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A REMARK ON THE CAUCHY PROBLEM FOR THE 2D GROSS-PITAEVSKII EQUATION WITH NONZERO DEGREE AT INFINITY

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Abstract. We prove global well-posedness for the Gross-Pitaevskii equation on the plane for classes of initial data having nonzero topological degree at infinity and therefore infinite Ginzburg-Landau energy. These classes allow us to consider arbitrary configurations of vortices as initial data. Our work follows recent results of Patrick Gérard [9] and Clément Gallo [4], where the finite energy regime is treated.

1. Introduction

In a recent paper, Patrick Gérard [9] has established global well-posedness of the Gross-Pitaevskii equation in $\mathbb{R}^N$, $N = 2$ or 3,

$$(GP) \qquad i\partial_t u + \Delta u = (|u|^2 - 1)u,$$

for initial data in the energy space. The energy in this case is given by the Ginzburg-Landau functional

$$\mathcal{E}(u) \equiv \int_{\mathbb{R}^2} e(u) := \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4}.$$

Equation (GP) is Hamiltonian, with Hamiltonian given by $\mathcal{E}$. The solutions constructed in [9] have finite, constant-in-time Ginzburg-Landau energy. Clément Gallo [4] proved additional properties of the flow as well as extensions to more general nonlinearities. One peculiarity of $\mathcal{E}$ and (GP) is that finite energy fields do not tend to zero at infinity, but have instead to stay close to the unit circle S^1.

In dimension two, the Gross-Pitaevskii equation possesses remarkable stationary solutions. These solutions, called vortices and labeled by an integer

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$d \in \mathbb{Z}^*$, have the special form

$$u_d(x) \equiv u_d(r,\theta) = f_d(r)\exp(id\theta),$$

where $f_d : \mathbb{R}^+ \to [0,1]$ satisfies

$$f_d'' + \frac{1}{r}f_d' - \frac{d^2}{r^2}f_d + f_d(1-f_d^2) = 0, \qquad f_d(0) = 0, \qquad f_d(+\infty) = 1.$$

It is known (see e.g. [10]) that $|\nabla u_d(x)| \sim d/|x|$ as $x \to +\infty$, so that

$$\int |\nabla u_d|^2 = +\infty. \tag{1.1}$$

On the other hand, the potential term remains bounded (actually $\int (1 - |u_d|^2)^2/4 = \pi d^2$), as well as the modulus part of the gradient: $\int |\nabla |u_d||^2 < +\infty$. Notice that u_d has winding number d at infinity, in the sense that for each radius $r > 0$ large enough (actually for any radius here) the map $\psi_r : \partial B_r \simeq S^1 \to S^1$ given by

$$x \mapsto \frac{u_d(x)}{|u_d(x)|}$$

has topological degree d. It can easily be proved that any continuous field which does not vanish outside a compact set and has a nonzero degree at infinity has infinite energy.

The purpose of this note is to address the Cauchy problem for (GP) for classes of initial data having a nonzero degree at infinity, and which thus do not belong to the energy space. In particular this will include perturbations of the aforementioned vortices. More precisely, we will consider the space

$$Y = \left\{ U \in L^\infty(\mathbb{R}^2, \mathbb{C}), \quad \nabla^k U \in L^2(\mathbb{R}^2), \forall k \geq 2 \right\},$$

its subset

$$\mathcal{V} = \left\{ U \in Y, \quad \nabla|U| \in L^2(\mathbb{R}^2), \quad (1 - |U|^2) \in L^2(\mathbb{R}^2) \right\},$$

and the set

$$Z = \mathcal{V} + H^1(\mathbb{R}^2).$$

Using regularization by convolution (see e.g. [4]), one realizes that $\{\mathcal{E}(u) < +\infty\} \subset Z$, and actually one has

$$Z \cap \dot{H}^1(\mathbb{R}^2) \equiv \{\mathcal{E}(u) < +\infty\}.$$

On the other hand, $u_d \in \mathcal{V}$ (this easily follows from the already-mentioned properties of u_d and from elliptic regularity); in particular, in view of (1.1)

$\mathcal{V}$ is different from the energy space. Moreover, if $U \in \mathcal{V}$, standard Sobolev embeddings yields $\nabla U \in L^\infty(\mathbb{R}^2)$.

Our main theorem is the following:

Theorem 1. *Let $u_0 \in Z$. There exists a unique solution $t \mapsto u(t)$ of (GP) such that $u(0) = u_0$ and $u(t) - u_0 \in \mathcal{C}^0(\mathbb{R}, H^1(\mathbb{R}^2))$.*

Since $t \mapsto u(t)$ belongs to the affine space $u_0 + H^1(\mathbb{R}^2)$, equation (GP) has a meaning for u, at least in the sense of distributions. Actually, choosing a decomposition

$$u_0 = U_0 + w_0,$$

where $U_0 \in \mathcal{V}$ and $w_0 \in H^1(\mathbb{R}^2)$, we will prove that there exists a unique solution $w \in \mathcal{C}^0(\mathbb{R}, H^1(\mathbb{R}^2))$ of

$$\begin{cases} i\partial_t w + \Delta w = f_{U_0}(w) \\ w(\cdot, 0) = w_0, \end{cases} \tag{1.2}$$

where

$$f_{U_0}(w) = -\Delta U_0 + (|U_0 + w|^2 - 1)(U_0 + w).$$

The function $u(t) = U_0 + w(t)$ is then the solution given in Theorem 1.

In the case $\nabla u_0 \in L^2(\mathbb{R}^2)$, the result is a consequence of [9] and [4]. Here, we only require $\nabla |u_0| \in L^2(\mathbb{R}^2)$, together with some less restrictive assumptions at infinity. From the technical point of view, our new assumption does not provide any major additional difficulty for proving local existence of solutions. For the global existence however, we cannot rely as in [9, 4] on the conservation of $\mathcal{E}(u_0)$, which may be infinite. Instead, we consider the quantity[1]

$$\tilde{\mathcal{E}}_{U_0}(w) = \int_{\mathbb{R}^2} \frac{|\nabla w|^2}{2} - \int_{\mathbb{R}^2} (\Delta U_0) \cdot w + \int_{\mathbb{R}^2} \frac{(|U_0 + w|^2 - 1)^2}{4}$$

for the perturbation w, and set $\mathcal{E}_{U_0}(u) = \tilde{\mathcal{E}}_{U_0}(w)$ for $u = U_0 + w$. This last quantity may be regarded as a renormalized energy for u. It does not seem to have an intrinsic meaning, since its definition relies heavily on the decomposition $u = U_0 + w$, which is not unique.

Theorem 2. *Let $u_0 = U_0 + w_0 \in Z$, and let $t \mapsto u(t)$ be the solution given in Theorem 1; then*

$$\frac{d}{dt} \mathcal{E}_{U_0}(u(t)) = 0 \qquad \text{on } \mathbb{R}.$$

[1] For complex numbers z and z', we denote by $z \cdot z'$ the scalar product $\mathrm{Re}(\bar{z} z')$.

Moreover, if $\tilde{u}_0 = U_0 + \tilde{w}_0 \in Z$ *and* $t \mapsto \tilde{u}(t)$ *is the corresponding solution given in Theorem* 1, *then*

$$\|u(t) - \tilde{u}(t)\|_{H^1(\mathbb{R}^2)} \leq C(t, U_0, \mathcal{E}_{U_0}(u_0))\|u_0 - \tilde{u}_0\|_{H^1(\mathbb{R}^2)}.$$

The definition of the renormalized energy $\mathcal{E}_{U_0}(u)$ was motivated as follows. For $R > 0$, it is natural to introduce the difference of energies

$$\Lambda(R, u) \equiv \int_{B(R)} [e(u) - \frac{|\nabla U_0|^2}{2}].$$

Expanding and integrating by parts, we are led to

$$\Lambda(R, u) = \int_{B(R)} \frac{|\nabla w|^2}{2} - (\Delta U_0) \cdot w + \frac{(|U_0 + w|^2 - 1)^2}{4} + \int_{\partial B(R)} \frac{\partial U_0}{\partial r} \cdot w. \quad (1.3)$$

Since w belongs to $H^1(\mathbb{R}^2)$, and hence to $L^2(\mathbb{R}^2)$, and since $|U_0|$ is bounded, it can be shown that, for a subsequence $R_n \to +\infty$, the boundary term on the right-hand side of (1.3) tends to zero, so that

$$\Lambda(R_n, u) \to \mathcal{E}_{U_0}(u).$$

As a matter of fact, if it is assumed moreover that, as for vortices, $|\nabla U_0(x)| \leq \frac{C}{|x|}$, then the full sequence converges and

$$\lim_{R \to +\infty} \Lambda(R, u) = \mathcal{E}_{U_0}(u), \quad (1.4)$$

providing therefore an alternative definition of the renormalized energy. This last definition is actually very similar to the one introduced in [12].

It is also worthwhile to notice that, although it plays the role of an energy, the renormalized energy may not be bounded from below. This fact is related to the subtle behavior of U_0 at infinity. In the case $U_0 = u_d$, it can be proved (see [1]) that $\mathcal{E}_{U_0}$ is not bounded from below, unless $|d| \leq 1$. For instance, the sequence

$$u_n(z) = \prod_{j=1}^{d} u_1(z - n \exp(2i\pi \frac{\theta}{d})),$$

when $d \geq 2$, has renormalized energy tending to minus infinity.

As mentioned, the space $Z \cap \dot{H}^1(\mathbb{R}^2) \equiv \{\mathcal{E}(u) < +\infty\}$ corresponds to the energy space for (GP) (see [9, 4]). It would be of interest to have a better understanding of Z, in particular introducing a topological point of view which should for instance recognize the degree at infinity.

To conclude, we would like to stress that the complete dynamics of (GP) exhibits a remarkable variety of special solutions and regimes. Besides the already mentioned stationary vortices, there are also soliton-like solutions, as well as traveling pairs of vortices [7, 2]. In the WKB limit the (GP) equation turns out to behave like a wave equation [3], whereas in other regimes it has scattering properties similar to those of the linear Schrödinger equation [6]. An important issue is to understand how these different modes interact and possibly excite each other (an example of radiating vortices is formally treated in [11]). Solving the Cauchy problem as is done here when vortices are present is a necessary preliminary step to addressing some of these issues.

2. Local existence

In this section, we prove local existence for (1.2) using a fixed-point argument for the map $w \mapsto \Phi(w)$, where

$$\Phi(w)(t) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} f_{U_0}(w(s))\, ds$$

and w_0 is given and fixed in $H^1(\mathbb{R}^2)$.

Proposition 1. *Let $T > 0$ and $0 < \gamma < 1$. For every $w \in \mathcal{C}^0((-T,T), H^1(\mathbb{R}^2))$, the function $\Phi(w)$ belongs to $\mathcal{C}^0((-T,T), H^1(\mathbb{R}^2))$. Moreover, for $T \leq 1$ and $\tilde{w} \in \mathcal{C}^0((-T,T), H^1(\mathbb{R}^2))$, if*

$$\sup_{t\in(-T,T)} \left(\|w(t)\|_{H^1(\mathbb{R}^2)} + \|\tilde{w}(t)\|_{H^1(\mathbb{R}^2)} \right) \leq R,$$

then

$$\begin{aligned} &\sup_{t\in(-T,T)} \|\Phi(w)(t) - \Phi(\tilde{w})(t)\|_{H^1(\mathbb{R}^2)} \\ &\leq C(\gamma, U_0)(1+R^2)\, T^\gamma \sup_{t\in(-T,T)} \|w(t) - \tilde{w}(t)\|_{H^1(\mathbb{R}^2)}. \end{aligned} \tag{2.1}$$

The proof relies mainly on estimates for $f_{U_0}(w)$, combined with classical Strichartz estimates.

Lemma 1. *Let $w \in H^1(\mathbb{R}^2)$ and $1 < r < 2$ be given. Then $f_{U_0}(w) = f_1(w) + f_2(w)$, where $f_1(w) \in L^2(\mathbb{R}^2)$, $f_2(w) \in L^r(\mathbb{R}^2)$, and*

$$\|f_1(w)\|_{L^2} \leq C(U_0)(1 + \|w\|_{H^1}^2),$$

$$\|f_2(w)\|_{L^r} \leq C(r, U_0)\|w\|_{H^1}(1 + \|w\|_{H^1}^2).$$

Proof. We write

$$f_1(w) = -\Delta U_0 + (|w+U_0|^2 - 1)U_0, \quad f_2(w) = (|w+U_0|^2 - 1)w$$

and expand

$$|U_0 + w|^2 - 1 = (|U_0|^2 - 1) + 2U_0 \cdot w + |w|^2.$$

Since by assumption $w \in H^1(\mathbb{R}^2)$, we have by Sobolev embedding $w \in L^p(\mathbb{R}^2)$ for any $2 \leq p < +\infty$ with $\|w\|_{L^p} \leq C(p)\|w\|_{H^1}$. Since by assumption $U_0 \in \mathcal{V}$, and hence $U_0 \in L^\infty(\mathbb{R}^2)$ and $(1 - |U_0|^2) \in L^2(\mathbb{R}^2)$, it follows that $(|U_0 + w|^2 - 1) \in L^2(\mathbb{R}^2)$. The conclusion follows by Hölder's inequality. □

Lemma 2. *Let $w \in H^1(\mathbb{R}^2)$ and $1 < r < 2$ be given. Then $\nabla f_{U_0}(w) = g_1(w) + g_2(w)$, where $g_1(w) \in L^2(\mathbb{R}^2)$, $g_2(w) \in L^r(\mathbb{R}^2)$, and*

$$\|g_1(w)\|_{L^2} \leq C(U_0)(1 + \|w\|^2_{H^1}),$$

$$\|g_2(w)\|_{L^r} \leq C(r, U_0)\|w\|_{H^1}(1 + \|w\|^2_{H^1}).$$

Proof. Differentiating, we write

$$\partial_{x_i} f_{U_0}(w) = -\Delta\partial_{x_i} U_0 + \partial_{x_i}(|U_0+w|^2-1)(U_0+w) + (|U_0+w|^2-1)\partial_{x_i}(U_0+w).$$

Expanding once more $|U_0 + w|^2 - 1 = (|U_0|^2 - 1) + 2U_0 \cdot w + |w|^2$, we set

$$\begin{aligned} g_1(w) =& - \Delta\partial_{x_i} U_0 + \partial_{x_i}(|U_0|^2 - 1)U_0 + \partial_{x_i}(2U_0 \cdot w)U_0 \\ &+ 2(\partial_{x_i} U_0 \cdot w)w + (|w|^2 + (|U_0|^2 - 1))\nabla U_0 \end{aligned}$$

and

$$\begin{aligned} g_2(w) =& \partial_{x_i}(|w|^2)(U_0 + w) + 2(U_0 \cdot \partial_{x_i} w)w + \partial_{x_i}(|U_0|^2 - 1)w \\ &+ (|U_0|^2 - 1 + |w|^2 + 2(U_0 \cdot w))\nabla w + 2(U_0 \cdot w)\nabla U_0. \end{aligned}$$

Since by assumption $|U_0|^2 - 1 \in L^2 \cap L^\infty$, we have $|U_0|^2 - 1 \in L^p$ for all $2 \leq p \leq +\infty$. By the Gagliardo-Nirenberg inequality (see Lemma A.1 of the Appendix),

$$\|\nabla U_0\|_{L^4(\mathbb{R}^2)} \leq \sqrt[4]{18}\, \|U_0\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)} \|\Delta U_0\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}, \tag{2.2}$$

and by Sobolev embedding $\|w\|_{L^p} \leq C(p)\|w\|_{H^1}$. The conclusion then follows using various Hölder's inequalities, and the assumption $U_0 \in \mathcal{V}$. □

Proof of Proposition 1. Recall that by the Strichartz estimates (see e.g. [8, 5]) we have for every $1 < r \leq 2$

$$\Big\| \int_0^t e^{i(t-s)\Delta} f(s, \cdot) ds \Big\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^2)} \leq C(r)\|f\|_{L_t^{\frac{2r}{3r-2}} L^r_x(\mathbb{R}\times\mathbb{R}^2)}.$$

It follows by Hölder's inequality that for every $T > 0$,

$$\Big\| \int_0^t e^{i(t-s)\Delta} f(s,\cdot)ds \Big\|_{L_t^\infty L_x^2((-T,T)\times\mathbb{R}^2)} \leq C(r) T^{\frac{3}{2}-\frac{1}{r}} \|f\|_{L_t^\infty L_x^r((-T,T)\times\mathbb{R}^2)}. \tag{2.3}$$

Set

$$\Phi_N(w) = -i \int_0^t e^{i(t-s)\Delta} f_{U_0}(w(s))ds.$$

Combining Lemma 1 and Lemma 2 with (2.3) for $f_1(w)$, $f_2(w)$, $g_1(w)$ and $g_2(w)$, we are led, for $1 < r < 2$ and $0 < T < 1$, to the estimate

$$\|\Phi_N(w)\|_{L_t^\infty H_x^1((-T,T)\times\mathbb{R}^2)} \leq C(r) T^{\frac{3}{2}-\frac{1}{r}} (1 + \|w\|^3_{L_t^\infty H_x^1((-T,T)\times\mathbb{R}^2)}). \tag{2.4}$$

Moreover, for $0 \leq |t|, |t'| < T$, we have

$$\Phi_N(w)(t') - \Phi_N(t) = (e^{i(t'-t)\Delta} - \mathrm{Id})\Phi_N(t) - i \int_t^{t'} e^{i(t'-s)\Delta} f_{U_0}(w(s))ds.$$

Computations similar to those above yield

$$\Big\| \int_t^{t'} e^{i(t'-s)\Delta} f_{U_0}(w(s))ds \Big\|_{H^1(\mathbb{R}^2)} \leq C(r)|t'-t|^{\frac{3}{2}-\frac{1}{r}}(1+\|w\|^3_{L_t^\infty H_x^1((-T,T)\times\mathbb{R}^2)}).$$

Combined with the fact that $t \mapsto e^{it\Delta}$ is a strongly continuous group on $H^1(\mathbb{R}^2)$, we therefore infer that $\Phi(w) \in \mathcal{C}^0((-T,T), H^1(\mathbb{R}^2))$.

For the second statement in Proposition 1, we write

$$\begin{aligned} &f_{U_0}(w) - f_{U_0}(\tilde{w}) \\ &= (|U_0 + w|^2 - 1)(w - \tilde{w}) + (|U_0 + w|^2 - |U_0 + \tilde{w}|^2)(U_0 + w) \\ &= (|U_0 + w|^2 - 1)(w - \tilde{w}) + (2U_0 \cdot (w - \tilde{w})) + |w|^2 - |\tilde{w}|^2)(U_0 + \tilde{w}). \end{aligned}$$

Decomposing each term as in Lemma 1 and Lemma 2 we obtain, if

$$\|w\|_{L^\infty((-T,T),H^1(\mathbb{R}^2))} + \|\tilde{w}\|_{L^\infty((-T,T),H^1(\mathbb{R}^2))} \leq R,$$

that

$$\|\Phi_N(w) - \Phi_N(\tilde{w})\|_{L^\infty((-T,T),H^1(\mathbb{R}^2))} \leq C(U_0)(1+R^2)\|w - \tilde{w}\|_{L^\infty((-T,T),H^1(\mathbb{R}^2))},$$

from which (2.1) follows. □

We are now in position to assert local existence:

Proposition 2. *Let $U_0 \in \mathcal{V}$ be given. For every $R > 0$ there exists a constant T^* depending only on U_0 and R such that for every $w_0 \in H^1(\mathbb{R}^2)$ satisfying $\|w_0\|_{H^1} \leq R$ equation (1.2) has a unique solution $w \in \mathcal{C}^0((-T^*,T^*), H^1(\mathbb{R}^2))$.*

Moreover, if $\|\tilde{w}_0\|_{H^1} \leq R$, *and* $\tilde{w}$ *denotes the corresponding solution to* (1.2), *then*

$$\|w - \tilde{w}\|_{L^\infty((-T^*,T^*),H^1(\mathbb{R}^2))} \leq 2\|w_0 - \tilde{w}_0\|_{H^1(\mathbb{R}^2)}.$$

Proof. The proof follows from the standard contraction mapping theorem applied to the map Φ. We first show that for T_0^* sufficiently small, Φ maps $B(2R)$, the ball of radius $2R$ centered at zero in $L^\infty((-T_O^*, T_0^*),\ H^1(\mathbb{R}^2))$, into itself. Indeed, for the linear part of Φ,

$$\|e^{it\Delta} w_0\|_{H^1(\mathbb{R}^2)} = \|w_0\|_{H^1(\mathbb{R}^2)} \leq R.$$

On the other hand, by (2.4) we have, for $w \in B(2R)$,

$$\|\Phi_N(w)\|_{L^\infty((-T_0^*,T_0^*),H^1(\mathbb{R}^2))} \leq C(U_0)(T_0^*)^{\frac{1}{2}}(1 + 8R^3) < R$$

if T_0^* is sufficiently small. For the contraction, we deduce from (2.1) that for $0 < T^* < T_0^*$,

$$\begin{aligned}&\|\Phi_N(\tilde{w}) - \Phi_N(w)\|_{L^\infty((-T^*,T^*),H^1(\mathbb{R}^2))}\\ &\leq C(U_0)(T^*)^{\frac{1}{2}}(1 + 4R^2)\|\tilde{w} - w\|_{L^\infty((-T^*,T^*),H^1(\mathbb{R}^2))}\end{aligned}$$

so that if T^* is sufficiently small

$$\|\Phi_N(\tilde{w}) - \Phi_N(w)\|_{L^\infty((-T^*,T^*),H^1(\mathbb{R}^2))} \leq \tfrac{1}{2}\|\tilde{w} - w\|_{L^\infty((-T^*,T^*),H^1(\mathbb{R}^2))}. \quad (2.5)$$

Finally, for the continuous dependence upon the initial datum, we write $w - \tilde{w}$ as $e^{it\Delta}(w_0 - \tilde{w}_0) + \Phi_N(w) - \Phi_N(\tilde{w})$, so that

$$\begin{aligned}&\|w - \tilde{w}\|_{L^\infty((-T^*,T^*),H^1(\mathbb{R}^2))}\\ &\leq \|w_0 - \tilde{w}_0\|_{H^1(\mathbb{R}^2)} + \|\Phi_N(\tilde{w}) - \Phi_N(w)\|_{L^\infty((-T^*,T^*),H^1(\mathbb{R}^2))},\end{aligned}$$

and the conclusion follows from (2.5). □

3. Global existence

In order to prove global existence, we will prove that the renormalized energy remains conserved. In a first step, we establish the previous statement for more regular solutions. In this direction, we begin with

Proposition 3. *Let* $U_0 \in \mathcal{V}$ *and* $w_0 \in H^2(\mathbb{R}^2)$. *There exists* $T_0 > 0$ *depending only on* U_0 *and* $\|w_0\|_{H^2(\mathbb{R}^2)}$ *and a unique solution* w *to* (1.2) *in* $\mathcal{C}^0((-T_0, T_0), H^2(\mathbb{R}^2))$.

Proof. Since $H^2(\mathbb{R}^2)$ is continuously embedded in $L^\infty(\mathbb{R}^2)$, the map $w \mapsto f_{U_0}(w)$ is locally Lipschitz on $H^2(\mathbb{R}^2)$, and the conclusion follows from standard (semi)group theory. □

Lemma 3. *Let $T > 0$ and $w \in \mathcal{C}^0((-T,T), H^2(\mathbb{R}^2))$ be a solution of* (1.2). *Then*

$$\frac{d}{dt}\tilde{\mathcal{E}}_{U_0}(w(t)) = 0 \qquad \text{for} \quad t \in (-T,T). \tag{3.1}$$

Moreover,

$$\|w(t)\|_{H^1(\mathbb{R}^2)} \leq \|w_0\|_{H^1(\mathbb{R}^2)} \exp(C|t|) \qquad \text{for} \quad t \in (-T,T), \tag{3.2}$$

where the constant C depends only on U_0 and $\|w_0\|_{H^1(\mathbb{R}^2)}$.

Proof. Invoking once more the embedding of $H^2(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$, we deduce that the term $f_{U_0}(w)$ belongs to $L^\infty_{\rm loc}((-T,T), L^2(\mathbb{R}^2))$. On the other hand, $\Delta w \in L^\infty_{\rm loc}((-T,T), L^2(\mathbb{R}^2))$, and therefore we deduce from (1.2) that $\partial_t w \in L^\infty_{\rm loc}((-T,T), L^2(\mathbb{R}^2))$. We may thus compute the derivative

$$\begin{aligned}\frac{d}{dt}\tilde{\mathcal{E}}_{U_0}(w(t)) &= \int_{\mathbb{R}^2} \nabla w \cdot \nabla \partial_t w + \int_{\mathbb{R}^2} f_{U_0}(w) \cdot \partial_t w \\ &= \int_{\mathbb{R}^2} (-\Delta w + f_{U_0}(w)) \cdot \partial_t w = \int_{\mathbb{R}^2} (i\partial_t w) \cdot \partial_t w = 0,\end{aligned}$$

which yields (3.1). For the second statement, by the Cauchy-Schwarz inequality we have, for any $w \in H^1(\mathbb{R}^2)$,

$$\tilde{\mathcal{E}}_{U_0}(w) \geq \frac{1}{2}\|\nabla w\|^2_{L^2(\mathbb{R}^2)} - C(U_0)\|w\|_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} \frac{(1-|U_0+w|^2)^2}{4},$$

so that

$$V(t) + \frac{1}{2}\|\nabla w\|^2_{L^2(\mathbb{R}^2)} \leq \tilde{\mathcal{E}}_{U_0}(w_0) + C(U_0)\|w\|_{L^2(\mathbb{R}^2)}, \tag{3.3}$$

where we have set $V(t) = \frac{1}{4}\int (1-|U_0+w(t)|^2)^2$. On the other hand, we may compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} |w|^2 &= 2\int_{\mathbb{R}^2} w \cdot \partial_t w \\ &= 2\int_{\mathbb{R}^2} iw \cdot [\Delta w - \Delta U_0 + (1-|U_0+w|^2)(U_0+w)] \\ &= -2\int_{\mathbb{R}^2} iw \cdot (\Delta U_0 + (1-|U_0+w|^2)U_0)\end{aligned}$$

so that

$$|\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R}^2)}| \leq C(U_0)\|w(t)\|_{L^2(\mathbb{R}^2)} + \|w(t)\|_{L^2(\mathbb{R}^2)} V(t). \tag{3.4}$$

Combining (3.3) and (3.4), we deduce

$$|\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R}^2)}| \le C(U_0, \tilde{\mathcal{E}}_{U_0}(w_0))(1 + \|w(t)\|^2_{L^2(\mathbb{R}^2)});$$

therefore,

$$(1 + \|w(t)\|^2_{L^2(\mathbb{R}^2)}) \le (1 + \|w_0\|^2_{L^2(\mathbb{R}^2)}) \exp(C|t|)$$

and the conclusion (3.2) then follows from (3.3). □

Lemma 4. *Let $w \in \mathcal{C}^0((-T, T), H^2(\mathbb{R}^2))$ be a solution to* (1.2)*. There exists a constant $T_1 > 0$ depending only on U_0 and $\|w_0\|_{H^1(\mathbb{R}^2)}$ such that, if $T \le T_1$,*

$$\|w\|_{L^\infty((-T,T),H^2(\mathbb{R}^2))} \le C(U_0, \|w_0\|_{H^1(\mathbb{R}^2)})\|w_0\|_{H^2(\mathbb{R}^2)}.$$

Proof. We consider the equation for $m = \Delta w$, namely

$$i\partial_t m + \Delta m = \Delta f_{U_0}(w).$$

We expand the right-hand side of this equation as $-\Delta^2 U_0 + (|U_0 + w|^2 - 1)\Delta(U_0 + w) + \Delta(|U_0 + w|^2 - 1)(U_0 + w) + 2\nabla(|U_0 + w|^2)\nabla(U_0 + w)$. As in the proof of Lemma 2, invoking various Hölder's inequalities and Sobolev embeddings, we obtain that for any $1 < r < 2$,

$$\|\Delta f_{U_0}(w)\|_{L^\infty(L^2+L^r)} \le C(U_0, \|w_0\|_{H^1})(1 + \|\Delta w\|_{L^\infty L^2}).$$

Therefore, it follows from the Strichartz inequality that

$$\begin{aligned}&\|m\|_{L^\infty((-T,T),L^2(\mathbb{R}^2))}\\ &\le \|\Delta w_0\|_{L^2(\mathbb{R}^2)} + C(U_0, \|w_0\|_{H^1(\mathbb{R}^2)})T^{\frac{1}{2}}(1 + \|m\|_{L^\infty((-T,T),L^2(\mathbb{R}^2))}).\end{aligned}$$

We choose T_1 so that $C(U_0, \|w_0\|_{H^1(\mathbb{R}^2)})T_1^{\frac{1}{2}} = \frac{1}{2}$, and we obtain

$$\|\Delta w\|_{L^\infty((-T,T),L^2(\mathbb{R}^2))} \le C(U_0, \|w_0\|_{H^1(\mathbb{R}^2)})(1 + \|\Delta w_0\|_{L^2(\mathbb{R}^2)}).$$

□

We may now prove

Proposition 4. *For $U_0 \in \mathcal{V}$ and $w_0 \in H^2(\mathbb{R}^2)$, equation* (1.2) *has a unique global solution $w \in \mathcal{C}^0(\mathbb{R}, H^2(\mathbb{R}^2))$.*

Proof. Let T^* be the maximal time of existence for (1.2) with initial datum w_0, and assume that $T^* < +\infty$. In view of Proposition 3, this implies that

$$\liminf_{t\to -T^*} \|w(t)\|_{H^2(\mathbb{R}^2)} + \liminf_{t\to T^*} \|w(t)\|_{H^2(\mathbb{R}^2)} = +\infty. \tag{3.5}$$

On the other hand, by (3.2),

$$\alpha := \sup_{t\in(-T^*,T^*)} \|w(t)\|_{H^1(\mathbb{R}^2)} < +\infty.$$

By Lemma 3, we obtain, for every $0 < T < T^*$,

$$\sup_{t\in[-T,T]} \|w(t)\|_{H^2(\mathbb{R}^2)} \leq C(U_0,\alpha)^{\frac{T^*}{T_1}+1}\|w_0\|_{H^2(\mathbb{R}^2)},$$

a contradiction with (3.5). □

Proof of Theorem 1. Notice first that, in view of Proposition 4, Theorem 1 has already been proved when the perturbation w_0 belongs to $H^2(\mathbb{R}^2)$. For the general case we proceed by approximation: let $w_0 \in H^1(\mathbb{R}^2)$ and $(w_0^n)_{n\in\mathbb{N}}$ be a sequence in $H^2(\mathbb{R}^2)$ such that $w_0^n \to w_0$ in $H^1(\mathbb{R}^2)$ as $n \to +\infty$. We denote by w^n the global solutions given by Proposition 4 and corresponding to the initial data w_0^n, and by w the solution given by Proposition 2 and corresponding to w_0. Let T^{**} denote the maximal time of existence of w. In view of Proposition 2, if $T^{**} < +\infty$ then necessarily

$$\sup_{t\in(-T^{**},T^{**})} \|w(t)\|_{H^1(\mathbb{R}^2)} = +\infty. \tag{3.6}$$

Assume that $T^{**} < +\infty$. We infer from Lemma 3 applied to each w_0^n that

$$R := \sup_{n\to+\infty} \max_{t\in[-T^{**},T^{**}]} \|w^n(t)\|_{H^1(\mathbb{R}^2)} \leq C\|w_0\|_{H^1(\mathbb{R}^2)},$$

where C depends only on U_0, $\|w_0\|_{H^1(\mathbb{R}^2)}$ and T^{**}. Invoking Proposition 2 with $\tilde{w}_0 = w_0^n$ and passing to the limit $n \to +\infty$ we obtain

$$\sup_{t\in(-T^*,T^*)} \|w(t)\|_{H^1(\mathbb{R}^2)} \leq R,$$

where $T^* \leq T^{**}$ is given in the statement of Proposition 2 and depends only on U_0 and R. After at most T^{**}/T^* shifts in time and further uses of Proposition 2, we finally deduce that

$$\sup_{t\in(-T^{**},T^{**})} \|w(t)\|_{H^1(\mathbb{R}^2)} \leq R < +\infty,$$

a contradiction with (3.6). Hence $T^{**} = +\infty$ and the proof is complete. □

Proof of Theorem 2. Conservation of (renormalized) energy as well as continuous dependence upon the initial datum have already been proved for $w_0 \in H^2(\mathbb{R}^2)$. As in the proof of Theorem 1, the general case follows by approximation. We omit the details. □

Appendix

A special case of the Gagliardo-Nirenberg inequality on $\mathbb{R}^N$ states that, for any function $u \in H^2(\mathbb{R}^N) \cap L^\infty(R^N)$, we have

$$\|\nabla u\|_{L^4(\mathbb{R}^N)} \leq (9N)^{\frac{1}{4}} \, \|u\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^N)} \|\Delta u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^N)}. \tag{3.7}$$

The proof of this inequality is actually elementary. By density, it suffices to prove it for $u \in C_c^\infty(\mathbb{R}^N)$. We write, for $i = 1, \dots, N$,

$$u_{x_i}^4 = u_{x_i}^3 u_{x_i}.$$

Integrating by parts on $\mathbb{R}^N$, we are led to

$$\int_{\mathbb{R}^N} u_{x_i}^4 = -\int_{\mathbb{R}^N} 3u_{x_i}^2 u_{x_i x_i} u.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\|u_{x_i}\|^4_{L^4(\mathbb{R}^N)} \leq 3\|u_{x_i}\|^2_{L^4(\mathbb{R}^N)} \|u_{x_i x_i}\|_{L^2(\mathbb{R}^N)} \|u\|_{L^\infty(\mathbb{R}^N)},$$

so that by summation

$$\sum_{i=1}^N \|u_{x_i}\|^4_{L^4(\mathbb{R}^N)} \leq 9\|\Delta u\|^2_{L^2(\mathbb{R}^N)} \|u\|^2_{L^\infty(\mathbb{R}^N)},$$

and the conclusion follows from the inequality of means. This result can be extended in our context to fields which do not necessarily tend to zero at infinity as follows.

Lemma A1. *For any $u \in L^\infty(\mathbb{R}^2)$ such that $\Delta u \in L^2(\mathbb{R}^2)$, we have*

$$\|\nabla u\|_{L^4(\mathbb{R}^2)} \leq 18^{\frac{1}{4}} \, \|u\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)} \|\Delta u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}.$$

Proof. Let $(\varrho_\varepsilon)_{\varepsilon>0}$ be a standard mollifier, and consider for $R > 1$ the cut-off function $\chi_R(\cdot) = \chi(\frac{\cdot}{R})$, where $0 \leq \chi \leq 1$ is a smooth function such that $\chi \equiv 1$ on $B(1)$ and $\chi \equiv 0$ outside $B(2)$. We set

$$u_\varepsilon = u * \varrho_\varepsilon, \qquad \text{and} \qquad u_{\varepsilon,R} = u_\varepsilon \chi_R.$$

Since $u_{\varepsilon,R} \in C_c^\infty(\mathbb{R}^2)$, we may apply the classical Gagliardo-Nirenberg inequality to assert that

$$\|\nabla u_{\varepsilon,R}\|_{L^4(\mathbb{R}^2)} \leq 18^{\frac{1}{4}} \, \|u_{\varepsilon,R}\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)} \|\Delta u_{\varepsilon,R}\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}.$$

We expand $\nabla u_{\varepsilon,R} = \nabla u_\varepsilon \chi_R + u_\varepsilon \nabla \chi_R$ and $\Delta u_{\varepsilon,R} = \Delta u_\varepsilon \chi_R + 2\nabla u_\varepsilon \nabla \chi_R + u_\varepsilon \Delta \chi_R$.
Clearly, we have, for some constant $C > 0$ depending only on χ,

$$\|\Delta u_\varepsilon \chi_R\|_{L^2(\mathbb{R}^2)} \leq \|\Delta u\|_{L^2(\mathbb{R}^2)}, \qquad \text{and} \qquad \| u_\varepsilon \Delta \chi_R\|_{L^2(\mathbb{R}^2)} \leq C\|u\|_{L^\infty} R^{-1}.$$

Similarly, we write

$$\begin{aligned}\|\nabla u_\varepsilon \chi_R\|_{L^4(\mathbb{R}^2)} &\leq \|\nabla(u_\varepsilon \chi_R)\|_{L^4(\mathbb{R}^2)} + \|u_\varepsilon \nabla \chi_R)\|_{L^4(\mathbb{R}^2)} \\ &\leq \|\nabla u_{\varepsilon,R}\|_{L^4(\mathbb{R}^2)} + C\|u\|_{L^\infty(\mathbb{R}^2)} R^{-\frac{1}{2}}.\end{aligned}$$

Combining the previous inequalities, we obtain, for $R > 1$,

$$\begin{aligned}&\|\nabla u_\varepsilon \chi_R\|_{L^4(\mathbb{R}^2)} \\ &\leq 18^{\frac{1}{4}} \|u\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)} \|\Delta u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)} + C(\|u\|_{L^\infty(\mathbb{R}^2)} R^{-\frac{1}{2}} + \|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^2 \setminus B(R))}).\end{aligned} \tag{3.8}$$

Next, we let ε be fixed and send R to $+\infty$. We claim that

$$\lim_{R\to+\infty} \|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^2\setminus B(R))} = 0. \tag{3.9}$$

Indeed, since $\Delta u_\varepsilon = \varrho_\varepsilon * \Delta u$ and $\Delta u \in L^2(\mathbb{R}^2)$, we have, for every $k \geq 2$,

$$\limsup_{|x|\to+\infty} \int_{B(x,1)} |D^k u_\varepsilon|^2 = 0.$$

Therefore, by Sobolev embedding

$$\limsup_{|x|\to+\infty} \operatorname{osc}(\nabla u_\varepsilon, B(x,1)) = 0,$$

and thus, for every $r > 0$,

$$\limsup_{|x|\to+\infty} \operatorname{osc}(\nabla u_\varepsilon, B(x,r)) = 0.$$

Since $u_\varepsilon \in L^\infty$, the claim follows. Passing then to the limit $R \to +\infty$ in (3.8), we are led to the inequality

$$\|\nabla u_\varepsilon\|_{L^4(\mathbb{R}^2)} \leq 18^{\frac{1}{4}} \|u\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)} \|\Delta u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}.$$

Finally, we let ε tend to zero to obtain the conclusion. □

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