Differential and Integral Equations Volume 20, Number 3 (2007), 293–308

ON THE FINITE-TIME BLOW-UP OF A NON-LOCAL PARABOLIC EQUATION DESCRIBING CHEMOTAXIS

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Abstract. The non-local parabolic equation

$$v_t = \Delta v + \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega \times (0,T)$$

associated with Dirichlet boundary and initial conditions is considered here. This equation is a simplified version of the full chemotaxis system. Let λ^* be such that the corresponding steady-state problem has no solutions for $\lambda > \lambda^*$, then it is expected that blow-up should occur in this case. In fact, for $\lambda > \lambda^*$ and any bounded domain $\Omega \subset \mathbf{R}^2$ it is proven, using Trudinger-Moser's inequality, that $\int_\Omega e^{v(x,t)}dx \to \infty$ as $t \to T_{max} \leq \infty$. Moreover, in this case, some properties of the blow-up set are provided. For the two-dimensional radially symmetric problem, i.e. when $\Omega = B(0,1)$, where it is known that $\lambda^* = 8\,\pi$, we prove that v blows up in finite time $T^* < \infty$ for $\lambda > 8\,\pi$ and this blow-up occurs only at the origin $r = 0$ (single-point blow-up, mass concentration at the origin).

1. Introduction

The full system of chemotaxis is composed of parabolic equations concerning $u = u(x,t)$ and $v = v(x,t)$, which stand for the particle density and the field distribution, respectively [12, 13, 28]. A typical case,

$$\begin{aligned} &\varepsilon u_t = \nabla \cdot (\nabla u - u \nabla v) \\ &\tau v_t = \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T) \\ &\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \end{aligned}$$

Accepted for publication: October 2006.
AMS Subject Classifications: 35B05, 35K57, 35K55, 35Q80.

$$\int_\Omega v(x,t)dx = 0 \qquad \text{for } t \in (0,T)$$
$$u|_{t=0} = u_0(x) \geq 0, \quad v|_{t=0} = v_0(x) \qquad \text{in } \Omega, \tag{1.1}$$

is called the Keller-Segel system, where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector, and ε, τ are positive constants [15]. It is associated with the Lagrange function,

$$L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2^2 - \langle v, u\rangle$$

provided with the constraint $\int_\Omega v = 0$, and is formulated by the model (B) - model (A) equations [29] :

$$\varepsilon u_t = \nabla \cdot u \nabla L_u(u,v)$$
$$\tau v_t = -L_v(u,v) \qquad \text{in } \Omega \times (0,T)$$
$$u\frac{\partial}{\partial\nu}L_u(u,v) = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T).$$

Here and henceforth, $\langle\ ,\ \rangle$ denotes the L^2-inner product.

Since $u = u(x,t) \geq 0$, follows from the maximum principle, this formulation results in

$$\|u(\cdot,t)\|_1 = \|u_0\|_1 = \lambda, \qquad \frac{d}{dt}L(u(\cdot,t), v(\cdot,t)) \leq 0,$$

and consequently, the stationary state is described by

$$\log u - v = \text{constant} \qquad \text{in } \Omega$$
$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \ \text{ on } \partial\Omega\ , \ \int_\Omega v = 0. \tag{1.2}$$

The first relation of (1.2) implies

$$u = \frac{\lambda e^v}{\int_\Omega e^v}, \tag{1.3}$$

where $\lambda = \|u\|_1$, and henceforth, we shall write $v = (-\Delta_{JL})^{-1}u$ for the second equation of (1.2). Thus, in the stationary state of (1.1), it holds that

$$\log u - (-\Delta_{JL})^{-1}u = \text{constant}, \quad u \geq 0 \quad \text{in } \Omega, \qquad \|u\|_1 = \lambda \tag{1.4}$$

and

$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right) \ \text{ in } \Omega, \ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega v = 0. \tag{1.5}$$

There is a duality in these problems describing stationary states, in fact (1.4) and (1.5) are equivalent to the variational problems

$$\delta\mathcal{F}(u) = 0, \qquad u \geq 0, \quad \|u\|_1 = \lambda$$

and

$$\delta\mathcal{J}_\lambda(v) = 0, \qquad \int_\Omega v = 0,$$

using free energy and field functionals,

$$\mathcal{F}(u) = L|_{v=(-\Delta_{JL})^{-1}u} = \int_\Omega u(\log u - 1) - \frac{1}{2}\left\langle (-\Delta_{JL}^{-1}u, u\right\rangle$$
$$\mathcal{J}_\lambda(v) = L|_{u=\frac{\lambda e^v}{\int_\Omega e^v}} = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v\right) + \lambda(\log\lambda - 1),$$

respectively. See [29] for more details and background of this structure.

Since (1.5) with $n = 2$ is provided with the quantized blow-up mechanism [20, 3, 16], a similar property to (1.1) has been suspected [4]. Actually, Herrero-Velázquez, [11], constructed a family of radially symmetric solutions satisfying

$$u(x,t) \rightharpoonup 8\pi\delta_0 + f(x)$$

in $\mathcal{M}(\overline{\Omega})$ as $t \uparrow T < +\infty$, where $\Omega = B(0,1) = \left\{x \in \mathbf{R}^2 : |x| < 1\right\}$ and $0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$.

The reduced case $(\varepsilon, \tau) = (1, 0)$ of (1.6) is called the simplified system. For this problem, the above profile has a generalization, and if $u = u(x,t)$ is a blow-up solution in finite time to

$$\begin{aligned}
&u_t = \nabla\cdot(\nabla u - u\nabla v)\\
&-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega\times(0,T)\\
&\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 \qquad \text{on } \partial\Omega\times(0,T)\\
&\int_\Omega v(x,t)dx = 0 \qquad \text{for } t\in(0,T)\\
&u|_{t=0} = u_0(x) > 0 \qquad \text{in } \Omega
\end{aligned} \tag{1.6}$$

with $n = 2$, then it develops delta-function singularities with the coefficients quantized by 8π or 4π inside Ω and on $\partial\Omega$, respectively [23, 25, 28]. This system (1.6) is also a typical form of the Smoluchowski equation [6], one of the fundamental equations of material transport subject to the second law of thermodynamics, and is put into the mean field hierarchy, see [29].

This paper studies the case $n = 2$ only. The above described quantized blow-up mechanism is regarded as a localization of the threshold of the total mass for the existence of the solution globally in time. For instance, Nagai [17] treated radially symmetric solutions to (1.6) and showed that $T = +\infty$ holds if $\lambda < 8\pi$, while $T < +\infty$ occurs if $\lambda > 8\pi$ and u_0 is sufficiently concentrated at the origin. Here and henceforth, $T = T_{\max} \in (0, +\infty]$ denotes the existence time of the solution. See [19, 2, 7, 18, 24] for this threshold phenomenon in the non-radially symmetric case.

The proof of $T = +\infty$ in this system is based on the Trudinger-Moser inequality, and is valid even in (1.1). There seems to be, however, no criterion of $T < +\infty$ valid in (1.1) up to now. To this connection, Wolansky [31] introduced the limiting system of $(\varepsilon, \tau) = (0, 1)$ as

$$\begin{aligned} &\varepsilon u_t = \nabla \cdot (\nabla u - u\nabla v) \\ &\tau v_t = \Delta v + u \qquad \text{in } \Omega \times (0,T) \\ &\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = v = 0 \qquad \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{1.7}$$

and showed the formation of collapse of radially symmetric solutions. Since in this case the first equation is transformed to (1.3), the original system is reduced to the non-local problem

$$v_t = \Delta v + \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega \times (0,T), \qquad v = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{1.8}$$

Then, his result is the following.

Theorem 1 ([31])**.** *If $\Omega = B(0,1) = \left\{x \in \mathbf{R}^2 : |x| < 1\right\}$, $v_0 = v_0(|x|)$ is smooth, and $\lambda \geq 8\pi$, then*

$$\frac{e^v}{\int_\Omega e^v} \rightharpoonup \delta_0(dx) \tag{1.9}$$

as $t \uparrow T$ in (1.8).

Although the case $T = +\infty$ is also included in the above theorem, we can actually show the following.

Theorem 2. *In the previous theorem, it holds that $T < +\infty$ if $\lambda > 8\pi$.*

This means that the formation of collapse with dis-quantized mass occurs in finite time in the limiting case $\varepsilon = 0$, in contrast with the other limiting case $\tau = 0$, in (1.7).

Theorem 2 is the main result of this work and is, according to our knowledge, the only finite-time blow-up result associated to problem (1.8).

In the case of the non-local problem

$$v_t = \Delta v + \frac{\lambda e^v}{\left(\int_\Omega e^v\right)^p} \quad \text{in } \Omega \times (0,T), \qquad v = 0 \quad \text{on } \partial\Omega \times (0,T)$$

with $0 < p < 1$, which models shear-banding phenomena, more results regarding blow-up are known. In this case the problem defines a gradient-like semiflow with its Lyapunov functional given by

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{\lambda}{p-1}\left(\int_\Omega e^v\right)^{1-p}$$

and blow-up can be proven using energy methods either in the case where $E = \emptyset$ (E : the set of the steady states) or for large enough initial data, see [1, 33]. For a nonlinearity that is positive, increasing, convex and growing superlinearly to infinity (different from the exponential) when the above problem doesn't admit a Lyapunov functional some blow-up results have been also proven. For this purpose Kaplan's method could be applied, see [14]. For problem (1.8) the methods in [1, 14, 33] cannot be applied. It seems that the exponent $p = 1$ is a kind of critical one, see also Section 4, and in this case we probably need more delicate arguments to prove blow-up.

This paper is composed of four sections. In the next section we present some preliminary results, and in Section 3 we give a proof of Theorem 2. Section 4 is devoted to concluding remarks.

2. Preliminaries

If $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, then the functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v\right) + \lambda(\log\lambda - 1)$$

is C^2 in $v \in H_0^1(\Omega)$, and we obtain the Trudinger-Moser inequality indicated by

$$\inf_{v \in H_0^1(\Omega)} \mathcal{J}_{8\pi}(v) > -\infty. \tag{2.1}$$

Equation (1.8) is the gradient flow of this functional,

$$v_t = -\delta\mathcal{J}_\lambda(v) \qquad \text{in } X = H_0^1(\Omega),$$

and hence it holds that

$$\frac{d}{dt}\mathcal{J}_\lambda(v(\cdot,t)) = -\|v_t(\cdot,t)\|_2^2. \tag{2.2}$$

The mapping

$$v \in X \mapsto \frac{e^v}{\int_\Omega e^v} \in X$$

is Lipschitz continuous on each bounded set of X, and therefore (1.8) is well posed in X; i.e., given an initial value $v_0 \in X$, we obtain a unique semi-group solution $v = v(\cdot, t) \in X$ locally in time, and henceforth the maximal time of its existence is denoted by $T = T_{\max} \in (0, +\infty]$, which is estimated from below by $\|\nabla v_0\|_2$. Then, from (2.1) and the parabolic regularity we obtain

$$T = +\infty \qquad \text{and} \qquad \sup_{t\geq 0} \|v(\cdot, t)\|_\infty < +\infty, \tag{2.3}$$

in the case of $\lambda < 8\pi$, and also

$$\lim_{t\uparrow T} \|v(\cdot, t)\|_\infty = +\infty, \tag{2.4}$$

if $T < +\infty$. In the latter case we can define

$$\mathcal{S} = \left\{x_0 \in \overline{\Omega} : \exists x_k \to x_0,\ \exists t_k \uparrow T,\ \ \text{s.t. } v(x_k, t_k) \to +\infty\right\} \neq \emptyset,$$

called the blow-up set. $\mathcal{S}$ can be also defined in the case where (2.4) is fulfilled for $T = \infty$. We obtain

$$v(\cdot, t) \geq \inf_\Omega v_0$$

by the comparison principle, and henceforth, $v_0 \geq 0$ is assumed without loss of generality.

The parabolic Brezis-Merle inequality [9, 32], on the other hand, is concerned with the linear parabolic equation

$$v_t = \Delta v + f(x, t) \qquad \text{in } \Omega \times (0, T), \quad v|_{\partial\Omega} = 0, \quad v|_{t=0} = 0,$$

where Ω is again a two-dimensional bounded domain with smooth boundary $\partial\Omega$. Its global form ensures that any $\delta > 0$ admits $p > 1$ and $C > 0$ such that

$$\sup_{t\in(0,T)} \|f(\cdot, t)\|_1 < 4\pi - \delta \quad \Rightarrow \quad \sup_{t\in(0,T)} \left\|e^{v(\cdot,t)}\right\|_p \leq C.$$

This induces the local version, i.e., given subdomains ω and $\hat{\omega}$ with $\omega \subset\subset \hat{\omega} \subset\subset \Omega$ and $\delta > 0$, we obtain $p > 1$ and $C > 0$ both determined by $\sup_{t\in(0,T)} \|f(\cdot, t)\|_1$ such that

$$\sup_{t\in(0,T)} \|f(\cdot, t)\|_{L^1(\hat{\omega})} < 4\pi - \delta \quad \Rightarrow \quad \sup_{t\in(0,T)} \left\|e^{v(\cdot,t)}\right\|_{L^p(\omega)} \leq C.$$

Regarding now the location of blow-up points in the case of a convex set we have the following.

Lemma 3. *If $T < +\infty$ and Ω is convex, then $\mathcal{S} \subset \Omega$.*

Proof. Let $v = v(x,t)$ be a solution to (1.8) with $v|_{t=0} = v_0(x)$. Having assumed $v_0 \geq 0$, we obtain $v = v(x,t) \geq 0$, and then

$$\sup_{t\in[0,T)} \|v(\cdot,t)\|_p < +\infty \tag{2.5}$$

follows for any $p \geq 1$ from the global parabolic Brezis-Merle inequality.

This v is a solution to

$$v_t = \Delta v + \sigma(t)e^v \quad \text{in } \Omega \times (0,T), \qquad v|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0(x)$$

for $\sigma(t) = \frac{\lambda}{\int_\Omega e^{v(\cdot,t)}}$ and the method of the moving plane, [8], is applicable. Using (2.5), we can apply the argument of [5], and consequently, there is an open set $\hat{\omega}$ containing $\partial\Omega$ and a constant $C > 0$ such that

$$\sup_{t\in[0,T)} \|v(\cdot,t)\|_{L^\infty(\Omega\cap\hat{\omega})} \leq C,$$

and therefore, $\mathcal{S} \subset \Omega$. □

Remark 2.1. Due to a uniform L^1-estimate of the solution of (1.8) the result of the above lemma can be extended to higher dimensions $n \geq 2$. In fact, multiplying the equation of (1.8) by the eigenfunction $\phi_1(x) > 0$ corresponding to the principal eigenvalue λ_1 of $-\Delta_D$ and integrating over Ω we obtain

$$\frac{d}{dt}\int_\Omega v(x,t)\,\phi_1(x)\,dx \leq -\lambda_1 \int_\Omega v(x,t)\,\phi_1(x)\,dx + \lambda M, \;\; 0 < t < T,$$

for $M = \max_{\overline{\Omega}} \phi_1(x) > 0$, which implies

$$\int_\Omega v(x,t)\,\phi_1(x)\,dx \leq C := C(v_0, \lambda_1, \lambda, M) \quad \text{for any} \quad 0 < t < T. \tag{2.6}$$

Since Ω is convex, using the method of moving planes, [8], we can find $\overline{\Omega}_0 \subset \Omega$ such that

$$\int_\Omega v(x,t)\,dx \leq \frac{k+1}{m}\int_{\Omega_0} v(x,t)\,\phi_1(x)\,dx \leq \frac{k+1}{m}\int_\Omega v(x,t)\,\phi_1(x)\,dx < C_1, \tag{2.7}$$

by (2.6) for any $0 < t < T$ and $m = \min_{\overline{\Omega}_0} \phi_1(x) > 0$. Using now estimate (2.7), which for $N = 2$ comes out from the parabolic version of the Brezis-Merle inequality, along with the ideas existing in [5] we derive the desired result. If $v = v(x,t)$ is radially symmetric and decreasing in $r = |x|$, and if (2.4) holds, then $\mathcal{S} = \{0\}$ by the same reasoning. We also note that in the case of $T_{\max} = +\infty$, any radially symmetric solution becomes decreasing in $r = |x|$ eventually, see [21].

The stationary problem to (1.8) is described by

$$-\Delta v_* = \frac{\lambda e^{v_*}}{\int_\Omega e^{v_*}} \quad \text{in } \Omega, \qquad v_* = 0 \quad \text{on } \partial\Omega. \tag{2.8}$$

Let $E = \{v_* : \text{solution to (2.8)}\}$ then we have the following lemma due to Wolansky, see Theorem 8 in [31]. Since we use the local well-posedness in time of (1.8) in $X = H_0^1(\Omega)$ and the Trudinger-Moser inequality, the proof is valid only to $n = 1, 2$. However, here a simpler proof is provided.

Lemma 4. *If $E = \emptyset$, then it holds that*

$$\lim_{t\uparrow T} \int_\Omega e^{v(\cdot,t)} = +\infty \tag{2.9}$$

and $\mathcal{S} \neq \emptyset$, where the case $T = +\infty$ is also included.

Proof. We follow the argument of [31]. First, if $\mathcal{S} = \emptyset$, then

$$\sup_{t\in[0,T)} \|v(\cdot,t)\|_\infty < +\infty.$$

This implies $T = +\infty$ and $E \neq \emptyset$ by a standard argument using Lyapunov functions [10], a contradiction.

Next, if

$$\liminf_{t\uparrow T} \int_\Omega e^{v(\cdot,t)} < +\infty,$$

then we obtain

$$\lim_{t\uparrow T} \mathcal{J}_\lambda(v(\cdot,t)) \geq \limsup_{t\uparrow T} \left\{-\lambda \log\left(\int_\Omega e^{v(\cdot,t)}\right)\right\} + \lambda(\log\lambda - 1) > -\infty \tag{2.10}$$

and

$$\frac{1}{2} \liminf_{t\uparrow T} \|\nabla v(\cdot,t)\|_2^2 \leq \mathcal{J}_\lambda(v_0) + \lambda \liminf_{t\uparrow T} \log\left(\int_\Omega e^{v(\cdot,t)}\right) - \lambda(\log\lambda + 1) < +\infty.$$

The latter relation guarantees $T = +\infty$ because of the well posedness in $X = H_0^1(\Omega)$ of (1.8). Furthermore, there are $\delta \in (0,1)$, $t_k \uparrow +\infty$, and $C > 0$ such that $t_{k+1} > t_k + 1$ and

$$\sup_{t\in(t_k,t_k+\delta)} \|\nabla v(\cdot,t)\|_2 \leq C.$$

This implies

$$\sum_{k=1}^{\infty}\int_{t_k}^{t_k+\delta}\|v_t(\cdot,t)\|_2^2\,dt \leq \int_0^{\infty}\|v_t(\cdot,t)\|_2^2\,dt < +\infty$$

by (2.10) and (2.2), and therefore,

$$\lim_{k\to\infty}\int_{t_k}^{t_k+\delta}\|v_t(\cdot,t)\|_2^2\,dt = 0.$$

Then, we obtain $t'_k \in (t_k, t_k+\delta)$ satisfying $\|v_t(\cdot,t'_k)\|_2 \to 0$. Since $\|\nabla v(\cdot,t'_k)\|_2 \leq C$, we obtain a subsequence, denoted by the same symbol, such that $v(\cdot,t'_k) \rightharpoonup {}^{\exists}w$ weakly in X. This implies $e^{v(\cdot,t_k)} \to e^w$ strongly in $L^p(\Omega)$, $p \geq 1$ by the Trudinger-Moser inequality, and then $w \in E$ follows, a contradiction. □

Now, we show the following:

Proof of Theorem 1. Let $\Omega = B(0,1)$, $v_0 = v_0(|x|) \geq 0$, and $\lambda \geq 8\pi$. We obtain $E = \emptyset$ in this case, and therefore, $\mathcal{S} \neq \emptyset$ and (2.9) follows. Since $v = v(|x|,t)$, any $x_0 \in \Omega \setminus \{0\}$ admits $0 < r \ll 1$ such that

$$\sup_{t\in(0,T)}\left\|\frac{\lambda e^{v(\cdot,t)}}{\int_\Omega e^{v(\cdot,t)}}\right\|_{L^1(B(x_0,2r))} < 4\pi.$$

Then, the local parabolic Brezis-Merle inequality to (1.8) guarantees

$$\sup_{t\in(0,T)}\left\|e^{v(\cdot,t)}\right\|_{L^p(B(x_0,r))} < +\infty$$

with $p > 1$, and therefore, it holds that

$$\sup_{t\in(0,T)}\left\|\frac{\lambda e^{v(\cdot,t)}}{\int_\Omega e^{v(\cdot,t)}}\right\|_{L^p(B(x_0,r))} < +\infty$$

since $v \geq 0$. This implies

$$\sup_{t\in(0,T)}\|v(\cdot,t)\|_{W^{2,p}(B(x_0,r/2))} < +\infty$$

from the local parabolic regularity, and hence $x_0 \notin \mathcal{S}$ by Sobolev's imbedding theorem. Thus, it holds that $\mathcal{S} = \{0\}$ and (1.9) follows. □

3. Proof of the main result

This section is devoted to the proof of Theorem 2. Thus, $v = v(|x|, t)$ denotes a solution to (1.8) for $\Omega = B(0,1)$ and $\lambda > 8\pi$, and we define u by (1.3). Then, we have $\nabla u = u\nabla v$ and therefore,

$$-\int_\Omega u\nabla\cdot\psi\,dx = \int_\Omega u\psi\cdot\nabla v\,dx \tag{3.1}$$

for any $\psi \in C^1(\overline{\Omega})^2$. Since

$$v_t = \Delta v + u \qquad \text{in } \Omega, \qquad v|_{\partial\Omega} = 0 \tag{3.2}$$

holds, by (1.8), we have

$$v = (-\Delta_D)^{-1}u - (-\Delta_D)^{-1}v_t,$$

where $w = (-\Delta_D)^{-1}u$ denotes the solution of the problem

$$-\Delta w = u \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

Thus, the right-hand side of (3.1) is equal to

$$\int_\Omega u\psi\cdot\nabla v\,dx = \int_\Omega u\psi\cdot\nabla\left\{(-\Delta_D)^{-1}u - (-\Delta_D)^{-1}v_t\right\}\,dx$$
$$= \frac{1}{2}\int\int_{\Omega\times\Omega}\rho(x,x')u(x,t)u(x',t)dxdx' - \int_\Omega u\psi\cdot\nabla(-\Delta_D)^{-1}v_t\,dx$$

for

$$\rho(x,x') = \psi(x)\cdot\nabla_x G(x,x') + \psi(x')\cdot\nabla_{x'}G(x,x'),$$

where $G = G(x,x')$ is the Green's function of $-\Delta_D$ satisfying $G(x',x) = G(x,x')$ and

$$G(x,x') = \frac{1}{2\pi}\log\frac{1}{|x-x'|} + K(x,x') \tag{3.3}$$

with $K \in C^{2+\theta}_{loc}(\overline{\Omega}\times\Omega\cup\Omega\times\overline{\Omega})$ for $0 < \theta < 1$ ([28]).

Here, we take $\psi(x) = x\varphi(|x|)$ for $\varphi = \varphi(|x|) \in C_0^\infty(\Omega)$ satisfying $0 \le \varphi \le 1$ and $\varphi = 1$ near $x = 0$. Then, $\nabla\cdot\psi|_{x=0} = 2$ and therefore,

$$\int_\Omega u\nabla\cdot\psi\,dx = 2\lambda + o(1) \tag{3.4}$$

holds as $t \uparrow T$ by (1.9). The relation (3.3), on the other hand, guarantees

$$\rho(x,x') = -\frac{1}{2\pi} + L(x,x')$$

with $L = L(x,x') \in C(\overline{\Omega} \times \overline{\Omega})$ satisfying $L(0,0) = 0$, and therefore,

$$\frac{1}{2}\int\int_{\Omega\times\Omega} \rho(x,x')u(x,t)u(x',t)dxdx' = -\frac{\lambda^2}{4\pi} + o(1)$$

as $t \uparrow T$. Thus, we obtain

$$-\int_\Omega u(\psi\cdot\nabla)(-\Delta_D)^{-1}v_t\,dx = \frac{\lambda^2}{4\pi} - 2\lambda + o(1) \tag{3.5}$$

as $t \uparrow T$ by (3.1) and (3.4).

The relation $V = (-\Delta_D)^{-1}v_t$ implies $(rV_r)_r = -rv_t$, and hence

$$rV_r(r,t) = -\int_0^r sv_t(s,t)ds$$

for $r = |x|$. This implies

$$\begin{aligned}(\psi\cdot\nabla)(-\Delta_D)^{-1}v_t &= \varphi(r)r\partial_r(-\Delta_D)^{-1}v_t = -\varphi(r)\int_0^r sv_t(s,t)ds \\ &= -\frac{\varphi(r)}{2\pi}\int_{B(0,r)} v_t(s,t)\,dx,\end{aligned}$$

and therefore,

$$-\int_\Omega u(\psi\cdot\nabla)(-\Delta_D)^{-1}v_t dx \le \frac{\lambda}{2\pi}\sup_r \varphi(r)\int_{B(0,r)} v_t dx \le \frac{\lambda}{2\pi}\|v_t\|_1$$

since $u \ge 0$ and $\|u\|_1 = \lambda$.

It holds that

$$\liminf_{t\uparrow T}\|v_t(\cdot,t)\|_1 \ge \frac{\lambda}{2} - 4\pi > 0 \tag{3.6}$$

by (3.5) and $\lambda > 8\pi$, and hence

$$\|v_t(\cdot,t)\|_2 \ge \delta \tag{3.7}$$

with $\delta > 0$ independent of $t \ge 1$, in case of $T = +\infty$.

Let $0 < \mu_1 < \mu_2 \le \ldots$ be the eigenvalues of $-\Delta_D$ with corresponding L^2-normalized eigenfunctions $\varphi_j = \varphi_j(x)$, $j = 1,2,...$; i.e.,

$$-\Delta\varphi_j = \mu_j\varphi_j \ \text{ in } \Omega, \quad \varphi_j = 0 \ \text{ on } \partial\Omega, \quad \|\varphi_j\|_2 = 1.$$

Then, we obtain

$$\mu_j \sim j \qquad \text{and} \qquad \|\varphi_j\|_\infty \le Cj^{1/4} \tag{3.8}$$

as $j \to \infty$, see [26]. If $T = +\infty$, there is $t_k \to +\infty$ such that

$$v(\cdot,t_k) \rightharpoonup {}^\exists v_* \text{ weakly in } L^2(\Omega) \text{ as } k \to \infty \tag{3.9}$$

by (2.5) with $p = 2$. Then, it follows that

$$g_j(t+t_k) = e^{-\mu_j t} g_j(t_k) + \int_0^t e^{-(t-s)\mu_j} \langle u(\cdot, s+t_k), \varphi_j \rangle \, ds$$

for $g_j(t) = \langle v(\cdot, t), \varphi_j \rangle$ by

$$\dot{g}_j = -\mu_j g_j + \langle u, \varphi_j \rangle ,$$

and therefore,

$$|\langle v(\cdot, t+t_k), \varphi_j \rangle| \leq e^{-\mu_j t} \|v(\cdot, t_k)\|_2 + \lambda \|\varphi_j\|_\infty \mu_j^{-1}.$$

Due to (2.5) and (3.8), the above relation implies

$$|\langle v(\cdot, t+t_k), \varphi_j \rangle| \leq A_j$$

for $t \geq 1$ and $k = 1, 2, \dots$ where $A_j = C(e^{-\alpha j} + \lambda j^{-\frac{3}{4}}) > 0$ for $j \gg 1$, $\alpha > 0$, and $C > 0$, satisfying $\sum_{j=1}^{\infty} A_j^2 < +\infty$. Now taking into account (1.9) and (3.9) we derive

$$\begin{aligned} \langle v(\cdot, t+t_k), \varphi_j \rangle &= g_j(t+t_k) \\ &\to e^{-\mu_j t} \langle v_*, \varphi_j \rangle + \int_0^t e^{-(t-s)\mu_j} \lambda \varphi_j(0) ds \\ &= e^{-\mu_j t} \langle v_*, \varphi_j \rangle + \frac{\lambda \varphi_j(0)}{\mu_j} (1 - e^{-\mu_j t}) \end{aligned}$$

as $k \to \infty$ uniformly in $t \geq 1$ for each j, and therefore,

$$v(\cdot, t+t_k) \to V(\cdot, t) \quad \text{as} \quad k \to \infty$$

in $L^2(\Omega)$ uniformly in $t \geq 1$, where

$$V(\cdot, t) = \sum_{j=1}^{\infty} \left\{ e^{-\mu_j t} \langle v_*, \varphi_j \rangle + \frac{\lambda \varphi_j(0)}{\mu_j} (1 - e^{-\mu_j t}) \right\} \varphi_j$$

converging in $L^2(\Omega)$. Similarly,

$$v_t(\cdot, t+t_k) \to W(\cdot, t) \quad \text{as} \quad k \to \infty$$

in $L^2(\Omega)$ uniformly in $t \geq 1$, where

$$W(\cdot, t) = \sum_{j=1}^{\infty} e^{-\mu_j t} \left\{ -\mu_j \langle v_*, \varphi_j \rangle + \lambda \varphi_j(0) \right\} \varphi_j = V_t.$$

Hence we derive $\|W(\cdot,t)\|_2^2 \geq \delta^2$ by (3.7). On the other hand, we obtain

$$\|W(\cdot,t)\|_2^2 = \sum_{j=1}^{\infty} e^{-2\mu_j t} \left| -\mu_j \langle v_*, \varphi_j \rangle + \lambda \varphi_j(0) \right|^2 \to 0$$

as $t \to +\infty$, a contradiction. □

4. Concluding remarks

This section is devoted to several remarks. First, the Green's function to $-\Delta_D$ for $\Omega = B(0,1)$ is given explicitly by

$$G(x,x') = -\frac{1}{4\pi} \log \left| \frac{z - z'}{1 - \overline{z} z'} \right|^2,$$

where z and z' are complex numbers corresponding to x and x', respectively. This implies

$$x \cdot \nabla_x G(x,x') + x' \cdot \nabla_{x'} G(x,x') = -\frac{1}{2\pi} \cdot \frac{1 - |\zeta|^2}{|1 - \zeta|^2}$$

for $\zeta = \overline{z} z'$. We can see that this function does not belong to $L^\infty(\Omega \times \Omega)$ in contrast with the corresponding function derived from the Green's function to $-\Delta_{JL}$ ([28]). There is a similar difficulty in (1.7), and the control of the boundary blow-up points has not been completed even in the case of $(\varepsilon, \tau) = (1, 0)$.

Next, if $0 < \lambda < 8\pi$, the stationary problem (2.8) admits a unique solution [27, 30]. Since (2.3) holds in this range of λ, the solution $v = v(x,t)$ to (1.8) converges uniformly to this steady solution. What is expected in the case that $\lambda = 8\pi$, $\Omega = B(0,1)$, and $v_0 = v_0(|x|)$ is that an infinite-time blow-up occurs for the solution v; i.e., $T = +\infty$ and $v(x,t) \to 4 \log \frac{1}{|x|}$ locally uniformly in $x \in \overline{\Omega} \setminus \{0\}$ as $t \uparrow +\infty$.

In the limiting case $\varepsilon = 0$ system (1.1) is described by

$$\begin{aligned} v_t &= \Delta v + \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) \quad \text{in } \Omega \times (0,T) \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{4.1}$$

with the smooth initial value $v_0 = v_0(x)$ satisfying $\int_\Omega v_0 \, dx = 0$. In the case that either $0 < \lambda < 4\pi$ or $0 < \lambda < 8\pi$, $\Omega = B(0,1)$, and $v_0 = v_0(|x|)$, we obtain (2.3) similarly to the full system, see [19, 2, 7]. In contrast to (2.8) the stationary problem (1.5) to (4.1) has the trivial solution $v = 0$ for any λ. Furthermore, multiple existence of the stationary (non-radially symmetric) solutions arises even when $0 < \lambda < 4\pi$ and $\Omega = B(0,1)$ ([22, 28]). On the other hand, it is proven that either $T = +\infty$ or (1.9) with $T < +\infty$ holds in (4.1), similarly to Theorem 2, for $\Omega = B(0,1)$, $v_0 = v_0(|x|)$ smooth, and $\lambda > 8\pi$.

If the former case occurs, then the ω-limit set contains a (radially symmetric) stationary solution, while there is a bifurcation of non-constant radially symmetric stationary solutions at $\lambda = \lambda_* > 8\pi$. Then, we can suspect that any $\lambda > 8\pi$ admits a radially symmetric stationary solution (possibly the trivial one $v = 0$), stable in the space of radially symmetric functions. Therefore, the possibility of (1.9) holding with $T < +\infty$ is left open in (4.1) even for $\lambda > 8\pi$, $\Omega = B(0,1)$ and $v_0 = v_0(|x|)$, in contrast to Theorem 2.

A simple blow-up criterion is obtained for the semilinear parabolic equation

$$v_t = \Delta v + |v|^{q-1} v \quad \text{in } \Omega \times (0,T), \qquad v = 0 \quad \text{on } \partial\Omega \times (0,T)$$

with $1 < q < \infty$. In fact, this equation admits the properties

$$\frac{d}{dt} J(v(\cdot,t)) \leq 0$$
$$\frac{1}{4}\frac{d}{dt} \|v(\cdot,t)\|_2^2 = -J(v(\cdot,t)) + \left(\frac{1}{2} - \frac{1}{q+1}\right) \|v(\cdot,t)\|_{q+1}^{q+1}$$

for

$$J(v) = \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{q+1}\|v\|_{q+1}^{q+1}.$$

Using the above, we can infer $T < +\infty$ by $J(v_0) \leq 0$. The same argument is valid with

$$v_t = \Delta v + \frac{\lambda e^v}{\left(\int_\Omega e^v\right)^p} \quad \text{in } \Omega \times (0,T), \qquad v = 0 \quad \text{on } \partial\Omega \times (0,T)$$

with $0 < p < 1$ ([1]), but this is not the case with (1.8).

In fact, (1.8) is written in the form (3.2) with $u = \frac{\lambda e^v}{\int_\Omega e^v}$ satisfying $\|u\|_1 = \lambda$. Setting $\mathcal{J}_\lambda(v) = L(u,v)$, where

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v\right) + \lambda(\log\lambda - 1)$$
$$L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2^2 - \langle v, u\rangle,$$

it then holds that

$$\frac{1}{4}\frac{d}{dt}\|v\|_2^2 = -L(u,v) + \int_\Omega u(\log u - 1) - \frac{1}{2}uv\ dx = -\mathcal{J}_\lambda(v) + K(u,v)$$

with

$$\begin{aligned} K(u,v) &= \int_\Omega u(\log u - 1) - \frac{1}{2}uv\ dx \\ &\geq K\big|_{u=\frac{\lambda e^{v/2}}{\int_\Omega e^{v/2}}} = -\lambda \log \int_\Omega e^{v/2} + \lambda(\log\lambda - 1); \end{aligned}$$

i.e.,

$$\frac{1}{4}\frac{d}{dt}\|v(\cdot,t)\|_2^2 \geq -\mathcal{J}_\lambda(v(\cdot,t)) - \lambda \log\left(\int_\Omega e^{v(\cdot,t)/2}\right) + \lambda(\log\lambda - 1).$$

In spite of

$$\frac{d}{dt}\mathcal{J}_\lambda(v(\cdot,t)) \leq 0,$$

the above inequality is not sufficient to guarantee $T < +\infty$ because of the negativity of the second term of the right-hand side.

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