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APPROXIMATION BY MEANS OF NONLINEAR INTEGRAL OPERATORS IN THE SPACE OF FUNCTIONS WITH BOUNDED φ−VARIATION

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Abstract. We study approximation problems by means of nonlinear convolution integral operators for functions belonging to BV_φ-spaces, i.e., functions with bounded φ-variation in the sense of Musielak-Orlicz. In particular, we obtain estimates and convergence results with respect to φ-variation. Introducing suitable Lipschitz classes that take into account the φ-variational functional, the problem of the rate of approximation is also considered.

1. Introduction

In [2] the problem of convergence with respect to classical variation for a family of nonlinear integral operators of the form

$$(T_w f)(s) = \int_{\mathbb{R}} K_w(t, f(s-t))dt, \ w > 0, \ s \in \mathbb{R},$$

where $K_w(t,u) = L_w(t)H_w(u)$, $t, u \in \mathbb{R}$, $w > 0$, is considered. We also studied the periodic case and the multidimensional one, together with some results concerning the order of approximation. For linear convolution integral operators see [4], where higher orders of approximation are also considered.

While, as is well known, the notion of variation was introduced by C. Jordan in [16] in connection with the convergence of Fourier series, in 1924 N. Wiener introduced in [31] the notion of quadratic variation, later generalized by L.C. Young in [32] and, together with E.R. Love, in [17] to a concept of variation of higher orders (p-variation, $p \geq 1$), still in connection with Fourier series and with applications to Riemann-Stieltjes integrals. Later

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on, in 1937 L.C. Young (see [33]) obtained a further generalization introducing the concept of φ-variation, φ being a φ-function (see Section 2).

The main developments of this concept were obtained by J. Musielak and W. Orlicz in 1959 ([23]) and by the Orlicz school in [29, 18, 20]; indeed this concept is known as the φ-variation in the sense of Musielak-Orlicz, and it extends the classical variation. Other results in this direction can be found in [15, 25, 13, 14, 26, 3, 21, 24, 10], while recent results on φ-variation in connection with estimates or convergence of integral operators can be found in [19, 6, 27, 28, 5, 7].

Working in the setting of functions with bounded variation, one of the main tools in order to get the convergence result is the integral representation of the variation for absolutely continuous functions.

Unfortunately, there does not exist here any integral representation for absolutely continuous functions in terms of φ-variation (as happens for the classical Jordan variation), which makes its use in approximation theory quite difficult. In order to avoid this problem, and to get the convergence results in Theorems 1 and 2, we need to state suitable lemmas.

The main convergence result (Theorem 3) states that, under suitable assumptions on the family of the kernels involved, there holds

$$\lim_{w\to+\infty} V_\varphi[\mu(T_w f - f)] = 0,$$

for some $\mu > 0$, f being a locally φ-absolutely continuous function. Furthermore, examples of kernels satisfying our assumptions are furnished (see Example 1).

Preliminary to this result are some estimates for our operators with respect to φ-variation and for the error of approximation $T_w f - f$. While in order to get estimates it is sufficient to assume that the function f is of bounded φ-variation, for the convergence result this condition is not sufficient, as is shown in Example 2, and we need local φ-absolute continuity for the function itself.

In Section 5, it is also proved (see Lemma 4) that the space of locally φ-absolutely continuous functions ($AC_\varphi^{loc}(\mathbb{R})$) is a closed subspace of $BV_\varphi(\mathbb{R})$ (the space of functions with bounded φ-variation). This result shows that the local φ-absolute continuity for the function f is also necessary for the convergence in φ-variation for our family of integral operators.

Apart from the classical singularity conditions on our kernels (K_w.1) and K_w.2)) and from the condition ($\star$), which is a generalization of a Hölder-type condition, we need to assume a convergence in φ-variation for the difference

$H_w(u) - u$. This last condition is almost natural in this setting, as is pointed out in Remark 1.

Finally, as a further application of the convergence result, in the last section we treat the problem of the order of approximation with respect to φ-variation for $T_w f - f$. In order to do this, we need to introduce suitable Lipschitz classes which take into account the φ-variational functional. In this respect, we prove that under some singularity assumption,

$$V_\varphi[\lambda(T_w f - f)] = O(w^{-\alpha}),$$

for some $\alpha, \lambda > 0$ and for sufficiently large $w > 0$, where f belongs to the Lipschitz class introduced and α comes from the singularity assumption. Examples of kernels satisfying the above singularity conditions are also furnished.

2. Notation and assumptions

Let X be the set of all measurable functions $f : \mathbb{R} \to \mathbb{R}$. Let Φ be the class of all functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

- (i) φ is a convex function on $\mathbb{R}_0^+$ with $\varphi(0) = 0$ and $\varphi(u) > 0$ for $u > 0$;
- (ii) $u^{-1}\varphi(u) \to 0$, as $u \to 0^+$.

From now on we shall assume that $\varphi \in \Phi$.

For a function $f \in X$ we denote by $V_\varphi[f]$ the Musielak-Orlicz φ-variation of f, namely

$$V_\varphi[f] = V_\varphi[f, \mathbb{R}] := \sup_D \sum_{i=1}^{n} \varphi(|f(s_i) - f(s_{i-1})|),$$

where $D = \{s_i\}_{i=0}^n$ denotes an increasing sequence in $\mathbb{R}$. By $BV_\varphi(\mathbb{R})$ we will denote the space of all functions with bounded φ-variation on $\mathbb{R}$ equipped with the functional $V_\varphi[f]$ (which is a semimodular; see [21, 5]), namely

$$BV_\varphi(\mathbb{R}) = \{f \in X : \lim_{\lambda \longrightarrow 0} V_\varphi[\lambda f] = 0\}.$$

Note that, by convexity of φ, the above space is equivalent to

$$BV_\varphi(\mathbb{R}) = \{f \in X : \exists \lambda > 0 \text{ s.t. } V_\varphi[\lambda f] < +\infty\}.$$

We recall that the following important properties of φ-variation hold (see [23]):

$$\textit{if } f_1, \dots, f_n \in X, \textit{ then } V_\varphi[\sum_{i=1}^{n} f_i] \le \frac{1}{n} \sum_{i=1}^{n} V_\varphi[n f_i]; \tag{I}$$

given a fixed partition $a = t_0 < t_1 < \cdots < t_n = b$ *of the interval* $[a, b]$*, if* $f(t_i) = 0$ *for* $i = 1, 2, \dots, n-1$*, then*

$$V_\varphi[\lambda f, [a,b]] \leq \frac{1}{2} \sum_{i=1}^{n} V_\varphi[2\lambda f, [t_{i-1}, t_i]], \tag{II}$$

for any constant $\lambda > 0$ *for which* $V_\varphi[\lambda f, [a,b]] < +\infty$*;*

$$\text{if } f \in X \text{ and } a < c < b, \text{ then } V_\varphi[f, [a,c]] + V_\varphi[f, [c,b]] \leq V_\varphi[f, [a,b]]; \tag{III}$$

$$\text{if } f \in X \text{ and } a < c < b, \text{ then } V_\varphi[f, [a,b]] \leq \frac{1}{2}\Big\{V_\varphi[2f, [a,c]] + V_\varphi[2f, [c,b]]\Big\}. \tag{IV}$$

We shall say that a family of functions $\{f_w\}_{w>0}$ is of *equibounded* φ*-variation* if it is of bounded φ-variation uniformly with respect to $w > 0$.

By $AC_\varphi^{loc}(\mathbb{R})$ we shall denote the subspace of $BV_\varphi(\mathbb{R})$ of the *locally* φ*-absolutely continuous functions*, namely the functions $f \in BV_\varphi(\mathbb{R})$ for which there exists $\lambda > 0$ such that the following property holds:

for every $\varepsilon > 0$ *and for every compact interval* $I = [a, b] \subset \mathbb{R}$*, there exists* $\delta > 0$ *such that*

$$\sum_{i=1}^{n} \varphi(\lambda |f(\beta_i) - f(\alpha_i)|) < \varepsilon,$$

for all finite collections of non-overlapping intervals $[\alpha_i, \beta_i] \subset I$*,* $i = 1, \dots, n$ *such that*

$$\sum_{i=1}^{n} \varphi(\beta_i - \alpha_i) < \delta.$$

We now introduce a family of kernel functions. Let $\{K_w\}_{w>0}$ be a family of measurable functions $K_w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$K_w(t, u) = L_w(t) H_w(u)$$

for every $t, u \in \mathbb{R}$, where $H_w : \mathbb{R} \to \mathbb{R}$ is such that $H_w(0) = 0$ and $L_w : \mathbb{R} \to \mathbb{R}$. We assume that H_w is a ψ-Lipschitz kernel for every $w > 0$; i.e., there exists $K > 0$ such that

$$|H_w(u) - H_w(v)| \leq K \ \psi(|u - v|) \tag{$\star$}$$

for every $u, v \in \mathbb{R}$, where $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a φ-function (this means, as usual, that φ is a nondecreasing function, $\varphi(u) = 0$ if and only if $u = 0$ and $\lim_{u\to+\infty} \varphi(u) = +\infty$ ([5])) and

$\mathbf{K_w.1}$) $L_w : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $L_w \in L^1(\mathbb{R})$, $\|L_w\|_1 \leq A$, for an absolute constant $A > 0$ and $\int_{\mathbb{R}} L_w(t)dt = 1$, for every $w > 0$;

$\mathbf{K_w.2}$) for any fixed $\delta > 0$, $\int_{|t|\geq\delta} |L_w(t)|\, dt \to 0$, as $w \to +\infty$;

$\mathbf{K_w.3}$) denoted by $G_w(u) := H_w(u) - u$, for every $u \in \mathbb{R}$, $w > 0$, there exists $\xi > 0$ such that

$$V_\varphi[\xi G_w, J] \to 0, \quad \text{as } w \to +\infty,$$

for every bounded interval $J \subset \mathbb{R}$.

In the following we will say that $\{K_w\}_{w>0} \subset \mathcal{K}_w$ if $(\star)$ and $K_w.i)$, $i = 1, 2, 3$ are satisfied.

Remark 1. We point out that the assumption $V_\varphi[\xi G_w, J] \to 0$ as $w \to +\infty$, for some constant $\xi > 0$, is quite natural in this nonlinear setting. Indeed, taking for the sake of simplicity $\xi = 1$, in the case of constant functions $(f(s) = u)$, the condition $V_\varphi[H_w(u) - u] \to 0$ as $w \to +\infty$ is implied by the convergence in φ-variation $V_\varphi[T_w f - f] \to 0$ as $w \to +\infty$, since

$$V_\varphi[T_w f - f] \;=\; V_\varphi\Big[\int_{\mathbb{R}} L_w(t) H_w(u)\, dt - u\Big] = V_\varphi[H_w(u) - u] \to 0$$

as $w \to +\infty$; therefore, for constant functions (which are obviously BV_φ-functions), the above condition becomes also necessary for the convergence in φ-variation of our operators. Hence, constant functions represent a set of test functions for the convergence in φ-variation, as happens for the classical singularity assumption on the convolution kernel, i.e., $\int_{\mathbb{R}} L_w(t)\, dt = 1$ for all $w > 0$, which means requiring the uniform convergence of $T_w f$ towards f, for any constant function f.

We shall now give an example of a family of kernel functions which satisfy the above assumptions.

Example 1. Let $K_w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be of the form

$$K_w(t, u) = L_w(t) H_w(u), \quad t, u \in \mathbb{R}, \; w > 0,$$

where

$$H_w(u) = \begin{cases} w \log\big(1 + \dfrac{u}{w}\big), & 0 \leq u < 1, \\ wu \log\big(1 + \dfrac{1}{w}\big), & u \geq 1, \end{cases}$$

and where we extend the definition of $H_w(u)$ to $u < 0$ by requiring H_w to be odd (see Figure 1). Moreover, we may take $\{L_w\}_{w>0}$ satisfying $K_w.1$) and $K_w.2$); i.e., $\{L_w\}_{w>0}$ is an approximate identity. Then it easy to see that

$(\star)$ is satisfied with $\psi(|u|) = |u|$ and $K = 1$ and, since $|G_w(u)|$ is increasing in $\mathbb{R}_0^+$, by convexity of φ, for every interval $J = [-M, M]$, $V_\varphi[G_w, J] \to 0$ as $w \to +\infty$; namely, $K_w.3)$ is satisfied (see [6, 2]).

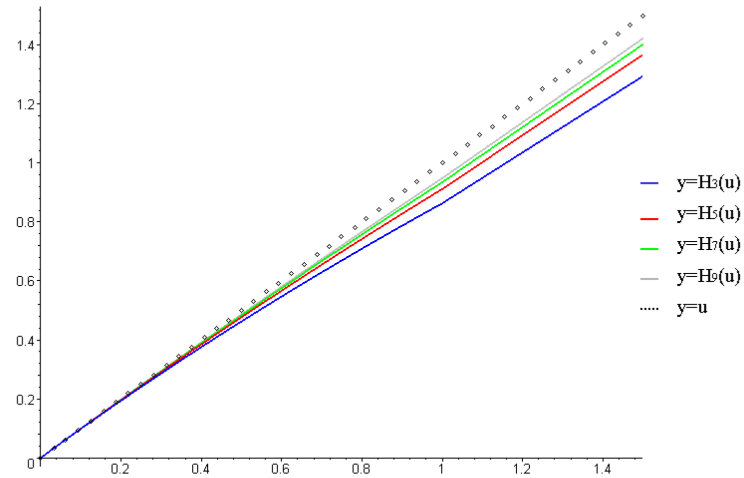

FIGURE 1

As concerns other examples of kernel functions for which $K_w.3)$ is satisfied, see [2].

Finally, as is usual in order to obtain estimates or convergence results by means of nonlinear integral operators with respect to Musielak-Orlicz φ-variation, we suppose the following growth condition on the φ-function ψ of the ψ-Lipschitz condition (see also [5]). From now on, we will assume that η is a φ-function.

Definition 1. Given a φ-function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, we say that the triple (φ, η, ψ) is properly directed if for every $\lambda \in (0, 1)$ there exists a constant $C_\lambda \in (0, 1)$ such that

$$\varphi[C_\lambda \psi(|g|)] \leq \eta[\lambda g] \qquad \text{for every } g \in X. \tag{$\star\star$}$$

3. ESTIMATES

We will consider the following family of nonlinear integral operators,

$$(T_w f)(s) = \int_{\mathbb{R}} K_w(t, f(s-t))dt, \; w > 0, \; s \in \mathbb{R}, \tag{+}$$

defined for every $f \in X$ for which $T_w f$ is well defined for every $s \in \mathbb{R}$, where $K_w(t,u) = L_w(t)H_w(u)$, $t, u \in \mathbb{R}$.

We first give an estimate in the sense of φ-variation for $T_w f$, $w > 0$.

Proposition 1. *Let $f \in BV_\eta(\mathbb{R})$. If $\{L_w\}_{w>0} \subset L^1(\mathbb{R})$, $(\star)$ is satisfied and the triple (φ, η, ψ) is properly directed, there exists a constant $\gamma > 0$ such that*

$$V_\varphi[\gamma T_w f] \leq V_\eta[\lambda f];$$

as a consequence T_w maps $BV_\eta(\mathbb{R})$ into $BV_\varphi(\mathbb{R})$.

Proof. Let $\{s_i\}_{i=0}^n$ be an increasing sequence in $\mathbb{R}$. Then, for $\gamma > 0$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \varphi(\gamma|(T_w f)(s_i) - (T_w f)(s_{i-1})|) \\
&= \sum_{i=1}^{n} \varphi\Big(\gamma\Big|\int_{\mathbb{R}} K_w(t, f(s_i - t))dt - \int_{\mathbb{R}} K_w(t, f(s_{i-1} - t))dt\Big|\Big) \\
&\leq \sum_{i=1}^{n} \ \varphi\Big(\gamma \int_{\mathbb{R}} |L_w(t)||H_w(f(s_i - t)) - H_w(f(s_{i-1} - t))|dt\Big).
\end{aligned}$$

By convexity of φ, Jensen's inequality and assumption $(\star)$, there holds

$$\begin{aligned}
&\sum_{i=1}^{n} \varphi(\gamma|(T_w f)(s_i) - (T_w f)(s_{i-1})|) \\
&\leq A^{-1} \int_{\mathbb{R}} |L_w(t)| \sum_{i=1}^{n} \varphi(\gamma A|H_w(f(s_i - t)) - H_w(f(s_{i-1} - t))|)dt \\
&\leq A^{-1} \int_{\mathbb{R}} |L_w(t)| \sum_{i=1}^{n} \varphi(\gamma A K \psi(|f(s_i - t) - f(s_{i-1} - t)|))dt,
\end{aligned}$$

where K is the constant of assumption $(\star)$ and A is the constant of assumption $K_w.1)$. Now, by $(\star\star)$, if γ is such that $\gamma A K \leq C_\lambda$, there holds

$$\begin{aligned}
&\sum_{i=1}^{n} \varphi(\gamma|(T_w f)(s_i) - (T_w f)(s_{i-1})|) \\
&\leq A^{-1} \int_{\mathbb{R}} |L_w(t)| \sum_{i=1}^{n} \eta(\lambda|f(s_i - t) - f(s_{i-1} - t)|)dt
\end{aligned}$$

$$\leq A^{-1} \int_{\mathbb{R}} |L_w(t)| V_\eta[\lambda f] dt \leq V_\eta[\lambda f].$$

Hence the thesis follows by the arbitrariness of the sequence $\{s_i\}_{i=0}^n$. □

We now establish an estimate with respect to φ-variation for the error of approximation $T_w f - f$.

In order to do it, we recall that for a function $f \in BV_\varphi(\mathbb{R})$ the V_φ-modulus of continuity of f is defined as

$$\omega_\varphi(f, \delta) := \sup_{|t| \leq \delta} V_\varphi[\tau_t f - f],$$

for $\delta > 0$, where $(\tau_t f)(s) := f(s-t)$, for every $s \in \mathbb{R}$, is the translation operator (see [22, 5]). From now on we shall denote by $g_w := (H_w \circ f)$, for every $w > 0$.

Proposition 2. *Let $f \in BV_\varphi(\mathbb{R})$. If $\{L_w\}_{w>0} \subset L^1(\mathbb{R})$, $K_w.1)$ and $(\star)$ are satisfied and the triple (φ, η, ψ) is properly directed, then for every $\mu, \delta > 0$ and for some bounded interval $J \subset \mathbb{R}$,*

$$V_\varphi[\mu(T_w f - f)] \leq$$
$$\frac{1}{2}\Big\{\omega_\varphi(2\mu A g_w, \delta) + A^{-1} V_\varphi\left[4\mu A g_w\right] \int_{|t|\geq\delta} |L_w(t)| dt + V_\varphi[2\mu A G_w, J]\Big\},$$

for every $w > 0$.

Proof. Let $\{s_0, \dots, s_n\}$ be an increasing sequence in $\mathbb{R}$. Then, for some constant $\mu > 0$ we have

$$\begin{aligned} I &:= \sum_{i=1}^n \varphi\left(\mu |(T_w f)(s_i) - f(s_i) - (T_w f)(s_{i-1}) + f(s_{i-1})|\right) \\ &= \sum_{i=1}^n \varphi\Big(\mu \Big| \int_{\mathbb{R}} \left[K_w(t, f(s_i - t)) - K_w(t, f(s_{i-1} - t))\right] dt - f(s_i) + f(s_{i-1}) \Big|\Big) \\ &= \sum_{i=1}^n \varphi\Big(\mu \Big| \int_{\mathbb{R}} L_w(t) \left[H_w(f(s_i - t)) - H_w(f(s_{i-1} - t)) - f(s_i) + f(s_{i-1})\right] dt \Big|\Big), \end{aligned}$$

and, by convexity of φ,

$$\begin{aligned} I \leq \sum_{i=1}^n \varphi\Big(\mu \int_{\mathbb{R}} &|L_w(t)| |H_w(f(s_i - t)) - H_w(f(s_i)) - H_w(f(s_{i-1} - t)) \\ &+ H_w(f(s_{i-1})) + H_w(f(s_i)) - f(s_i) - H_w(f(s_{i-1})) + f(s_{i-1})| dt\Big) \end{aligned}$$

$$\leq \frac{1}{2}\Big\{ \sum_{i=1}^{n} \varphi(2\mu \int_{\mathbb{R}} |L_w(t)||H_w(f(s_i - t))$$
$$- H_w(f(s_i)) - H_w(f(s_{i-1} - t)) + H_w(f(s_{i-1}))|dt)$$
$$+ \sum_{i=1}^{n} \varphi\Big(2\mu \int_{\mathbb{R}} |L_w(t)||H_w(f(s_i)) - f(s_i) - H_w(f(s_{i-1})) + f(s_{i-1})|dt\Big)\Big\}.$$

Now, applying $K_w.1$) and Jensen's inequality,

$$I \leq \frac{1}{2}\Big\{A^{-1} \int_{\mathbb{R}} |L_w(t)| \sum_{i=1}^{n} \varphi(2\mu A|H_w(f(s_i - t))$$
$$- H_w(f(s_i)) - H_w(f(s_{i-1} - t)) + H_w(f(s_{i-1}))|)dt$$
$$+ A^{-1} \int_{\mathbb{R}} |L_w(t)| \sum_{i=1}^{n} \varphi(2\mu A|H_w(f(s_i)) - f(s_i)$$
$$- H_w(f(s_{i-1})) + f(s_{i-1})|)dt\Big\} := \frac{1}{2}\ (I_1 + I_2).$$

We first study I_1. There holds, for any fixed $\delta > 0$,

$$I_1 \leq A^{-1} \int_{\mathbb{R}} |L_w(t)|\ V_\varphi\left[2\mu A\left|(H_w \circ f)(\cdot - t) - (H_w \circ f)(\cdot)\right|\right] dt$$
$$= A^{-1}\Big\{ \int_{0\leq|t|\leq\delta} + \int_{|t|\geq\delta} \Big\}|L_w(t)|V_\varphi[2\mu A|(H_w \circ f)(\cdot - t) - (H_w \circ f)(\cdot)|]dt$$
$$:= I_1^1 + I_1^2.$$

For I_1^1 there holds, by $K_w.1$),

$$I_1^1 \leq A^{-1}\omega_\varphi(2\mu A g_w, \delta) \int_{0\leq|t|\leq\delta} |L_w(t)|dt \leq \omega_\varphi(2\mu A g_w, \delta).$$

For I_1^2, by property (I),

$$I_1^2 \leq A^{-1} \int_{|t|\geq\delta} |L_w(t)|V_\varphi\left[2\mu A\left|(H_w \circ f)(\cdot - t) - (H_w \circ f)(\cdot)\right|\right] dt$$
$$\leq \frac{1}{2}A^{-1} \int_{|t|\geq\delta} |L_w(t)|\Big\{V_\varphi\left[4\mu A(H_w \circ f)(\cdot - t)\right] + V_\varphi\left[4\mu A(H_w \circ f)(\cdot)\right]\Big\}dt$$
$$= A^{-1}V_\varphi\left[4\mu A g_w\right] \int_{|t|\geq\delta} |L_w(t)|dt.$$

Finally, for I_2, since $f \in BV_\varphi(\mathbb{R})$, in particular f is bounded; then there exists a bounded interval $J \subset \mathbb{R}$ such that

$$I_2 \leq A^{-1} \int_{\mathbb{R}} |L_w(t)| V_\varphi[2\mu A G_w, J] dt$$
$$= A^{-1} \|L_w\|_1 \, V_\varphi[2\mu A G_w, J] \leq V_\varphi[2\mu A G_w, J].$$

Hence, the thesis follows by the arbitrariness of the sequence $\{s_i\}_{i=0}^n$. $\square$

4. Convergence results

We now give some preliminary results that will be used in the proof of the main convergence theorem.

Lemma 1. *If $f \in AC_\varphi^{loc}(\mathbb{R})$, then there exists $\lambda > 0$ such that for every $\varepsilon > 0$ there exist $a, b \in \mathbb{R}$ and $\delta > 0$ with the property that, if $D = \{t_0 = a, \dots, t_n = b\}$ is a partition of $[a, b]$ with $t_i - t_{i-1} < \delta$, then*

(a) $V_\varphi[\lambda f, (-\infty, a]] < \varepsilon$ *and* $V_\varphi[\lambda f, [b, +\infty)] < \varepsilon$;

(b) $\sum_{i=1}^n V_\varphi[\lambda f, [t_{i-1}, t_i]] < \varepsilon$;

(c) *the step function* $\nu : \mathbb{R} \to \mathbb{R}$ *defined as*

$$\nu(t) = \begin{cases} f(a), \ t < a, \\ f(t_{i-1}), \ t_{i-1} \leq t < t_i, \ i = 1, \dots, n, \\ f(b), \ t \geq b, \end{cases}$$

is such that $V_\varphi[\lambda(\nu - f)] < \varepsilon$.

Proof. Since $f \in AC_\varphi^{loc}(\mathbb{R})$, and so by definition $f \in BV_\varphi(\mathbb{R})$, then there exists $\overline{\lambda} > 0$ such that $V_\varphi[\overline{\lambda} f] < +\infty$ and there holds

$$V_\varphi[\overline{\lambda} f] = \lim_{n \to +\infty} V_\varphi[\overline{\lambda} f, [-c_n, c_n]],$$

where $(c_n)_n$ is an increasing sequence in $\mathbb{R}^+$. By property (III),

$$V_\varphi[\overline{\lambda} f, (-\infty, -c_n]] + V_\varphi[\overline{\lambda} f, [c_n, +\infty)] \leq V_\varphi[\overline{\lambda} f, \mathbb{R}] - V_\varphi[\overline{\lambda} f, [-c_n, c_n]];$$

hence,

$$\lim_{n \to +\infty} V_\varphi[\overline{\lambda} f, (-\infty, -c_n]] = \lim_{n \to +\infty} V_\varphi[\overline{\lambda} f, [c_n, +\infty)] = 0,$$

and the assertion obviously follows.

Property (b) is a direct consequence of the fact that, by hypothesis, $f \in AC_\varphi^{loc}(\mathbb{R})$ (see 2.1 of [23]).

About (c), let $\lambda > 0$ be such that $4\lambda < \overline{\lambda}$ and $V_\varphi[4\lambda(f-\nu), [a,b]] < \varepsilon$ (this is a consequence of Theorem 2.21 in [23]). Then, applying property (IV),

$$V_\varphi[\lambda(f-\nu), \mathbb{R}] \leq \frac{1}{2}\Big\{V_\varphi[2\lambda(f-\nu), (-\infty, a]] + \frac{1}{2}V_\varphi[4\lambda(f-\nu), [a,b]] + \frac{1}{2}V_\varphi[4\lambda(f-\nu), [b, +\infty)]\Big\}.$$

Now, since $\nu(t) = f(a)$, for every $t \leq a$, and $\nu(t) = f(b)$, for every $t \geq b$,

$$V_\varphi[2\lambda(f-\nu), (-\infty, a]] = V_\varphi[2\lambda f, (-\infty, a]] < \varepsilon$$

and

$$V_\varphi[4\lambda(f-\nu), [b, +\infty)] = V_\varphi[4\lambda f, [b, +\infty)] < \varepsilon,$$

by (a). Hence, we conclude that $V_\varphi[\lambda(f-\nu), \mathbb{R}] < \varepsilon$. □

Lemma 2. *Let $f \in AC_\varphi^{loc}(\mathbb{R})$ and let $\{H_w\}_{w>0}$ be such that $K_w.3)$ is satisfied. Then there exists $\lambda > 0$ such that, for every $\varepsilon > 0$ and for every $a, b \in \mathbb{R}$ there exist $\bar{w} > 0$ and a step function $\nu : \mathbb{R} \to \mathbb{R}$ for which*

$$V_\varphi[\lambda((H_w \circ f) - \nu), [a,b]] < \varepsilon$$

uniformly with respect to $w \geq \bar{w}$.

Proof. The proof is analogous to the proof of Lemma 2 in [6], with ν defined as in Lemma 1. □

Lemma 3. *Let $f \in BV_\eta(\mathbb{R})$. If $\{H_w\}_{w>0}$ is such that $(\star)$ is satisfied and (φ, η, ψ) is properly directed, then for every $\lambda > 0$ there exists a constant $\mu > 0$ such that*

$$V_\varphi[\mu(H_w \circ f), J] \leq V_\eta[\lambda f, J],$$

for every $w > 0$ and $J \subset \mathbb{R}$. Hence the family $\{H_w \circ f\}_{w>0}$ is of equibounded φ-variation on every $J \subset \mathbb{R}$.

Proof. The proof is similar to that of Lemma 3 in [6]. □

Theorem 1. *If $f \in AC_\varphi^{loc}(\mathbb{R})$, there exists $\lambda > 0$ such that*

$$\lim_{t \to 0} V_\varphi[\lambda(\tau_t \nu - \nu)] = 0,$$

where ν is defined as in Lemma 1.

Proof. We shall first prove that $\lim_{t \to 0^-} V_\varphi[\lambda(\tau_t \nu - \nu)] = 0$.

Let $\varepsilon > 0$ be fixed. From Lemma 1 there exist $\overline{\lambda} > 0$ and $a, b \in \mathbb{R}$ such that, for every partition $D = \{t_i\}_{i=0}^n$ of $[a, b]$ with $t_i - t_{i-1} < \delta$,

$$\sum_{i=1}^{n} V_\varphi[\overline{\lambda} f, [t_{i-1}, t_i]] < 2\varepsilon.$$

Define now $\beta := \max\{t_{i-1} - t_i,\ i = 1, \dots, n\}$ and note that, if $\beta < t < 0$, $t_{i-1} < t_{i-1} - t < t_i$, and so $\nu(t_{i-1} - t) = \nu(t_{i-1})$, for every $i = 1, \dots, n$; moreover, $(\tau_t \nu - \nu)(x) = 0$ for $x \le a$ or $x \ge b$ and so $V_\varphi[\lambda(\tau_t \nu - \nu)] = V_\varphi[\lambda(\tau_t \nu - \nu), [a, b]]$. Therefore, by property (II) there holds, for $\lambda > 0$,

$$V_\varphi[\lambda(\tau_t \nu - \nu)] \le \frac{1}{2} \sum_{i=1}^{n} V_\varphi[2\lambda(\tau_t \nu - \nu), [t_{i-1}, t_i]]. \tag{4.1}$$

Let now $\Pi = \{\tau_i\}_{i=0}^m$ be a partition of the interval $[t_{i-1}, t_i]$, for $i = 1, \dots, n$. By convexity of φ we have

$$\begin{aligned} &\sum_{i=1}^{m} \varphi\left(2\lambda |(\tau_t \nu - \nu)(\tau_i) - (\tau_t \nu - \nu)(\tau_{i-1})|\right) \\ &\le \frac{1}{2} \Big\{ \sum_{i=1}^{m} \varphi\left(4\lambda |(\tau_t \nu)(\tau_i) - (\tau_t \nu)(\tau_{i-1})|\right) + \sum_{i=1}^{m} \varphi\left(4\lambda |\nu(\tau_{i-1}) - \nu(\tau_i)|\right) \Big\}. \end{aligned}$$

Passing to the supremum on every partition of $[t_{i-1}, t_i]$,

$$V_\varphi[\lambda(\tau_t \nu - \nu)] \le \frac{1}{4} \sum_{i=1}^{n} \Big\{ V_\varphi[4\lambda \tau_t \nu, [t_{i-1}, t_i]] + V_\varphi[4\lambda \nu, [t_{i-1}, t_i]] \Big\}.$$

We shall now prove that, for $t \in (\beta, 0)$,

$$V_\varphi[4\lambda \tau_t \nu, [t_{i-1}, t_i]] \le V_\varphi[4\lambda \nu, [t_{i-1}, t_i]]. \tag{4.2}$$

Indeed, it is clear that $V_\varphi[4\lambda \tau_t \nu, [t_{i-1}, t_i]] \le V_\varphi[4\lambda \nu, [t_{i-1} - t, t_i - t]]$. Moreover, let $\Pi' = \{\tau_i'\}_{i=0}^k$ be a partition of $[t_{i-1} - t, t_i - t]$. If ℓ is such that $\tau_\ell' \le t_i < \tau_{\ell+1}'$, then

$$\begin{aligned} \sum_{j=1}^{k} \varphi\left(4\lambda |\nu(\tau_j') - \nu(\tau_{j-1}')|\right) &= \sum_{j=1}^{\ell} \varphi\left(4\lambda |\nu(\tau_j') - \nu(\tau_{j-1}')|\right) \\ &\quad + \sum_{j=\ell+2}^{k} \varphi\left(4\lambda |\nu(\tau_j') - \nu(\tau_{j-1}')|\right) + \varphi\left(4\lambda |\nu(\tau_{\ell+1}') - \nu(\tau_\ell')|\right) \\ &= \varphi\left(4\lambda |\nu(t_i) - \nu(t_{i-1})|\right) \le V_\varphi[4\lambda \nu, [t_{i-1}, t_i]], \end{aligned}$$

and hence (4.2) follows.

In conclusion, if $\lambda > 0$ is such that $4\lambda < \overline{\lambda}$, for $t \in (\beta, 0)$,

$$V_\varphi[\lambda(\tau_t\nu - \nu)] \leq \frac{1}{2}\sum_{i=1}^{n} V_\varphi[4\lambda\nu, [t_{i-1}, t_i]] \leq \frac{1}{2}\sum_{i=1}^{n} V_\varphi[4\lambda f, [t_{i-1}, t_i]] < \varepsilon.$$

Let us now prove that $\lim_{t\to 0^+} V_\varphi[\lambda(\tau_t\nu - \nu)] = 0$. Define $\delta := \min\{t_i - t_{i-1},\ i \geq 2\}$, and let $t \in (0, \delta)$. First note that $(\tau_t\nu - \nu)(x) = 0$ if $x \leq a$ or $x \geq b + \delta$, $V_\varphi[\lambda(\tau_t\nu - \nu), [t_{n-1}, b + \delta]] = V_\varphi[\lambda(\tau_t\nu - \nu), [t_{n-1}, b]]$ since $(\tau_t\nu - \nu)(x)$ is 0 or equal to $(f(t_{n-1}) - f(b))$ for $b \leq x < b + \delta$ and therefore $V_\varphi[\lambda(\tau_t\nu - \nu)] = V_\varphi[\lambda(\tau_t\nu - \nu), [a, b]]$. Hence, as before, by property (II) (4.1) holds.

In order to estimate $\sum_{i=1}^{n} V_\varphi[2\lambda(\tau_t\nu - \nu), [t_{i-1}, t_i]]$ it is sufficient to follow the same proof as before, replacing (4.2) with the following:

$$V_\varphi[4\lambda\tau_t\nu, [t_{i-1}, t_i]] \leq V_\varphi[4\lambda\nu, [t_{i-2}, t_{i-1}]], \tag{4.3}$$

for every $i = 2, \ldots, n$ and for $t \in (0, \delta)$.

To prove (4.3) note that, as before, for every $t > 0$, $V_\varphi[4\lambda\tau_t\nu, [t_{i-1}, t_i]] \leq V_\varphi[4\lambda\nu, [t_{i-1} - t, t_i - t]]$. Moreover, if $t \in (0, \delta)$, then $t_{i-1} < t_i - t < t_i$; hence, $\nu(t_i - t) = \nu(t_{i-1})$, for every $i \geq 2$. Let then $t \in (0, \delta)$ and let $\Pi^{''} := \{\tau_i^{''}\}_{i=0}^{p}$ be a partition of $[t_{i-1} - t, t_i - t]$. If ℓ is such that $\tau_\ell^{''} \leq t_{i-1} < \tau_{\ell+1}^{''}$, then

$$\begin{aligned}
\sum_{j=1}^{p} \varphi\left(4\lambda|\nu(\tau_j^{''}) - \nu(\tau_{j-1}^{''})|\right) &= \sum_{j=1}^{\ell} \varphi\left(4\lambda|\nu(\tau_j^{''}) - \nu(\tau_{j-1}^{''})|\right) \\
&\quad + \sum_{j=\ell+2}^{p} \varphi\left(4\lambda|\nu(\tau_j^{''}) - \nu(\tau_{j-1}^{''})|\right) + \ \varphi\left(4\lambda|\nu(\tau_{\ell+1}^{''}) - \nu(\tau_\ell^{''})|\right) \\
&= \varphi\left(4\lambda|\nu(t_{i-1}) - \nu(t_{i-2})|\right) \leq V_\varphi[4\lambda\nu, [t_{i-2}, t_{i-1}]],
\end{aligned}$$

and so (4.3) follows by the arbitrariness of the partition $\Pi^{''}$. Therefore,

$$\begin{aligned}
V_\varphi[\lambda(\tau_t\nu - \nu)] &\leq \frac{1}{2}\sum_{i=2}^{n} V_\varphi[4\lambda\nu, [t_{i-2}, t_{i-1}]] + \frac{1}{4}V_\varphi[4\lambda\tau_t\nu, [a, t_1]] \\
&\quad + \frac{1}{4}V_\varphi[4\lambda\nu, [t_{n-1}, b]] \\
&= \frac{1}{2}\sum_{i=2}^{n} V_\varphi[4\lambda\nu, [t_{i-2}, t_{i-1}]] + \frac{1}{4}V_\varphi[4\lambda\nu, [t_{n-1}, b]],
\end{aligned}$$

taking into account the fact that $\tau_t\nu$ is constant on $[a,t_1]$. Then, if $4\lambda < \overline{\lambda}$,

$$V_\varphi[\lambda(\tau_t\nu-\nu)] \le \frac{1}{2}\sum_{i=1}^{n} V_\varphi[4\lambda\nu,[t_{i-1},t_i]] \le \frac{1}{2}\sum_{i=1}^{n} V_\varphi[4\lambda f,[t_{i-1},t_i]] < \varepsilon,$$

from (b) of Lemma 1, and so the thesis follows. □

Theorem 2. *Let $f \in AC^{loc}_\varphi(\mathbb{R}) \cap BV_\eta(\mathbb{R})$, let $\{H_w\}_{w>0}$ be such that $(\star)$ is satisfied and let (φ,η,ψ) be properly directed. Then there exist $\bar{w}>0$ and $\lambda>0$ such that*

$$\lim_{\delta\to 0}\omega_\varphi(\lambda g_w,\delta)=0,$$

uniformly with respect to $w \ge \bar{w}$, with $g_w = H_w \circ f$.

Proof. We now prove that $\lim_{t\to 0} V_\varphi[\lambda(\tau_t g_w - g_w)] = 0$, for some $\lambda>0$ and for sufficiently large $w>0$, since the assertion becomes an easy consequence.

From Lemma 1, since $f \in BV_\eta(\mathbb{R})$, for a fixed $\varepsilon>0$ there exist $a,b\in\mathbb{R}$, $\overline{\lambda}>0$ such that $V_\eta[\overline{\lambda}f,(-\infty,a]] < \varepsilon$ and $V_\eta[\overline{\lambda}f,[b,+\infty)] < \varepsilon$. From Lemma 3, to each $\overline{\lambda}$ there corresponds $\overline{\mu}>0$ such that

$$V_\varphi[\overline{\mu}g_w,(-\infty,a]] \le V_\eta[\overline{\lambda}f,(-\infty,a]] < \varepsilon$$

and

$$V_\varphi[\overline{\mu}g_w,[b,+\infty)] \le V_\eta[\overline{\lambda}f,[b,+\infty)] < \varepsilon.$$

Choose now $d,h,\bar{a},\bar{b}\in\mathbb{R}$ such that $h<\bar{a}<a$, $b<\bar{b}<d$, and let ν be the step function defined in Lemma 1. Moreover, let t be such that $\max\{\bar{a}-a,\bar{b}-d\} < t < \min\{\bar{b}-b,\bar{a}-h\}$. Note that, in this case, $s-t<a$ if $s<\bar{a}$, $s-t>b$ if $s>\bar{b}$ and $h<s-t<d$ if $\bar{a}<s<\bar{b}$. Then, by properties (IV) and (I), for $\lambda>0$ such that $\lambda<\frac{\overline{\mu}}{8}$,

$$\begin{aligned}
V_\varphi[\lambda(\tau_t g_w - g_w)] &\le \frac{1}{2}V_\varphi[2\lambda(\tau_t g_w - g_w),(-\infty,\bar{a}]] \\
&\quad + \frac{1}{4}V_\varphi[4\lambda(\tau_t g_w - g_w),[\bar{a},\bar{b}]] + \frac{1}{4}V_\varphi[4\lambda(\tau_t g_w - g_w),[\bar{b},+\infty)] \\
&\le \frac{1}{4}V_\varphi[4\lambda\tau_t g_w,(-\infty,\bar{a}]] + \frac{1}{4}V_\varphi[4\lambda g_w,(-\infty,\bar{a}]] + \frac{1}{4}V_\varphi[4\lambda(\tau_t g_w - g_w),[\bar{a},\bar{b}]] \\
&+ \frac{1}{8}V_\varphi[8\lambda\tau_t g_w,[\bar{b},+\infty)] + \frac{1}{8}V_\varphi[8\lambda g_w,[\bar{b},+\infty)] \\
&\le \frac{1}{4}V_\varphi[4\lambda(\tau_t g_w - g_w),[\bar{a},\bar{b}]] + \frac{1}{4}\{2V_\eta[\overline{\lambda}f,(-\infty,a]] + V_\eta[\overline{\lambda}f,[b,+\infty)]\} \\
&< \frac{1}{4}V_\varphi[4\lambda(\tau_t g_w - g_w),[\bar{a},\bar{b}]] + \frac{3}{4}\varepsilon.
\end{aligned}$$

Moreover, again by (I),

$$V_\varphi[4\lambda(\tau_t g_w - g_w), [\bar{a}, \bar{b}]] \leq \frac{1}{3}\{V_\varphi[12\lambda\tau_t(g_w - \nu), [\bar{a}, \bar{b}]] + V_\varphi[12\lambda(\tau_t\nu - \nu), [\bar{a}, \bar{b}]] + V_\varphi[12\lambda(\nu - g_w), [\bar{a}, \bar{b}]]\}$$
$$\leq \frac{1}{3}\{2V_\varphi[12\lambda(g_w - \nu), [h, d]] + V_\varphi[12\lambda(\tau_t\nu - \nu), [h, d]]\} := \frac{1}{3}(2I_1 + I_2).$$

Now, for sufficiently large $w > 0$ and sufficiently small $\lambda > 0$, by Lemma 2, the fact that $I_1 < \frac{3}{4}\,\varepsilon$ and by Theorem 1, $I_2 < \frac{3}{2}\,\varepsilon$, for t small enough; hence, we conclude that $V_\varphi[\lambda(\tau_t g_w - g_w)] < \varepsilon$. □

We are now ready to prove our main convergence result.

Theorem 3. *If $f \in AC^{loc}_\varphi(\mathbb{R}) \cap BV_\eta(\mathbb{R})$, $\{K_w\}_{w>0} \subset \mathcal{K}_w$ and the triple (φ, η, ψ) is properly directed, then there exists $\mu > 0$ such that*

$$\lim_{w\to+\infty} V_\varphi[\mu(T_w f - f)] = 0.$$

Proof. We first note that $(T_w f - f) \in BV_\varphi(\mathbb{R})$ since, by Proposition 1, $T_w f \in BV_\varphi(\mathbb{R})$, since $f \in BV_\eta(\mathbb{R})$. By Proposition 2, for every $\mu, \delta > 0$,

$$V_\varphi[\mu(T_w f - f)] \leq \frac{1}{2}\Big\{\omega_\varphi(2\mu A g_w, \delta) + A^{-1}V_\varphi\left[4\mu A g_w\right]\int_{|t|\geq\delta}|L_w(t)|dt + V_\varphi[2\mu A G_w, J]\Big\},$$

for some bounded interval $J \subset \mathbb{R}$. Now, by Theorem 2, for every $\varepsilon > 0$ there exist $\lambda > 0$ and $\delta > 0$ such that, for sufficiently large $w > 0$, $\omega_\varphi(2\mu A g_w, \delta) < \varepsilon$, for $0 < \mu < \frac{\lambda}{2A}$.

Moreover, for any fixed $\delta > 0$, by K_w.2) and by Lemma 3, for every $\lambda > 0$ there exists $\mu > 0$ such that

$$A^{-1}V_\varphi\left[4\mu A g_w\right]\int_{|t|\geq\delta}|L_w(t)|dt \leq \varepsilon A^{-1}V_\eta[\lambda f],$$

for sufficiently large $w > 0$. Finally, if μ is such that $2\mu A < \xi$, by assumption K_w.3), for w large enough, $V_\varphi[2\mu A G_w, J] \leq \varepsilon$. In conclusion, for sufficiently large $w > 0$ and if μ is sufficiently small,

$$V_\varphi[\mu(T_w f - f)] \leq \varepsilon\Big(1 + \frac{1}{2A}V_\eta[\lambda f]\Big).$$

Hence, the thesis follows since $f \in BV_\eta(\mathbb{R})$ and by the arbitrariness of $\varepsilon > 0$. □

Example 2. Here we remark that Theorem 3 does not hold in general for $f \in BV_\varphi(\mathbb{R}) \setminus AC^{loc}_\varphi(\mathbb{R})$. Take, as example, $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(z) = \begin{cases} 1, & |z| \leq 1, \\ 0, & |z| > 1. \end{cases}$$

First of all we note that $V_\varphi[\tau_t f - f] \not\longrightarrow 0$ as $t \to 0$ and so $V_\varphi[\tau_t \nu - \nu] \not\longrightarrow 0$ as $t \to 0$, where ν is defined as in Lemma 1. Let us now consider the Poisson-Cauchy kernel

$$L_w(t) = \sqrt{\frac{2}{\pi}} \, \frac{w}{1 + w^2 t^2}, \ w > 0, \ t \in \mathbb{R},$$

and take $K_w(t,u) = L_w(t)u$, i.e., K_w is a linear kernel (in this case one can take $\varphi = \eta$ in condition $(\star\star)$, since $\psi(|u|) = |u|$). Then

$$(T_w f)(s) = \sqrt{\frac{2}{\pi}} \, [\arctan(w(s+1)) - \arctan(w(s-1))], \ s \in \mathbb{R},$$

and therefore by property 1.03 of [23], there holds

$$\begin{aligned} V_\varphi[T_w f - f, \mathbb{R}] &\geq V_\varphi[T_w f - f, (-\infty, -1]] = V_\varphi[T_w f, (-\infty, -1]] \\ &= \varphi\Big((T_w f)(-1) - \lim_{s \to -\infty} (T_w f)(s)\Big) \\ &= \varphi\Big(- \sqrt{\frac{2}{\pi}} \arctan(-2w)\Big). \end{aligned}$$

Therefore,

$$\liminf_{w \to +\infty} V_\varphi[T_w f - f, \mathbb{R}] \geq \varphi\Big(\sqrt{\frac{\pi}{2}}\Big),$$

and so $V_\varphi[T_w f - f, \mathbb{R}] \not\longrightarrow 0$ as $w \to +\infty$.

5. Final considerations of the convergence result

In this section we show that in the convergence result (Theorem 3) the condition that $f \in AC^{loc}_\varphi(\mathbb{R})$ is not only sufficient, but, in general, becomes also necessary for the convergence of $T_w f - f$ with respect to the φ-variation. Indeed it is possible to state the following:

Lemma 4. *The space $AC^{loc}_\varphi(\mathbb{R})$ is a closed subspace of $BV_\varphi(\mathbb{R})$ with respect to the φ-variation functional.*

Proof. Let $\{f_w\}_{w>0}$ be a sequence of $AC^{loc}_\varphi(\mathbb{R})$-functions which converges with respect to the φ-variation to f; i.e., there exists $\mu > 0$ such that for every $\varepsilon > 0$ there is a $\bar{w} > 0$ for which $V_\varphi[\mu(f_w - f)] < \varepsilon$, for every

$w \geq \bar{w}$. Now, since $\{f_w\}_{w>0} \subset AC_\varphi^{loc}(\mathbb{R})$, there exists a constant $\bar{\lambda} > 0$ such that the following property holds: for every $\varepsilon > 0$ and for any fixed compact interval $I = [a,b] \subset \mathbb{R}$, there is a $\delta > 0$ such that for every finite collection $\{[a_i, b_i],\ i = 1, \ldots, k\} \subset I$ of non-overlapping intervals for which $\sum_{i=1}^k \varphi(b_i - a_i) < \delta$, there holds

$$\sum_{i=1}^{k} \varphi(\overline{\lambda}|f_w(b_i) - f_w(a_i)|) < \varepsilon.$$

So, taking $\{[a_i, b_i],\ i = 1, \ldots, k\}$ a finite collection of I with the above property, we may write, by convexity of φ, for $w \geq \bar{w}$,

$$\begin{aligned}
&\sum_{i=1}^{k} \varphi(\lambda|f(b_i) - f(a_i)|) \\
&\leq \sum_{i=1}^{k} \varphi\Big(\lambda\Big[|f(b_i) - f_w(b_i) - (f(a_i) - f_w(a_i))| + |f_w(b_i) - f_w(a_i)|\Big]\Big) \\
&\leq \frac{1}{2}\sum_{i=1}^{k} \varphi(2\lambda|(f - f_w)(b_i) - (f - f_w)(a_i)|) + \frac{1}{2}\sum_{i=1}^{k} \varphi(2\lambda|f_w(b_i) - f_w(a_i)|) \\
&\leq \frac{1}{2} V_\varphi[\mu(f_w - f)] + \frac{1}{2}\sum_{i=1}^{k} \varphi(\overline{\lambda}|f_w(b_i) - f_w(a_i)|) \leq \varepsilon,
\end{aligned}$$

choosing $0 < \lambda \leq \min\left\{\frac{\overline{\lambda}}{2}, \frac{\mu}{2}\right\}$.

Now, by the arbitrariness of the collection $\{[a_i, b_i],\ i = 1, \ldots, k\}$, we obtain that $f \in AC_\varphi^{loc}(\mathbb{R})$, and so $AC_\varphi^{loc}(\mathbb{R})$ is a closed subspace of $BV_\varphi(\mathbb{R})$. □

Now it is well known that linear convolution integral operators (i.e., having kernels of the form $K_w(t,u) = L_w(t) \cdot u$) are absolutely continuous if, for example, the kernels have the same property (as happens for the classical approximate identities).

In the case of nonlinear integral operators of the form (+), we may prove the following:

Proposition 3. *If* $\{L_w\}_{w>0} \subset AC_\varphi^{loc}(\mathbb{R})$, $(\star)$ *is satisfied and* $(\psi \circ |f|) \in L^1(\mathbb{R})$, *then* $T_w f \in AC_\varphi^{loc}(\mathbb{R})$.

Proof. We may write, putting $t = s - z$,

$$(T_w f)(s) = \int_{\mathbb{R}} K_w(t, f(s-t))\, dt = \int_{\mathbb{R}} L_w(t) H_w(f(s-t))\, dt$$

$$= \int_{\mathbb{R}} L_w(s-z) H_w(f(z))\, dz.$$

Now, let us take as $\delta > 0$ the number in the definition of local absolute continuity of $\{L_w\}_{w>0}$. So, given a finite collection $\{[a_i, b_i],\ i = 1, \ldots, k\}$ of non-overlapping intervals for which

$$\sum_{i=1}^{k} \varphi(b_i - a_i) < \delta,$$

we have, since $H_w(0) = 0$, applying Jensen's inequality,

$$\begin{aligned}
&\sum_{i=1}^{k} \varphi(\lambda |(T_w f)(b_i) - (T_w f)(a_i)|) \\
&= \sum_{i=1}^{k} \varphi\left(\lambda \left| \int_{\mathbb{R}} [L_w(b_i - z) H_w(f(z)) - L_w(a_i - z) H_w(f(z))]\, dz \right| \right) \\
&\leq \sum_{i=1}^{k} \varphi\left(\lambda \int_{\mathbb{R}} |H_w(f(z))|\ |L_w(b_i - z) - L_w(a_i - z)|\, dz \right) \\
&\leq \sum_{i=1}^{k} \varphi\left(K\lambda \int_{\mathbb{R}} \psi(|f(z)|) |L_w(b_i - z) - L_w(a_i - z)|\, dz \right) \\
&\leq \frac{1}{A} \int_{\mathbb{R}} \psi(|f(z)|) \sum_{i=1}^{k} \varphi\left(K\lambda A\ |L_w(b_i - z) - L_w(a_i - z)|\right)\, dz,
\end{aligned}$$

for some constant A with $A \geq \| \psi \circ |f| \|_{L^1(\mathbb{R})}$ and for $\lambda > 0$. Above, K is the constant of assumption $(\star)$.

Now, since $\{L_w\}_{w>0} \subset AC_{\varphi}^{loc}(\mathbb{R})$ and

$$\sum_{i=1}^{k} \varphi(b_i - z - (a_i - z)) < \delta,$$

we have

$$\sum_{i=1}^{k} \varphi(\lambda |(T_w f)(b_i) - (T_w f)(a_i)|) \leq \frac{\varepsilon}{A} \| \psi \circ |f| \|_{L^1(\mathbb{R})} \leq \varepsilon,$$

for a suitable $\lambda > 0$. Therefore, by the arbitrariness of the collection $\{[a_i, b_i],\ i = 1, \ldots, k\}$, the assertion follows. $\square$

6. Order of approximation

We shall now turn to the problem of the order of approximation for our family of nonlinear integral operators with respect to the convergence in φ-variation.

For a fixed $\alpha > 0$ we say that L_w is an *α-singular kernel* if, for every $\delta > 0$,

$$\int_{|t|\geq\delta} |L_w(t)|dt = O(w^{-\alpha}), \text{ as } w \to +\infty. \tag{6.1}$$

Let $\mathcal{T}$ be the class of all functions $\tau : \mathbb{R} \to \mathbb{R}_0^+$ which are measurable, continuous at 0, such that $\tau(0) = 0$ and $\tau(t) > 0$, for $t \neq 0$. For $\tau \in \mathcal{T}$, denoting by $\mathcal{H} = \{H_w\}_{w>0}$, we define the Lipschitz class $V_\varphi Lip^{\mathcal{H}}(\tau)$ as

$$V_\varphi Lip^{\mathcal{H}}(\tau) = \Big\{ f \in AC_\varphi^{loc}(\mathbb{R}) : \ \exists \sigma > 0 \ \text{ s.t.}$$

$$V_\varphi\left[\sigma\left|(H_w \circ f)(\cdot - t) - (H_w \circ f)(\cdot)\right|\right] = O(\tau(t)), \quad \text{as } t \to 0 \Big\}$$

uniformly with respect to $w > 0$.

The following result establishes the order of approximation for $(T_w f - f)$.

Theorem 4. *Let $\{K_w\}_{w>0} \subset \mathcal{K}_w$ be such that L_w is α-singular, for every $w > 0$. Suppose that $f \in BV_\eta(\mathbb{R}) \cap V_\varphi Lip^{\mathcal{H}}(\tau)$, $\tau \in \mathcal{T}$; if there exists $\gamma > 0$ such that*

$$V_\varphi[\gamma\ G_w, J] = O(w^{-\alpha}), \quad \textit{as } w \to +\infty, \ \textit{for every bounded interval } J \subset \mathbb{R}, \tag{6.2}$$

and there exists $\delta > 0$ such that

$$\int_{0\leq|t|\leq\delta} |L_w(t)|\tau(t)\, dt = O(w^{-\alpha}), \quad \textit{as } w \to +\infty, \tag{6.3}$$

then

$$V_\varphi[\lambda(T_w f - f)] = O(w^{-\alpha}),$$

for sufficiently large $w > 0$ and for some $\lambda > 0$.

Proof. For the sake of simplicity, we may take the constants γ and σ of assumption (6.2) and of the class $V_\varphi Lip^{\mathcal{H}}(\tau)$ respectively, equal to 1. Following the proof of Proposition 2, for every $\lambda, \delta > 0$ and for some bounded interval $J \subset \mathbb{R}$, it is easy to achieve the following estimate, for every $w > 0$:

$$V_\varphi[\lambda(T_w f - f)]$$

$$\leq \frac{1}{2A}\left\{ \int_{0\leq|t|\leq\delta} + \int_{|t|\geq\delta} \right\} |L_w(t)|\; V_\varphi\left[2\lambda A\,|(H_w\circ f)(\cdot - t) - (H_w\circ f)(\cdot)|\right] dt$$

$$+ \frac{1}{2}V_\varphi[2\lambda A G_w, J] := I_1 + I_2 + I_3.$$

Now, since $f \in V_\varphi Lip^{\mathcal{H}}(\tau)$, if $\lambda < \frac{1}{2A}$, there exist $N, \bar{\delta} > 0$ such that

$$V_\varphi\left[2\lambda A\,|(H_w\circ f)(\cdot - t) - (H_w\circ f)(\cdot)|\right] \leq N\tau(t)$$

for every $t \in [-\bar{\delta}, \bar{\delta}]$, and so, by (6.3),

$$I_1 \leq \frac{N}{2A}\int_{0\leq|t|\leq\delta} |L_w(t)|\tau(t)dt = O(w^{-\alpha}),$$

for sufficiently large $w > 0$ and for a suitable $\delta > 0$. Moreover, by (6.2), $I_3 = \frac{1}{2}V_\varphi[2\lambda A G_w, J] = O(w^{-\alpha})$, as $w \to +\infty$.

For I_2, by (I) there holds that

$$I_2 \leq \frac{1}{2A}V_\varphi[4\lambda A g_w]\int_{|t|\geq\delta} |L_w(t)|dt,$$

where $g_w = H_w \circ f$. Now, since $f \in BV_\eta(\mathbb{R})$, by Lemma 3 there exist $M > 0$, $\mu > 0$ (depending on M) such that $V_\varphi[\mu g_w] \leq M$, for every $w > 0$. Finally,

$$\int_{|t|\geq\delta} |L_w(t)|dt = O(w^{-\alpha}), \quad \text{as } w \to +\infty,$$

since L_w is an α-singular kernel, and therefore $I_2 = O(w^{-\alpha})$, as $w \to +\infty$, for $\lambda < \frac{\mu}{4A}$. Therefore, the assertion follows taking $0 < \lambda < \min\left\{\frac{1}{2A}, \frac{\mu}{4A}\right\}$. □

Remarks 2. (a) In [1] it is proved that if the absolute moment of order $\alpha > 0$ of the functions L_w, defined as usual by

$$m(L_w, \alpha) := \int_{\mathbb{R}} |t|^\alpha |L_w(t)|\, dt,$$

is such that $m(L_w, \alpha) = O(w^{-\alpha})$, as $w \to +\infty$, then (6.1) holds and (6.3) of Theorem 4 is satisfied with $\tau(t) = |t|^\alpha$. Moreover, it is not difficult to prove that for the family of kernels of Example 1 assumption (6.2) of Theorem 4 is satisfied, taking into account the convexity of φ. Hence, assuming the classical condition that $m(L_w, \alpha) = O(w^{-\alpha})$ as $w \to +\infty$, the family of kernels of Example 1 provides an example of kernels to which our theory can be applied; other examples of kernels can be found in [2]. Furthermore, as is well known, there are several examples of kernel functions which satisfy the condition $m(L_w, \alpha) = O(w^{-\alpha})$, such as the Abel-Poisson kernels, the Fejér-Korovkin kernels, the integral means and others (see [1] and [2]).

(b) We point out that it is possible to provide a straightforward generalization of the above result proving that the error of approximation $T_w f - f$ is $O(\xi(w^{-1}))$, as $w \to +\infty$, where $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous at $u = 0$, with $\xi(0) = 0$ and $\xi(u) > 0$ for $u > 0$. In this case it is sufficient to replace $w^{-\alpha}$ by $\xi(w^{-1})$ in (6.1) and in (6.2) and (6.3) of Theorem 4.

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