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ON A CLASS OF CRITICAL HEAT EQUATIONS WITH AN INVERSE SQUARE POTENTIAL

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Abstract. In this paper, we study a class of parabolic equations with critical Sobolev exponents and Hardy terms. Using Moser-type iteration, we characterize the asymptotic behavior of solutions at singular points. By means of critical point theory and the potential well method, we prove both global existence and finite-time blow-up depending on the initial datum.

1. Introduction and main results

Let $\Omega \subset R^N (N \geq 3)$ be an open bounded domain with smooth boundary $\partial\Omega,\ 0 \in \Omega,\ 2^* = \frac{2N}{N-2}$. We are concerned with the following semilinear parabolic problem

$$\begin{cases} u_t - \Delta u - \mu \frac{u}{|x|^2} = \lambda u + |u|^{2^*-2}u & in \ \ \Omega \times (0,T), \\ u(x,t) = 0 & on \ \ \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) & in \ \ \Omega, \end{cases} \tag{1.1}$$

where $\lambda \geq 0,\ 0 \leq \mu < \bar{\mu} = (\frac{N-2}{2})^2$ and $u_0 \in H_0^1(\Omega)$.

Let X be a Banach space, endowed with the norm $\|\cdot\|_X$. An operator A in X is $m-dissipative$ if

(i) $\|u - \lambda Au\|_X \geq \|u\|_X$ for all $u \in D(A)$ and all $\lambda > 0$;
(ii) for all $\lambda > 0$ and all $f \in X$, there exists $u \in D(A)$ such that $u - \lambda Au = f$, where $D(A)$ denotes the domain of definition of A in X.

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Set $A = -\Delta - \frac{\mu}{|x|^2}$ $(0 \leq \mu < \bar{\mu})$. Then by means of the Hille-Yosida-Phillips theorem, the linear operator generates an analytic semigroup of bounded linear operators $\{e^{-tA}\}_{t\geq 0}$ in $X = L^2(\Omega)$. Moreover, for $\beta \geq 0$, we denote by X^β the fractional powers of A.

$u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ is said to be a weak solution of problem (1.1) if u satisfies

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}(|u(s)|^{2^*-2}u(s) + \lambda u(s))ds.$$

In a remarkable paper, P. Baras and A. Goldstein [5] considered the following particular case (see [1, 5, 24] for more details and extensions):

$$\begin{cases} u_t - \Delta u = \mu \frac{u}{|x|^2} & in \;\; \Omega \times (0,\infty), \\ u(x,t) = 0 & on \;\; \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x) \in L^2(\Omega) & in \;\; \Omega, \end{cases} \tag{1.2}$$

and proved that

(i) If $\mu \leq \bar{\mu}$, then problem (1.2) has a solution.

(ii) If $\mu > \bar{\mu}$, then problem (1.2) has no local solution for any $u_0 > 0$.

There is a great literature on the existence of global solutions and blow-up for the following problem (see [16, 17, 21, 22] for examples):

$$\begin{cases} u_t - \Delta u = |u|^{p-2}u & in \;\; \Omega \times (0,T), \\ u(x,t) = 0 & on \;\; \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) & in \;\; \Omega, \end{cases} \tag{1.3}$$

where $p > 1$. It is well known that there exist choices of u_0 for which the corresponding solutions tend to zero as $t \to \infty$ and other choices for which the solutions blow-up in finite time (see [12] for example). Recently, Z. Tan [19], R. Ikehata and T. Suzuki [14] considered critical problem (1.1) with $\lambda = \mu = 0$. By means of the potential well method, they established the existence of global solutions and studied the asymptotic behavior of solutions which heavily depend on the initial data.

However, as far as we know, there are few results on problem (1.1). Equation (1.1) can serve as a model for many problems coming from quantum mechanics, chemistry, cosmology, astrophysics and differential geometry. The inverse square potential $\frac{1}{|x|^2}$ arises in many fields (see [10, 13]). For example, it arises in point-dipole interactions in molecular physics [11], where the interactions between the charge of the electron and the dipole moment of the molecule gives rise to long-distance forces and to the presence of an inverse square potential in the Schrödinger equation for the wave function of the electron. In addition, equation (1.1) is closely related to the Yamabe problem on the sphere $\mathbb{S}^N$. Indeed, by means of the stereographic projection, we may identify R^N with $\mathbb{S}^N$ and endow $\mathbb{S}^N$ with a metric whose scalar curvature is singular at the north pole; the problem of finding a conformal metric with prescribed scalar curvature 1 leads to solving equation (1.1) (see [2]).

Problem (1.1) is closely related to its stationary case:

$$\begin{cases} -\Delta u - \mu\frac{u}{|x|^2} = \lambda u + |u|^{2^*-2}u & in\ \Omega, \\ u(x) = 0 & on\ \partial\Omega. \end{cases} \tag{1.4}$$

It is well known that the nontrivial solutions of problem (1.4) are equivalent to the nonzero critical points of the energy functional

$$I_{\lambda,\mu}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \mu\frac{|u|^2}{|x|^2} - \lambda|u|^2)dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx \quad u \in H_0^1(\Omega). \tag{1.5}$$

E. Jannelli [15] considered problem (1.4) and proved that if $0 < \mu \leq \bar{\mu} - 1$, then problem (1.4) admits a positive solution for all $\lambda \in (0, \lambda_{1,\mu})$; if $\bar{\mu} - 1 < \mu < \bar{\mu}$, and $\Omega = B_1(0)$, then there exists $\mu^* \in (0, \lambda_{1,\mu})$ such that problem (1.4) admits a positive solution if and only if $\lambda \in (\mu^*, \lambda_{1,\mu})$, where $\lambda_{1,\mu}$ is the first eigenvalue of the positive operator $-\Delta - \frac{\mu}{|x|^2}$ with Dirichlet boundary condition.

The functional $I_{\lambda,\mu} \in C^1(H_0^1(\Omega), R)$ is said to satisfy the $(P.S.)_c$ condition if any sequence $\{u_n\} \subset H_0^1(\Omega)$ such that as $n \longrightarrow \infty$

$$I_{\lambda,\mu}(u_n) \to c \ , \ dI_{\lambda,\mu}(u_n) \to 0 \quad \text{strongly in} \quad H^{-1}(\Omega)$$

contains a subsequence converging in $H_0^1(\Omega)$ to a critical point of $I_{\lambda,\mu}$.

Set $D^{1,2}(R^N) = \{u \in L^{2^*}(R^N) : |\nabla u| \in L^2(R^N)\}$. For all $\mu \in [0, \bar{\mu})$, $\bar{\mu} = (\frac{N-2}{2})^2$, we define the constant

$$S_\mu := \inf_{u \in D^{1,2}(R^N)\setminus\{0\}} \frac{\int_{R^N}(|\nabla u|^2 - \mu\frac{u^2}{|x|^2})dx}{(\int_{R^N}|u|^{2^*}dx)^{\frac{2}{2^*}}}.$$

From [9, 15], S_μ is independent of any $\Omega \subset R^N$ in the sense that if

$$S_\mu(\Omega) := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega(|\nabla u|^2 - \mu\frac{u^2}{|x|^2})dx}{(\int_\Omega |u|^{2^*}dx)^{\frac{2}{2^*}}},$$

then $S_\mu(\Omega) = S_\mu(R^N) = S_\mu$.

Letting $\gamma = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}$, $\gamma' = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}$, S. Terracini [20] proved that for $\epsilon > 0$

$$U_{\mu,\epsilon}(x) = \frac{(4\epsilon^2 N(\bar{\mu} - \mu)/(N-2))^{\frac{N-2}{4}}}{(\epsilon^2 |x|^{\frac{\gamma'}{\sqrt{\bar{\mu}}}} + |x|^{\frac{\gamma}{\sqrt{\bar{\mu}}}})^{\sqrt{\bar{\mu}}}} \tag{1.6}$$

satisfies

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2} u & in\ R^N \backslash \{0\}, \\ u \longrightarrow 0 & as\ |x| \longrightarrow \infty. \end{cases} \tag{1.7}$$

From Theorem B in [6], all the positive solutions of problem (1.7) must have the form of $U_{\mu,\epsilon}$. Moreover, $U_{\mu,\epsilon}$ achieves S_μ.

By the Hardy inequality (see [1]):

$$\int_\Omega \frac{|u|^2}{|x|^2} dx \leq \frac{1}{\bar{\mu}} \int_\Omega |\nabla u|^2 dx \quad \forall u \in H_0^1(\Omega),$$

we easily derive that if $\mu \in [0, \bar{\mu})$, the norm $\big(\int_\Omega (|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2}) dx\big)^{\frac{1}{2}}$ is equivalent to the usual norm in $H_0^1(\Omega)$.

Set

$$c_{\lambda,\mu} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda,\mu}(\gamma(t)),$$

where $\Gamma = \big\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ I_{\lambda,\mu}(\gamma(1)) < 0\big\}$, and

$$\mathcal{O}^+ = \big\{u \in H_0^1(\Omega) : I_{\lambda,\mu}(u) < c_{\lambda,\mu}, \quad \langle dI_{\lambda,\mu}(u), u\rangle > 0\big\},$$

$$\mathcal{O}^- = \big\{u \in H_0^1(\Omega) : I_{\lambda,\mu}(u) < c_{\lambda,\mu}, \quad \langle dI_{\lambda,\mu}(u), u\rangle < 0\big\},$$

$$\mathcal{N} = \big\{u \in H_0^1(\Omega) \backslash \{0\} : \langle dI_{\lambda,\mu}(u), u\rangle = 0\big\}.$$

Next, we denote the solution of (1.1) corresponding to initial datum u_0 by $u(t) = u(x, t; u_0)$.

Our main results are the following:

Theorem 1.1. *Assume that $\mu \in [0, \bar{\mu})$, $\lambda \in R$. Then for any $\epsilon \in (0, \frac{T}{2})$, each solution u of problem* (1.1) *satisfies*

$$|u(x,t)| \leq C|x|^{-\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}} \qquad \forall (x,t) \in \Omega \backslash \{0\} \times [\epsilon, T - \epsilon], \tag{1.8}$$

where C depends on ϵ, T. In particular, if $\lambda \geq 0$ and u is positive, then there exists an $m_0 = m_0(\epsilon, T) > 0$ such that

$$u(x,t) \geq m_0 |x|^{-\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}} \qquad \forall (x,t) \in B_{\rho_0}(0) \backslash \{0\} \times [\epsilon, T), \tag{1.9}$$

where $\rho_0 > 0$ is small enough such that $\overline{B_{\rho_0}(0)} \subset \Omega$.

Remark 1.2. Theorem 1.1 shows the characterization of the local asymptotic behavior of solutions for problem (1.1). Especially, Theorem 1.1 gives the exact behavior of positive solutions for (1.1) near points $\{0\} \times [\epsilon, T-\epsilon]$.

Theorem 1.3. *If there exists $t_0 \geq 0$ such that $I_{\lambda,\mu}(u(t_0)) \leq 0$, then $u(t)$ blows up in finite time.*

Define

$$\lambda_*(\mu) = \min_{\varphi \in H_0^1(\Omega)} \int_\Omega \frac{|\nabla \varphi|^2}{|x|^{2(\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu})}} dx \Big/ \int_\Omega \frac{|\varphi|^2}{|x|^{2(\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu})}} dx.$$

Then $\lambda_*(\mu) \in (0, \lambda_{1,\mu})$ if $\mu \in (\bar{\mu}-1, \bar{\mu})$ (see [15]).

Suppose that

(H) $\lambda = 0$ and $\mu \in [0, \bar{\mu})$ or $\lambda \in (\lambda^*, \lambda_{1,\mu})$, where

$$\begin{cases} \lambda^* = 0 & if \quad \mu \in [0, \bar{\mu}-1], \\ \lambda^* = \lambda_*(\mu) & if \quad \mu \in (\bar{\mu}-1, \bar{\mu}). \end{cases}$$

It is well known that $0 < c_{\lambda,\mu} < \frac{1}{N} S_\mu^{\frac{N}{2}}$ if $\lambda \in (\lambda^*, \lambda_{1,\mu})$ and $c_{\lambda,\mu} = \frac{1}{N} S_\mu^{\frac{N}{2}}$ if $\lambda = 0$ and $\mu \in [0, \bar{\mu})$ (see [15]).

Theorem 1.4. *Assume that (H) holds and $u_0 \in \mathcal{O}^+$. Then problem (1.1) admits one global solution $u(t)$. Moreover, $u(t)$ converges to 0 in the strong topology of $H_0^1(\Omega)$. Precisely, there exists a positive number α such that*

$$\|\nabla u(t)\|_{L^2(\Omega)} = O(e^{-\alpha t}) \quad as \quad t \longrightarrow \infty. \tag{1.10}$$

Set $\bar{\lambda} = \big(1 - (\frac{2}{N})^{\frac{2}{N-2}}\big)\lambda_{1,\mu}$. Then we have the following from Theorem 1.4:

Corollary 1.5. *Assume $\lambda \in [0, \bar{\lambda}]$ and $\mu \in [0, \bar{\mu}-1]$. If $\int_\Omega (|\nabla u_0|^2 - \mu \frac{|u_0|^2}{|x|^2}) dx \leq 2c_{\lambda,\mu}$, then the same results in Theorem 1.4 hold.*

Remark 1.6. In Corollary 1.5, we restrict μ on the interval $[0, \bar{\mu}-1]$, not on $[0, \bar{\mu})$, because we cannot compare with the values of $\bar{\lambda}$ and $\lambda_*(\mu)$ for $\mu \in (\bar{\mu}-1, \bar{\mu})$.

Theorem 1.7. *Assume that (H) holds and $u_0 \in \mathcal{O}^-$. Then the solution $u(t)$ of (1.1) blows up in finite time.*

Theorem 1.8. *Let $\mu \in [0, \bar{\mu})$ and $\lambda \in R$. If $u(t)$ is a global solution of (1.1), then there exists a divergent time sequence t_m and a critical point $\bar{u}$ of $I_{\lambda,\mu}$, k sequences of positive numbers $\{r_m^j\}_m (1 \leq j \leq k)$, l sequences of positive numbers $\{R_m^j\}_m (1 \leq j \leq l)$ and l sequences of points $\{x_m^j\}_m (1 \leq j \leq l)$ in Ω which converge to $\bar{x}^j \in \overline{\Omega}$, such that as $m \longrightarrow \infty$*

(i) $\quad I_{\lambda,\mu}(u(t_m)) = I_{\lambda,\mu}(\bar u) + \frac{k_1}{N}S_0^{\frac{N}{2}} + \frac{l_1}{N}S_\mu^{\frac{N}{2}} + o(1),$
where $k_1,\ l_1$ *are positive integers;*
(ii) $u(t_m) = \bar u + \sum_{j=1}^{k}(r_m^j)^{\frac{N-2}{2}}u^j(r_m^j x) + \sum_{j=1}^{l}(R_m^j)^{\frac{N-2}{2}}v^j(R_m^j(x - x_m^j)) + \omega_m,$
where $\|\omega_m\|_{H_0^1(\Omega)} \longrightarrow 0;$ $r_m^j \longrightarrow \infty$ $(1 \le j \le k)$ $R_m^j \longrightarrow \infty$ $(1 \le j \le l)$ *as* $m \longrightarrow \infty$*; and* $u^j\,(1 \le j \le k),$ $v^j\,(1 \le j \le l)$ *satisfy the following problems "at infinity" respectively*

$$-\Delta u = |u|^{2^*-2}u \qquad u \in D^{1,2}(R^N),$$

and

$$-\Delta u - \mu\frac{u}{|x|^2} = |u|^{2^*-2}u \qquad u \in D^{1,2}(R^N).$$

Remark 1.9. It seems reasonable to restrict $\mu < \bar\mu$. If $\mu \ge \bar\mu$, the functional framework becomes delicate because the Hardy inequality fails to provide the coercivity of $-\Delta - \frac{\mu}{|x|^2}$ in $H_0^1(\Omega)$. In addition, the energy inequality (see (4.8) below) no longer holds, which generates much difficulty in proving the global existence of solutions for (1.1). However, for $1 \le q < 2$, the improved Hardy-Poincaré inequality holds (see [24]):

$$\int_\Omega \big(|\nabla u|^2 - \bar\mu\frac{|u|^2}{|x|^2}\big)dx \ge C(q)\|u\|^2_{W_0^{1,q}(\Omega)} \qquad \forall u \in H_0^1(\Omega).$$

It maybe possible to consider the well posedness of (1.1) ($\mu = \bar\mu$) in the Hilbert space H (see [24]) with the norm: $\|u\|_H = \big(\int_\Omega \big(|\nabla u|^2 - \bar\mu\frac{|u|^2}{|x|^2}\big)dx\big)^{\frac{1}{2}}$.

Throughout this paper, we denote the norm of $L^\infty(0,T;H_0^1(\Omega))$ by

$$\|u\|_{L^\infty(0,T;H_0^1(\Omega))} = ess \sup_{0<t<T}\big(\int_\Omega |\nabla u(t)|^2dx\big)^{\frac{1}{2}};$$

the norm of $L^\infty(0,T;L^l(\Omega))(1 \le l < \infty)$ by

$$\|u\|_{L^\infty(0,T;L^l(\Omega))} = ess \sup_{0<t<T}\big(\int_\Omega |u(t)|^l dx\big)^{\frac{1}{l}}$$

and positive constants (possibly different from line to line) by $C, C_1, C_2, \cdots$.

2. Exact asymptotic behavior of solutions for (1.1) at singular points

Before giving the proof of Theorem 1.1, we introduce a preliminary lemma, which is a parabolic generalization of Theorem 2.3 in [18].

Lemma 2.1. *Assume that* $V \in L^\infty(0,T; L^{\frac{N}{2}}(\Omega))$ *and* $g \in L^\infty(0,T;L^q(\Omega))$, $q \geq 2$. *If* $u \in L^\infty(0,T;H_0^1(\Omega))$, $u_t \in L^\infty(0,T;H^{-1}(\Omega))$ *is a weak solution of*

$$\begin{cases} u_t - \Delta u - \mu\frac{u}{|x|^2} - V(x,t)u + \nu u = g & in\ \Omega \times (0,T), \\ u(x,t) = 0 & on\ \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) & in\ \Omega, \end{cases}$$

where $u_0 \in H_0^1(\Omega)$, ν *is such that the linear operator on the left-hand side is positive, then for any sufficiently small number* $\epsilon_0 > 0$

$$u \in L^\infty(\epsilon_0, T-\epsilon_0; L^p(\Omega)) \qquad \forall p < p_{lim} = 2^* \min\Big\{\frac{q}{2}, \frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu} - \sqrt{\bar\mu - \mu}}\Big\}.$$

Proof. The proof is similar to that of Theorem 2.3 in [18]. For the reader's convenience, we give a sketch of it. Since $0 \leq \mu < \bar\mu = \big(\frac{N-2}{2}\big)^2$ and $V \in L^\infty(0,T;L^{\frac{N}{2}}(\Omega))$, the operator $-\Delta - \frac{\mu}{|x|^2} + V(x,t) - \nu$ is positive for large ν. We can assume that g and u_0 are positive; the general case is obtained by decomposing $g = g^+ - g^-$, $u_0 = u_0^+ - u_0^-$, where $v^+ = \max\{v, 0\}$, $v^- = \max\{-v,\ 0\}$.

Let $W_k(x,t) = \min\big\{\frac{\mu}{|x|^2} + V(x,t),\ k\big\}$ and u_k be the unique solution of

$$\begin{cases} (u_k)_t - \Delta u_k - W_k(x,t)u_k + \nu u_k = g & in\ \ \Omega \times (0,T), \\ u_k(x,t) = 0 & on\ \ \partial\Omega \times [0,T), \\ u_k(x,0) = u_0(x) & in\ \ \Omega. \end{cases} \tag{2.1}$$

Clearly, u_k is positive. Set $u_k^n = \min\{u_k,\ n\}$ and assume that $u_k \in L^p(\Omega \times (0,T))$ for some $p \in [2,q]$. Note that $(u_k^n)^{p-1} \in L^\infty(0,T;H_0^1(\Omega))$. So, from (2.1), we obtain for any $\epsilon_0 \in (0, \frac{T}{2})$

$$\begin{aligned} &\frac{1}{p}\int_\Omega (u_k^n(T-\epsilon_0))^p dx + \frac{4(p-1)}{p^2}\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |\nabla\big((u_k^n)^{\frac{p}{2}}\big)|^2 dxdt \\ &\leq \int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega \Big(\frac{\mu}{|x|^2} + |V(x,t)|\Big)(u_k^n)^p dxdt + k\int\int_{Q_{\epsilon_0}\bigcap\{u_k > n\}} (u_k)^p dxdt \\ &\quad + \int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega g(x,t)(u_k^n)^{p-1}dxdt + \frac{1}{p}\int_\Omega (u_k^n(\epsilon_0))^p dx \\ &\leq C\|g\|_{L^\infty(\epsilon_0,T-\epsilon_0;L^q(\Omega))}\|u_k^n\|^{p-1}_{L^p(\Omega\times(\epsilon_0,T-\epsilon_0))} \\ &\quad + \Big(\frac{\mu}{\bar\mu} + \epsilon\Big)\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |\nabla\big((u_k^n)^{\frac{p}{2}}\big)|^2 dxdt \end{aligned}$$

$$+ C(\epsilon)\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |u_k^n|^p dxdt + k\int\int_{Q_{\epsilon_0}\bigcap\{u_k>n\}}(u_k)^p dxdt,$$

where $Q_{\epsilon_0} = \Omega\times[\epsilon_0, T-\epsilon_0]$, and we have used the fact that for each $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that, for any $v \in H_0^1(\Omega)$,

$$\begin{aligned}\int_\Omega |V(x,t)|v^2 dx &\le \|V(x,t)\|_{L^\infty(\epsilon_0,T-\epsilon_0;L^{\frac{N}{2}}(\Omega))}\|v\|^2_{L^{2^*}(\Omega)}\\ &\le \epsilon\int_\Omega |\nabla v|^2 dx + C(\epsilon)\int_\Omega v^2 dx.\end{aligned}$$

A direct calculation shows that, if $p < \frac{2p_{lim}}{2^*}$, then $\frac{\mu}{\bar\mu} < \frac{4(p-1)}{p^2}$. Hence, choosing ϵ sufficiently small and using the Sobolev inequality, we infer that

$$\begin{aligned}\|u_k^n\|^p_{L^\infty(\epsilon_0,T-\epsilon_0;L^{\frac{2^*p}{2}}(\Omega))} &\le C\Big(\|g\|^p_{L^\infty(\epsilon_0,T-\epsilon_0;L^q(\Omega))}\\ &\quad + \|u_k^n\|^p_{L^\infty(\epsilon_0,T-\epsilon_0;L^p(\Omega))} + k\int\int_{Q_{\epsilon_0}\bigcap\{u_k>n\}}(u_k)^p dxdt\Big).\end{aligned}$$

Since by assumption $u_k \in L^p(\Omega\times(\epsilon_0, T-\epsilon_0))$, taking the limit as $n\to+\infty$ yields

$$\|u_k^n\|^p_{L^\infty(\epsilon_0,T-\epsilon_0;L^{\frac{2^*p}{2}}(\Omega))} \le C\Big(\|g\|^p_{L^\infty(\epsilon_0,T-\epsilon_0;L^q(\Omega))} + \|u_k^n\|^p_{L^\infty(\epsilon_0,T-\epsilon_0;L^p(\Omega))}\Big),$$

so that $u_k \in L^\infty(\epsilon_0, T-\epsilon_0; L^{\frac{2^*p}{2}}(\Omega))$. Since the above estimate is uniform with respect to k, we deduce that $u \in L^\infty(\epsilon_0, T-\epsilon_0; L^{\frac{2^*p}{2}}(\Omega))$. Starting with $p = 2$, we can use recursively the above process to improve the integrability order of u, as long as $p < \frac{2p_{lim}}{2^*}$. Then we obtain the desired result. □

Corollary 2.2. *In view of Lemma* 2.1, *for any sufficiently small number* $\epsilon_0 > 0$*, each solution* u *of* (1.1) *satisfies*

$$u \in L^\infty(\epsilon_0, T-\epsilon_0; L^p(\Omega)), \quad \forall p < p_{lim} = \frac{2N}{N-2-2\sqrt{\bar\mu-\mu}}.$$

Proof of Theorem 1.1. Set $v(x,t) = |x|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}u(x,t)$, where $u(x,t)$ is one solution of (1.1). We claim that $v \in L^\infty(0,T; H_0^1(\Omega, |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dx))$. Indeed, by the Hardy inequality, we obtain

$$\begin{aligned}&\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|\nabla v|^2 dx\\ &= \int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\Big||x|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}\nabla u - (\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})|x|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}-2}xu\Big|^2 dx\end{aligned}$$

$$\leq 2\int_\Omega \Big(|\nabla u|^2 + (\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})^2\frac{|u|^2}{|x|^2}\Big)dx \leq\ C\int_\Omega |\nabla u|^2dx,$$

which implies that $v \in L^\infty(0,T;H_0^1(\Omega,|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dx))$ due to $u\in L^\infty(0,T;H_0^1(\Omega))$.

After a direct calculation, we deduce that v satisfies

$$\begin{cases} |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}v_t - div(|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\nabla v) & \\ = |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*-2}v + \lambda|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}v & in\ \ \Omega\backslash\{0\}\times(0,T) \\ v(x,t)=0 & on\ \ \partial\Omega\times[0,T) \\ v(x,0)=|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}u_0(x) & in\ \ \Omega. \end{cases} \tag{2.5}$$

From Corollary 2.2, we can assume

$$\sup_{t\in[\epsilon_0,T-\epsilon_0]}\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*}|v|^{2(s-1)}dx<\infty$$

for some $s>1$. Then, from (2.5), we have

$$\begin{aligned} &\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}v_t v|v|^{2(s-1)}dxdt \\ &\qquad + \int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\nabla v\cdot\nabla(v|v|^{2(s-1)})dxdt \\ &= \int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*}|v|^{2(s-1)}dxdt \\ &\qquad + \lambda\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s}dxdt. \end{aligned} \tag{2.6}$$

Now we compute each term in (2.6).

$$\begin{aligned} &\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}v_t v|v|^{2(s-1)}dxdt \\ &= \frac{1}{2s}\int_{\epsilon_0}^{T-\epsilon_0}\frac{d}{dt}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s}dxdt \\ &= \frac{1}{2s}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v(x,T-\epsilon_0)|^{2s}dx \\ &\qquad - \frac{1}{2s}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v(x,\epsilon_0)|^{2s}dx, \end{aligned} \tag{2.7}$$

$$\begin{aligned}&\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\nabla v\cdot\nabla(v|v|^{2(s-1)})dxdt\\&=(2s-1)\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2(s-1)}|\nabla v|^2dxdt.\end{aligned}\tag{2.8}$$

Inserting (2.7) and (2.8) into (2.6), we get

$$\begin{aligned}&(2s-1)\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2(s-1)}|\nabla v|^2dxdt\\&\le\frac{1}{2s}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v(x,\epsilon_0)|^{2s}dx\\&\qquad+\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*}|v|^{2(s-1)}dxdt\\&+\lambda\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s}dxdt.\end{aligned}\tag{2.9}$$

Now we recall Caffarelli-Kohn-Nirenberg's inequality (see [7]):

$$\Big(\int_\Omega |x|^{-bp}|w|^pdx\Big)^{\frac{2}{p}}\le C_{a,b}\int_\Omega |x|^{-2a}|\nabla w|^2dx\quad\forall w\in H_0^1(\Omega,|x|^{-2a}dx),\tag{2.10}$$

where $-\infty<a<\frac{N-2}{2},\ a\le b\le a+1, p=\frac{2N}{N-2+2(b-a)}$ and $C_{a,b}$ is a positive constant depending on a,b.

Taking $a=b=\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}<\frac{N-2}{2}$ in (2.10), then $p=2^*$. Choosing $w=v|v|^{s-1}$, together with (2.9), we derive that

$$\begin{aligned}&\int_{\epsilon_0}^{T-\epsilon_0}\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|v|v|^{s-1}\big|^{2^*}dx\Big)^{\frac{2}{2^*}}dt\\&\le C_{a,a}\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu_i})}|\nabla(v|v|^{s-1})|^2dxdt\\&\le 2C_{a,a}s^2\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2(s-1)}|\nabla v|^2dxdt\\&\le C\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v(x,\epsilon_0)|^{2s}dx\\&\qquad+Cs\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*}|v|^{2(s-1)}dxdt\end{aligned}$$

$$
+ Cs\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s}dxdt. \tag{2.11}
$$

By using the mean value theorem, we infer from (2.11) that

$$
\begin{aligned}
&\sup_{t\in[\epsilon_0,T-\epsilon_0]}\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|v|v|^{s-1}\big|^{2^*}dx\Big)^{\frac{2}{2^*}}\\
\leq\ & C(T)s\sup_{t\in[\epsilon_0,T-\epsilon_0]}\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*}|v|^{2(s-1)}dx \\
&+C(T)s\sup_{t\in[\epsilon_0,T-\epsilon_0]}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s}dx.
\end{aligned} \tag{2.12}
$$

Since $u\in L^\infty(\epsilon_0,T-\epsilon_0;L^p(\Omega))$, for all $p<\frac{2N}{N-2-2\sqrt{\bar\mu-\mu}}$ (see Corollary 2.2), we may choose $\frac{N}{2}<q<\frac{N(N-2)}{2\left(N-2-2\sqrt{\bar\mu-\mu}\right)}$, then $(2^*-2)q<\frac{2N}{N-2-2\sqrt{\bar\mu-\mu}}$ and $2<\frac{2q}{q-1}<2^*$. Therefore, we deduce that, for any $\epsilon>0$,

$$
\begin{aligned}
&\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*}|v|^{2(s-1)}dx=\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|u|^{2^*-2}|v|^{2s}dx\\
&\leq \|u\|^{2^*-2}_{L^{(2^*-2)q}(\Omega)}\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}|v|^s\|^2_{L^{\frac{2q}{q-1}}(\Omega)}\\
&\leq \|u\|^{2^*-2}_{L^{(2^*-2)q}(\Omega)}\times\big(\epsilon\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}|v|^s\|_{L^{2^*}(\Omega)}\\
&\qquad+C(N,q)\epsilon^{-\frac{N}{2q-N}}\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}|v|^s\|_{L^2(\Omega)}\big)^2\\
&\leq C\epsilon^2\Big(\int_\Omega|x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*s}dx\Big)^{\frac{2}{2^*}}+C\epsilon^{-\frac{2N}{2q-N}}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s}dx.
\end{aligned} \tag{2.13}
$$

Inserting (2.13) into (2.12), we obtain

$$
\begin{aligned}
&\sup_{t\in[\epsilon_0,T-\epsilon_0]}\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big|v|v|^{s-1}\big|^{2^*}dx\Big)^{\frac{2}{2^*}}\\
\leq\ & Cs\epsilon^2\sup_{t\in[\epsilon_0,T-\epsilon_0]}\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*s}dx\Big)^{\frac{2}{2^*}}\\
&+Cs\epsilon^{-\frac{2N}{2q-N}}\sup_{t\in[\epsilon_0,T-\epsilon_0]}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s}dxdx\\
&+C(T)s\sup_{t\in[\epsilon_0,T-\epsilon_0]}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s}dx.
\end{aligned} \tag{2.14}
$$

Taking $\epsilon = \frac{1}{\sqrt{2Cs}}$ in (2.14), we conclude that

$$\sup_{t\in[\epsilon_0,T-\epsilon_0]}\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*s}dx\Big)^{\frac{2}{2^*}}$$
$$\leq Cs^\alpha \sup_{t\in[\epsilon_0,T-\epsilon_0]}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s}dx, \tag{2.15}$$

where $\alpha = \frac{2q}{2q-N} > 0$. Define the sequence $s_n = \left(\frac{2^*}{2}\right)^n \quad n = 1, 2, \cdots$ and take $s = s_n$ in (2.15), then

$$\begin{aligned}
&\sup_{t\in[\epsilon_0,T-\epsilon_0]}\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s_{n+1}}dx\Big)^{\frac{1}{2s_{n+1}}}\\
\leq\ & \Big(Cs_n^\alpha\Big)^{\frac{1}{2s_n}}\sup_{t\in[\epsilon_0,T-\epsilon_0]}\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s_n}dx\Big)^{\frac{1}{2s_n}}\\
\leq\ & \cdots\\
\leq\ & C^{\sum\limits_{n=1}^{\infty}\frac{1}{2s_n}}\prod_{n=1}^{\infty}s_n^{\frac{\alpha}{2s_n}}\sup_{t\in[\epsilon_0,T-\epsilon_0]}\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2^*}dx\Big)^{\frac{1}{2^*}}\\
\leq\ & C\sup_{t\in[\epsilon_0,T-\epsilon_0]}\Big(\int_\Omega |u|^{2^*}dx\Big)^{\frac{1}{2^*}} \leq C.
\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}
&\|v\|_{L^\infty(\epsilon_0,T-\epsilon_0;L^{2s_{n+1}}(\Omega))}\\
\leq\ & (diam\Omega)^{\frac{2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}{2s_{n+1}}}\sup_{t\in[\epsilon_0,T-\epsilon_0]}\Big(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{2s_{n+1}}dx\Big)^{\frac{1}{2s_{n+1}}}\\
\leq\ & C(diam\Omega)^{\frac{2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}{2s_{n+1}}}.
\end{aligned}$$

Note that $s_{n+1}\to\infty$ as $n\to\infty$. Letting $n\to\infty$ in the above inequality, we infer that $\|v\|_{L^\infty(\epsilon_0,T-\epsilon_0;L^\infty(\Omega))}\leq C$, which implies that (1.8) holds.

Now we prove (1.9). Note that, at this point, only positive solutions will be considered.

Set $0 < t_0 < \rho_0$, and $n(s) = \min\limits_{|x|=s}\min\limits_{\epsilon_0\leq t<T} v(x,t)$, $t_0 \leq s \leq \rho_0$, such that

$$n(t_0) = At_0^{-2\sqrt{\bar\mu-\mu_i}} + B, \quad n(\rho_0) = A\rho_0^{-2\sqrt{\bar\mu-\mu_i}} + B,$$

where

$$A = \frac{n(\rho_0)-n(t_0)}{\rho_0^{-2\sqrt{\bar\mu-\mu}} - t_0^{-2\sqrt{\bar\mu-\mu}}}, \quad B = \frac{n(\rho_0)t_0^{-2\sqrt{\bar\mu-\mu}} - n(t_0)\rho_0^{-2\sqrt{\bar\mu-\mu}}}{t_0^{-2\sqrt{\bar\mu-\mu}} - \rho_0^{-2\sqrt{\bar\mu-\mu}}}.$$

It is easy to verify that

$$-div\big(|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\nabla(|x|^{-2\sqrt{\bar\mu-\mu}})\big)=0 \qquad \forall x\in\Omega\backslash\{0\}. \tag{2.16}$$

Combining (2.5) with (2.16), we get

$$\begin{aligned} & |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\big(v-(A|x|^{-2\sqrt{\bar\mu-\mu}}+B)\big)_t \\ & -div\big(|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\nabla\big(v-(A|x|^{-2\sqrt{\bar\mu-\mu}}+B)\big)\big) \\ = \; & |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}v^{2^*-1}+\lambda|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}v>0 \ \text{ in } \ \Omega\backslash\{0\}\times[\epsilon_0,T). \end{aligned}$$

By the choice of A and B, we have

$$v(x,t)\ge A|x|^{-2\sqrt{\bar\mu-\mu}}+B,$$

for all $(x,t)\in\big(\partial\big(B_{\rho_0}(0)\backslash B_{t_0}(0)\big)\times[\epsilon_0,T)\big)\bigcup\big(\overline{B_{\rho_0}(0)}\backslash B_{t_0}(0)\times\{\epsilon_0\}\big)$. Therefore, by the maximum principle, we obtain

$$\begin{aligned} v(x,t) & \ge A|x|^{-2\sqrt{\bar\mu-\mu}}+B \\ & = \frac{|x|^{-2\sqrt{\bar\mu-\mu}}-\rho_0^{-2\sqrt{\bar\mu-\mu}}}{t_0^{-2\sqrt{\bar\mu-\mu}}-\rho_0^{-2\sqrt{\bar\mu-\mu}}}n(t_0)+\frac{t_0^{-2\sqrt{\bar\mu-\mu}}-|x|^{-2\sqrt{\bar\mu-\mu}}}{t_0^{-2\sqrt{\bar\mu-\mu}}-\rho_0^{-2\sqrt{\bar\mu-\mu}}}n(\rho_0) \\ & \ge \frac{|x|^{2\sqrt{\bar\mu-\mu}}-t_0^{2\sqrt{\bar\mu-\mu}}}{|x|^{2\sqrt{\bar\mu-\mu}}-t_0^{2\sqrt{\bar\mu-\mu}}\rho_0^{-2\sqrt{\bar\mu-\mu}}|x|^{2\sqrt{\bar\mu-\mu}}}n(\rho_0) \end{aligned}$$

for all $(x,t)\in B_{\rho_0}(0)\backslash B_{t_0}(0)\times[\epsilon_0,T)$. Letting $t_0\to 0$, we conclude that

$$v(x,t)\ge n(\rho_0)=\min_{|x|=\rho_0}\min_{\epsilon_0\le t<T}v(x,t)>0,\quad \forall(x,t)\in B_{\rho_0}(0)\backslash\{0\}\times[\epsilon_0,T).$$

3. Blow-up of solutions for problem (1.1) with initial negative energy

Proof of Theorem 1.3. We argue by contradiction that $u(t)$ is a global solution of (1.1). Note that $I_{\lambda,\mu}(u(t_0))\le 0$ and

$$\begin{aligned} \int_{t_0}^t\int_\Omega|u_s(s)|^2dxds+\frac12\int_\Omega(|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2}-\lambda|u(t)|^2)dx \\ -\frac{1}{2^*}\int_\Omega|u(t)|^{2^*}dx=I_{\lambda,\mu}(u(t_0)). \end{aligned}$$

Set $g(t)=\int_\Omega|u(t)|^2dx$. Then

$$\begin{aligned} \frac{d}{dt}g(t) & =\int_\Omega u(t)u_t(t)dx \\ & =-2\int_\Omega(|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2}-\lambda|u(t)|^2)dx+2\int_\Omega|u(t)|^{2^*}dx \end{aligned}$$

$$= 4\int_{t_0}^{t}\int_\Omega |u_s(s)|^2 dxds + 2(1-\frac{2}{2^*})\int_\Omega |u(t)|^{2^*}dx - 4I_{\lambda,\mu}(u(t_0)) > 0$$
$$\geq 2(1-\frac{2}{2^*})\int_\Omega |u(t)|^{2^*}dx. \tag{3.1}$$

Furthermore, we get for any $t \geq t_0$

$$g(t) \geq g(t_0) = \int_\Omega |u(t_0)|^2 dx \triangleq m_1 > 0.$$

Therefore, by (3.1) and choosing $\alpha \in (1, \frac{2^*}{2})$, we deduce that for any $t \geq t_0$

$$-\frac{1}{\alpha-1}\frac{d}{dt}g^{1-\alpha}(t) = g^{-\alpha}(t)\frac{d}{dt}g(t) \geq 2(1-\frac{2}{2^*})g^{-\alpha}(t)\int_\Omega |u(t)|^{2^*}dx$$
$$\geq Cg^{-\alpha}(t)\Big(\int_\Omega |u(t)|^2dx\Big)^{\frac{2^*}{2}} = Cg^{\frac{2^*}{2}-\alpha}(t) \geq Cm_1^{\frac{2^*}{2}-\alpha}.$$

Hence, we infer that for any $t \geq t_0$

$$0 < \Big(\int_\Omega |u(t)|^2 dx\Big)^{1-\alpha} = g^{1-\alpha}(t)$$
$$\leq g^{1-\alpha}(t_0) + C(\alpha-1)m_1^{\frac{2^*}{2}-\alpha}t_0 - C(\alpha-1)m_1^{\frac{2^*}{2}-\alpha}t,$$

which is a contradiction for $t \geq t_0$ sufficiently large.

4. Existence of global solutions for problem (1.1) with initial datum in $\mathcal{O}^+$

By means of analytic semigroup theory, we first give local existence of solutions for problem (1.1) .

Proposition 4.1. *Assume that (H) holds. Then, for all $u_0 \in H_0^1(\Omega)$, there exists $T \in (0,\infty]$ such that problem (1.1) admits a unique solution $u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$, which becomes a classical solution for $t > 0$. Moreover, suppose $T^* = T^*(u_0)$ denotes the maximal existence time of u and $T^* < \infty$. Then*

$$\lim_{t\to T^*} \|u(t)\|_{X^{1+\epsilon}} = \infty \quad \textit{for all sufficiently small } \epsilon > 0. \tag{4.1}$$

Remark 4.2. If $T^* < \infty$, we say that T^* is the blow-up time, whereas if $T^* = \infty$, we say that u is a global solution. It is still an open question whether (4.1) can be reduced to

$$\lim_{t\to T^*} \|u(t)\|_{H_0^1(\Omega)} = \infty \quad if \quad T^* < \infty. \tag{4.2}$$

It seems that (4.2) does not hold because it is shown in [3] that parabolic problems with critical growth are not uniformly well posed.

Proof of Proposition 4.1. Choose $\beta = \frac{N}{N+2} \in (\frac{1}{2}, 1)$. Then, for each $u_0 \in H_0^1(\Omega)$, there exists a positive number $T > 0$ such that problem (1.1) has a unique solution u satisfying

$$
\begin{array}{ll}
(i) & u \in C([0,T]; D(A^\beta)) \cap C([0,T]; H_0^1(\Omega)), \\
(ii) & u(t) = e^{-tA}u_0 + \int_0^t e^{-t(t-s)A} f(u(s))ds, \\
(iii) & \lim\limits_{t\to 0} t^{\beta-\frac{1}{2}} \|A^\beta u(t)\|_{L^2(\Omega)} = 0,
\end{array}
$$

where $f(u) = |u|^{2^*-2}u + \lambda u$. Since its proof is similar to that of Proposition 2.4 in [14], we omit its details here. Moreover, since $u \in C([0,T]; H_0^1(\Omega))$, we have $\Delta u \in C([0,T]; H^{-1}(\Omega))$ and $|u|^{2^*-2}u + \lambda u \in C([0,T]; H^{-1}(\Omega))$. Hence $u_t \in C([0,T]; H^{-1}(\Omega))$. Set $T^* = \sup\{T :$ the solution u of (1.1) satisfies conditions (i)-(iii) in the above$\}$. According to Proposition 3 in [4], f is an $\epsilon-$ regular map relative to $(H_0^1(\Omega),\ H^{-1}(\Omega))$ for all $\epsilon \in (0, \frac{N-2}{2(N+2)})$. Then, by Proposition 1 in [4], we conclude that (4.1) holds. □

Proof of Theorem 1.4. Let $u_0 \in \mathcal{O}^+$ and $u(t) = u(x,t;u_0)$ be the unique solution established in Proposition 4.1. Then, after a direct calculation, we get for any $t \in [0, T^*)$,

$$
\frac{d}{dt} I_{\lambda,\mu}(u(t)) = -\int_\Omega u_t^2(t)dx < 0, \tag{4.3}
$$

which implies that $t \mapsto I_{\lambda,\mu}(u(t))$ is strictly decreasing. Thus,

$$
I_{\lambda,\mu}(u(t)) \le I_{\lambda,\mu}(u_0) < c_{\lambda,\mu}. \tag{4.4}
$$

We claim that $\mathcal{O}^+$ is invariant under the flow defined in (1.1); that is, $u(t) \in \mathcal{O}^+$ for all $t > 0$. In fact, suppose there exists $t^* \in (0, T^*)$ such that $u(t^*) \notin \mathcal{O}^+$. Then, from (4.4), we infer that

$$
\langle dI_{\lambda,\mu}(u(t^*)), u(t^*)\rangle \le 0. \tag{4.5}
$$

Noting that $t \mapsto \langle dI_{\lambda,\mu}(u(t)), u(t)\rangle$ is continuous, there exists $t_0 \in (0, t^*]$ such that

$$
\langle dI_{\lambda,\mu}(u(t_0)), u(t_0)\rangle = 0.
$$

Hence, $u(t_0) \equiv 0$ in Ω, or $u(t_0) \in \mathcal{N}$. If $u(t_0) \equiv 0$ in Ω, then by the uniqueness of $u(t)$, we conclude that $u(t) \equiv 0$ for any $t \in [t_0, T^*)$. Thus $u(t)$ is global by extending to 0 for all $t \ge T^*$, and so $I_{\lambda,\mu}(u(t)) > 0$ for any $t \ge 0$ due to Theorem 1.3. But $I_{\lambda,\mu}(u(t_0)) = 0$, which is a contradiction.

Therefore, $u(t_0) \in \mathcal{N}$. Observing that $c_{\lambda,\mu} = \inf_{\gamma\in\Gamma}\max_{t\in[0,T]} I_{\lambda,\mu}(\gamma(t)) = \inf_{u\in\mathcal{N}} I_{\lambda,\mu}(u)$ (see [23]), we derive that

$$c_{\lambda,\mu} \le I_{\lambda,\mu}(u(t_0)) < I_{\lambda,\mu}(u_0) < c_{\lambda,\mu},$$

which is also a contradiction.

From the above arguments, we infer that $u(t) \in \mathcal{O}^+$ for all $t \in [0, T^*)$. From (1.1), we have for any $t \in [0, T^*)$

$$\int_0^t \|u_s(s)\|^2_{L^2(\Omega)} ds + I_{\lambda,\mu}(u(t)) = I_{\lambda,\mu}(u_0) < c_{\lambda,\mu}. \tag{4.6}$$

In addition,

$$\begin{aligned} I_{\lambda,\mu}(u(t)) &= \frac{1}{N}\int_\Omega (|\nabla u(t)|^2 - \mu\frac{|u(t)|^2}{|x|^2} - \lambda|u(t)|^2)dx + \frac{1}{2^*}\langle dI_{\lambda,\mu}(u(t)), u(t)\rangle \\ &> \frac{1}{N}\int_\Omega (|\nabla u(t)|^2 - \mu\frac{|u(t)|^2}{|x|^2} - \lambda|u(t)|^2)dx \\ &\ge \frac{1}{N}(1-\frac{\lambda}{\lambda_{1,\mu}})(1-\frac{\mu}{\bar{\mu}})\int_\Omega |\nabla u(t)|^2 dx > 0. \end{aligned} \tag{4.7}$$

Inserting (4.7) into (4.6), we get

$$\int_0^t \|u_s(s)\|^2_{L^2(\Omega)} ds + \frac{1}{N}(1-\frac{\lambda}{\lambda_{1,\mu}})(1-\frac{\mu}{\bar{\mu}})\int_\Omega |\nabla u(t)|^2 dx < c_{\lambda,\mu} < \frac{1}{N}S_\mu^{\frac{N}{2}}. \tag{4.8}$$

Especially, for any $t \in [0, T^*)$

$$\int_0^t \|u_s(s)\|^2_{L^2(\Omega)} ds < \frac{1}{N}S_\mu^{\frac{N}{2}} \quad \text{and} \quad \int_\Omega |\nabla u(t)|^2 dx < \frac{\bar{\mu}\lambda_{1,\mu}}{(\lambda_{1,\mu}-\lambda)(\bar{\mu}-\mu)}S_\mu^{\frac{N}{2}}, \tag{4.9}$$

which implies that $u(t) = u(x, t; u_0)$ is a global solution of (1.1).

From (4.7), we have

$$\int_0^\infty \int_\Omega u_t^2(t)dxdt = I_{\lambda,\mu}(u_0) - I_{\lambda,\mu}(u(t)) \le I_{\lambda,\mu}(u_0). \tag{4.10}$$

Combining (4.9) with (4.10), we may choose a divergent sequence, still denoted by t, such that as $t \to \infty$

$$\int_\Omega u_t^2(t)dx \to 0, \quad u(t) \rightharpoonup v \text{ weakly in } H_0^1(\Omega) \text{ and } u(t) \to v \text{ a.e. in } \Omega. \tag{4.11}$$

For any $\varphi \in H_0^1(\Omega)$, we infer from (1.1) that

$$\int_\Omega u_t(t)\varphi dx + \int_\Omega (\nabla u(t)\nabla\varphi - \mu\frac{u(t)\varphi}{|x|^2})dx = \int_\Omega (|u(t)|^{2^*-2}u(t) + \lambda u(t))\varphi dx. \tag{4.12}$$

By (4.11), we obtain by taking $t \to \infty$ in (4.12)

$$\int_\Omega (\nabla v\nabla\varphi - \mu\frac{v\varphi}{|x|^2})dx = \int_\Omega (|v|^{2^*-2}v + \lambda v)\varphi dx,$$

which implies that $v \in H_0^1(\Omega)$ is a solution of (1.4). In particular,

$$\langle dI_{\lambda,\mu}(v), v\rangle = 0. \tag{4.13}$$

We claim that $v \equiv 0$ in Ω. In fact, if $v \not\equiv 0$, then from (4.13), $v \in \mathcal{N}$, and so

$$I_{\lambda,\mu}(v) \geq c_{\lambda,\mu}. \tag{4.14}$$

On the other hand, since

$$\int_\Omega u(t)u_t(t)dx = -\langle dI_{\lambda,\mu}(u(t)), u(t)\rangle,$$

we conclude from (4.11) and (4.13) that

$$\lim_{t\to\infty}\langle dI_{\lambda,\mu}(u(t)), u(t)\rangle = 0 = \langle dI_{\lambda,\mu}(v), v\rangle. \tag{4.15}$$

Thus, we deduce from (4.15) that

$$\begin{aligned} I_{\lambda,\mu}(v) &= \frac{1}{N}\int_\Omega (|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2})dx - \frac{\lambda}{N}\int_\Omega |v|^2 dx \\ &\leq \lim_{t\to\infty}\frac{1}{N}\int_\Omega (|\nabla u(t)|^2 - \mu\frac{|u(t)|^2}{|x|^2})dx - \frac{\lambda}{N}\lim_{t\to\infty}\int_\Omega |u(t)|^2 dx \\ &= \lim_{t\to\infty} I_{\lambda,\mu}(u(t)) \leq I_{\lambda,\mu}(u_0) < c_{\lambda,\mu}, \end{aligned}$$

which contradicts (4.14). Hence, from (4.15), we deduce that as $t \to \infty$

$$\begin{aligned} \int_\Omega (|\nabla u(t)|^2 - \mu\frac{|u(t)|^2}{|x|^2})dx &= \int_\Omega |u(t)|^{2^*}dx + o(1) \\ &\leq S_\mu^{-\frac{2^*}{2}}\Big(\int_\Omega (|\nabla u(t)|^2 - \mu\frac{|u(t)|^2}{|x|^2})dx\Big)^{\frac{2^*}{2}} + o(1). \end{aligned}$$

So,

$$\int_\Omega (|\nabla u(t)|^2 - \mu\frac{|u(t)|^2}{|x|^2})dx\Big(1 - S_\mu^{-\frac{2^*}{2}}\big(\int_\Omega (|\nabla u(t)|^2 - \mu\frac{|u(t)|^2}{|x|^2})dx\big)^{\frac{2}{N-2}}\Big) \leq o(1),$$

which implies

$$\lim_{t\to\infty}\int_\Omega (|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2})dx=0, \tag{4.16}$$

or

$$\liminf_{t\to\infty}\int_\Omega (|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2})dx\geq S_\mu^{\frac{N}{2}}. \tag{4.17}$$

If (4.17) holds, we infer from (4.15) that

$$\begin{aligned}&\frac{1}{N}S_\mu^{\frac{N}{2}}>c_{\lambda,\mu}>I_{\lambda,\mu}(u_0)\\ &\geq\frac{1}{N}\lim_{t\to\infty}\int_\Omega (|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2})dx+\frac{1}{2^*}\lim_{m\to\infty}\langle dI_{\lambda,\mu}(u(t)),u(t)\rangle\geq\frac{1}{N}S_\mu^{\frac{N}{2}}.\end{aligned}$$

This is a contradiction. Thus, as $t\to\infty$

$$u(t)\longrightarrow 0\ \ \textit{strongly in}\ \ H_0^1(\Omega). \tag{4.18}$$

Now we prove that (1.10) holds. From (4.6) and (4.7), we get

$$\int_\Omega (|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2})dx<NI_{\lambda,\mu}(u_0)+\lambda\int_\Omega |u(t)|^2dx. \tag{4.19}$$

By using (4.18) and the Sobolev inequality, we have

$$\lim_{t\to\infty}\int_\Omega |u(t)|^2dx=0.$$

Thus, there exists $t_0>0$ such that for all $t\geq t_0$

$$NI_{\lambda,\mu}(u_0)+\lambda\int_\Omega |u(t)|^2dx\leq Nc_{\lambda,\mu}. \tag{4.20}$$

Therefore, from (4.19) and (4.20), we obtain

$$\begin{aligned}&\int_\Omega |u(t)|^{2^*}dx\leq S_\mu^{-\frac{2^*}{2}}\Big(\int_\Omega (|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2})dx\Big)^{\frac{2^*}{2}}\\ &<S_\mu^{-\frac{2^*}{2}}\Big(NI_{\lambda,\mu}(u_0)+\lambda\int_\Omega |u(t)|^2dx\Big)^{\frac{2^*}{2}-1}\int_\Omega (|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2})dx\\ &\leq S_\mu^{-\frac{2^*}{2}}\big(Nc_{\lambda,\mu}\big)^{\frac{2^*}{2}-1}\int_\Omega (|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2})dx,\end{aligned}$$

from which we get

$$\big(1-S_\mu^{-\frac{2^*}{2}}\big(Nc_{\lambda,\mu}\big)^{\frac{2^*}{2}-1}\big)\int_\Omega (|\nabla u(t)|^2-\mu\frac{|u(t)|^2}{|x|^2})dx$$

$$\leq \langle dI_{\lambda,\mu}(u(t)), u(t)\rangle + \lambda \int_\Omega |u(t)|^2 dx. \tag{4.21}$$

Noting that $\left(1 - S_\mu^{-\frac{2^*}{2}} \left(Nc_{\lambda,\mu}\right)^{\frac{2^*}{2}-1}\right) \in (0,1)$, and from (4.7), we have for any $t \geq t_0$

$$\begin{aligned}\int_t^\infty \langle dI_{\lambda,\mu}(u(s)), u(s)\rangle ds &= \frac{1}{2}\int_\Omega |u(t)|^2 dx \\ \leq \frac{1}{2\lambda_{1,\mu}} \int_\Omega (|\nabla u(t)|^2 - \mu \frac{|u(t)|^2}{|x|^2}) dx &\leq C I_{\lambda,\mu}(u(t)).\end{aligned} \tag{4.22}$$

Therefore, for any $t \geq t_0$, we get from (4.21) and (4.22) that

$$\begin{aligned}\int_t^\infty I_{\lambda,\mu}(u(s)) ds &\leq \frac{1}{2}\int_t^\infty \int_\Omega (|\nabla u(s)|^2 - \mu \frac{|u(s)|^2}{|x|^2}) dx ds \\ &\leq C \int_t^\infty \langle dI_{\lambda,\mu}(u(s)), u(s)\rangle ds + C \int_t^\infty \int_\Omega |u(s)|^2 dx ds \\ &\leq C I_{\lambda,\mu}(u(t)) + C \int_t^\infty \int_\Omega |u(s)|^2 dx ds.\end{aligned} \tag{4.23}$$

Note that

$$\begin{aligned}&\int_t^\infty \langle dI_{\lambda,\mu}(u(s)), u(s)\rangle ds \\ &\geq \left(1 - \frac{\lambda}{\lambda_{1,\mu}}\right) S_\mu \int_t^\infty \left(\int_\Omega |u(s)|^{2^*} dx\right)^{\frac{2}{2^*}} ds - \int_t^\infty \int_\Omega |u(s)|^{2^*} dx ds \\ &\geq \int_t^\infty (C_1 \|u(s)\|^2_{L^{2^*}(\Omega)} - C_2 \|u(s)\|^{2^*}_{L^{2^*}(\Omega)}) ds.\end{aligned} \tag{4.24}$$

From (4.18), we infer that $\|u(t)\|_{L^{2^*}(\Omega)} \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, there exists $t_1 > t_0$ such that for any $t \geq t_1$, we obtain from (4.22) and (4.24) that

$$\begin{aligned}I_{\lambda,\mu}(u(t)) &\geq C \int_t^\infty \langle dI_{\lambda,\mu}(u(s)), u(s)\rangle ds \geq C_3 \int_t^\infty \|u(s)\|^2_{L^{2^*}(\Omega)} ds \\ &\geq C_4 \int_t^\infty \|u(s)\|^2_{L^2(\Omega)} ds.\end{aligned} \tag{4.25}$$

Inserting (4.25) into (4.23), we conclude that for any $t \geq t_1$

$$\int_t^\infty I_{\lambda,\mu}(u(s)) ds \leq C I_{\lambda,\mu}(u(t)).$$

After a straightforward computation, we derive that for any $t \geq t_1$

$$\int_t^{\infty} I_{\lambda,\mu}(u(s))ds \leq C(t_1)e^{-Ct}. \tag{4.26}$$

Note that

$$I_{\lambda,\mu}(u(t+1)) \leq \int_t^{t+1} I_{\lambda,\mu}(u(s))ds < \int_t^{\infty} I_{\lambda,\mu}(u(s))ds,$$

which, together with (4.26), implies that (1.10) holds. □

Proof of Corollary 1.5. By the Sobolev inequality, we have

$$\int_\Omega |u_0(x)|^{2^*}dx \leq S_\mu^{-\frac{2^*}{2}}\Big(\int_\Omega(|\nabla u_0|^2-\frac{|u_0|^2}{|x|^2})dx\Big)^{\frac{2^*}{2}} \leq \Big(\frac{2c_{\lambda,\mu}}{S_\mu}\Big)^{\frac{2^*}{2}} \leq \Big(\frac{2}{N}S_\mu^{\frac{N-2}{2}}\Big)^{\frac{2^*}{2}}.$$

Thus we obtain

$$\begin{aligned}
&\int_\Omega(|\nabla u_0|^2 - \frac{|u_0|^2}{|x|^2} - \lambda|u_0|^2)dx \geq (1-\frac{\lambda}{\lambda_{1,\mu}})\int_\Omega(|\nabla u_0|^2 - \frac{|u_0|^2}{|x|^2})dx\\
&\geq (1-\frac{\lambda}{\lambda_{1,\mu}})S_\mu\Big(\int_\Omega |u_0|^{2^*}dx\Big)^{\frac{2}{2^*}}\\
&> (1-\frac{\lambda}{\lambda_{1,\mu}})S_\mu\Big(\frac{2}{N}S_\mu^{\frac{N-2}{2}}\Big)^{(\frac{2}{2^*}-1)\frac{2^*}{2}}\int_\Omega |u_0|^{2^*}dx\\
&= (1-\frac{\lambda}{\lambda_{1,\mu}})\Big(\frac{2}{N}\Big)^{-\frac{2}{N-2}}\int_\Omega |u_0|^{2^*}dx \geq \int_\Omega |u_0|^{2^*}dx,
\end{aligned} \tag{4.27}$$

where we have used the result: $(1-\frac{\lambda}{\lambda_{1,\mu}})\big(\frac{2}{N}\big)^{-\frac{2}{N-2}} \geq 1$ for any $\lambda \in [0,\bar{\lambda}]$.

Note that

$$I_{\lambda,\mu}(u_0) < \frac{1}{2}\int_\Omega(|\nabla u_0|^2 - \frac{|u_0|^2}{|x|^2})dx \leq c_{\lambda,\mu},$$

which, together with (4.27), implies that $u_0 \in \mathcal{O}^+$. Therefore, it follows from Theorem 1.4 that the proof is complete. □

5. Blow-up of solutions for problem (1.1) with initial datum in $\mathcal{O}^-$

In this section, by means of the invariance of $\mathcal{O}^-$, we prove the blow-up of solutions if the initial datum is in $\mathcal{O}^-$. That is, we prove Theorem 1.7.

Proof of Theorem 1.7. Let $u(t) = u(x,t;u_0)$ be the unique solution established in Proposition 4.1. We claim that $\mathcal{O}^-$ is invariant under the flow defined in (1.1); that is, $u(t) \in \mathcal{O}^-$ for all $t \in (0, T^*)$. In fact, if there exists $t^{**} \in (0, T^*)$ such that $u(t^{**}) \notin \mathcal{O}^-$, then, from (4.4), we infer that

$$\langle dI_{\lambda,\mu}(u(t^{**})), u(t^{**})\rangle \geq 0.$$

Noting that $t \mapsto \langle dI_{\lambda,\mu}(u(t)), u(t)\rangle$ is continuous and $u_0 \in \mathcal{O}^-$, there exists $t_2 \in (0, t^{**}]$ such that

$$\langle dI_{\lambda,\mu}(u(t_2)), u(t_2)\rangle = 0.$$

Hence, $u(t_2) \equiv 0$ in Ω, or $u(t_2) \in \mathcal{N}$. If $u(t_2) \equiv 0$ in Ω, then by the uniqueness of $u(t)$ for (1.1), we infer that $u(t) \equiv 0$ for any $t \in [t_2, T^*)$. Thus $u(t)$ can be regarded as a global solution by extending to 0 for all $t \geq T^*$, and so $I_{\lambda,\mu}(u(t)) > 0$ for any $t \geq 0$ due to Theorem 1.3. But $I_{\lambda,\mu}(u(t_2)) = 0$, which is a contradiction. Therefore, $u(t_2) \in \mathcal{N}$. By the definition of $c_{\lambda,\mu}$, $I_{\lambda,\mu}(u(t_0)) \geq c_{\lambda,\mu}$, which also contradicts (4.4).

We argue by contradiction that $u(t)$ exists for all $t \geq 0$. Then we claim that

$$\lim_{t\to\infty} \|u(t)\|_{L^2(\Omega)} = \infty. \tag{5.1}$$

Note that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t)|^2 dx = -\langle dI_{\lambda,\mu}(u(t)), u(t)\rangle > 0.$$

We infer that $t \mapsto \|u(t)\|_{L^2(\Omega)}$ is strictly increasing, and so

$$\lim_{t\to\infty} \|u(t)\|_{L^2(\Omega)} = c \in (0, \infty]. \tag{5.2}$$

From (4.10), we may select a divergent sequence, still denoted by t, such that

$$\lim_{t\to\infty} \int_\Omega u_t^2(t)dx = 0. \tag{5.3}$$

Thus, if $c < \infty$ in (5.2), from (5.3), we infer that as $t \to \infty$

$$\begin{aligned} |\langle -dI_{\lambda,\mu}(u(t)), u(t)\rangle| &= \Big|\int_\Omega u(t)u_t(t)dx\Big| \\ &\leq \|u(t)\|_{L^2(\Omega)}\|u_t(t)\|_{L^2(\Omega)} \leq C\|u_t(t)\|_{L^2(\Omega)} \to 0. \end{aligned} \tag{5.4}$$

Note that $I_{\lambda,\mu}(u(t)) < c_{\lambda,\mu}$. From (5.4), we deduce that $\|u(t)\|_{H_0^1(\Omega)} \leq C$. Then we can select a subsequence, still denoted by t, such that as $t \to \infty$

$$\int_\Omega u_t^2(t)dx \to 0, \quad u(t) \rightharpoonup \bar{u} \text{ weakly in } H_0^1(\Omega). \text{ Moreover, } dI_{\lambda,\mu}(\bar{u}) = 0.$$

As in the proof of Theorem 1.4, we also infer that $\bar{u} \equiv 0$ in Ω, and $u(t) \to 0$ strongly in $H_0^1(\Omega)$. Furthermore, we have

$$0 < c = \lim_{t\to\infty} \|u(t)\|_{L^2(\Omega)} \le \lambda_{1,0}^{-\frac{1}{2}} \lim_{t\to\infty} \|\nabla u(t)\|_{L^2(\Omega)} = 0,$$

which is a contradiction. Hence, (5.1) holds.

Set $h(t) = \int_\Omega |u(t)|^2 dx$. Then, by the Hölder inequality, we have

$$\begin{aligned} &-\frac{2}{2^*-2}\frac{d}{dt}h^{1-\frac{2^*}{2}}(t) = h'(t)h^{-\frac{2^*}{2}}(t) = 2\Big(\int_\Omega |u(t)|^2 dx\Big)^{-\frac{2^*}{2}} \\ &\quad\times\Big(\int_\Omega |u(t)|^{2^*} dx - \int_\Omega (|\nabla u(t)|^2 - \mu\frac{|u(t)|^2}{|x|^2} - \lambda|u(t)|^2) dx\Big) \\ &\ge C_5 - C_6\Big(\int_\Omega |\nabla u(t)|^2 dx\Big)^{1-\frac{2^*}{2}}. \end{aligned} \tag{5.5}$$

By using (5.1) and the Poincaré inequality, we infer that

$$\lim_{t\to\infty} \int_\Omega |\nabla u(t)|^2 dx = \infty.$$

Thus there exists large $\bar{t} > 0$ such that, for any $t \ge \bar{t}$,

$$C_5 - C_6\Big(\int_\Omega |u(t)|^2 dx\Big)^{-\frac{2^*}{2}} \ge C_7. \tag{5.6}$$

Inserting (5.6) into (5.5), we get, for any $t \ge \bar{t}$,

$$0 < h(t) \le h(\bar{t}) + \frac{2^*-2}{2}C_7\bar{t} - \frac{2^*-2}{2}C_7 t,$$

which is a contradiction if $t > 0$ is sufficiently large. □

6. Representation of Palais-Smale sequences related to global solutions of (1.1)

In this section, by critical point theory, we characterize the asymptotic behavior of global solutions and the distribution of the corresponding energy as $t \to \infty$.

Proof of Theorem 1.8. Let $u(t)$ be a global solution of (1.1). Then by Theorem 1.3, $I_{\lambda,\mu}(u(t)) > 0$ for all $t > 0$. Using (4.10), we may select a divergent sequence t_m, such that

$$\lim_{m\to\infty} \int_\Omega u_t^2(t_m) dx = 0. \tag{6.1}$$

Moreover, we get

$$I_{\lambda,\mu}(u(t_m)) < I_{\lambda,\mu}(u_0). \tag{6.2}$$

Observe that for any $\varphi \in H_0^1(\Omega)$

$$\begin{aligned}
&|\langle dI_{\lambda,\mu}(u(t_m)), \varphi\rangle| \\
&= \Big| \int_\Omega (\nabla u(t_m)\nabla\varphi - \mu\frac{u(t_m)\varphi}{|x|^2} - \lambda u(t_m)\varphi)dx - \int_\Omega |u(t_m)|^{2^*-2}u(t_m)\varphi dx\Big| \\
&= \Big| \int_\Omega u_t(t_m)\varphi dx\Big| \le \lambda_{1,0}^{-\frac{1}{2}}\|u_t(t_m)\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}.
\end{aligned}$$

Thus, we infer that as $m \to \infty$

$$\|dI_{\lambda,\mu}(u(t_m))\|_{H^{-1}(\Omega)} \le \lambda_{1,0}^{-\frac{1}{2}}\|u_t(t_m)\|_{L^2(\Omega)} \to 0,$$

which together with (6.2) implies that $\{u(t_m)\}$ is a Palais-Smale sequence of the functional $I_{\lambda,\mu}$. By using Theorem 2.1 in [8], we complete the proof of Theorem 1.8. □

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