = \mathcal{D}_- f$. In the same way we can prove that $F^* = \mathcal{D}^+ f$. Because of $(F_*, F^*) \in S$, we obtain that $\mathcal{D} f = (\mathcal{D}_- f, \mathcal{D}_+ f) \in S, f \in \Delta$. □

Remark 3.1. From the definition of the operator $\mathcal{D}$ it follows that every solution (in $S(X)$) of the equation $\mathcal{D} y = h$, where $h \in S(X)$, is a viscosity solution (in the sense of [1-3]) of the equation

$$H(x, y(x),\ \text{grad}\ y(x)) = h(x). \tag{3.6}$$

The converse statement is true for Hamiltonians such that the viscosity solutions of the corresponding Hamilton-Jacobi equation define an operator: if f is a viscosity solution of (3.6) and is not a viscosity solution of any equation

$$H(x, y(x), \text{ grad } y(x)) = h(x) + \alpha$$

with $\alpha \in \mathbb{R}, \alpha \neq 0$, then $f \in S(X)$ implies $f \in \Delta$ and $\mathcal{D}f = h$.

This justifies our notation of $\mathcal{D}$ as "the viscose extension".

Proof of Theorem 3.1. It is sufficient to choose for every $f \in \Delta$ a sequence $(\varphi_n)_n, \varphi_n \in C^1_{\min}$ such that $\varphi_n \to f$ and $\mathcal{D}\varphi_n \to (\mathcal{D}_- f, \mathcal{D}_+ f)$, where both limits are in $S(X)$.

Let $f \in \Delta$ and put

$$Y_1 = \{x \in X : \partial_- f_*(x) \neq \phi\}, \qquad Y_2 = \{x \in X : \partial_+ f^*(x) \neq \phi\}.$$

For every $n \in \mathbb{N}$, let us choose a finite $\frac{1}{n^2}$-net $\{\alpha_{in} \mid i = 1, \dots, m_n\}$ in X, i.e. X belongs to the union of balls of radius $\frac{1}{n^2}$ centered at α_{in}. Let us put

$$\varphi_n(x) = \min_{1 \leq i \leq m_n} (f_*(\alpha_{in}) + n||x - \alpha_{in}||^2).$$

For every n, the function φ_n belongs to $C^1_{\min}(X)$ and $\varphi_n \to f$ in $S(X)$.

Let X_n be the set of points of differentiability of the function φ_n.

The definition of the subdifferential, the construction of the net $\{\alpha_{in}\}$, Proposition 3.2, and the uniform continuity of H imply the following properties:

i) for all $x \in Y_1, a \in \partial_- f(x), \mu > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, there exists $x' \in X_n$ for which the inequalities

$$||x - x'|| < \mu \text{ and } |H(x, a) - \mathcal{D}\varphi_n(x')| < \mu$$

hold ;

ii) for all $\mu > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N, x' \in X_n$ there exists $x \in Y_1, a \in \partial_- f(x)$ for which the inequalities

$$||x - x'|| < \mu \text{ and } |H(x, a) - \mathcal{D}\varphi_n(x')| < \mu$$

hold.

iii) We want to prove the following property:

for all $\mu > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N, x' \in X_n$ there exists $x \in Y_2, b \in \partial_+ f(x)$ for which the inequalities

$$||x - x'|| < \mu \text{ and } \mathcal{D}\varphi_n(x') \leq H(x, b) + \mu$$

hold.

iv) The graph of φ_n is a union of pieces of parabolas of the type $\wedge_{m,\ell,n} : w \to m + n||w - \ell||^2$. Let $x' \in X_n$ and $\wedge_{m_0,\ell_0,n}$ be such that $(x', \varphi_n(x'))$ belongs to it. If in some neighborhood of the point $\ell_0 \in X$ the inequality

$$\wedge_{m_0,\ell_0,n} \geq f^*$$

is true, then grad $\varphi_n(x') \in \partial_+ f^*(\ell_0)$ and as a consequence we have

$$\begin{aligned} H(x', \text{grad } \varphi_n(x')) = \mathcal{D}\varphi_n(x') &\leq H(\ell_0, \text{grad } \varphi_n(x')) + \mu_n \\ &\leq \sup_{b \in \partial_+ f^*(\ell_0)} H(\ell_0, b) + \mu_n \end{aligned}$$

for some μ_n. Noting that $\mu_n \to 0$ as $n \to \infty$ we can put $x = \ell_0$ in iii). If it is not the case we have the following alternative : either $\wedge_{m',\ell',n} \geq f^*$ and $\wedge_{m',\ell',n} = f^*(z_n)$ at some point z_n sufficiently close to the point x' and for some pair (m', ℓ') sufficiently close to the pair (m_0, ℓ_0) ; or it is not true at any point z_n sufficiently close to the point x'.

v) In the first case, one can repeat the preceding argument and obtain the property iii) with $x = z_n$ instead of $x = \ell_0$.

vi) In the second case choose the vectors $\eta_1, \ldots, \eta_k$ from $\mathbb{R}^n$ such that their convex hull Conv$(\eta_1, \ldots, \eta_k)$ contains 0. For every fixed η_i consider the family of functions $\wedge_{m,\ell_0+\lambda\eta_i,n}$, where $m \in \mathbb{R}$ and $\ell_0 + \lambda\eta_i \in X$. If m is sufficiently large, then $\wedge_{m,\ell_0+\lambda\eta_i,n} > f^*$ in X. Letting m tend to m_0 $(m > m_0)$ and λ to 0 $(\lambda > 0)$ we have for every n a pair (m_n, λ_n) and a point $c_i \in X$ such that $\wedge_{m_i,\ell_0+\lambda_i\eta_i,n}(c_i) = f^*(c_i)$ and the inequality $\wedge_{m_i,\ell_0+\lambda_i\eta_i,n} \geq f^*$ is satisfied in some neighborhood of the point c_i. Let us assume $\xi_i =$ grad $\wedge_{m_i,\ell_0+\lambda_i\eta_i,n}(x_n)$. Taking into account the choice of $\eta_1, \ldots, \eta_k$ we obtain that

$$\text{grad } \varphi_n(x_n) \in \text{ Conv}(\xi_1, \ldots, \xi_k).$$

Hence, because of the uniform continuity of the function H and of its convexity in the second argument, there are the next inequalities

$$\begin{aligned} H(x_n, \text{grad } \varphi_n(x_n)) &\leq \max_{1 \leq i \leq k} H(c_i, \xi_i) + \frac{\mu}{2} = H(c_{i_0}, \xi_{i_0} + \frac{\mu}{2} \\ &\leq \sup_{b \in \partial_+ f^*(c_{i_0})} H(c_{i_0}, b) + \mu \end{aligned}$$

for fixed μ and for sufficiently large $n \in \mathbb{N}$. So iii) is true for $x = c_{i_0}$. The properties i) and ii) imply that the sequence $(\mathcal{D}\varphi_n)_n$ converges to $\mathcal{D}f$ in the metric m_- on $S(X)$. The property iii) implies that $(\mathcal{D}\varphi_n)_n$ converges to $\mathcal{D}f$ in the metric m_+. In view of Proposition 3.1 the sequence $(\mathcal{D}\varphi_n)_n$ converges to $\mathcal{D}f$ in $S(X)$ (i.e., in the metric s). This proves the theorem. $\square$

4. Spaces with graph metric

Let $H : X \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function (Hamiltonian), $\mathcal{D}$ be an operator in $S(X)$ with the domain Δ that is the viscose extension (as in Proposition 3.3) of an operator $f \to H(\cdot, \mathrm{grad}\, f(\cdot))$. Introduce in Δ the following "graph metric" $s_H(f,g) = s(f,g) + s(\mathcal{D}f, \mathcal{D}g)$. Since $\mathcal{D}$ is a closed operator (Proposition 3.1), Δ endowed with this metric becomes a complete metric space. Denote this space by $S_H(X)$. We say, as usual, that the inclusion of $S_H(X)$ in $S(X)$ is compact, if the image of any bounded subset of $S_H(X)$ is a precompact subset of $S(X)$.

Theorem 4.1. *Let $H : X \times \mathbb{R}^n \to \mathbb{R}$ be a uniformly continuous Hamiltonian convex in the second argument that satisfies in the first argument the next Lipschitz condition :*

$$\exists C \in \mathbb{R} \text{ such that } \forall p \in \mathbb{R}^n, \forall (x', x'') \in X \times X$$

the inequality

$$|H(x', p) - H(x'', p)| \leq C(1 + ||p||)||x' - x''||$$

is valid. Assume also that the Hamiltonian H satisfies the following condition: for all $x \in X$, $p \in \mathbb{R}^n$, $p \neq 0$ $q \in \mathbb{R}^n$, the function $\lambda \to H(x, \lambda p + q)$ from $\mathbb{R}$ to $\mathbb{R}$ is not bounded. Then the inclusion of $S_H(X)$ in $S(X)$ is compact.

Proof. First, we remark that if A is a compact subset of $S(X)$, then the following property holds: for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $f \in A$ we have $m_-(T_n^- f, f) < \varepsilon$, $m_+(T_n^+ f, f) < \varepsilon$ (here $m_-(g,f)$ (respectively $m_+(g,f)$) is the Hausdorff distance between epigraphs (respectively hypographs) of g_* and f_* (respectively of g^* and f^*)). In fact, let $\varepsilon > 0$ and $\{f^{(i)} : i = 1, \dots, m_i\}$ be a finite $\dfrac{\varepsilon}{3}$ net : $A \subset \bigcup_i B(f^{(i)} \frac{\varepsilon}{3})$. Let $f \in B(f^{(i)}, \frac{\varepsilon}{3})$.

Since for every $i = 1, \dots, m_i, T_n^- f^{(i)} \to f^{(i)}$ in $S(X)$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $m_-(T_n^- f^{(i)}, f^{(i)}) \leq s(T_n^- f^{(i)}, f^{(i)}) < \frac{\varepsilon}{3}$ for any $n > N$. Then

$$m_-(T_n^- f, f) \leq m_-(T_n^- f, T_n^- f^{(i)}) + m_-(T_n^- f^{(i)}, f^{(i)}) + m_-(f^{(i)}, f) \leq \varepsilon$$

if $n > N$. The same reasons are valid for m_+.

To prove the theorem we argue by reductio ad absurdum. Suppose that A is a closed ball in $S_H(X)$ and that its image in $S(X)$ is not a precompact subset. By the previous property and by Theorem 4.1 there exist a sequence $(f_n)_n$ of functions from $C^1_{\min}(X) \cap A$ and $\varepsilon_0 > 0$ such that the inequality $\max(m_-(T_n^- f_n, f_n), m_+(T_n^+ f_n, f_n)) > \varepsilon_0$ is valid for any $n \in \mathbb{N}$.

We can suppose that $m_-(T_n^- f_n, f_n) > \varepsilon_0$ (if it is not the case, then $m_+(T_n^+ f_n, f_n) > \varepsilon_0$ and the corresponding reasonings are the same ones). By the definition of the metric m_- it means that there are sequences $(\eta_i > 0)_i$ and $(y_i)_i$ of reals, $(x_i = (x_{i1}, \ldots, x_{in}))_i$ of points from X $\lim_{i\to\infty} \eta_i = 0, \eta_i$-net in

$$A_i = \prod_{j=1}^{n} [x_{ij} - \varepsilon_0, x_{ij} + \varepsilon_0] \subset X$$

consisting of points $\{w_{i1}, \ldots, w_{ik_i}\}$ such that $f_i(x_i) > y_i + \frac{\varepsilon_0}{2}$ and $f_i(w_{ij}) < y_i - \frac{\varepsilon_0}{2}, j = 1, \ldots, k_i$.

Passing on to converging sequences and reducing ε_0 (if it is necessary), one can assume that the sequence $(x_i)_i$ converges to some $\overline{x} \in X$ and that the sequence of sets $(A_i)_i$ and the sequence $(y_i)_i$ become stabilized for sufficiently large $i \in \mathbb{N} :\ A_i = A, y_i = y$.

Any function f_i belongs to $C^1_{\min}(X)$. Hence we have that for all $L \in \mathbb{R}, \xi \in \mathbb{R}^n, ||\xi|| = 1, \lambda > 0, N \in \mathbb{N}$ there exists $n > N, \xi'_n \in \mathbb{R}^n, ||\xi'_n|| = 1, x'_n \in X$, where x'_n is a point of differentiability of f_n, such that

$$||\xi'_n - \xi|| < \lambda, ||x - x'_n|| < \lambda, < \text{grad } f_n(x'_n), \xi'_n > > L. \tag{4.1}$$

The hypotheses on the Hamiltonian in the theorem imply that for every $M > 0$ and $x_0 \in X$ there are a neighborhood $\mathcal{U}$ of the point x_0, a strictly convex cone K in $\mathbb{R}^n$ and $p_0 \in \mathbb{R}^n$ such that for all $y \in \mathcal{U}$ the following inclusion

$$\{p : |H(y, p)| < M\} \subset K + p_0$$

is true.

Hence there is a vector $\zeta \in \mathbb{R}^n, ||\zeta|| = 1$, with the following property : there exists$\nu > 0$ such that for every vector $\zeta' \in \mathbb{R}^n, ||zeta'|| = 1$, the inequality $||\zeta - \zeta'|| < \nu$ implies the inequality $< v - p_0, \zeta' >\leq 0$ for every $v \in K$.

The functions f_i belong to the bounded ball A in $S_H(X)$, hence there exists $M > 0$ such that for all $n \in \mathbb{N}$, the inequality

$$|H(x, \text{grad } f_i(x))| < M$$

is valid in any point x of differentiability of f_n.

Using this M we shall construct a neighborhood $\mathcal{U}$ of the point x, a cone K, p_0 and ζ as has been described above. $B = \{p \in \mathbb{R}^n : p = \text{grad } f_n(w)$, where w is a point of differentiability of $f_n, w \in \mathcal{U}, n \in \mathbb{N}\} \subset K + p_0$. Hence for any p from the subset B and for any ζ' sufficiently close to ζ the inequality $< p - p_0, \zeta' >< 0$ holds.

The last inequality contradicts the inequality (4.1) with $\xi = \zeta$ which can be rewritten in the form

$$< \text{grad } f_n(x'_n), \zeta'_n > \; > \; < p_0, \zeta' >= L.$$

The theorem is proved. □

The following application of the compactness is usual in functional analysis.

Corollary 4.1. *Let a Hamiltonian H satisfy the conditions of Theorem* 4.1, *$\mathcal{D}$ be the viscose extension of the operator $f \to H(\cdot, grad\ f(\cdot)), \Delta$ be its domain and $\wedge$ be a closed bounded subset in $S(X)$. Then*:

i) for all $h \in S(X)$ the subset of all solutions from $\wedge$ of the equation $\mathcal{D}y = h$ is compact;

ii) the image $\mathcal{D}(\wedge \cap \Delta)$ of the subset $\wedge \cap \Delta$ is a closed subset in $S(X)$.

Proof. i) Let $M \in \mathbb{R}$ be such that $s(f, g) \leq M$ for any f and g from $\wedge$. If $\mathcal{D}f = \mathcal{D}g$ then $s_H(f, g) = s(f, g) \leq M$. The subset $\{f : \mathcal{D}|f = h\} \cap \wedge$ is closed because $\mathcal{D}$ is a closed operator in $S(X)$. Hence this subset is a compact subset by virtue of Theorem 4.1.

ii) Let $(h_i)_i$ be a sequence converging to $h, h_i \in \mathcal{D}(\wedge \cap \Delta)$. Let $h_i = \mathcal{D}f_i$. The subset $\{s_H(f_i, f_j) : (i, j) \in \mathbb{N}^2\} \subset \mathbb{R}$ is bounded. Hence, by virtue of Theorem 4.1 the subset $\{f_i : i \in \mathbb{N}\}$ is precompact in $S(X)$. Let us choose a converging subsequence $(f_{i_j})_j$. Then $h = \mathcal{D}(\lim_{j\to\infty} f_{i_j})$, because $\mathcal{D}$ is a closed operator. □

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