

**ON A VON KÁRMÁN PLATE SYSTEM
WITH FREE BOUNDARY AND BOUNDARY CONDITIONS
OF MEMORY TYPE**

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Abstract. In this paper we study the existence of weak and strong global solutions and uniform decay of the energy to a von Kármán system for Kirchhoff plates equations with thermal effects and memory conditions working at the boundary. We show that the dissipation produced by the memory effect does not depend on the present values of the temperature gradient. That is, we show that the dissipation produced by memory effect is strong enough to produce exponential decay of the solution provided the relaxation functions also decay exponentially. When the relaxation functions decay polynomially, we show that the solution decays polynomially with the same rate.

1. INTRODUCTION

The main purpose of this work is to study the existence, uniqueness and asymptotic behavior of the solutions to a von Kármán system for Kirchhoff plates equations in the presence of thermal effects with free boundary and boundary conditions of memory type. To introduce this model we need some notation. Let Ω be an open bounded subset of \mathbb{R}^2 with regular boundary Γ . Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal vector on Γ and by $\tau = (-\nu_2, \nu_1)$ the corresponding unit tangent vector. Finally, by the brackets $[\cdot, \cdot]$ we denote the binary differential operator given by

$$[u, v] := \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2}.$$

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Taking into account this notation, we consider the following initial boundary-value problem

$$u_{tt} + \Delta^2 u + \alpha \Delta \theta = [u, v] \quad \text{in } \Omega \times (0, \infty), \quad (1.1)$$

$$\theta_t - \Delta \theta - \alpha \Delta u_t = 0 \quad \text{in } \Omega \times (0, \infty), \quad (1.2)$$

$$\Delta^2 v = -[u, u] \quad \text{in } \Omega \times (0, \infty), \quad (1.3)$$

$$v = \frac{\partial v}{\partial \nu} = \theta = 0 \quad \text{in } \Gamma \times (0, \infty), \quad (1.4)$$

$$\frac{\partial u}{\partial \nu} + \int_0^t g_1(t-s)(\mathfrak{B}_1 u(s) + \rho_1 \frac{\partial u}{\partial \nu}(s)) ds = 0 \quad \text{on } \Gamma \times (0, \infty), \quad (1.5)$$

$$u - \int_0^t g_2(t-s)(\mathfrak{B}_2 u(s) + \alpha \frac{\partial \theta}{\partial \nu}(s) - \rho_2 u(s)) ds = 0 \quad \text{on } \Gamma \times (0, \infty), \quad (1.6)$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \theta(0, x) = \theta_0(x) \quad \text{in } \Omega. \quad (1.7)$$

Here, by $\mathfrak{B}_1, \mathfrak{B}_2$ we are denoting the following differential operators

$$\mathfrak{B}_1 u = \Delta u + (1 - \mu) B_1 u, \quad \mathfrak{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1 - \mu) \frac{\partial B_2 u}{\partial \tau},$$

where $\mu \in]0, \frac{1}{2}[$ is Poisson's ratio and

$$\begin{aligned} B_1 u &= 2\nu_1 \nu_2 \frac{\partial^2 u}{\partial x \partial y} - \nu_1^2 \frac{\partial^2 u}{\partial y^2} - \nu_2^2 \frac{\partial^2 u}{\partial x^2}, \\ B_2 u &= (\nu_1^2 - \nu_2^2) \frac{\partial^2 u}{\partial x \partial y} + \nu_1 \nu_2 \left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} \right). \end{aligned}$$

In (1.1)-(1.3), u denotes the transversal displacement, v the Airy's stress function of the vibrating plate and θ the difference of temperature. In (1.5)-(1.6) the relaxation functions $g_1, g_2 \in C^1(0, \infty)$ are positive and nondecreasing and α, ρ_1, ρ_2 are positive constants. Also, let us assume that there exists $x_0 \in \mathbb{R}^2$ such that $\Gamma = \{x \in \Gamma : \nu(x) \cdot (x - x_0) > 0\}$. Denoting by $m(x) := x - x_0$, the compactness of Γ implies that there exists $\delta_0 > 0$ such that $m(x) \cdot \nu(x) \geq \delta_0 > 0$, for all $x \in \Gamma$.

The Von Kármán system with dissipation was studied by different authors. All of them consider essentially three types of dissipative mechanisms:

(a) the frictional dissipation, obtained by introducing a frictional damping that can be acting either on the boundary or in a neighborhood of the boundary;

(b) the thermal dissipation, which is obtained by coupling the parabolic heat equation;

(c) finally, the viscoelastic dissipation given by the memory effects as in [20, 21].

The frictional damping is the simplest dissipative mechanism when one is working either in the whole domain Ω or over a strategic part of the domain (locally). It was proved by [1, 2, 6, 10, 12, 14, 16, 23] that the first-order energy decays exponentially to zero as time goes to infinity.

The thermal dissipation, produced by the Fourier law for the temperature, is more interesting than the frictional damping, from the physical and mathematical point of view. Here the boundary conditions play an important role, not because of their influence on the asymptotic behavior, but because of the formidable changes of the mathematical aspects, making more difficult the asymptotic analysis. The exponential stability of solutions was proved in [4, 5, 16].

Finally, the memory effect on the stress tensor produces a suitable dissipative mechanism which depends on the relaxation function (see [20, 21]). They proved that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation function; that is, when the relaxation function decays exponentially, the corresponding solution also decays exponentially. On the other hand, when the relaxation function decays polynomially, the solution decays polynomially with the same rate.

The contribution of the present paper is that thermal effects are being considered and only the memory dissipation is active on the free boundary.

As is well known, the presence of thermal effects provides some sort of dissipation for the energy. Thus, it should be expected that in order to achieve the uniform decays for the energy, one may not need the additional mechanical dissipation. In fact, this is the case for many Von Kármán equations in the presence of thermal effects, where uniform rate decays were proved without any mechanical dissipation, see [4, 5, 16].

In the classical linear theory of thermoelasticity Fourier's law is used to describe the heat conduction in the body. This theory has two principal shortcomings. First, it is unable to account for a memory effect which may prevail in some materials, particularly at low temperatures. Second, the corresponding parabolic part of the system predicts an unrealistic result: The thermal disturbance at one point of the body instantly falls everywhere in the body. The observations lead one to believe that for materials with memory, Fourier's law is not a good model, and we have to look for another more general constitutive assumption relating the heat flux to the material's thermal history. Here instead of Fourier's law we follow the Gurtin-Pipkin model for the heat transmission(see [8]).

Being like this, with the objective of completing the obtained results in [4, 5, 16], our paper will work with memory effect acting only in the boundary following the Gurtin-Pipkin model for the heat transmission. The main difference is that this “dissipation” does not depend on the present values of the gradient temperature; it is of nonlocal nature, given by two time convolutions of the relaxation functions on the free boundary. We show that the energy of system (1.1)-(1.7) decays uniformly in time, with rates depending on the rate of decay of the relaxation functions. More precisely, we show that the energy decays exponentially to zero provided g_1, g_2 decay exponentially to zero. When g_1, g_2 decay polynomially, we show that the corresponding energy of system (1.1)-(1.7) also decays polynomially to zero with the same rate of decay. This means that the memory effect produces strong dissipation capable of making a uniform rate of decay for the energy.

The method used is based on the construction of a suitable functional \mathcal{L} satisfying

$$\frac{d}{dt}\mathcal{L}(t) \leq -C_1\mathcal{L}(t) + C_2e^{-\gamma_1 t} \quad \text{or} \quad \frac{d}{dt}\mathcal{L}(t) \leq -C_1\mathcal{L}(t)^{1+\frac{1}{\gamma_2}} + \frac{C_2}{(1+t)^{\alpha+1}},$$

with positive constants C_1, C_2, γ_1 and γ_2 .

Let us define the following bilinear form

$$a(u, v) = \int_{\Omega} \left\{ \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} + \mu \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} \right) + 2(1-\mu) \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} \right\} dx dy.$$

The following Green’s formula will be often used.

Lemma 1.1. *For any $u \in H^4(\Omega)$ and $v \in H^2(\Omega)$, we have*

$$\int_{\Omega} (\Delta^2 u) v dx = a(u, v) + \int_{\Gamma} \left\{ (\mathfrak{B}_2 u) v - (\mathfrak{B}_1 u) \frac{\partial v}{\partial \nu} \right\} d\Gamma. \quad (1.8)$$

Proof. See [14, 15]. □

The notation we use in this paper is standard and can be found in Lion’s book [19]. In the sequel, by C (sometime C_1, C_2, \dots) we denote various positive constants which do not depend on t or on the initial data. The organization of this paper is as follows: In section 2 we establish the existence, regularity and uniqueness of strong solutions for the problem (1.1)-(1.7). We use the Galerkin approximation, energy methods introduced by Lions [19] and Lemma 2.4, a result critical to the proof of uniqueness of weak solution (see [11]). Finally in sections 3 and 4 we study the stability of the strong

solutions for the system (1.1)-(1.7). We show that the dissipation is strong enough to produce exponential (or polynomial) decay of solutions, provided the relaxation functions also decay exponentially (or polynomially). We use the technique of the multipliers introduced by Komornik [13] and Lions [19] coupled with some technical lemmas and some technical ideas.

2. EXISTENCE, UNIQUENESS AND REGULARITY

In this section we shall establish existence and uniqueness of solutions for the system (1.1)-(1.7); moreover we shall prove that the solution is regular provided the initial data are also regular. To facilitate our analysis, we introduce the following binary operators

$$\begin{aligned}(g \square \varphi)(t) &= \int_0^t g(t-s)|\varphi(t) - \varphi(s)|^2 ds, \\ (g * \varphi)(t) &= \int_0^t g(t-s)\varphi(s) ds,\end{aligned}$$

where $*$ is the convolution product. An important relation between these two binary operators is given by the following lemma

Lemma 2.1. *For $g, \varphi \in C^1([0, \infty[: \mathbb{R})$ we have*

$$(g * \varphi)\varphi_t = \frac{1}{2}g' \square \varphi - \frac{1}{2}g(t)|\varphi|^2 - \frac{1}{2} \frac{d}{dt} \left[g \square \varphi - \left(\int_0^t g(s) ds \right) |\varphi|^2 \right].$$

Proof. Differentiating the term $g \square \varphi$ we obtain

$$\begin{aligned}\frac{d}{dt} g \square \varphi &= \int_0^t g'(t-s)|\varphi(t) - \varphi(s)|^2 ds \\ &\quad - 2 \int_0^t g(t-s)\varphi_t(t)\varphi(s) ds + \left(\int_0^t g(t-s) ds \right) \frac{d}{dt} |\varphi(t)|^2,\end{aligned}$$

from which it follows that

$$\begin{aligned}2 \int_0^t g(t-s)\varphi_t(t)\varphi(s) ds &= -\frac{d}{dt} \left\{ g \square \varphi - \int_0^t g(t-s) ds |\varphi(t)|^2 \right\} \\ &\quad + \int_0^t g'(t-s)|\varphi(t) - \varphi(s)|^2 ds - g(t)|\varphi(t)|^2.\end{aligned}$$

The proof is now complete. \square

Let us use boundary condition (1.5)-(1.6) to find an appropriate expression for $\mathfrak{B}_1 u$, $\mathfrak{B}_2 u$ on the boundary. Differentiating the equations (1.5) and (1.6)

we arrive at following Volterra equations

$$\begin{aligned} \left(\mathfrak{B}_1 u + \rho_1 \frac{\partial u}{\partial \nu} \right) + \frac{1}{g_1(0)} g_1' * \left(\mathfrak{B}_1 u + \rho_1 \frac{\partial u}{\partial \nu} \right) &= -\frac{1}{g_1(0)} \frac{\partial u_t}{\partial \nu}, \\ \left(\mathfrak{B}_2 u + \alpha \frac{\partial \theta}{\partial \nu} - \rho_2 u \right) + \frac{1}{g_2(0)} g_2' * \left(\mathfrak{B}_2 u + \alpha \frac{\partial \theta}{\partial \nu} - \rho_2 u \right) &= \frac{1}{g_2(0)} u_t. \end{aligned}$$

Inverting Volterra's integral operator we get

$$\begin{aligned} \mathfrak{B}_1 u + \rho_1 \frac{\partial u}{\partial \nu} &= -\frac{1}{g_1(0)} \left\{ \frac{\partial u_t}{\partial \nu} + k_1 * \frac{\partial u_t}{\partial \nu} \right\}, \\ \mathfrak{B}_2 u + \alpha \frac{\partial \theta}{\partial \nu} - \rho_2 u &= \frac{1}{g_2(0)} \{ u_t + k_2 * u_t \}, \end{aligned}$$

where the resolvent kernels k_i , $i = 1, 2$ are given by the solutions of

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i'.$$

Denoting by $\eta_1 = \frac{1}{g_1(0)}$ and $\eta_2 = \frac{1}{g_2(0)}$ we can rewrite $\mathfrak{B}_1 u$, $\mathfrak{B}_2 u$ as

$$\mathfrak{B}_1 u = -\rho_1 \frac{\partial u}{\partial \nu} - \eta_1 \left\{ \frac{\partial u_t}{\partial \nu} + k_1(0) \frac{\partial u}{\partial \nu} - k_1(t) \frac{\partial u_0}{\partial \nu} + k_1' * \frac{\partial u}{\partial \nu} \right\}, \quad (2.1)$$

$$\mathfrak{B}_2 u = -\alpha \frac{\partial \theta}{\partial \nu} + \rho_2 u + \eta_2 \{ u_t + k_2(0)u - k_2(t)u_0 + k_2' * u \}. \quad (2.2)$$

Since we are interested in relaxation functions of exponential or polynomial type and the boundary conditions (2.1)-(2.2) involve the resolvent kernels k_i , we want to know whether k_i has the same decay properties or not. The following lemma answers this question. Let h be a relaxation function and k its resolvent kernel; that is,

$$k(t) - k * h(t) = h(t). \quad (2.3)$$

Lemma 2.2. *If h is a positive continuous function, then k is also a positive continuous function. Moreover:*

- (1) *If there exist positive constants C_0 and γ with $C_0 < \gamma$ such that $h(t) \leq C_0 e^{-\gamma t}$, then the function k satisfies $k(t) \leq \frac{C_0(\gamma - \epsilon)}{\gamma - \epsilon - C_0} e^{-\epsilon t}$, for all $0 < \epsilon < \gamma - C_0$.*
- (2) *Given $p > 1$, let us denote*

$$C_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p} ds.$$

If there exists a positive constant C_0 with $C_0 C_p < 1$ such that $h(t) \leq C_0(1+t)^{-p}$, then the function k satisfies $k(t) \leq \frac{C_0}{1-C_0 C_p}(1+t)^{-p}$.

Proof. Note that $k(0) = h(0) > 0$. Now, we take $t_0 = \inf\{t \in \mathbb{R}^+ : k(t) = 0\}$, so $k(t) > 0$ for all $t \in [0, t_0[$. If $t_0 \in \mathbb{R}^+$, from equation (2.3) we get that $-k * h(t_0) = h(t_0)$ but this is a contradiction. Therefore $k(t) > 0$ for all $t \in \mathbb{R}_0^+$. Now, let us fix ϵ , such that $0 < \epsilon < \gamma - C_0$ and denote by

$$k_\epsilon(t) := e^{\epsilon t} k(t), \quad h_\epsilon(t) := e^{\epsilon t} h(t).$$

Multiplying equation (2.3) by $e^{\epsilon t}$ we get $k_\epsilon(t) = h_\epsilon(t) + k_\epsilon * h_\epsilon(t)$, hence

$$\begin{aligned} \sup_{s \in [0, t]} k_\epsilon(s) &\leq \sup_{s \in [0, t]} h_\epsilon(s) + \left(\int_0^\infty C_0 e^{(\epsilon - \gamma)s} ds \right) \sup_{s \in [0, t]} k_\epsilon(s) \\ &\leq C_0 + \frac{C_0}{(\gamma - \epsilon)} \sup_{s \in [0, t]} k_\epsilon(s). \end{aligned}$$

Therefore, $k_\epsilon(t) \leq \frac{C_0(\gamma - \epsilon)}{\gamma - \epsilon - C_0}$, which implies our first assertion. To show the second part let us introduce the following notation

$$k_p(t) := (1+t)^p k(t), \quad h_p(t) := (1+t)^p h(t).$$

Multiplying equation (2.3) by $(1+t)^p$ we get

$$k_p(t) = h_p(t) + \int_0^t k_p(t-s)(1+t-s)^{-p}(1+t)^p h(s) ds$$

hence

$$\sup_{s \in [0, t]} k_p(s) \leq \sup_{s \in [0, t]} h_p(s) + C_0 C_p \sup_{s \in [0, t]} k_p(s) \leq C_0 + C_0 C_p \sup_{s \in [0, t]} k_p(s).$$

Therefore, $k_p(t) \leq \frac{C_0}{1-C_0 C_p}$, which proves our second assertion. \square

Remark. The finiteness of the constant C_p can be found in [24, Lemma 7.4].

Due to the above lemma, in the remainder of this paper we shall use conditions (2.1)-(2.2) instead of (1.5)-(1.6). Some important properties for the brackets binary operator is given in the following lemmas.

Lemma 2.3. *Let u, v and w be functions in $H^2(\Omega)$, such that $v \in H_0^2(\Omega)$, where Ω is an open bounded and connected subset of \mathbb{R}^2 with smooth boundary. Then we have that*

$$\int_{\Omega} w[v, u] dx = \int_{\Omega} v[w, u] dx.$$

Proof. See [14]. \square

Let G be the inverse of the biharmonic operator with Dirichlet boundary condition; that is, $G(f) = w$ where w is the solution of

$$\Delta^2 w = f \quad \text{in } \Omega, \quad w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma.$$

Lemma 2.4. (i) The map $(u, v) \rightarrow [u, v]$ is bounded from $H^2(\Omega) \times H^3(\Omega) \rightarrow L^2(\Omega)$. (ii) The map $(u, w) \rightarrow G(-[u, w])$ is bounded from $H^2(\Omega) \times H^2(\Omega) \rightarrow H^3(\Omega) \cap W^{2,\infty}(\Omega) \cap W^{4,1}(\widehat{\Omega})$, for $\widehat{\Omega} \subset \Omega$.

Proof. See the addendum to [11]. \square

Remark. Lemma 2.4 is a result critical to the proof of uniqueness in [11].

Definition 2.1. We say that the triple (u, v, θ) is a weak solution of equation (1.1)-(1.7) when $u, u_t \in L^\infty(0, T : H^2(\Omega) \times H^1(\Omega))$, $v \in L^\infty(0, T : H_0^2(\Omega))$, $\theta \in L^\infty(0, T : L^2(\Omega))$ and satisfy

$$\begin{aligned} & - \int_0^T \int_\Omega u_t \psi_t dx dt + \int_0^T a(u, \psi) dt + \alpha \int_0^T \int_\Omega \theta \Delta \psi dx dt \\ & = \int_0^T \int_\Omega [u, \psi] v dx dt + \int_\Omega u_1 \psi(\cdot, 0) dx \\ & \quad - \eta_1 \int_0^T \int_\Gamma \left\{ \frac{\partial u_t}{\partial \nu} + \left(\frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u}{\partial \nu} - k_1(t) \frac{\partial u_0}{\partial \nu} + k_1' * \frac{\partial u}{\partial \nu} \right\} \frac{\partial \psi}{\partial \nu} d\Gamma dt \\ & \quad - \eta_2 \int_0^T \int_\Gamma \left\{ u_t + \left(\frac{\rho_2}{\eta_2} + k_2(0) \right) u - k_2(t) u_0 + k_2' * u \right\} \psi d\Gamma dt, \end{aligned} \quad (2.4)$$

$$\begin{aligned} & - \int_0^T \int_\Omega \theta \phi_t dx dt + \int_0^T \int_\Omega \nabla \theta \cdot \nabla \phi dx dt - \alpha \int_0^T \int_\Omega \nabla u \cdot \nabla \phi_t dx dt \\ & = \int_\Omega \theta(0) \phi(0) dx + \alpha \int_\Omega \nabla u(0) \cdot \nabla \phi(0) dx \end{aligned} \quad (2.5)$$

$$\int_0^T \int_\Omega \Delta v \Delta \varphi dx dt = - \int_0^T \int_\Omega [u, \varphi] u dx dt, \quad (2.6)$$

for any functions $\varphi \in H_0^2(\Omega)$, $\psi \in C^1([0, T] : H^2(\Omega))$ such that $\psi(\cdot, T) = 0$ and $\phi \in C^1([0, T] : H_0^1(\Omega))$ such that $\phi(\cdot, T) = 0$.

Note that the bilinear form given by

$$(u, w) \mapsto a(u, w) + \int_\Gamma \left(\rho_1 \frac{\partial u}{\partial \nu} \frac{\partial w}{\partial \nu} + \rho_2 u w \right) d\Gamma$$

is strictly coercive on $H^2(\Omega)$. Let us introduce the energy functional

$$\begin{aligned} E(t, u, \theta, v) &:= \frac{1}{2} \int_{\Omega} |u_t|^2 dx + \frac{1}{2} a(u, u) + \frac{1}{2} \int_{\Omega} |\theta|^2 dx + \frac{1}{4} \int_{\Omega} |\Delta v|^2 dx \\ &+ \frac{1}{2} \int_{\Gamma} \left(\rho_1 \left| \frac{\partial u}{\partial \nu} \right| + \rho_2 |u|^2 \right) d\Gamma + \frac{\eta_1}{2} \int_{\Gamma} \left(k_1(t) \left| \frac{\partial u}{\partial \nu} \right|^2 - k_1' \square \frac{\partial u}{\partial \nu} \right) d\Gamma \\ &+ \frac{\eta_2}{2} \int_{\Gamma} \left(k_2(t) |u|^2 - k_2' \square u \right) d\Gamma. \end{aligned}$$

Now, we are able to prove the existence of weak solutions for von Kármán systems with thermal effects and boundary conditions of memory type (1.1)-(1.7).

Theorem 2.1. *Let $k_i \in C^2(0, \infty)$ be such that $k_i, -k_i', k_i'' \geq 0$ for $i = 1, 2$. If the initial data $(u_0, u_1, \theta_0) \in H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ then there exists a unique weak solution for the system (1.1)-(1.7).*

Proof. The main idea is to use the Galerkin method. Let us denote by $\{w_j : j \in \mathbb{N}\}$ a basis to $H^2(\Omega)$ and $\{z_j : j \in \mathbb{N}\}$ a basis to $H_0^1(\Omega) \cap H^2(\Omega)$. Let us take $u^m(\cdot, t) \in W_m$ and $\theta^m(\cdot, t) \in Z_m$, where W_m and Z_m are generated by w_1, w_2, \dots, w_m and z_1, z_2, \dots, z_m respectively. Standard results on ordinary differential equations guarantee that there exists only one local solution

$$u^m(\cdot, t) = \sum_{j=1}^m h_{j,m}(t) w_j(\cdot), \quad \theta^m(\cdot, t) = \sum_{j=1}^m f_{j,m}(t) z_j(\cdot)$$

of the approximate system

$$\begin{aligned} &\int_{\Omega} u_{tt}^m w_j dx + a(u^m, w_j) - \alpha \int_{\Omega} \nabla \theta^m \cdot \nabla w_j dx - \int_{\Omega} [u^m, G(-[u^m, u^m])] w_j dx \\ &= -\eta_1 \int_{\Gamma} \left\{ \frac{\partial u_t^m}{\partial \nu} + \left(\frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u^m}{\partial \nu} - k_1(t) \frac{\partial u_{0,m}}{\partial \nu} + k_1' * \frac{\partial u^m}{\partial \nu} \right\} \frac{\partial w_j}{\partial \nu} d\Gamma \\ &\quad - \eta_2 \int_{\Gamma} \left\{ u_t^m + \left(\frac{\rho_2}{\eta_2} + k_2(0) \right) u^m - k_2(t) u_{0,m} + k_2' * u^m \right\} w_j d\Gamma, \end{aligned} \quad (2.7)$$

$$\int_{\Omega} \theta_t^m z_j dx + \int_{\Omega} \theta^m \cdot \nabla z_j dx + \alpha \int_{\Omega} \nabla u_t^m \cdot \nabla z_j dx = 0 \quad (2.8)$$

for $j = 1, \dots, m$ with initial data

$$\begin{aligned} u^m(\cdot, 0) &= u_{0,m} \rightarrow u_0 \quad \text{in } H^2(\Omega), \\ u_t^m(\cdot, 0) &= u_{1,m} \rightarrow u_1 \quad \text{in } L^2(\Omega), \\ \theta^m(\cdot, 0) &= \theta_{0,m} \rightarrow \theta_0 \quad \text{in } L^2(\Omega). \end{aligned}$$

We define v^m as

$$v^m(t) = G(-[u^m(t), u^m(t)]), \quad \text{with } v^m \in H_0^2(\Omega),$$

where

$$\int_{\Omega} \Delta v^m(t) \Delta w_j dx = - \int_{\Omega} [u^m(t), u^m(t)] w_j dx. \quad (2.9)$$

The extension of these solutions to the whole interval $[0, T]$, $0 < T < \infty$, is a consequence of the estimate which we are going to prove below.

A priori estimate. Multiplying the equations (2.7) and (2.8) by $h'_{j,m}$ and $f_{j,m}$, respectively, and summing up the product result in $j = 1, 2, \dots, m$ we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_t^m|^2 dx + \frac{1}{2} \frac{d}{dt} a(u^m, u^m) + \frac{1}{2} \int_{\Omega} |\theta^m|^2 dx \\ & - \int_{\Omega} [u^m, u_t^m] v^m dx + \int_{\Omega} |\nabla \theta^m|^2 dx \\ & = -\eta_1 \int_{\Gamma} \left\{ \left| \frac{\partial u_t^m}{\partial \nu} \right|^2 + \left(\frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u^m}{\partial \nu} \frac{\partial u_t^m}{\partial \nu} \right. \\ & \quad \left. - k_1(t) \frac{\partial u_{0,m}}{\partial \nu} \frac{\partial u_t^m}{\partial \nu} + k_1' * \frac{\partial u^m}{\partial \nu} \frac{\partial u_t^m}{\partial \nu} \right\} d\Gamma \\ & - \eta_2 \int_{\Gamma} \left\{ |u_t^m|^2 + \left(\frac{\rho_2}{\eta_2} + k_2(0) \right) u^m u_t^m - k_2(t) u_{0,m} u_t^m + k_2' * u^m u_t^m \right\} d\Gamma. \end{aligned} \quad (2.10)$$

Using Lemma 2.1 we have

$$\begin{aligned} & \int_{\Gamma} k_1' * \frac{\partial u^m}{\partial \nu} \frac{\partial u_t^m}{\partial \nu} d\Gamma = -\frac{1}{2} k_1'(t) \int_{\Gamma} \left| \frac{\partial u^m}{\partial \nu} \right|^2 d\Gamma + \frac{1}{2} \int_{\Gamma} k_1'' \square \frac{\partial u^m}{\partial \nu} d\Gamma \\ & - \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \left[k_1' \square \frac{\partial u^m}{\partial \nu} - \left(\int_0^t k_1'(s) ds \right) \left| \frac{\partial u^m}{\partial \nu} \right|^2 \right] d\Gamma, \\ & \int_{\Gamma} k_2' * u^m u_t^m d\Gamma = -\frac{1}{2} k_2'(t) \int_{\Gamma} |u^m|^2 d\Gamma + \frac{1}{2} \int_{\Gamma} k_2'' \square u^m d\Gamma \\ & - \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \left[k_2' \square u^m - \left(\int_0^t k_2'(s) ds \right) |u^m|^2 \right] d\Gamma. \end{aligned}$$

Noting that

$$\int_{\Omega} [u^m, u_t^m] v^m dx = \frac{1}{2} \int_{\Omega} \frac{d}{dt} ([u^m, u^m]) v^m dx = -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta v^m|^2 dx$$

and taking into account the last equalities we obtain

$$\frac{d}{dt}E(t, u^m, \theta^m, v^m) + \int_{\Omega} |\nabla \theta^m|^2 dx \leq CE(0, u^m, \theta^m, v^m).$$

Integrating it over $[0, t]$ and taking into account the definition of the initial data of u^m and θ^m we conclude that

$$E(t, u^m, \theta^m, v^m) + \int_0^t \int_{\Omega} |\nabla \theta^m|^2 dx dt \leq C, \forall t \in [0, T], \forall m \in \mathbb{N}.$$

From the above estimate follows

$$\begin{aligned} u^m &\rightharpoonup u && \text{weak-star in } L^\infty(0, T; H^2(\Omega)), \\ u_t^m &\rightharpoonup u_t && \text{weak-star in } L^\infty(0, T; L^2(\Omega)), \\ v^m &\rightharpoonup v && \text{weak-star in } L^\infty(0, T; H_0^2(\Omega)), \\ \theta^m &\rightharpoonup \theta && \text{weak-star in } L^\infty(0, T; L^2(\Omega)), \\ \nabla \theta^m &\rightharpoonup \nabla \theta && \text{weak in } L^2(0, T; L^2(\Omega)). \end{aligned}$$

Multiplying equation (2.7) by $\beta \in C^1([0, T])$, such that $\beta(T) = 0$ and integrating over $[0, T]$ we have

$$\begin{aligned} & - \int_0^T \int_{\Omega} u_t^m w_j \beta_t dx dt + \int_0^T a(u^m, w_j) \beta dt - \alpha \int_0^T \int_{\Omega} \theta^m \beta \Delta w_j dx dt \\ & = \int_0^T \int_{\Omega} [u^m, w_j] v^m \beta dx dt - \int_{\Omega} u_{0,m} w_j \beta_t(0) dx + \int_{\Omega} u_{1,m} w_j \beta(0) dx \\ & - \eta_1 \int_0^T \int_{\Gamma} \left\{ \frac{\partial u_t^m}{\partial \nu} + \left(\frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u^m}{\partial \nu} - k_1(t) \frac{\partial u_{0,m}}{\partial \nu} + k_1' * \frac{\partial u^m}{\partial \nu} \right\} \frac{\partial w_j}{\partial \nu} \beta d\Gamma dt \\ & - \eta_2 \int_0^T \int_{\Gamma} \left\{ u_t^m + \left(\frac{\rho_2}{\eta_2} + k_2(0) \right) u^m - k_2(t) u_{0,m} + k_2' * u^m \right\} w_j \beta dx dt. \end{aligned}$$

Since $[u^m, w_j] \rightharpoonup [u, w_j]$ weakly in $L^2(0, T; L^2(\Omega))$ and $u^m \rightarrow u$ strongly in $L^2(0, T; L^2(\Omega))$ and using the density of the set $\{w_j \beta : j \in \mathbb{N}, \beta \in C^1([0, T])\}$ in $C^0([0, T]; H^2(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ we show (2.4).

Similarly, multiplying (2.8) by β we have

$$\begin{aligned} & - \int_0^T \int_{\Omega} \theta^m z_j \beta_t dx dt + \int_0^T \int_{\Omega} \nabla \theta^m \cdot \nabla z_j \beta dx dt - \alpha \int_0^T \int_{\Omega} \nabla u^m \cdot \nabla z_j \beta dx dt \\ & = \int_{\Omega} \theta_{0,m} z_j \beta(0) dx + \alpha \int_{\Omega} \nabla u_{0,m} \cdot \nabla z_j \beta(0) dx; \end{aligned}$$

letting $m \rightarrow \infty$ we have (2.5). Using equation (2.9) with $\varphi \in H_0^2(\Omega)$ and integrating over $[0, T]$, we have

$$\int_0^T \int_{\Omega} \Delta v^m \Delta \varphi dx dt = - \int_0^T \int_{\Omega} [u^m, u^m] \varphi dx dt.$$

Since $[u^m, u^m]$ converges in the sense of distributions when $m \rightarrow \infty$, we get (2.6). This completes the proof of the existence.

Uniqueness. Let (u^1, θ^1, v^1) and (u^2, θ^2, v^2) be two solutions of (1.1)-(1.6) with the same initial data and take $(u, \theta, v) := (u^1 - u^2, \theta^1 - \theta^2, v^1 - v^2)$. In this conditions (u, θ, v) has null initial data and satisfies

$$\begin{aligned} & \frac{d}{dt} E_0(t, u, \theta, v) \tag{2.11} \\ &= \int_{\Omega} ([u, v^1] u_t + [u^2, v] u_t) dx - \frac{\eta_1}{2} \int_{\Gamma} \left(2 \left| \frac{\partial u_t}{\partial \nu} \right|^2 + k_1'' \square \frac{\partial u}{\partial \nu} - k_1'(t) \left| \frac{\partial u}{\partial \nu} \right|^2 \right) d\Gamma \\ & - \frac{\eta_2}{2} \int_{\Gamma} \left(2 |u_t|^2 + k_2'' \square u - k_2'(t) |u|^2 \right) d\Gamma - \int_{\Omega} |\nabla \theta|^2 dx, \end{aligned}$$

where

$$\begin{aligned} & E_0(t, u, \theta, v) \\ &:= \frac{1}{2} \int_{\Omega} |u_t|^2 dx + \frac{1}{2} a(u, u) + \frac{1}{2} \int_{\Gamma} \left(\rho_1 \left| \frac{\partial u}{\partial \nu} \right|^2 + \rho_2 |u|^2 \right) d\Gamma + \frac{1}{2} \int_{\Omega} |\theta|^2 dx \\ & + \frac{\eta_1}{2} \int_{\Gamma} \left(k_1(t) \left| \frac{\partial u}{\partial \nu} \right|^2 - k_1' \square \frac{\partial u}{\partial \nu} \right) d\Gamma + \frac{\eta_2}{2} \int_{\Gamma} (k_2(t) |u|^2 - k_2' \square u) d\Gamma. \end{aligned}$$

Now, we will estimate the first term of the right-hand side of (2.11). Following remark 0.2 in the addendum to [11] and the second part of Lemma 2.4 we get

$$\begin{aligned} & \int_{\Omega} ([u, v^1] u_t + [u^2, v] u_t) dx \leq \| [u, v^1] \|_{L^2} \| u_t \|_{L^2} + \| [u^2, v] \|_{L^2} \| u_t \|_{L^2} \\ & \leq C \left(\| u \|_{H^2} \| v^1 \|_{W^{2,\infty}} \| u_t \|_{L^2} + \| u^2 \|_{H^2} \| v \|_{W^{2,\infty}} \| u_t \|_{L^2} \right) \\ & \leq C \left(\| u \|_{H^2} \| v^1 \|_{H^2}^2 \| u_t \|_{L^2} + \| u^2 \|_{H^2} \| v \|_{H^2}^2 \| u_t \|_{L^2} \right). \end{aligned}$$

Since $\Delta^2 v = -[u, u^1 + u^2]$ and using the same argument as above we obtain

$$\begin{aligned} & \int_{\Omega} ([u, v^1] u_t + [u^2, v] u_t) dx \\ & \leq C \left(\| u \|_{H^2} \| u^1 \|_{H^2}^2 \| u_t \|_{L^2} + \| u^2 \|_{H^2} \| u \|_{H^2} \| u^1 + u^2 \|_{H^2} \| u_t \|_{L^2} \right) \end{aligned}$$

$$\leq C\|u\|_{H^2}\|u_t\|_{L^2}.$$

Substitution of the last inequalities into (2.11) implies

$$\begin{aligned} & \frac{d}{dt}E_0(t, u, \theta, v) \\ & \leq C\left\{ \int_{\Omega} |u_t|^2 dx + a(u, u) \right\} - \frac{\eta_1}{2} \int_{\Gamma} \left(2 \left| \frac{\partial u_t}{\partial \nu} \right|^2 + k_1'' \square \frac{\partial u}{\partial \nu} - k_1'(t) \left| \frac{\partial u}{\partial \nu} \right|^2 \right) d\Gamma \\ & \quad - \frac{\eta_2}{2} \int_{\Gamma} \left(2 |u_t|^2 + k_2'' \square u - k_2'(t) |u|^2 \right) d\Gamma - \int_{\Omega} |\nabla \theta|^2 dx. \end{aligned}$$

Integrating this inequality over $[0, t]$, taking into account that the initial data are null and applying Gronwall's inequality our conclusion follows. \square

To show the regularity result we will use the following lemma:

Lemma 2.5. *Suppose that $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\Gamma)$ and $h \in H^{\frac{3}{2}}(\Gamma)$. Then any solution of*

$$a(u, w) = \int_{\Omega} f w dx + \int_{\Gamma} g w d\Gamma + \int_{\Gamma} h \frac{\partial w}{\partial \nu} d\Gamma, \quad \forall w \in H^2(\Omega)$$

satisfies $u \in H^4(\Omega)$.

Proof. See [18]. \square

The regularity of the solution is established in the next theorem.

Theorem 2.2. *Let $k_i \in C^2(0, \infty)$ be such that $k_i, -k_i', k_i'' \geq 0$ for $i = 1, 2$. If the initial data (u_0, u_1, θ_0) belongs to $H^4(\Omega) \times H^2(\Omega) \times (H_0^1(\Omega) \cap H^2(\Omega))$ and satisfies the compatibility conditions*

$$\mathfrak{B}_1 u_0 = -\rho_1 \frac{\partial u_0}{\partial \nu} - \eta_1 \frac{\partial u_1}{\partial \nu}, \quad \mathfrak{B}_2 u_0 = -\alpha \frac{\partial \theta_0}{\partial \nu} + \rho_2 u_0 + \eta_2 u_1 \quad \text{on } \Gamma, \quad (2.12)$$

then the solution of (1.1)-(1.7) has the following regularity

$$\begin{aligned} & (u, u_t, \theta) \in C([0, T] : H^2(\Omega) \cap H^4(\Omega) \times H^2(\Omega) \times (H_0^1(\Omega) \cap H^2(\Omega))) \\ & v \in C([0, T]; H_0^2(\Omega) \cap H^4(\Omega)). \end{aligned}$$

Proof. Differentiating equation (2.7) with respect to time, multiplying by $h_{j,m}''(t)$ and summing up the product results in j we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_{tt}^m|^2 dx + \frac{1}{2} \frac{d}{dt} a(u_t^m, u_t^m) + \alpha \int_{\Omega} \theta_t^m \Delta u_{tt}^m dx \\ & = \int_{\Omega} [u_t^m, v^m] u_{tt}^m dx + \int_{\Omega} [u^m, v_t^m] u_{tt}^m dx \end{aligned}$$

$$\begin{aligned}
& -\eta_1 \int_{\Gamma} \left\{ \frac{\partial u_{tt}^m}{\partial \nu} + \left(\frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u_t^m}{\partial \nu} + k_1' * \frac{\partial u_t^m}{\partial \nu} \right\} \frac{\partial u_{tt}^m}{\partial \nu} d\Gamma \\
& -\eta_2 \int_{\Gamma} \left\{ u_{tt}^m + \left(\frac{\rho_2}{\eta_2} + k_2(0) \right) u_t^m + k_2' * u_t^m \right\} u_{tt}^m d\Gamma. \quad (2.13)
\end{aligned}$$

On the other hand, differentiating (2.8) with respect to time, multiplying the product result by $f'_{j,m}(t)$ and summing up in j yields

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\theta_t^m|^2 dx + \int_{\Omega} |\nabla \theta_t^m|^2 dx = \alpha \int_{\Omega} \theta_t^m \Delta u_{tt}^m dx. \quad (2.14)$$

Since

$$\int_{\Omega} [u_t^m, v^m] u_{tt}^m dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} [u_t^m, u_t^m] v^m dx - \frac{1}{2} \int_{\Omega} [u_t^m, u_t^m] v_t^m dx, \quad (2.15)$$

substituting (2.14) and (2.15) in (2.13) and noting that

$$\begin{aligned}
& -\frac{1}{2} \int_{\Omega} [u_t^m, u_t^m] v_t^m dx + \int_{\Omega} [u_t^m, v_t^m] u_{tt}^m dx \\
& = -\frac{3}{2} \int_{\Omega} [u_t^m, u_t^m] v_t^m dx - \frac{1}{4} \frac{d}{dt} \int_{\Omega} |\Delta v_t^m|^2 dx,
\end{aligned}$$

we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_{tt}^m|^2 dx + \frac{1}{2} \frac{d}{dt} a(u_t^m, u_t^m) + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\theta_t^m|^2 dx \\
& + \frac{1}{4} \frac{d}{dt} \int_{\Omega} |\Delta v_t^m|^2 dx - \frac{1}{2} \frac{d}{dt} \int_{\Omega} [u_t^m, u_t^m] v^m dx + \int_{\Omega} |\nabla \theta_t^m|^2 dx \\
& = -\frac{3}{2} \int_{\Omega} [u_t^m, u_t^m] v_t^m dx \\
& - \eta_1 \int_{\Gamma} \left\{ \frac{\partial u_{tt}^m}{\partial \nu} + \left(\frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u_t^m}{\partial \nu} + k_1' * \frac{\partial u_t^m}{\partial \nu} \right\} \frac{\partial u_{tt}^m}{\partial \nu} d\Gamma \\
& - \eta_2 \int_{\Gamma} \left\{ u_{tt}^m + \left(\frac{\rho_2}{\eta_2} + k_2(0) \right) u_t^m + k_2' * u_t^m \right\} u_{tt}^m d\Gamma.
\end{aligned}$$

Using Lemma 2.1 we obtain

$$\begin{aligned}
& \frac{d}{dt} \left\{ E(t, u_t^m, \theta_t^m, v_t^m) - \frac{1}{2} \int_{\Omega} [u_t^m, u_t^m] v^m dx \right\} + \int_{\Omega} |\nabla \theta_t^m|^2 dx \\
& = -\frac{3}{2} \int_{\Omega} [u_t^m, u_t^m] v_t^m dx - \frac{\eta_1}{2} \int_{\Gamma} \left\{ 2 \left| \frac{\partial u_{tt}^m}{\partial \nu} \right|^2 - k_1'(t) \left| \frac{\partial u_t^m}{\partial \nu} \right|^2 + k_1'' \square \frac{\partial u_t^m}{\partial \nu} \right\} d\Gamma \\
& - \frac{\eta_2}{2} \int_{\Gamma} \left\{ 2 |u_{tt}^m|^2 - k_2'(t) |u_t^m|^2 + k_2'' \square u_t^m \right\} d\Gamma. \quad (2.16)
\end{aligned}$$

Integrating this identity over $[0, t]$ we obtain

$$\begin{aligned}
& E(t, u_t^m, \theta_t^m, v_t^m) + \int_0^t \int_{\Omega} |\nabla \theta_t^m|^2 dx dt \\
& \leq E(0, u_t^m, \theta_t^m, v_t^m) + \frac{1}{2} \int_{\Omega} [u_t^m, u_t^m] v_t^m dx \\
& \quad - \frac{1}{2} \int_{\Omega} [u_t^m(0), u_t^m(0)] v_t^m(0) dx - \frac{3}{2} \int_0^t \int_{\Omega} [u_t^m, v_t^m] u_t^m dx dt.
\end{aligned} \tag{2.17}$$

Using Lemma 2.4 and a first-order estimate we have

$$\begin{aligned}
& \left| \int_{\Omega} [u_t^m, v_t^m] u_t^m dx \right| \leq \|[u_t^m, v_t^m]\|_{L^2} \|u_t^m\|_{L^2} \\
& \leq C \|u_t^m\|_{H^2} \|v_t^m\|_{W^{2,\infty}} \|u_t^m\|_{L^2} \leq C \|u_t^m\|_{H^2} \|u^m\|_{H^2}^2 \|u_t^m\|_{L^2} \\
& \leq \epsilon \|u_t^m\|_{H^2}^2 + C_{\epsilon} \|u^m\|_{H^2}^4 \|u_t^m\|_{L^2}^2 \leq \epsilon \|u_t^m\|_{H^2}^2 + C_{\epsilon}.
\end{aligned}$$

Similarly,

$$\left| -\frac{1}{2} \int_{\Omega} [u_t^m(0), u_t^m(0)] v_t^m(0) dx \right| \leq \epsilon \|u_t^m(0)\|_{H^2}^2 + C_{\epsilon}$$

and

$$\begin{aligned}
& \left| -\int_{\Omega} [u_t^m, v_t^m] u_t^m dx \right| \leq \|[u_t^m, v_t^m]\|_{L^2} \|u_t^m\|_{L^2} \\
& \leq C \|u_t^m\|_{H^2} \|v_t^m\|_{W^{2,\infty}} \|u_t^m\|_{L^2} \leq C \|u_t^m\|_{H^2}^2 \|u^m\|_{H^2} \|u_t^m\|_{L^2} \leq C \|u_t^m\|_{H^2}^2,
\end{aligned}$$

where C_{ϵ} and C are constants positive that depend on u_0 and u_1 . By substitution of these three estimates into (2.17) and using Gronwall's inequality we conclude that (u_{tt}^m, u_t^m) is bounded in $L^{\infty}(0, T; L^2(\Omega) \times H^2(\Omega))$, $(\theta_t^m, \nabla \theta_t^m)$ is bounded in $L^{\infty}(0, T; L^2(\Omega) \times L^2(\Omega))$, and v_t^m is bounded in $L^{\infty}(0, T; H_0^2(\Omega))$. Using equation (2.8) and taking a projection over $Z_m = [z_1, \dots, z_m]$, we conclude that

$$P_m(\Delta \theta^m) = \theta_t^m - \alpha P_m(\Delta u_t^m),$$

and therefore, $\Delta \theta^m$ is bounded in $L^{\infty}(0, T; L^2(\Omega))$. The estimate above implies that $(u_{tt}^m, u_t^m, \theta_t^m) \rightarrow (u_{tt}, u_t, \theta_t)$ weak-star in $L^{\infty}(0, T; L^2(\Omega) \times H^2(\Omega) \times L^2(\Omega))$. By Lemma 2.4 we have that $[u, u] \in L^2(\Omega)$, then, using (1.3), we obtain $\Delta^2 v \in L^2(\Omega)$, which implies that $v \in H^4(\Omega)$. From equation (1.2), using $\theta_t, \Delta u_t \in L^2(\Omega)$ and the elliptic regularity once more we have $\theta \in L^{\infty}(0, T; H^2(\Omega))$. Integrating by parts equation (2.4) with respect to

time we get

$$\begin{aligned} a(u, w) &= - \int_{\Omega} \{u_{tt} - [u, v]\} w dx + \int_{\Omega} \nabla \theta \cdot \nabla w dx \\ &\quad - \eta_2 \int_{\Gamma} \{u_t + k_2(0)u - k_2(t)u_0 + k_2' * u\} w d\Gamma \\ &\quad + \eta_1 \int_{\Gamma} \left\{ - \frac{\partial u_t}{\partial \nu} - k_1(0) \frac{\partial u}{\partial \nu} + k_1(t) \frac{\partial u_0}{\partial \nu} - k_1' * \frac{\partial u}{\partial \nu} \right\} \frac{\partial w}{\partial \nu} d\Gamma, \end{aligned}$$

for any $w \in H^2(\Omega)$. From Lemma 2.5, we get that $u \in L^\infty(0, T; H^4(\Omega))$.

To prove the uniqueness of solutions of problem (1.1)-(1.7) we use the method of energy introduced by Lions [19], coupled with Gronwall's inequality and the hypothesis introduced in the paper.

3. EXPONENTIAL DECAY

In this section we show that the solution of system (1.1)-(1.7) decays exponentially provided the resolvent kernels satisfy

$$k_i(0) > 0, k_i'(t) \leq -C_1 k_i(t), k_i''(t) \geq -C_2 k_i'(t), \quad i = 1, 2 \quad (3.1)$$

for any $t \geq 0$ and some positive constants C_1, C_2 .

Let us denote by $E(t) := E(t, u, \theta, v)$. It is easy to verify that any strong solution of the system (1.1)-(1.7) has the following dissipative property

$$\begin{aligned} \frac{d}{dt} E(t) &= - \frac{\eta_1}{2} \int_{\Gamma} \left(2 \left| \frac{\partial u_t}{\partial \nu} \right|^2 + k_1'' \square \frac{\partial u}{\partial \nu} - k_1'(t) \left| \frac{\partial u}{\partial \nu} \right|^2 - 2k_1(t) \frac{\partial u_0}{\partial \nu} \frac{\partial u_t}{\partial \nu} \right) d\Gamma \\ &\quad - \frac{\eta_2}{2} \int_{\Gamma} \left(2 |u_t|^2 + k_2'' \square u - k_2'(t) |u|^2 - 2k_2(t) u_0 u_t \right) d\Gamma - \int_{\Omega} |\nabla \theta|^2 dx, \end{aligned}$$

from which follows, by using Young's inequality, that

$$\begin{aligned} \frac{d}{dt} E(t) &\leq - \frac{\eta_1}{2} \int_{\Gamma} \left(\left| \frac{\partial u_t}{\partial \nu} \right|^2 + k_1'' \square \frac{\partial u}{\partial \nu} - k_1'(t) \left| \frac{\partial u}{\partial \nu} \right|^2 - k_1^2(t) \left| \frac{\partial u_0}{\partial \nu} \right|^2 \right) d\Gamma \quad (3.2) \\ &\quad - \frac{\eta_2}{2} \int_{\Gamma} \left(|u_t|^2 + k_2'' \square u - k_2'(t) |u|^2 - k_2^2(t) |u_0|^2 \right) d\Gamma - \int_{\Omega} |\nabla \theta|^2 dx. \end{aligned}$$

The following identity will be used later.

Lemma 3.1. *For every $\varphi \in H^4(\Omega)$ we have*

$$\begin{aligned} \int_{\Omega} (m \cdot \nabla \varphi) \Delta^2 \varphi dx &= a(\varphi, \varphi) + \int_{\Gamma} \left[(\mathfrak{B}_2 \varphi) m \cdot \nabla \varphi - (\mathfrak{B}_1 \varphi) \frac{\partial}{\partial \nu} (m \cdot \nabla \varphi) \right] d\Gamma \\ &\quad + \frac{1}{2} \int_{\Gamma} m \cdot \nu \left[\left(\frac{\partial^2 \varphi}{\partial x^2} \right)^2 + \left(\frac{\partial^2 \varphi}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 \varphi}{\partial x^2} \frac{\partial^2 \varphi}{\partial y^2} + 2(1 - \mu) \left(\frac{\partial^2 \varphi}{\partial x \partial y} \right)^2 \right] d\Gamma. \end{aligned}$$

Proof. See [14]. □

Let us denote

$$\mathcal{N}_1(t) := \frac{1}{2} \int_{\Omega} |u_t|^2 dx + \frac{1}{2} a(u, u) + \frac{1}{2} \int_{\Gamma} \left(\rho_1 \left| \frac{\partial u}{\partial \nu} \right| + \rho_2 |u|^2 \right) d\Gamma + \frac{1}{4} \int_{\Omega} |\Delta v|^2 dx.$$

Let us introduce the functional

$$\psi(t) = \int_{\Omega} \left(m \cdot \nabla u + \frac{1}{2} u \right) u_t dx.$$

The following lemma plays an important role in the construction of the desired functional.

Lemma 3.2. *Any strong solution of (1.1)-(1.7) satisfies*

$$\begin{aligned} & \frac{d}{dt} \psi(t) \\ & \leq -\lambda_1 \mathcal{N}_1(t) + C \int_{\Gamma} \left\{ \left| \frac{\partial u_t}{\partial \nu} \right|^2 + k_1^2(t) \left| \frac{\partial u}{\partial \nu} \right|^2 + |k_1' \diamond \frac{\partial u}{\partial \nu}|^2 + k_1^2(t) \left| \frac{\partial u_0}{\partial \nu} \right|^2 \right\} d\Gamma \\ & + C \int_{\Gamma} \left\{ |u_t|^2 + k_2^2(t) |u|^2 + |k_2' \diamond u|^2 + k_2^2(t) |u_0|^2 \right\} d\Gamma + C_{\epsilon} \int_{\Omega} |\nabla \theta|^2 dx, \end{aligned}$$

for some positive constants λ_1 , C , C_{ϵ} . Here, the binary operator \diamond is given by

$$(k \diamond h)(t) := \int_0^t k(t-s)(h(t) - h(s)) ds.$$

Proof. Differentiating ψ , using equation (1.1) and taking $\varphi = u$ in Lemma 3.1 we get

$$\begin{aligned} \frac{d}{dt} \psi(t) &= \int_{\Omega} \left(m \cdot \nabla u_t + \frac{1}{2} u_t \right) u_t dx + \int_{\Omega} \left(m \cdot \nabla u + \frac{1}{2} u \right) u_{tt} dx \\ &= \frac{1}{2} \int_{\Gamma} m \cdot \nu |u_t|^2 d\Gamma - \frac{1}{2} \int_{\Omega} |u_t|^2 dx - \frac{3}{2} a(u, u) - \frac{1}{2} \int_{\Omega} |\Delta v|^2 dx \\ &\quad - \frac{1}{2} \int_{\Gamma} m \cdot \nu \left[\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} + 2(1-\mu) \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] d\Gamma \\ &\quad - \int_{\Gamma} \left\{ \mathfrak{B}_2 u \left(m \cdot \nabla u + \frac{1}{2} u \right) - \mathfrak{B}_1 u \frac{\partial}{\partial \nu} \left(m \cdot \nabla u + \frac{1}{2} u \right) \right\} d\Gamma \\ &\quad + \int_{\Omega} (m \cdot \nabla u) [u, v] dx - \alpha \int_{\Omega} \left(m \cdot \nabla u + \frac{1}{2} u \right) \Delta \theta dx. \end{aligned} \tag{3.3}$$

Now, let us calculate the term $\int_{\Omega} (m \cdot \nabla u)[u, v] dx$. According to Lemma 2.3 we have

$$\begin{aligned} \int_{\Omega} [u, v] m \cdot \nabla u dx &= \int_{\Omega} [m \cdot \nabla u, u] v dx = \frac{1}{2} \int_{\Omega} \operatorname{div}([u, u] m) v + [u, u] v dx \\ &= -\frac{1}{2} \int_{\Omega} [u, u] m \cdot \nabla v dx + \int_{\Omega} [u, u] v dx = \frac{1}{2} \int_{\Omega} \Delta^2 v (m \cdot \nabla v) dx - \int_{\Omega} \Delta^2 v v dx \\ &= -\frac{1}{2} \int_{\Omega} |\Delta v|^2 dx - \frac{1}{2} \int_{\Gamma} m \cdot \nu |\Delta v|^2 d\Gamma. \end{aligned} \quad (3.4)$$

On the other hand, since the bilinear form $a(u, w) + \int_{\Gamma} \left(\rho_1 \frac{\partial u}{\partial \nu} \frac{\partial w}{\partial \nu} + \rho_2 u w \right) d\Gamma$ is strictly coercive on $H^2(\Omega)$ there exists a constant $C > 0$ such that

$$\int_{\Gamma} \left\{ \left| \frac{\partial}{\partial \nu} (m \cdot \nabla u) \right|^2 + |m \cdot \nabla u|^2 \right\} d\Gamma \quad (3.5)$$

$$\begin{aligned} &\leq C \left\{ a(u, u) + \int_{\Gamma} \left(\rho_1 \left| \frac{\partial u}{\partial \nu} \right|^2 + \rho_2 |u|^2 \right) d\Gamma \right\} \\ &\quad + C \int_{\Gamma} \frac{m \cdot \nu}{\delta_0} \left[\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} + 2(1-\mu) \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] d\Gamma, \\ &\int_{\Omega} (m \cdot \nabla u + \frac{1}{2} u) \Delta \theta dx \leq \epsilon C \|u\|_{H^2}^2 + C_{\epsilon} \int_{\Omega} |\nabla \theta|^2 dx, \end{aligned} \quad (3.6)$$

with $\epsilon > 0$. Using (3.4)-(3.6), fixing ϵ small enough and using the same arguments as in J. E. Lagnese ([14], pages 117-118) and I. Lasiecka ([16], page 1809), the inequality (3.3) becomes

$$\begin{aligned} \frac{d}{dt} \psi(t) &\leq \frac{1}{2} \int_{\Gamma} m \cdot \nu |u_t|^2 d\Gamma - \frac{1}{2} \int_{\Omega} |u_t|^2 dx - a(u, u) \\ &\quad - \int_{\Omega} |\Delta v|^2 dx - \lambda_0 \int_{\Gamma} \left(\rho_1 \left| \frac{\partial u}{\partial \nu} \right|^2 + \rho_2 |u|^2 \right) d\Gamma \\ &\quad + C \int_{\Gamma} \left\{ \left| \mathfrak{B}_1 u + \rho_1 \frac{\partial u}{\partial \nu} \right|^2 + \left| \mathfrak{B}_2 u - \rho_2 u \right|^2 \right\} d\Gamma + \epsilon C \|u\|_{H^2}^2 + C_{\epsilon} \int_{\Omega} |\nabla \theta|^2 dx, \end{aligned}$$

where λ_0 is a small positive constant. Using the trace theorem and noting that the boundary conditions (2.1)-(2.2) can be written as

$$\begin{aligned} \mathfrak{B}_1 u &= -\rho_1 \frac{\partial u}{\partial \nu} - \eta_1 \left\{ \frac{\partial u_t}{\partial \nu} + k_1(t) \frac{\partial u}{\partial \nu} - k_1'(t) \diamond \frac{\partial u}{\partial \nu} - k_1(t) \frac{\partial u_0}{\partial \nu} \right\}, \\ \mathfrak{B}_2 u &= \rho_2 u + \eta_2 \{ u_t + k_2(t) u - k_2'(t) \diamond u - k_2(t) u_0 \}, \end{aligned}$$

our conclusion follows. \square

Lemma 3.3. *Let f be a real positive function of class C^1 . If there exist positive constants γ_0, γ_1 and C_0 such that*

$$f'(t) \leq -\gamma_0 f(t) + C_0 e^{-\gamma_1 t},$$

then there exist positive constants γ and C such that

$$f(t) \leq (f(0) + C)e^{-\gamma t}.$$

Proof. First, let us suppose that $\gamma_0 < \gamma_1$. Define $F(t)$ by

$$F(t) := f(t) + \frac{C_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}.$$

Then

$$F'(t) = f'(t) - \frac{\gamma_1 C_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \leq -\gamma_0 F(t).$$

Integrating from 0 to t we arrive at

$$F(t) \leq F(0)e^{-\gamma_0 t} \quad \Rightarrow \quad f(t) \leq \left(f(0) + \frac{C_0}{\gamma_1 - \gamma_0}\right)e^{-\gamma_0 t}.$$

Now, we shall assume that $\gamma_0 \geq \gamma_1$. In these conditions we get

$$f'(t) \leq -\gamma_1 f(t) + C_0 e^{-\gamma_1 t} \quad \Rightarrow \quad [e^{\gamma_1 t} f(t)]' \leq C_0.$$

Integrating from 0 to t we obtain $f(t) \leq (f(0) + C_0 t)e^{-\gamma_1 t}$. Since $t \leq (\gamma_1 - \epsilon)e^{(\gamma_1 - \epsilon)t}$ for any $0 < \epsilon < \gamma_1$ we conclude that $f(t) \leq [f(0) + C_0(\gamma_1 - \epsilon)]e^{-\epsilon t}$. This completes the proof. \square

Let us introduce the functional

$$\mathcal{L}(t) = NE(t) + \psi(t),$$

with $N > 0$. It is not difficult to see that $\mathcal{L}(t)$ satisfies

$$q_0 E(t) \leq \mathcal{L}(t) \leq q_1 E(t), \tag{3.7}$$

for some positive constants q_0, q_1 . Now we are in position to show the main result of this paper.

Theorem 3.1. *Let us suppose that the initial data $(u_0, u_1, \theta_0) \in H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. If the resolvent kernels k_1, k_2 satisfy the condition (3.1) and ρ_1, ρ_2 are positive constants, then there exist positive constants C, γ such that $E(t) \leq Ce^{-\gamma t}E(0)$, for all $t \geq 0$.*

Proof. We shall prove this theorem for strong solutions, that is, for solutions with initial data $(u^0, u^1, \theta_0) \in H^4(\Omega) \times H^2(\Omega) \times (H_0^1(\Omega) \cap H^2(\Omega))$ satisfying the compatibility conditions (2.12). Our conclusion follows by using standard density arguments. Using hypothesis (3.1) and the Poincaré inequality in the inequality (3.2) we have

$$\begin{aligned} \frac{d}{dt}E(t) &\leq -\frac{\eta_1 C}{2} \int_{\Gamma} \left(\left| \frac{\partial u_t}{\partial \nu} \right|^2 - k_1' \square \frac{\partial u}{\partial \nu} + k_1(t) \left| \frac{\partial u}{\partial \nu} \right|^2 - k_1^2(t) \left| \frac{\partial u_0}{\partial \nu} \right|^2 \right) d\Gamma \\ &\quad - \frac{\eta_2 C}{2} \int_{\Gamma} \left(\left| u_t \right|^2 - k_2' \square u + k_2(t) \left| u \right|^2 - k_2^2(t) \left| u_0 \right|^2 \right) d\Gamma \\ &\quad - C \int_{\Omega} |\theta|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla \theta|^2 dx. \end{aligned}$$

On the other hand, from Lemma 3.2, we obtain

$$\begin{aligned} \frac{d}{dt}\psi(t) &\leq -\lambda_1 \mathcal{N}_1(t) + C \int_{\Gamma} \left\{ \left| \frac{\partial u_t}{\partial \nu} \right|^2 + k_1(t) \left| \frac{\partial u}{\partial \nu} \right|^2 - k_1' \square \frac{\partial u}{\partial \nu} + k_1^2(t) \left| \frac{\partial u_0}{\partial \nu} \right|^2 \right\} d\Gamma \\ &\quad + C \int_{\Gamma} \left\{ \left| u_t \right|^2 + k_2(t) \left| u \right|^2 - k_2' \square u + k_2^2(t) \left| u_0 \right|^2 \right\} d\Gamma, \end{aligned}$$

from which it follows that, for N large

$$\frac{d}{dt}\mathcal{L}(t) \leq -\lambda_2 E(t) + cNR^2(t)E(0),$$

where $R(t) := k_2(t) + k_1(t)$ and λ_2 is a small positive constant. In view of inequality (3.7) we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\lambda_2}{q_1} \mathcal{L}(t) + cNR^2(t)E(0).$$

Applying Lemma 3.3 we arrive at

$$\mathcal{L}(t) \leq \{\mathcal{L}(0) + C\}e^{-\gamma t},$$

for some $C, \gamma > 0$. From (3.7) our conclusion follows. \square

4. POLYNOMIAL RATE OF DECAY

Here our attention will be focused on the uniform rate of decay when k_1 and k_2 decay polynomially like $(1+t)^{-p}$. In this case we will show that the solution also decays polynomially with the same rate. Let us consider the following hypothesis, for $i = 1, 2$,

$$0 < k(0), \quad k_i'(t) \leq -b_1 k_i^{1+\frac{1}{p}}(t), \quad k_i''(t) \geq b_2 [-k_i'(t)]^{1+\frac{1}{p+1}}, \quad (4.1)$$

where $p > 1$ and b_1, b_2 are positive constants. The following lemmas will play an important role in the sequel.

Lemma 4.1. *Let m and h be integrable functions, $0 \leq r < 1$ and $q > 0$. Then, for $t \geq 0$*

$$\begin{aligned} & \int_0^t |m(t-s)h(s)| ds \\ & \leq \left(\int_0^t |m(t-s)|^{1+\frac{1-r}{q}} |h(s)| ds \right)^{\frac{q}{q+1}} \left(\int_0^t |m(t-s)|^r |h(s)| ds \right)^{\frac{1}{q+1}}. \end{aligned}$$

Proof. In fact, let us take

$$v(s) := |m(t-s)|^{1-\frac{r}{q+1}} |h(s)|^{\frac{q}{q+1}}, \quad w(s) := |m(t-s)|^{\frac{r}{q+1}} |h(s)|^{\frac{1}{q+1}}.$$

Applying Hölder's inequality to $|m(s)h(s)| = v(s)w(s)$ with exponents $\delta = \frac{q}{q+1}$ for v and $\delta^* = q+1$ for w our conclusion follows. \square

Lemma 4.2. *Let us denote by $(\phi_1, \phi_2) = (\frac{\partial u}{\partial \nu}, u)$ where u is a solution of (1.1)–(1.7). Then, for $p > 1$, $0 \leq r < 1$ and $t \geq 0$, we have*

$$\begin{aligned} & \left(\int_{\Gamma} |k'_i| \square \phi_i d\Gamma \right)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}} \\ & \leq 2 \left(\int_0^t |k'_i(s)|^r ds \|\phi_i\|_{L^\infty(0,t;L^2(\Gamma))}^2 \right)^{\frac{1}{(1-r)(p+1)}} \int_{\Gamma} |k'_i|^{1+\frac{1}{p+1}} \square \phi_i d\Gamma, \end{aligned}$$

while for $r = 0$ we get

$$\begin{aligned} & \left(\int_{\Gamma} |k'_i| \square \phi_i d\Gamma \right)^{\frac{p+2}{p+1}} \\ & \leq 2 \left(\int_0^t \|\phi_i(s, \cdot)\|_{L^2(\Gamma)}^2 ds + t \|\phi_i(s, \cdot)\|_{L^2(\Gamma)}^2 \right)^{p+1} \int_{\Gamma} |k'_i|^{1+\frac{1}{p+1}} \square \phi_i d\Gamma, \end{aligned}$$

for $i = 1, 2$.

Proof. The above inequalities are a consequence of Lemma 4.1 for

$$m(s) := |k'_i(s)|, \quad h(s) := \int_{\Gamma} |\phi_i(t, x) - \phi_i(s, x)|^2 d\Gamma, \quad q := (1-r)(p+1).$$

This concludes our assertion. \square

Lemma 4.3. *Let $f \geq 0$ be a differentiable function satisfying*

$$f'(t) \leq -\frac{C_1}{f(0)^{\frac{1}{\alpha}}} f(t)^{1+\frac{1}{\alpha}} + \frac{C_2}{(1+t)^\beta} f(0), \quad \forall t \geq 0,$$

for some positive constants C_1, C_2 , α and β such that $\beta \geq \alpha + 1$. Then there exists a constant $C > 0$ such that

$$f(t) \leq \frac{C}{(1+t)^\alpha} f(0), \quad \forall t \geq 0.$$

Proof. Let us denote $F(t) = f(t) + \frac{2C_2}{\alpha}(1+t)^{-\alpha}f(0)$. Differentiating this function we have that

$$\begin{aligned} F'(t) &= f'(t) - 2C_2(1+t)^{-(\alpha+1)}f(0) \\ &\leq -\frac{C_1}{f(0)^{\frac{1}{\alpha}}}f(t)^{1+\frac{1}{\alpha}} - C_2(1+t)^{-(\alpha+1)}f(0) \\ &\leq -\frac{C}{f(0)^{\frac{1}{\alpha}}}\left[f(t)^{1+\frac{1}{\alpha}} + \left(\frac{f(0)}{(1+t)^\alpha}\right)^{1+\frac{1}{\alpha}}\right] \leq -\frac{C}{F(0)^{\frac{1}{\alpha}}}F(t)^{1+\frac{1}{\alpha}}. \end{aligned}$$

From this it follows that

$$F(t) \leq \frac{F(0)}{(1+Ct)^\alpha} \leq \frac{C}{(1+t)^\alpha}f(0).$$

Therefore, $f(t) \leq \frac{C}{(1+t)^\alpha}f(0)$. This complete the proof. \square

Theorem 4.1. *Let us suppose that the initial data $(u_0, u_1, \theta_0) \in H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. If the resolvent kernels k_1, k_2 satisfy condition (4.1) and ρ_1, ρ_2 are positive constants then there is a positive constant C such that*

$$E(t) \leq \frac{C}{(1+t)^{p+1}}E(0).$$

Proof. We shall prove this theorem for strong solutions, that is, for solutions with initial data $(u^0, u^1, \theta_0) \in H^4(\Omega) \times H^2(\Omega) \times (H_0^1(\Omega) \cap H^2(\Omega))$ satisfying the compatibility conditions (2.12). Our conclusion follows by using standard density arguments. Using hypothesis (4.1), inequality (3.2) can be written as

$$\begin{aligned} &\frac{d}{dt}E(t) \\ &\leq -\frac{\eta_1 C}{2} \int_{\Gamma} \left(\left| \frac{\partial u_t}{\partial \nu} \right|^2 + [-k'_1]^{1+\frac{1}{p+1}} \square \frac{\partial u}{\partial \nu} + k_1^{1+\frac{1}{p}}(t) \left| \frac{\partial u}{\partial \nu} \right|^2 - k_1^2(t) \left| \frac{\partial u_0}{\partial \nu} \right|^2 \right) d\Gamma \\ &\quad - \frac{\eta_2 C}{2} \int_{\Gamma} \left(|u_t|^2 + [-k'_2]^{1+\frac{1}{p+1}} \square u + k_2^{1+\frac{1}{p}}(t) |u|^2 - k_2^2(t) |u_0|^2 \right) d\Gamma \\ &\quad - C \int_{\Omega} |\theta|^2 dx + C_\epsilon \int_{\Omega} |\nabla \theta|^2 dx, \end{aligned} \tag{4.2}$$

for some positive constant C . Using (4.1) again, there exists another positive constant $C > 0$ such that

$$\left| k'_1 \diamond \frac{\partial u}{\partial \nu} \right|^2 \leq C[-k'_1]^{1+\frac{1}{p+1}} \square \frac{\partial u}{\partial \nu}, \quad \left| k'_2 \diamond u \right|^2 \leq C[-k'_2]^{1+\frac{1}{p+1}} \square u.$$

Using this estimate in Lemma 3.2 we obtain

$$\begin{aligned} & \frac{d}{dt} \psi(t) \\ & \leq -\lambda_1 \mathcal{N}(t) + C \int_{\Gamma} \left\{ |u_t|^2 + k_2^{1+\frac{1}{p}}(t) |u|^2 + [-k'_2]^{1+\frac{1}{p+1}} \square u + k_2^2(t) |u_0|^2 \right\} d\Gamma \\ & + C \int_{\Gamma} \left\{ \left| \frac{\partial u_t}{\partial \nu} \right|^2 + k_1^{1+\frac{1}{p}}(t) \left| \frac{\partial u}{\partial \nu} \right|^2 + [-k'_1]^{1+\frac{1}{p+1}} \square \frac{\partial u}{\partial \nu} + k_1^2(t) \left| \frac{\partial u_0}{\partial \nu} \right|^2 \right\} d\Gamma. \end{aligned} \quad (4.3)$$

Taking N large, from (4.2)-(4.3) we arrive at

$$\begin{aligned} \frac{d}{dt} \mathcal{L}(t) & \leq -\lambda_2 \mathcal{N}(t) - \lambda_1 \int_{\Gamma} \left([-k'_1]^{1+\frac{1}{p+1}} \square \frac{\partial u}{\partial \nu} + [-k'_2]^{1+\frac{1}{p+1}} \square u \right) d\Gamma \\ & + CNR^2(t)E(0), \end{aligned} \quad (4.4)$$

for some $\lambda_2 > 0$ where

$$\mathcal{N}(t) = \mathcal{N}_1(t) + \int_{\Omega} |\theta|^2 dx.$$

Let us fix $0 < r < 1$ such that $\frac{1}{p+1} < r < \frac{p}{p+1}$. In this condition from hypothesis (4.1) we have

$$\int_0^{\infty} [-k'_i]^r \leq C \int_0^{\infty} \frac{1}{(1+t)^{r(p+1)}} < \infty \quad \text{for } i = 1, 2.$$

Using this estimate in Lemma 4.2 we get

$$\int_{\Gamma} [-k'_1]^{1+\frac{1}{p+1}} \square \frac{\partial u}{\partial \nu} d\Gamma \geq CE(0)^{-\frac{1}{(1-r)(p+1)}} \left(\int_{\Gamma} [-k'_1] \square \frac{\partial u}{\partial \nu} d\Gamma \right)^{1+\frac{1}{(1-r)(p+1)}}, \quad (4.5)$$

$$\int_{\Gamma} [-k'_2]^{1+\frac{1}{p+1}} \square u d\Gamma \geq CE(0)^{-\frac{1}{(1-r)(p+1)}} \left(\int_{\Gamma} [-k'_2] \square u d\Gamma \right)^{1+\frac{1}{(1-r)(p+1)}}. \quad (4.6)$$

On the other hand, since the energy is bounded we have

$$\mathcal{N}(t)^{1+\frac{1}{(1-r)(p+1)}} \leq CE(0)^{\frac{1}{(1-r)(p+1)}} \mathcal{N}(t). \quad (4.7)$$

Substituting (4.5)-(4.7) into (4.4) we arrive at

$$\frac{d}{dt} \mathcal{L}(t) \leq -CE(0)^{-\frac{1}{(1-r)(p+1)}} \mathcal{N}(t)^{1+\frac{1}{(1-r)(p+1)}} + CNR^2(t)E(0)$$

$$-CE(0)^{-\frac{1}{(1-r)(p+1)}} \times \left\{ \left(\int_{\Gamma} [-k'_1] \square u d\Gamma \right)^{1+\frac{1}{(1-r)(p+1)}} + \left(\int_{\Gamma} [-k'_2] \square v d\Gamma \right)^{1+\frac{1}{(1-r)(p+1)}} \right\}.$$

Taking into account inequality (3.7) we obtain

$$\frac{d}{dt} \mathcal{L}(t) \leq -\frac{C}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}} \mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + CNR^2(t)E(0). \quad (4.8)$$

Therefore, from Lemma 4.3 we conclude that

$$\mathcal{L}(t) \leq \frac{C}{(1+t)^{(1-r)(p+1)}} \mathcal{L}(0). \quad (4.9)$$

Since $(1-r)(p+1) > 1$ we get, for $t \geq 0$, the following estimations

$$t \|u(t, \cdot)\|_{L^2(\Gamma)}^2 + t \left\| \frac{\partial u(t, \cdot)}{\partial \nu} \right\|_{L^2(\Gamma)}^2 \leq Ct \mathcal{L}(t) < \infty,$$

$$\int_0^t \|u(s, \cdot)\|_{L^2(\Gamma)}^2 + \left\| \frac{\partial u(s, \cdot)}{\partial \nu} \right\|_{L^2(\Gamma)}^2 \leq C \int_0^\infty \mathcal{L}(t) < \infty.$$

Using the above estimates in Lemma 4.2 with $r = 0$, we get

$$\int_{\Gamma} [-k'_1]^{1+\frac{1}{p+1}} \square \frac{\partial u}{\partial \nu} d\Gamma \geq \frac{C}{E(0)^{\frac{1}{p+1}}} \left(\int_{\Gamma} [-k'_1] \square \frac{\partial u}{\partial \nu} d\Gamma \right)^{1+\frac{1}{p+1}},$$

$$\int_{\Gamma} [-k'_2]^{1+\frac{1}{p+1}} \square u d\Gamma \geq \frac{C}{E(0)^{\frac{1}{p+1}}} \left(\int_{\Gamma} [-k'_2] \square u d\Gamma \right)^{1+\frac{1}{p+1}}.$$

Using these inequalities and the same arguments as in the derivation of (4.8), we have

$$\frac{d}{dt} \mathcal{L}(t) \leq -\frac{C}{\mathcal{L}(0)^{\frac{1}{p+1}}} \mathcal{L}(t)^{1+\frac{1}{p+1}} + CNR^2(t)E(0).$$

Applying Lemma 4.3 we obtain $\mathcal{L}(t) \leq \frac{C}{(1+t)^{p+1}} \mathcal{L}(0)$. Using inequality (3.7) we conclude that $E(t) \leq \frac{C}{(1+t)^{p+1}} E(0)$, which completes the proof. \square

Remark. Already for the system of (magneto-thermo) elasticity with memory type boundary conditions it was shown in [22] that a merely polynomial kernel cannot lead to an exponential decay result for the energy in general. In a similar manner, we can prove that the decay rate for polynomial kernels for problem (1.1)-(1.7) cannot be of exponential type.

Final Comments. The techniques used in [4, 5, 16, 17] (semigroup) in general are not applied to the system (1.1)-(1.7) due to the convolution terms that appear in the free boundary. However, for the system below

$$\begin{aligned} u_{tt} + \Delta^2 u + \alpha \Delta \theta &= [u, v] && \text{in } \Omega \times (0, \infty), \\ \theta_t - \Delta \theta - \alpha \Delta u_t &= 0 && \text{in } \Omega \times (0, \infty), \\ \Delta^2 v &= -[u, u] && \text{in } \Omega \times (0, \infty), \\ v = \frac{\partial v}{\partial \nu} = \theta &= 0 && \text{on } \Gamma \times (0, \infty), \\ \frac{\partial u}{\partial \nu} + \int_0^\infty g_1(s) \left(\mathfrak{B}_1 u(s) + \rho_1 \frac{\partial u}{\partial \nu}(s) \right) ds &= 0 && \text{on } \Gamma \times (0, \infty), \\ u - \int_0^\infty g_2(s) \left(\mathfrak{B}_2 u(s) + \alpha \frac{\partial \theta}{\partial \nu}(s) - \rho_2 u(s) \right) ds &= 0 && \text{on } \Gamma \times (0, \infty), \\ u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x), \quad \theta(0, x) = \theta_0(x) && \text{in } \Omega, \end{aligned}$$

known as a system with fading memory, the semigroup techniques can be used and in this case the authors conjecture that the thermal effect is fundamental to obtain the stabilization of the system. This is an important open problem.

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