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CONSTANT-SIGN AND SIGN-CHANGING SOLUTIONS OF A NONLINEAR EIGENVALUE PROBLEM INVOLVING THE p-LAPLACIAN

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Abstract. For a certain range of the eigenvalue parameter we prove a new multiple and sign-changing solutions theorem. The novelties of our paper are twofold. First, unlike recent papers in the field we do not assume jumping nonlinearities and allow a rather general growth condition on the nonlinearity involved. Second, our approach strongly relies on a combined use of variational and topological arguments (e.g. critical points, mountain–pass theorem, second deformation lemma, variational characterization of the first and second eigenvalue of the p-Laplacian) on the one hand, and comparison principles for nonlinear differential inequalities, in particular, the existence of extremal constant-sign solutions, on the other hand.

1. Introduction

We consider the nonlinear eigenvalue problem: Find $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\lambda > 0$ such that

$$\begin{cases} -\Delta_p u = f(x,u,\lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$, $1 < p < +\infty$, and $f : \Omega \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, is a given function. In problem (1.1) we have the negative p-Laplacian operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$

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($\frac{1}{p} + \frac{1}{p'} = 1$) acting as

$$\langle -\Delta_p u, v\rangle = \int_\Omega \|\nabla u(x)\|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$

Our aim is to provide an existence result of multiple solutions for all values of the parameter λ in an interval $(0, \lambda_0)$, with $\lambda_0 > 0$, guaranteeing that for any such λ there exist at least three nontrivial solutions of problem (1.1), two of them having opposite constant sign and the third one being sign-changing (or nodal). More precisely, we demonstrate that under suitable assumptions there exist a smallest positive solution, a greatest negative solution, and a sign-changing solution between them, whereas the notions *smallest* and *greatest* refer to the underlying natural partial ordering of functions. The present paper continues the works of Jin [8] (where $p = 2$ and $f(x, s, \lambda)$ is Hölder continuous with respect to $(x, s) \in \overline{\Omega} \times \mathbb{R}$ for every fixed λ) and of Motreanu-Motreanu-Papageorgiou [12]. In these cited works one obtains three nontrivial solutions, two of which being of opposite constant sign, but without knowing that the third one changes sign. In the present paper we derive the new information of having, in addition, a sign-changing solution by strengthening the unilateral condition for the right-hand side of the equation in (1.1) at zero. We also mention that the existence of three nontrivial solutions, two of them being of opposite constant signs and the third one changing sign, has been obtained by Carl–Perera [5] for problem (1.1) not depending on the parameter λ, and assuming different assumptions on f which, in particular, are related to the Fucik spectrum of the p-Laplacian, see also, e.g., [11]. In some sense, the main theorem of [5] and our result are complementary because in [5] a growth condition up to the order $|s|^{p-1}$ is allowed, while here we deal with order $|s|^r$ where $r > p - 1$, see H(f)(ii). This general growth condition that we are going to impose on the function $s \mapsto f(x, s, \lambda)$ causes a further difficulty in our approach of problem (1.1). Usually, problem (1.1) is studied by looking for the critical points of the associated Euler functional, that is,

$$E(u, \lambda) = \frac{1}{p}\, \|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} f(x, s, \lambda)\, ds\, dx$$

whenever $u \in W_0^{1,p}(\Omega)$ and $\lambda \in (0, \bar{\lambda})$. Unlike in the above cited papers [5, 11] and also various recent papers in the field such as, e.g., [2, 9, 10, 13, 14, 17, 18, 19], here we work with "non-jumping" nonlinearities having a general growth condition that does not even allow the functional $E(\cdot, \lambda)$ to be well defined, so the common variational method does not apply. Our approach strongly relies on a combined use of variational and topological arguments (e.g. critical points, mountain–pass theorem, second deformation lemma) on the one hand, and comparison principles for nonlinear differential inequalities, in particular, the existence of extremal constant-sign solutions, on the other hand. The main tools for the latter are, e.g., the sub-supersolution method, truncation techniques, and the strong maximum principle due to Vazquez [16].

The rest of the paper is organized as follows. Section 2 contains our hypotheses and an example. Section 3 deals with the existence of extremal (with respect to the underlying natural partial ordering of functions) constant-sign solutions. Section 4 presents our main result on the multiple and sign-changing solutions.

2. Hypotheses and example

Let $L^q(\Omega)_+$, $1 \le q \le +\infty$, denote the positive cone of $L^q(\Omega)$ given by

$$L^q(\Omega)_+ = \{v \in L^q(\Omega) : v(x) \ge 0 \text{ for a.a. } x \in \Omega\}.$$

We impose the following hypotheses on the nonlinearity $f(x,s,\lambda)$ in problem (1.1):

H(f) $f : \Omega\times\mathbb{R}\times(0,\bar\lambda) \to \mathbb{R}$, with $\bar\lambda > 0$, is a function such that $f(x,0,\lambda) = 0$ for a.a. $x \in \Omega$, whenever $\lambda \in (0,\bar\lambda)$, and

(i) for all $\lambda \in (0,\bar\lambda)$, $f(\cdot,\cdot,\lambda)$ is Carathéodory (that is, $f(\cdot,s,\lambda)$ is measurable for all $s \in \mathbb{R}$ and $f(x,\cdot,\lambda)$ is continuous for almost all $x \in \Omega$);

(ii) there are constants $c > 0$, $r > p-1$, and functions $a(\cdot,\lambda) \in L^\infty(\Omega)_+$ ($\lambda \in (0,\bar\lambda)$) with $\|a(\cdot,\lambda)\|_\infty \to 0$ as $\lambda \downarrow 0$ such that

$$|f(x,s,\lambda)| \le a(x,\lambda) + c|s|^r \quad \text{for a.a. } x \in \Omega \text{ and all } (s,\lambda) \in \mathbb{R}\times(0,\bar\lambda);$$

(iii) for all $\lambda \in (0,\bar\lambda)$ there exist constants $\mu_0 = \mu_0(\lambda) > \lambda_2$, $\nu_0 = \nu_0(\lambda) > \mu_0$ and a set $\Omega_\lambda \subset \Omega$ with $\Omega\setminus\Omega_\lambda$ of Lebesgue measure zero such that

$$\mu_0 < \liminf_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \limsup_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \nu_0$$

uniformly with respect to $x \in \Omega_\lambda$;

(iv) there exists a constant $b > 0$ such that for almost all $x \in \Omega$, all $\lambda \in (0,\bar\lambda)$ and all $s \in \mathbb{R}$ with $|s| < b$ we have $f(x,s,\lambda)s \ge 0$.

In H(f)(iii), λ_2 denotes the second eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. We point out that assumption H(f)(iv) does not follow from H(f)(iii) because the constant $b > 0$ must be independent of λ. As mentioned in the Introduction the strengthening with respect to [12] (see also [8]) of the unilateral condition for the right-hand side f in (1.1), which will enable us to obtain, in addition, sign-changing solutions consists in adding the part involving the limit superior in H(f)(iii).

Let us provide an example where all the assumptions formulated in H(f) are fulfilled.

Example 2.1. For the sake of simplicity we drop the x dependence for the function f in the right-hand side of equation (1.1). The function $f : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ given by

$$f(s,\lambda) = \lambda \arctan\Big(\frac{\lambda+\lambda_2}{\lambda}|s|^{p-2}s\Big) + c|s|^{r-1}s \quad \text{for all } (s,\lambda) \in \mathbb{R} \times (0,+\infty),$$

with $c > 0$ and $r > p-1$, satisfies hypotheses H(f). Next we give an example of a function $f : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ verifying assumptions H(f) which is generally not odd with respect to s:

$$f(s,\lambda) = \begin{cases} \lambda \arctan\Big(a_1 \frac{\lambda+\lambda_2}{\lambda}|s|^{p-2}s\Big) + c_1\,|s|^{r_1-1}s & \text{if } s \le 0 \\ \lambda \arctan\Big(a_2 \frac{\lambda+\lambda_2}{\lambda} s^{p-1}\Big) + c_2\, s^{r_2} & \text{if } s > 0, \end{cases}$$

with $\lambda > 0$, $a_1 \ge 1$, $a_2 \ge 1$, $c_1 > 0$, $c_2 > 0$, $r_1 > p-1$, $r_2 > p-1$.

3. Constant-sign solutions

The operator $-\Delta_p$ on $W_0^{1,p}(\Omega)$ is maximal monotone and coercive, therefore there exists $e \in W_0^{1,p}(\Omega)$ such that $-\Delta_p\, e = 1$. With $s^- = \max\{-s,0\}$ for $s \in \mathbb{R}$, and using $-e^- \in W_0^{1,p}(\Omega)$ as a test function, we see that

$$\|\nabla e^-\|_p^p = \langle -\Delta_p\, e, -e^- \rangle = -\int_\Omega e^-(x)\,dx \le 0,$$

which implies that $e \ge 0$. From the nonlinear regularity theory (cf., e.g., [7, Theorem 1.5.6]) we have $e \in C_0^1(\overline{\Omega})$. Then from the nonlinear strong maximum principle (see [16]) we infer that $e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Here $\operatorname{int}(C_0^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) \,:\, u(x) \ge 0,\ \forall x \in \Omega\}$ in the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) \,:\, u(x) = 0,\ \forall x \in \partial\Omega\}$, given by

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \{u \in C_0^1(\overline{\Omega}) \,:\, u(x) > 0,\ \forall x \in \Omega, \text{ and } \frac{\partial u}{\partial n}(x) < 0,\ \forall x \in \partial\Omega\},$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega$.

Lemma 3.1. *Let the data c, r, and $a(\cdot,\lambda)$ be as in* H(f)(ii)*. Then for every constant $\theta > 0$ there is $\lambda_0 \in (0,\bar{\lambda})$ with the property that if $\lambda \in (0,\lambda_0)$ we can choose $\xi_0 = \xi_0(\lambda) \in (0,\theta)$ such that*

$$c(\xi_0\|e\|_\infty)^r + \|a(\cdot,\lambda)\|_\infty < \xi_0^{p-1}. \tag{3.1}$$

Proof. On the contrary there would exist a constant $\theta > 0$ and a sequence $\lambda_n \downarrow 0$ as $n \to \infty$ such that

$$c(\xi\|e\|_\infty)^r + \|a(\cdot,\lambda_n)\|_\infty \geq \xi^{p-1} \quad \text{for all } n \in \mathbb{N} \text{ and } \xi \in (0,\theta).$$

Letting $n \to \infty$ we get $c\|e\|_\infty^r \xi^{r-p+1} \geq 1$ for all $\xi \in (0,\theta)$ because we have $\|a(\cdot,\lambda)\|_\infty \to 0$ as $\lambda \downarrow 0$. Since $r > p-1$, a contradiction is achieved as $\xi \downarrow 0$. Therefore (3.1) holds true. □

We denote by λ_1 the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and by φ_1 the eigenfunction of $(-\Delta_p, W_0^{1,p}(\Omega))$ corresponding to λ_1 satisfying

$$\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+) \quad \text{and} \quad \|\varphi_1\|_p = 1.$$

Lemma 3.2. *Assume* H(f)(i), (ii) *and the following weaker form of hypothesis* H(f)(iii)*: For all $\lambda \in (0,\bar{\lambda})$ there exist $\mu_0 = \mu_0(\lambda) > \lambda_1$ and $\Omega_\lambda \subset \Omega$ with $\Omega \setminus \Omega_\lambda$ of Lebesgue measure zero such that*

$$\mu_0 < \liminf_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \quad \textit{uniformly with respect to } x \in \Omega_\lambda.$$

Fix a constant $\theta > 0$ and consider the corresponding number $\lambda_0 \in (0,\bar{\lambda})$ obtained in Lemma 3.1. Then for any $\lambda \in (0,\lambda_0)$ the function $\overline{u} = \xi_0 e \in$ $\operatorname{int}(C_0^1(\overline{\Omega})_+)$*, with $\xi_0 \in (0,\theta)$ given by Lemma* 3.1*, is a supersolution for problem* (1.1)*, and the function $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is a subsolution of problem* (1.1) *provided the number $\varepsilon > 0$ is sufficiently small.*

Proof. For a fixed $\lambda \in (0,\lambda_0)$, from (3.1) and H$(f)$(ii) we derive

$$-\Delta_p\,\overline{u} = \xi_0^{p-1} > \|a(\cdot,\lambda)\|_\infty + c\|\overline{u}\|_\infty^r \geq f(\cdot,\overline{u}(\cdot),\lambda),$$

which says that $\overline{u} = \xi_0 e$ is a supersolution for problem (1.1).

On the other hand, by hypothesis we can find $\mu = \mu(\lambda) > \lambda_1$ and $\delta = \delta(\lambda) > 0$ such that

$$\mu < \frac{f(x,s,\lambda)}{|s|^{p-2}s} \quad \text{for a.a. } x \in \Omega \text{ and all } 0 < |s| \leq \delta. \tag{3.2}$$

Choose $\varepsilon \in (0, \frac{\delta}{\|\varphi_1\|_\infty})$. Then by (3.2) we have

$$-\Delta_p(\varepsilon\varphi_1) = \lambda_1\varepsilon^{p-1}\varphi_1^{p-1} < \mu\varepsilon^{p-1}\varphi_1^{p-1} < f(x,\varepsilon\varphi_1(x),\lambda) \quad \text{for a.a. } x \in \Omega,$$

which ensures that $\underline{u} = \varepsilon\varphi_1$ is a subsolution of problem (1.1). $\square$

The following result which asserts the existence of two solutions of problem (1.1) having opposite constant sign and being extremal plays an important role in the proof of the existence of sign-changing solutions.

Theorem 3.1. *Assume* H(f)(i), (ii), (iv) *and the following weaker form of* H(f)(iii)*: For all* $\lambda \in (0, \bar{\lambda})$ *there exist constants* $\mu_0 = \mu_0(\lambda) > \lambda_1$, $\nu_0 = \nu_0(\lambda) > \mu_0$ *and a set* $\Omega_\lambda \subset \Omega$ *with* $\Omega \setminus \Omega_\lambda$ *of Lebesgue measure zero such that*

$$\mu_0 < \liminf_{s \to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \limsup_{s \to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \nu_0 \tag{3.3}$$

uniformly with respect to $x \in \Omega_\lambda$*. Then there exists a number* $\lambda_0 \in (0, \bar{\lambda})$ *with the property that if* $\lambda \in (0, \lambda_0)$ *there is a constant* $\xi_0 = \xi_0(\lambda) \in (0, \frac{b}{\|e\|_\infty})$ *such that problem* (1.1) *has a least positive solution* $u_+ = u_+(\lambda) \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *in the order interval* $[0, \xi_0 e]$ *and a greatest negative solution* $u_- = u_-(\lambda) \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *in the order interval* $[-\xi_0 e, 0]$.

Proof. Since the proof of the existence of the greatest negative solution follows the same lines, we only provide the arguments for the existence of the least positive solution.

Applying Lemma 3.2 for $\theta = \frac{b}{\|e\|_\infty}$ we find $\lambda_0 \in (0, \bar{\lambda})$ as therein. Fix $\lambda \in (0, \lambda_0)$. Lemma 3.2 ensures that $\overline{u} = \xi_0 e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is a supersolution for problem (1.1), with $\xi_0 \in (0, \frac{b}{\|e\|_\infty})$ given by Lemma 3.1, and $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is a subsolution for problem (1.1) if $\varepsilon > 0$ is small enough. Passing eventually to a smaller $\varepsilon > 0$, we may assume that $\varepsilon\varphi_1 \le \xi_0 e$. Then by the method of sub-supersolution we know that in the order interval $[\varepsilon\varphi_1, \xi_0 e]$ there is a least solution $u_\varepsilon = u_\varepsilon(\lambda) \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ of problem (1.1) (see [3, 4]).

We thus obtain that for every positive integer n sufficiently large there is a least solution $u_n \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ of problem (1.1) in the order interval $[\frac{1}{n}\varphi_1, \xi_0 e]$. Clearly, we have

$$u_n \downarrow u_+ \ \text{ pointwise,} \tag{3.4}$$

with some function $u_+ : \Omega \to \mathbb{R}$ satisfying $0 \le u_+ \le \xi_0 e$. First we claim that

$$u_+ \ \text{ is a solution of problem (1.1).} \tag{3.5}$$

Taking into account that u_n solves (1.1), and the fact that u_n belongs to the order interval $[0, \xi_0 e]$, from H(f)(ii) we see that

$$\|\nabla u_n\|_p^p = \int_\Omega f(x, u_n(x), \lambda) u_n(x)\, dx \le \int_\Omega (a(x,\lambda) + c\xi_0^r e(x)^r)\xi_0 e(x)\, dx,$$

which implies the boundedness of the sequence (u_n) in $W_0^{1,p}(\Omega)$. Then due to (3.4) we have that $u_+ \in W_0^{1,p}(\Omega)$ as well as

$$u_n \rightharpoonup u_+ \text{ in } W_0^{1,p}(\Omega)\ , \quad u_n \to u_+ \text{ in } L^p(\Omega) \text{ and a.e. in } \Omega. \tag{3.6}$$

Since u_n solves problem (1.1) one has

$$\langle -\Delta_p u_n, \varphi\rangle = \int_\Omega f(x, u_n(x), \lambda)\varphi(x)\, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{3.7}$$

Setting $\varphi = u_n - u_+$ in (3.7) gives

$$\langle -\Delta_p u_n, u_n - u_+\rangle = \int_\Omega f(x, u_n(x), \lambda)(u_n(x) - u_+(x))\, dx. \tag{3.8}$$

As already noticed the sequence $(f(\cdot, u_n(\cdot), \lambda)$ is uniformly bounded on Ω, so (3.6) and (3.8) yield

$$\lim_{n\to\infty} \langle -\Delta_p u_n, u_n - u_+\rangle = 0.$$

The S_+-property of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ implies

$$u_n \to u_+ \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{3.9}$$

The strong convergence in (3.9) and Lebesgue's dominated convergence theorem permit us to pass to the limit in (3.7) that results in (3.5).

By (3.5) and the nonlinear regularity theory (cf., e.g., Theorem 1.5.6 in [7]) it turns out $u_+ \in C_0^1(\overline{\Omega})$. The choice of ξ_0 guarantees that

$$0 \le u_+(x) \le \xi_0 e(x) \le b \quad \text{for a.e. } x \in \Omega.$$

Thus, from (3.5) and assumption H(f)(iv), we get

$$-\Delta_p u_+(x) = f(x, u_+(x), \lambda) \ge 0 \ \text{ for a.a. } x \in \Omega.$$

Applying the nonlinear strong maximum principle (cf. [16]) we conclude that either $u_+ = 0$ or $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

We claim that

$$u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+). \tag{3.10}$$

Assume on the contrary that $u_+ = 0$. Then (3.4) becomes

$$u_n(x) \downarrow 0 \ \text{ for all } x \in \Omega. \tag{3.11}$$

Since $u_n \geq (1/n)\varphi_1$ we may consider

$$\tilde{u}_n = \frac{u_n}{\|\nabla u_n\|_p} \quad \text{for all } n.$$

Along a relabeled subsequence we may suppose

$$\tilde{u}_n \rightharpoonup \tilde{u} \ \text{ in } W_0^{1,p}(\Omega)\ , \quad \tilde{u}_n \to \tilde{u} \ \text{ in } L^p(\Omega) \text{ and a.e. in } \Omega, \tag{3.12}$$

for some $\tilde{u} \in W_0^{1,p}(\Omega)$. Moreover, one can find a function $w \in L^p(\Omega)_+$ such that $|\tilde{u}_n(x)| \leq w(x)$ for almost all $x \in \Omega$. Relation (3.7) reads

$$\langle -\Delta_p \tilde{u}_n, \varphi\rangle = \int_\Omega \frac{f(x,u_n(x),\lambda)}{\|\nabla u_n\|_p^{p-1}}\varphi\, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{3.13}$$

Setting $\varphi = \tilde{u}_n - \tilde{u}$ leads to

$$\langle -\Delta_p \tilde{u}_n, \tilde{u}_n - \tilde{u}\rangle = \int_\Omega \frac{f(x,u_n(x),\lambda)}{\|\nabla u_n\|_p^{p-1}}(\tilde{u}_n - \tilde{u})\, dx. \tag{3.14}$$

By H(f)(iii) we know that there exist constants $c_0 = c_0(\lambda) > \lambda_1$ and $\alpha = \alpha(\lambda) > 0$ such that

$$|f(x,s,\lambda)| \leq c_0|s|^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| < \alpha,$$

while H(f)(ii) entails

$$|f(x,s,\lambda)| \leq a(x,\lambda) + c|s|^r \leq \left(\frac{\|a(\cdot,\lambda)\|_\infty}{\alpha^r} + c\right)|s|^r$$

for almost all $x \in \Omega$ and for all $|s| \geq \alpha$. Combining the two estimates gives

$$|f(x,s,\lambda)| \leq c_0|s|^{p-1} + c_1|s|^r \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R} \tag{3.15}$$

with a constant $c_1 = c_1(\lambda) > 0$. Since $u_n \in [\frac{1}{n}\varphi_1, \xi_0 e]$, $r > p-1$ and (3.15) holds, there exists a constant $C > 0$ such that

$$\frac{|f(x,u_n(x),\lambda)|}{u_n(x)^{p-1}} \leq C \quad \text{for a.a. } x \in \Omega \text{ and for all } n. \tag{3.16}$$

We see from (3.16) that

$$\frac{|f(x,u_n(x),\lambda)|}{\|\nabla u_n\|_p^{p-1}}|\tilde{u}_n(x) - \tilde{u}(x)| \leq Cw(x)^{p-1}(w(x) + |\tilde{u}(x)|) \ \text{ for a.a. } x \in \Omega.$$

Then, because the right-hand side of the above inequality is in $L^1(\Omega)$, by means of (3.12) and (3.16) we can apply Lebesgue's dominated convergence theorem to get

$$\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n(x),\lambda)}{\|\nabla u_n\|_p^{p-1}}(\tilde{u}_n - \tilde{u})\, dx = 0.$$

Consequently, from (3.14) we obtain

$$\lim_{n\to\infty}\langle -\Delta_p \tilde{u}_n, \tilde{u}_n - \tilde{u}\rangle = 0.$$

The S_+-property of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ implies

$$\tilde{u}_n \to \tilde{u} \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty\ . \tag{3.17}$$

On the basis of (3.13) and (3.17) it follows that

$$\langle -\Delta_p \tilde{u}, \varphi\rangle = \lim_{n\to\infty}\int_\Omega \frac{f(x, u_n(x), \lambda)}{\|\nabla u_n\|_p^{p-1}}\varphi\, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{3.18}$$

Notice from (3.16) that

$$\frac{|f(x, u_n(x), \lambda)|}{\|\nabla u_n\|_p^{p-1}}|\varphi(x)| \le Cw(x)^{p-1}|\varphi(x)| \tag{3.19}$$

for almost all $x \in \Omega$ and for all $\varphi \in W_0^{1,p}(\Omega)$. We are thus allowed to apply Fatou's lemma which in conjunction with (3.11), (3.12) and (3.3) ensures

$$\lim_{n\to\infty}\int_\Omega \frac{f(x, u_n(x), \lambda)}{\|\nabla u_n\|_p^{p-1}}\varphi(x)\, dx = \lim_{n\to\infty}\int_\Omega \frac{f(x, u_n(x), \lambda)}{u_n(x)^{p-1}}\tilde{u}_n(x)^{p-1}\varphi(x)\, dx$$
$$\ge \int_\Omega \liminf_{n\to\infty}\left(\frac{f(x, u_n(x), \lambda)}{u_n(x)^{p-1}}\tilde{u}_n(x)^{p-1}\varphi(x)\right) dx \ge \mu_0\int_\Omega \tilde{u}(x)^{p-1}\varphi(x)\, dx,$$

for all $\varphi \in W_0^{1,p}(\Omega)_+ := W_0^{1,p}(\Omega) \cap L^p(\Omega)_+$. Thanks to (3.18) we obtain

$$\langle -\Delta_p \tilde{u}, \varphi\rangle \ge \mu_0\int_\Omega \tilde{u}(x)^{p-1}\varphi(x)\, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega)_+. \tag{3.20}$$

Owing to (3.19) we may once again use Fatou's lemma, so according to (3.11), (3.12) and the last part of (3.3), we find

$$\lim_{n\to\infty}\int_\Omega \frac{f(x, u_n(x), \lambda)}{\|\nabla u_n\|_p^{p-1}}\varphi(x)\, dx = \lim_{n\to\infty}\int_\Omega \frac{f(x, u_n(x), \lambda)}{u_n(x)^{p-1}}\tilde{u}_n(x)^{p-1}\varphi(x)\, dx$$
$$\le \int_\Omega \limsup_{n\to\infty}\left(\frac{f(x, u_n(x), \lambda)}{u_n(x)^{p-1}}\tilde{u}_n(x)^{p-1}\varphi(x)\right) dx \le \nu_0\int_\Omega \tilde{u}(x)^{p-1}\varphi(x)\, dx$$

for all $\varphi \in W_0^{1,p}(\Omega)_+$. Then (3.18) ensures

$$\langle -\Delta_p \tilde{u}, \varphi\rangle \le \nu_0\int_\Omega \tilde{u}(x)^{p-1}\varphi(x)\, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega)_+. \tag{3.21}$$

Combining (3.20) and (3.21) results in

$$\mu_0\tilde{u}^{p-1} \le -\Delta_p\tilde{u} \le \nu_0\tilde{u}^{p-1} \quad \text{a.e. in } \Omega, \tag{3.22}$$

which guarantees that we will have $\tilde{u} \in L^\infty(\Omega)$ (see [7], Theorem 1.5.5). Since by (3.22) we know that $\Delta_p \tilde{u} \in L^\infty(\Omega)$, we are in a position to address Theorem 1.5.6 in [7], which provides $\tilde{u} \in C^{1,\beta}(\overline{\Omega})$ with some $\beta \in (0,1)$. This regularity up to the boundary, the fact that $\tilde{u} \geq 0$ almost everywhere in Ω and (3.22) enable us to refer to the strong maximum principle (see Theorem 5 of Vázquez [16]). Recalling that $\tilde{u}$ does not vanish identically on Ω (because $\|\nabla \tilde{u}\|_p = 1$) we deduce that $\tilde{u}(x) > 0$ for all $x \in \Omega$ and $\frac{\partial \tilde{u}}{\partial n}(x) < 0$ for all $x \in \partial\Omega$ which amounts to saying $\tilde{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Consequently, there exist constants $k_0 > 0$ and $k_1 > 0$ such that

$$k_0 \varphi_1 \leq \tilde{u} < k_1 \varphi_1 \quad \text{a.e. in } \Omega. \tag{3.23}$$

Following [1] let us denote

$$I(u,v) = \langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \rangle - \langle -\Delta_p v, \frac{u^p - v^p}{v^{p-1}} \rangle$$

whenever $(u,v) \in D_I$, where

$$D_I = \{(w_1, w_2) \in (W_0^{1,p}(\Omega))^2 : w_i \geq 0 \text{ and } \frac{w_i}{w_j} \in L^\infty(\Omega) \text{ for } i,j \in \{1,2\}\}.$$

Relation (3.23) justifies the assertion that $(k_1\varphi_1, \tilde{u}) \in D_I$. Then Proposition 1 of Anane [1] implies $I(k_1\varphi_1, \tilde{u}) \geq 0$. On the other hand a direct computation based on (3.23) and (3.22) shows

$$\begin{aligned} I(k_1\varphi_1, \tilde{u}) &= \langle -\Delta_p (k_1\varphi_1), \frac{(k_1\varphi_1)^p - \tilde{u}^p}{(k_1\varphi_1)^{p-1}} \rangle - \langle -\Delta_p \tilde{u}, \frac{(k_1\varphi_1)^p - \tilde{u}^p}{\tilde{u}^{p-1}} \rangle \\ &\leq (\lambda_1 - \mu_0) \int_\Omega ((k_1\varphi_1)^p - \tilde{u}^p) dx < 0. \end{aligned}$$

This contradiction proves that the claim in (3.10) holds true.

In view of (3.5) it remains to establish that u_+ is the least positive solution of problem (1.1) in the interval $[0, \overline{u}]$. Let $u \in W_0^{1,p}(\Omega)$ be a positive solution to (1.1) in $[0, \overline{u}]$. Since $u \in L^\infty(\Omega)$, then (1.1) and H(f)(ii) allow us to deduce that $-\Delta_p u \in L^\infty(\Omega)$. Using Theorem 1.5.6 of [7] leads to $u \in C_0^1(\overline{\Omega})$. Then, as u is a solution to (1.1) and $u \in [0, \overline{u}]$, with $\|\overline{u}\|_\infty < b$, by means of hypothesis H(f)(iv) we are able to apply the strong maximum principle. So we get $u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, hence $u \in [\frac{1}{n}\varphi_1, \overline{u}]$ for n sufficiently large. The fact that u_n is the least solution of (1.1) in $[\frac{1}{n}\varphi_1, \overline{u}]$ ensures $u_n \leq u$. Taking into account (3.4), we obtain $u_+ \leq u$. This completes the proof. □

4. Sign-changing solution

Our main result is now stated.

Theorem 4.1. *Under hypotheses* H(f)(i)-(iv) *there exists a number* $\lambda_0 \in (0, \bar{\lambda})$ *with the property that if* $\lambda \in (0, \lambda_0)$, *then problem* (1.1) *has a (positive) solution* $u_+ = u_+(\lambda) \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, *a (negative) solution* $u_- = u_-(\lambda) \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *and a nontrivial sign-changing solution* $u_0 = u_0(\lambda) \in C_0^1(\overline{\Omega})$ *satisfying* $\|u_+\|_\infty < b$, $\|u_-\|_\infty < b$, $\|u_0\|_\infty < b$.

Proof. Consider the positive number λ_0 given by Theorem 3.1 and fix $\lambda \in (0, \lambda_0)$. Let $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $u_- \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ be the two extremal solutions determined in Theorem 3.1. We introduce on $\Omega \times \mathbb{R}$ the truncation functions

$$\tau_+(x,s) = \begin{cases} 0 & \text{if } s \le 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x), \end{cases}$$

$$\tau_-(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \ge 0, \end{cases}$$

$$\tau_0(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x), \end{cases}$$

and then define the following associated functionals:

$$E_+(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \tau_+(x,s), \lambda)\, ds\, dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

$$E_-(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \tau_-(x,s), \lambda)\, ds\, dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

$$E_0(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \tau_0(x,s), \lambda)\, ds\, dx, \quad \forall u \in W_0^{1,p}(\Omega).$$

It is clear that $E_+, E_-, E_0 \in C^1(W_0^{1,p}(\Omega))$.

We observe that if v is a critical point of E_+, then

$$\begin{aligned}&\langle -\Delta_p v + \Delta_p u_+, (v-u_+)^+\rangle \\ &= \int_\Omega (f(x, \tau_+(x, v(x)), \lambda) - f(x, u_+(x), \lambda))(v-u_+)^+\, dx = 0\end{aligned}$$

which implies $v \le u_+$. Similarly, it follows that $v \ge 0$. This leads to:

$$v \text{ is a critical point of } E_+ \implies 0 \le v(x) \le u_+(x) \text{ for a.a. } x \in \Omega. \tag{4.1}$$

Since the function E_+ is coercive and weakly lower semicontinuous, there exists a global minimizer $z_+ \in W_0^{1,p}(\Omega)$ of it. Using (3.2), it is seen that

$$E_+(z_+) = \inf_{W_0^{1,p}(\Omega)} E_+ < 0,$$

so $z_+ \neq 0$. Relation (4.1) shows that z_+ is a nontrivial solution of problem (1.1) belonging to the order interval $[0, u_+]$. Via assumption H(f)(iv) we may apply the strong maximum principle which ensures $z_+ > 0$ on Ω. In view of the minimality property of u_+ as stated in Theorem 3.1, it follows that $z_+ = u_+$. In fact, u_+ is the unique global minimizer of E_+.

Since $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, there exists a neighborhood V of u_+ in the space $C_0^1(\overline{\Omega})$ such that $V \subset C_0^1(\overline{\Omega})_+$. Therefore $E_0 = E_+$ on V, which guarantees that u_+ is a local minimizer of E_0 on $C_0^1(\overline{\Omega})$. It follows that u_+ is also a local minimizer of E_0 on the space $W_0^{1,p}(\Omega)$ (see [7, pages 655-656]). Employing the functional E_- and proceeding as in the case of u_+, we establish that u_- is a local minimizer of E_0 on $W_0^{1,p}(\Omega)$.

As in the case of (4.1), we verify that every critical point of E_0 belongs to the set $\{u \in W_0^{1,p}(\Omega) : u_-(x) \leq u(x) \leq u_+(x) \text{ a.e. } x \in \Omega\}$, which implies that every critical point of E_0 is a solution to problem (1.1). The functional E_0 is coercive and weakly lower semicontinuous, with $\inf_{W_0^{1,p}(\Omega)} E_0 < 0$. Thus E_0 has a global minimizer $y_0 \in W_0^{1,p}(\Omega)$ with $y_0 \neq 0$. The above properties ensure that y_0 is a nontrivial solution of problem (1.1) belonging to the order interval $[u_-, u_+]$. Assume $y_0 \neq u_+$ and $y_0 \neq u_-$. We claim that y_0 changes sign. Indeed, if not, y_0 would have constant sign, for instance $y_0 \geq 0$ almost everywhere on Ω. Using assumption H(f)(iv) we may apply the strong maximum principle which leads to $y_0 > 0$ on Ω. This is impossible because it contradicts the minimality property of the solution u_+ as given by Theorem 3.1. According to the claim, we obtain the conclusion of the theorem setting $u_0 = y_0$.

Thus, the proof reduces to considering the cases $y_0 = u_+$ or $y_0 = u_-$. To make a choice, suppose $y_0 = u_+$. We may also admit that u_- is a strict local minimizer of E_0. This is true since on the contrary we would find (infinitely many) critical points x_0 of E_0 belonging to the order interval $[u_-, u_+]$ which are different from 0, u_-, u_+, and if x_0 does not change sign, taking into account the strong maximum principle, the extremality properties of the solutions u_-, u_+ given in Theorem 3.1 will be contradicted. A straightforward argument allows us then to find $\rho \in (0, \|u_+ - u_-\|)$ such that

$$E_0(u_+) \leq E_0(u_-) < \inf\{E_0(u) : u \in \partial B_\rho(u_-)\}, \tag{4.2}$$

where $\partial B_\rho(u_-) = \{u \in W_0^{1,p}(\Omega) : \|u - u_-\| = \rho\}$. Relation (4.2) in conjunction with the Palais-Smale condition (which holds for E_0 due to its coercivity) enables us to apply the mountain pass theorem to the functional E_0 (see, e.g., [15]). In this way we get $u_0 \in W_0^{1,p}(\Omega)$ satisfying $E_0'(u_0) = 0$ and

$$\inf\{E_0(u) \,:\, u \in \partial B_\rho(u_-)\} \leq E_0(u_0) = \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} E_0(\gamma(t)), \tag{4.3}$$

where

$$\Gamma = \{\gamma \in C([-1,1], W_0^{1,p}(\Omega)) :\ \gamma(-1) = u_-\,,\ \gamma(1) = u_+\}.$$

We infer from (4.2) and (4.3) that $u_0 \neq u_-$ and $u_0 \neq u_+$.

The next step in the proof is to show that

$$E_0(u_0) < 0. \tag{4.4}$$

By the equality in (4.3), it suffices to produce a path $\hat\gamma \in \Gamma$ such that

$$E_0(\hat\gamma(t)) < 0 \text{ for all } t \in [-1,1]. \tag{4.5}$$

Let $S = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p(\Omega)}$, where $\partial B_1^{L^p(\Omega)} = \{u \in L^p(\Omega) : \|u\|_p = 1\}$, and $S_C = S \cap C_0^1(\overline{\Omega})$ be endowed with the topologies induced by $W_0^{1,p}(\Omega)$ and $C_0^1(\overline{\Omega})$, respectively. We set

$$\Gamma_{0,C} = \{\gamma \in C([-1,1], S_C) \,:\, \gamma(-1) = -\varphi_1,\ \gamma(1) = \varphi_1\}.$$

Making use of the first inequality in assumption H(f)(iii), we fix numbers $\mu > \lambda_2$ and $\delta > 0$ such that (3.2) holds, and then let $\rho_0 \in (0, \mu - \lambda_2)$. We recall the following variational expression for λ_2 given by Cuesta–de Figueiredo–Gossez [6]:

$$\lambda_2 = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \|\nabla u\|_p^p\,, \tag{4.6}$$

where

$$\Gamma_0 = \{\gamma \in C([-1,1], S) \,:\, \gamma(-1) = -\varphi_1,\ \gamma(1) = \varphi_1\}.$$

By (4.6) there exists $\gamma \in \Gamma_0$ such that

$$\max_{t\in[-1,1]} \|\nabla\gamma(t)\|_p^p < \lambda_2 + \frac{\rho_0}{2}.$$

Choose some number r with $0 < r \leq (\lambda_2 + \rho_0)^{\frac{1}{p}} - (\lambda_2 + \frac{\rho_0}{2})^{\frac{1}{p}}$. The density of S_C in S implies that $\Gamma_{0,C}$ is dense in Γ_0, so there is $\gamma_0 \in \Gamma_{0,C}$ satisfying

$$\max_{t\in[-1,1]} \|\nabla\gamma(t) - \nabla\gamma_0(t)\|_p < r.$$

Then the choice of r establishes

$$\max_{t\in[-1,1]} \|\nabla\gamma_0(t)\|_p^p < \lambda_2 + \rho_0. \tag{4.7}$$

The boundedness of the set $\gamma_0([-1,1])(\overline{\Omega})$ in $\mathbb{R}$ ensures the existence of some $\varepsilon_1 > 0$ such that

$$\varepsilon_1|u(x)| \leq \delta \quad \text{for all } x \in \Omega \text{ and all } u \in \gamma_0([-1,1]). \tag{4.8}$$

Since $u_+, -u_- \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ (see Theorem 3.1), for every $u \in \gamma_0([-1,1])$ and any bounded neighborhood V_u of u in $C_0^1(\overline{\Omega})$ there exist positive numbers h_u and j_u such that

$$u_+ - \frac{1}{h}v \in \operatorname{int}(C_0^1(\overline{\Omega})_+) \quad \text{and} \quad -u_- + \frac{1}{j}v \in \operatorname{int}(C_0^1(\overline{\Omega})_+),$$

whenever $h \geq h_u$, $j \geq j_u$, and $v \in V_u$. This fact and the compactness of $\gamma_0([-1,1])$ in $C_0^1(\overline{\Omega})$ allow us to determine a number $\varepsilon_0 > 0$ for which one has

$$u_-(x) \leq \varepsilon u(x) \leq u_+(x) \quad \text{for all } x \in \Omega,\ u \in \gamma_0([-1,1]),\ \varepsilon \in (0, \varepsilon_0). \tag{4.9}$$

We now focus on the continuous path $\varepsilon\gamma_0$ in $C_0^1(\overline{\Omega})$ joining $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$ with a fixed constant ε satisfying $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$. By (4.9), (4.7), (4.8), (3.2) with $\mu > \lambda_2$ and taking into account the choice of ρ_0 as well as $\gamma_0([-1,1]) \subset \partial B_1^{L^p(\Omega)}$ we obtain

$$\begin{aligned} E_0(\varepsilon\gamma_0(t)) &= \frac{\varepsilon^p}{p}\|\nabla\gamma_0(t)\|_p^p - \int_\Omega \int_0^{\varepsilon\gamma_0(t)(x)} f(x, \tau_0(x,s), \lambda)\,ds\,dx \\ &= \frac{\varepsilon^p}{p}\|\nabla\gamma_0(t)\|_p^p - \int_\Omega \int_0^{\varepsilon\gamma_0(t)(x)} f(x, s, \lambda)\,ds\,dx \\ &\leq \frac{\varepsilon^p}{p}(\lambda_2 + \rho_0 - \mu) < 0 \quad \text{for all } t \in [-1,1]. \end{aligned} \tag{4.10}$$

At this point we apply the second deformation lemma (see, e.g., [7, page 366]) to the C^1 functional $E_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$. Towards this let us denote

$$c_+ = c_+(\lambda) = E_+(\varepsilon\varphi_1), \quad m_+ = m_+(\lambda) = E_+(u_+), \quad \text{and}$$
$$E_+^{c_+} = \{u \in W_0^{1,p}(\Omega) : E_+(u) \leq c_+\}.$$

It was already shown that u_+ is the unique global minimizer of E_+, so we have $m_+ < c_+$. Taking into account (4.1), E_+ has no critical values in the interval $(m_+, c_+]$ (for, otherwise, the minimality of the positive solution u_+ of (1.1) would be contradicted). Using also the fact that the functional E_+ satisfies the Palais-Smale condition (because it is coercive), the second deformation lemma can be applied to E_+ yielding a continuous mapping $\eta \in C([0,1] \times E_+^{c+}, E_+^{c+})$ such that $\eta(0,u) = u$ and $\eta(1,u) = u_+$ for all $u \in E_+^{c+}$, as well as $E_+(\eta(t,u)) \leq E_+(u)$ whenever $t \in [0,1]$ and $u \in E_+^{c+}$. Introducing $\gamma_+ : [0,1] \to W_0^{1,p}(\Omega)$ by

$$\gamma_+(t) := (\eta(t, \varepsilon\varphi_1))^+ := \max\{\eta(t, \varepsilon\varphi_1), 0\}$$

for all $t \in [0,1]$, it is seen that γ_+ is a continuous path in $W_0^{1,p}(\Omega)$ joining $\varepsilon\varphi_1$ and u_+. (Note the mapping $w \mapsto w^+$ is continuous from $W_0^{1,p}(\Omega)$ into itself.) The properties of the deformation η imply

$$E_0(\gamma_+(t)) = E_+(\gamma_+(t)) \leq E_+(\eta(t, \varepsilon\varphi_1)) = E_+(\varepsilon\varphi_1) = E_0(\varepsilon\varphi_1) < 0 \quad (4.11)$$

for all $t \in [0,1]$. Similarly, applying the second deformation lemma to the functional E_-, we construct a continuous path $\gamma_- : [0,1] \to W_0^{1,p}(\Omega)$ joining u_- and $-\varepsilon\varphi_1$ such that

$$E_0(\gamma_-(t)) < 0 \quad \text{for all } t \in [0,1]. \quad (4.12)$$

The union of the curves γ_-, $\varepsilon\gamma_0$ and γ_+ gives rise to a path $\hat{\gamma} \in \Gamma$. We see from (4.12), (4.10) and (4.11) that (4.5) is satisfied. Hence (4.4) holds, so $u_0 \neq 0$. Recalling that the critical points of E_0 are in the order interval $\{u \in W_0^{1,p}(\Omega) \ : \ u_-(x) \leq u(x) \leq u_+(x) \text{ a.e. } x \in \Omega\}$ we derive that u_0 is a nontrivial solution of (1.1) distinct from u_- and u_+, with $u_- \leq u_0 \leq u_+$. By the nonlinear regularity theory we have that $u_0 \in C_0^1(\overline{\Omega})$. The extremality properties of the constant sign solutions u_- and u_+ as described in Theorem 3.1 force u_0 to be sign-changing. This completes the proof. □

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