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LARGE-TIME BEHAVIOR OF A ONE-DIMENSIONAL MONOCHARGED PLASMA

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Abstract. A collisionless plasma is described by the Vlasov-Poisson system. This system is considered in one space dimension in the case when there is one species of ions. Using the fact that the time evolution preserves a certain partial order of points in phase space, two theorems on large-time asymptotics are established.

1. Introduction

Consider the Vlasov-Poisson system:

$$\begin{cases} \partial_t f + v\partial_x f + E(t,x)\partial_v f = 0 \\ \rho(t,x) = \int f(t,x,v)dv, \\ E(t,x) = \frac{1}{2}\int_{-\infty}^{x} \rho d\tilde{x} - \frac{1}{2}\int_{x}^{\infty} \rho d\tilde{x}. \end{cases} \tag{1.1}$$

Here $t \geq 0$ is time, $x \in \mathbb{R}$ is position, $v \in \mathbb{R}$ is momentum, and f gives the number density in phase space of charged particles. It is assumed that all particles have mass one and charge one. The effect of collisions is neglected, as are relativistic effects. The initial condition,

$$f(0,x,v) = f_0(x,v) \quad \text{for } (x,v) \in \mathbb{R}^2,$$

is assumed where $f_0 \in C^1(\mathbb{R}^2)$ is nonnegative and compactly supported.

Define

$$\overset{\circ}{m} = \iint f_0(x,v)dv\,dx,$$

$$f_\infty(t,x,v) = 2t^{-2} I_{[-\frac{1}{4}\overset{\circ}{m}t^2, \frac{1}{4}\overset{\circ}{m}t^2]}(x)\delta\Big(v - \frac{2x}{t}\Big),$$

$$\rho_\infty(t,x) = 2t^{-2} I_{[-\frac{1}{4}\overset{\circ}{m}t^2, \frac{1}{4}\overset{\circ}{m}t^2]}(x),$$

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$$E_\infty(t,x) = \begin{cases} -\frac{1}{2}\overset{\circ}{m}, & x \le -\frac{1}{4}\overset{\circ}{m}t^2 \\ 2t^{-2}x, & -\frac{1}{4}\overset{\circ}{m}t^2 \le x \le \frac{1}{4}\overset{\circ}{m}t^2 \\ \frac{1}{2}\overset{\circ}{m}, & \frac{1}{4}\overset{\circ}{m}t^2 \le x. \end{cases}$$

The large-time behavior of solutions of (1.1) are related to $(f_\infty, \rho_\infty, E_\infty)$. This was pointed out in [3] by use of a rescaling method. In [2] it was shown rigorously that the rescaling used in [3] results in a system with a steady global attractor ($(f_\infty, \rho_\infty, E_\infty)$ scaled). However, it was also shown in [2] that for any $t > 0$

$$\|E_\infty(t) - E_\infty(t+1)\|_{L^2} \ge C, \tag{1.2}$$

so $(f_\infty, \rho_\infty, E_\infty)$ is not globally asymptotically stable for the original unscaled system. In this paper it will be shown that for a large class of solutions

$$\|E(t) - E_\infty(t)\|_{L^\infty} \le Ct^{-1} \tag{1.3}$$

for large t. Note that (1.2) uses the L^2 norm and (1.3) uses the L^∞ norm. The estimate

$$\|\rho(t)\|_{L^\infty} \le Ct^{-1}$$

was established in [2]. Here $\|\rho(t)\|_{L^\infty} \le Ct^{-2}\ln t$ will be established (for large t) for a large class of solutions.

Before stating the two main theorems, two definitions must be made:

1. $(x_1, v_1) \ge (x_2, v_2)$ holds if $x_1 \le x_2$ and $v_1 \ge v_2$.
2. Let $S \subset \mathbb{R}^2$. S is called a slanted set if for every $(x_1, v_1) \in S$ and $(x_2, v_2) \in S$ with $(x_1, v_1) \ge (x_2, v_2)$, $[x_1, x_2] \times [v_2, v_1] \subset S$ holds.

Theorem 1.1 *Assume that $f_0 \in C^1(\mathbb{R}^2)$ is nonnegative and compactly supported. Further assume that there exists $F_0 > 0$ such that for any $F \in (0, F_0]$ $\{(x,v) : f_0(x,v) \ge F\}$ is a slanted set. Then there exists $C > 0$ such that*

$$\rho(t,x) \le Ct^{-2}\ln t$$

for all $x \in \mathbb{R}$ and $t \ge 3$.

Theorem 1.2. *Assume that $f_0 \in C^1(\mathbb{R}^2)$ is nonnegative and compactly supported. Further assume that the support of f_0 is a slanted set. Then there exists $C > 0$ such that*

$$\|E(t) - E_\infty(t)\|_{L^\infty} \le Ct^{-1}$$

for all $t \ge 3$.

The characteristics for (1.1), $(X(s,t,x,v), V(s,t,x,v))$, are defined by

$$\frac{dX}{ds} = V, \qquad X(t,t,x,v) = x$$
$$\frac{dV}{ds} = E(s,X), \qquad V(t,t,x,v) = v.$$

Then for all $(t,x,v) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}$

$$0 \le f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v)) \le \sup f_0.$$

Also

$$\iint f(t,x,v)dv\,dx = \iint f_0(y,w)dw\,dy = \overset{\circ}{m}$$

and $-\frac{1}{2}\overset{\circ}{m} \le E(t,x) \le \frac{1}{2}\overset{\circ}{m}$. The initial data, f_0, is always assumed to be nonnegative, continuously differentiable, and compactly supported.

The global existence of smooth solutions is known even in the much more difficult three-dimensional case, [1,6,8,10,12,13]. In the plasma physics case (in three dimensions) some decay is known as $t \to \infty$, [4,5,7,9,11].

As in [2], C denotes a positive generic constant which may change from line to line and may depend on the initial data, f_0.

2. Properties of Slanted Sets

Before proving the theorems, some properties of slanted sets and their time evolution will be developed.

Lemma 2.1. *Let* $0 \le s_0 \le t$. *If* $(x_1,v_1) \ge (x_2,v_2)$, *then*

$$(X(s_0,t,x_1,v_1), V(s_0,t,x_1,v_1)) \ge (X(s_0,t,x_2,v_2), V(s_0,t,x_2,v_2)). \tag{2.1}$$

If

$$X(s_0,t,x_1,v_1) < X(s_0,t,x_2,v_2) \quad and \quad V(s_0,t,x_1,v_1) \le V(s_0,t,x_2,v_2),$$

then $x_1 < x_2$ *and* $v_1 \le v_2$.

Proof. Suppose $x_1 < x_2$ and $v_1 > v_2$ first. Define

$$T = \inf\Big\{s \ge 0 : X(\tau,t,x_1,v_1) \le X(\tau,t,x_2,v_2)$$
$$\text{and } V(\tau,t,x_1,v_1) \ge V(\tau,t,x_2,v_2) \text{ for all } \tau \in [s,t]\Big\}.$$

Then for $s \in [T,t]$ $X(s,t,x_1,v_1) \le X(s,t,x_2,v_2)$ so $E(s,X(s,t,x_1,v_1)) \le E(s,X(s,t,x_2,v_2))$ and hence

$$V(T,t,x_1,v_1) - V(T,t,x_2,v_2) = v_1 - v_2$$

$$-\int_T^t \left(E(s,X(s,t,x_1,v_1)) - E(s,X(s,t,x_2,v_2))\right) ds \geq v_1 - v_2 > 0.$$

Similarly,

$$\begin{aligned} &X(T,t,x_1,v_1) - X(T,t,x_2,v_2) \\ &= x_1 - x_2 - \int_T^t \left(V(s,t,x_1,v_1) - V(s,t,x_2,v_2)\right) ds \leq x_1 - x_2 < 0. \end{aligned}$$

If $T > 0$ a continuity argument leads to a contradiction, hence $T = 0$ and (2.1) follows.

Now if $(x_1, v_1) \geq (x_2, v_2)$, (2.1) follows by continuity with respect to initial conditions.

Next suppose

$$X(s_0,t,x_1,v_1) < X(s_0,t,x_2,v_2) \ \text{ and } \ V(s_0,t,x_1,v_1) < V(s_0,t,x_2,v_2) \tag{2.2}$$

and define

$$\begin{aligned} T \quad = \quad & \sup \{ s \in [s_0, t] : X(\tau,t,x_1,v_1) \leq X(\tau,t,x_2,v_2) \text{ and} \\ & V(\tau,t,x_1,v_1) \leq V(\tau,t,x_2,v_2) \text{ for all } \tau \in [s_0, s] \}. \end{aligned}$$

Then

$$\begin{aligned} &V(T,t,x_2,v_2) - V(T,t,x_1,v_1) = V(s_0,t,x_2,v_2) - V(s_0,t,x_1,v_1) \\ &+ \int_{s_0}^T \left(E(\tau,X(\tau,t,x_2,v_2)) - E(\tau,X(\tau,t,x_1,v_1))\right) d\tau > 0 \end{aligned}$$

and

$$X(T,t,x_2,v_2) - X(T,t,x_1,v_1) \geq X(s_0,t,x_2,v_2) - X(s_0,t,x_1,v_1) > 0. \tag{2.3}$$

It follows that $T = t$ and hence $v_2 > v_1$ and $x_2 > x_1$. If (2.2) is relaxed to $V(s_0,t,x_1,v_1) \leq V(s_0,t,x_2,v_2)$, then $x_2 > x_1$ and $v_2 \geq v_1$ follow by continuity with respect to initial conditions and (2.3).

Lemma 2.2. *Let S_0 be a slanted set and $t \geq 0$, then the set*

$$S(t) = \{(X(t,0,y,w), V(t,0,y,w)) : (y,w) \in S_0\}$$

is slanted.

Proof. Let $(x_1,v_1) \in S(t)$ and $(x_2,v_2) \in S(t)$ with $(x_1,v_1) \geq (x_2,v_2)$. Consider any $(x_3,v_3) \in [x_1,x_2] \times [v_2,v_1]$. Then $(x_1,v_1) \geq (x_3,v_3) \geq (x_2,v_2)$. Let

$$(y_k,w_k) = (X(0,t,x_k,v_k), V(0,t,x_k,v_k))$$

for $k = 1, 2, 3$. By Lemma 2.1 $(y_1, w_1) \geq (y_3, w_3) \geq (y_2, w_2)$. Since $(y_1, w_1) \in S_0, (y_2, w_2) \in S_0$, and S_0 is a slanted set, $[y_1, y_2] \times [w_2, w_1] \subset S_0$, and hence $(y_3, w_3) \in S_0$. Therefore $(x_3, v_3) \in S(t)$. It follows that $[x_1, x_2] \times [v_2, v_1] \subset S(t)$ and that $S(t)$ is slanted.

Lemma 2.3. *Let S_0 be a closed slanted set and C_0 a connected component of S_0. Then C_0 is closed. Assume C_0 is bounded and define*

$$\begin{aligned} y_1 &= \min \{y : (y, w) \in C_0 \text{ for some } w\}, \\ w_1 &= \min \{w : (y_1, w) \in C_0\}, \\ y_2 &= \max \{y : (y, w) \in C_0 \text{ for some } w\}, \\ w_2 &= \max \{w : (y_2, w) \in C_0\}. \end{aligned}$$

Then

$$C_0 \subset [y_1, y_2] \times [w_1, w_2] \tag{2.4}$$

and

$$S_0 \subset ((-\infty, y_1] \times (-\infty, w_1]) \cup C_0 \cup ([y_2, \infty) \times [w_2, \infty)). \tag{2.5}$$

Proof. To show that C_0 is closed, let O_1 and O_2 be disjoint open sets with

$$O_1 \cup O_2 \supset \text{ closure } C_0. \tag{2.6}$$

Then $O_1 \cup O_2 \supset C_0$ and C_0 is connected so either $O_1 \supset C_0$ or $O_2 \supset C_0$. Without loss of generality consider $O_1 \supset C_0$. Then closure $O_1 \supset$ closure C_0 and hence (closure C_0) $\cap O_2 = \emptyset$. By (2.6), closure $C_0 \subset O_1$ follows. Since S_0 is closed, closure $C_0 \subset S_0$. It follows that $C_0 =$ closure C_0.

Suppose $(y, w) \in S_0$ and $y_1 \leq y \leq y_2$. Suppose $w > w_2$. Then $(y, w) \geq (y_2, w_2), (y, w) \in S_0$, and $(y_2, w_2) \in C_0 \subset S_0$ so $[y, y_2] \times [w_2, w] \subset S_0$. But now $([y, y_2] \times [w_2, w]) \cup C_0$ is a connected subset of S_0 that properly includes C_0, which is a contradiction. Hence $w \leq w_2$ follows. $w_1 \leq w$ may be shown similarly so $S_0 \cap ([y_1, y_2] \times \mathbb{R}) \subset [y_1, y_2] \times [w_1, w_2]$. But $C_0 \subset S \cap ([y_1, y_2]) \times \mathbb{R})$ so (2.4) follows.

Suppose $(y, w) \in S_0$ and $y_2 < y$. Suppose $w \leq w_2$. Then $(y_2, w_2) \geq (y, w), (y, w) \in S_0$, and $(y_2, w_2) \in C_0 \subset S_0$ so $[y_2, y] \times [w, w_2] \subset S_0$. But now $([y_2, y] \times [w, w_2]) \cup C_0$ is a connected subset of S_0 that properly includes C_0, which is a contradiction. Hence $w > w_2$.

If $(y, w) \in S_0$ and $y < y_1$ then $w < w_1$ follows as above, and (2.5) now follows.

Lemma 2.4. *Let S_0 be the support of f_0 and assume S_0 is a slanted set. Let $C_0, y_1, w_1, y_2,$ and w_2 be as in Lemma 2.3. Let $t \geq 0$ and define*

$$S(t) = \{(X(t, 0, y, w), V(t, 0, y, w)) : (y, w) \in S_0\}$$

$$C(t) = \{(X(t,0,y,w), V(t,0,y.w)) : (y,w) \in C_0\}.$$

Note that $S(t)$ is a closed slanted set and that $C(t)$ is a connected component of $S(t)$, so Lemma 2.3 applies to $S(t)$ and $C(t)$: Define

$$\begin{aligned} x_1 &= \min\{x : (x,v) \in C(t) \text{ for some } v\}, \\ v_1 &= \min\{v : (x_1,v) \in C(t)\}, \\ x_2 &= \max\{x : (x,v) \in C(t) \text{ for some } v\}, \\ v_2 &= \max\{v : (x_2,v) \in C(t)\}, \end{aligned}$$

then by Lemma 2.3 we have $C(t) \subset [x_1, x_2] \times [v_1, v_2]$ and

$$S(t) \subset ((-\infty, x_1] \times (-\infty, v_1]) \times C(t) \times ([x_2, \infty) \times [v_2, \infty)). \tag{2.7}$$

Moreover, if we also define

$$m_1 = \int_{-\infty}^{y_1} \int f_0 dw\, dy \quad \textit{and} \quad m_2 = \int_{-\infty}^{y_2} \int f_0 dw\, dy,$$

then

$$\begin{aligned} x_1 &= X(t,0,y_1,w_1) = y_1 + w_1 t + \frac{1}{2}\Big(m_1 - \frac{1}{2}\overset{\circ}{m}\Big)t^2 \\ v_1 &= V(t,0,y_1,w_1) = w_1 + (m_1 - \frac{1}{2}\overset{\circ}{m})t \\ x_2 &= X(t,0,y_2,w_2) = y_2 + w_2 t + \frac{1}{2}\Big(m_2 - \frac{1}{2}\overset{\circ}{m}\Big)t^2 \\ v_2 &= V(t,0,y_2,w_2) = w_2 + (m_2 - \frac{1}{2}\overset{\circ}{m})t. \end{aligned}$$

Proof. For any $(y,w) \in C_0$ we have by (2.4) that $y_1 \le y \le y_2$ and $w_1 \le w \le w_2$, and hence Lemma 2.1 yields

$$X(t,0,y_1,w_1) \le X(t,0,y,w) \le X(t,0,y_2,w_2)$$

and

$$V(t,0,y_1,w_1) \le V(t,0,y,w) \le V(t,0,y_2,w_2).$$

Hence,

$$C(t) \subset [X(t,0,y_1,w_1), X(t,0,y_2,w_2)] \times [V(t,0,y_1,w_1), V(t,0,y_2,w_2)].$$

But $(y_1,w_1) \in C_0$ and $(y_2,w_2) \in C_0$ so $(X(t,0,y_1,w_1), V(t,0,y_1,w_1)) \in C(t)$ and $(X(t,0,y_2,w_2), V(t,0,y_2,w_2)) \in C(t)$ so it follows that

$$(X(t,0,y_1,w_1), V(t,0,y_1,w_1)) = (x_1, v_1) \tag{2.8}$$

and

$$(X(t,0,y_2,w_2),V(t,0,y_2,w_2))=(x_2,v_2).$$

Let $S_0^1=S_0\cap((-\infty,y_1]\times(-\infty,w_1])$ and

$$S^1(t)=\{(X(t,0,y,w),V(t,0,y,w)):(y,w)\in S_0^1\}.$$

By Lemma 2.1 and (2.8) it follows that

$$S^1(t)=S(t)\cap((-\infty,x_1]\times(-\infty,v_1]).$$

Hence, using (2.7), we have

$$\begin{aligned}E(t,X(t,0,y_1,w_1))&=\int_{S^1(t)}\int f(t,x,v)dv\,dx-\frac{1}{2}\mathring{m}\\&=\int_{S_0^1}\int f_0(y,w)dw\,dy-\frac{1}{2}\mathring{m}=m_1-\frac{1}{2}\mathring{m}.\end{aligned}$$

Since $t\geq 0$ is arbitrary, it follows that

$$V(t,0,y_1,w_1))=w_1+(m_1-\frac{1}{2}\mathring{m})t$$

and

$$X(t,0,y_1,w_1)=y_1+w_1t+\frac{1}{2}\Big(m_1-\frac{1}{2}\mathring{m}\Big)t^2.$$

It may be shown similarly that

$$V(t,0,y_2,w_2)=w_2+\Big(m_2-\frac{1}{2}\mathring{m}\Big)t$$

and

$$X(t,0,y_2,w_2)=y_2+w_2t+\frac{1}{2}\Big(m_2-\frac{1}{2}\mathring{m}\Big)t$$

so the proof is complete.

Consider the following example:

$$f_0(x,v)=\begin{cases}(x-1)^2(x-2)^2(v-1)^2(v-2)^2\\ \qquad\qquad\qquad\text{if } 1\leq x\leq 2 \text{ and } 1\leq v\leq 2\\(x+1)^2(x+2)^2(v+1)^2(v+2)^2\\ \qquad\qquad\qquad\text{if } -2\leq x\leq -1 \text{ and } -2\leq v\leq -1\\0 \qquad\qquad\qquad\text{otherwise.}\end{cases}$$

Then

$$\int_1^2\int_1^2 f_0dv\,dx=\int_{-2}^{-1}\int_{-2}^{-1}f_0dv\,dx=\frac{1}{2}\mathring{m},$$

where

$$\overset{\circ}{m} = \iint f_0 dv\, dx.$$

By Lemma 2.4 $V(t,0,1,1) = 1$, $X(t,0,1,1) = 1+t$, $V(t,0,-1,-1) = -1$, $X(t,0,-1,-1) = -1-t$, and $f(t,x,v) = 0$ for

$$X(t,0,-1,-1) \leq x \leq X(t,0,1,1).$$

Hence, $\rho(t,x) = E(t,x) = 0$ for $-1-t \leq x \leq 1+t$. For $|x| \leq 1+t$ and $1+t \leq \frac{1}{4}\overset{\circ}{m}t^2$ this yields $\rho_\infty(t,x) - \rho(t,x) = 2t^{-2}$ and

$$E_\infty(t,x) - E(t,x) = 2t^{-2}x.$$

Therefore, for $1+t \leq \frac{1}{4}\overset{\circ}{m}t^2$ and $p \in [1,\infty]$,

$$\|\rho_\infty(t) - \rho(t)\|_{L^p} \geq Ct^{-2+\frac{1}{p}} \quad \text{and} \quad \|E_\infty(t) - E(t)\|_{L^p} \geq Ct^{-1+\frac{1}{p}}.$$

Thus, the estimate of Theorem 1.2 is sharp.

3. Proofs of the Theorems

The following lemma is a major ingredient in the proof of Theorem 1.1.

Lemma 3.1. *Let $F > 0$ and $t > 0$ and $Z : [0,t] \to \mathbb{R}$ be twice continuously differentiable. Assume that $Z(t) = 0$, $\dot{Z}(t) < 0$, and $\ddot{Z} \geq FZ(-\dot{Z})$ holds on $[0,t]$. Then*

$$-\dot{Z}(t) < \frac{\pi^2}{2Ft^2}.$$

Proof. Note that

$$\frac{d}{ds}\Big(\dot{Z} + \frac{1}{2}FZ^2\Big) \geq 0$$

so for $s \in [0,t]$

$$\dot{Z}(t) = \dot{Z}(t) + \frac{1}{2}FZ^2(t) \geq \dot{Z}(s) + \frac{1}{2}FZ^2(s).$$

Let $A = \sqrt{-\dot{Z}(t)}$ and $B = \sqrt{\frac{1}{2}F}$, then

$$-A^2 \geq \dot{Z}(s) + B^2Z^2(s)$$

and hence

$$\frac{d}{ds}\Big[\tan^{-1}\Big(\frac{BZ(s)}{A}\Big)\Big] = \frac{AB\dot{Z}(s)}{A^2 + B^2Z^2(s)} \leq -AB.$$

Therefore, since $Z(t) = 0$,

$$\tan^{-1}\left(\frac{BZ(s)}{A}\right) \geq AB(t-s), \qquad \frac{\pi}{2} > \tan^{-1}\left(\frac{BZ(0)}{A}\right) \geq ABt,$$

and

$$\frac{\pi^2}{2Ft^2} = \left(\frac{\pi}{2Bt}\right)^2 > A^2 = -\dot{Z}(t),$$

completing the proof.

Proof of Theorem 1.1. Consider $t > 0$ and x with $\rho(t,x) > 0$. Define

$$\tilde{F}_0 = \min\{F_0, \tfrac{1}{2}\max_x f(t,x,v)\}, \quad v_0 = \inf\{v : f(t,x,v) \geq \tilde{F}_0\}, \tag{3.1}$$

and $v_k = \sup\{v : f(t,x,v) \geq 2^{1-k}\tilde{F}_0\}$ for $k = 1, 2, \ldots$. Then $v_0 < v_1 < v_2 < \cdots$. Also define (for $0 \leq s \leq t$)

$$S_k(s) = \left\{(y,w) : f(s,y,w) \geq 2^{1-k}\tilde{F}_0\right\}$$

for $k = 1$ and $(X_k(s), V_k(s)) = (X(s,t,x,v_k), V(s,t,x,v_k))$ for $k \geq 0$. For $k \geq 1$ $(x, v_{k-1}) \in S_k(t)$ and $(x, v_k) \in S_k(t)$ so $(X_{k-1}(s), V_{k-1}(s)) \in S_k(s)$ and $(X_k(s), V_k(s)) \in S_k(s)$. Since $(x, v_k) \geq (x, v_{k-1})$, for $s \in [0,t]$ Lemma 2.1 implies that

$$(X_k(s), V_k(s)) \geq (X_{k-1}(s), V_{k-1}(s)).$$

By Lemma 2.2, $S_k(s)$ is a slanted set so

$$[X_k(s), X_{k-1}(s)] \times [V_{k-1}(s), V_k(s)] \subset S_k(s)$$

and hence

$$\begin{aligned} E(s, X_{k-1}(s)) - E(s, X_k(s)) &= \int_{X_k(s)}^{X_{k-1}(s)} \int f(s,y,w)\,dw\,dy \\ &\geq \int_{X_{k(s)}}^{X_{k-1}(s)} \int_{V_{k-1}(s)}^{V_{k(s)}} 2^{1-k}\tilde{F}_0\,dw\,dy. \end{aligned}$$

Letting $Z = X_{k-1} - X_k$, we have $Z(t) = x - x = 0$, $\dot{Z}(t) = v_{k-1} - v_k < 0$, and

$$\ddot{Z}(s) = E(s, X_{k-1}(s)) - E(s, X_k(s)) \geq 2^{1-k}\tilde{F}_0 Z(s)(-\dot{Z}(s)).$$

By Lemma 3.1 we have

$$v_k - v_{k-1} = -\dot{Z}(t) < \frac{\pi^2}{2(2^{1-k}\tilde{F}_0)t^2}. \tag{3.2}$$

Now for any integer $N \geq 2$, (3.2) yields

$$\int_{v_0}^{v_N} f(t,x,v)dv \leq \int_{v_0}^{v_1} (\max_w f(t,x,w))dv + \sum_{k=2}^{N} \int_{v_{k-1}}^{v_k} 2^{2-k}\tilde{F}_0 dv$$
$$\leq \left(\max_w f(t,x,w)\right) \frac{\pi^2}{2\tilde{F}_0 t^2} + \sum_{k=2}^{N} \left(2^{2-k}\tilde{F}_0\right) \frac{\pi^2}{2^{2-k}\tilde{F}_0 t^2}$$
$$\leq C\left(\tilde{F}_0^{-1} \max_v f(t,x,v) + N\right) t^{-2}.$$

From (3.1) we have

$$\tilde{F}_0^{-1} \max_v f(t,x,v) \leq \max\left\{2, F_0^{-1}\|f_0\|_{L^\infty}\right\} \leq C$$

so

$$\int_{v_0}^{v_N} f(t,x,v)dv \leq CNt^{-2}. \tag{3.3}$$

In the proof of Theorem 2.1 of [2] it is shown that for any t, x, v

$$\partial_v X(0,t,x,v) \leq -t.$$

Hence,

$$\int_{v_N}^{\infty} f(t,x,v)dv = \int_{v_N}^{\infty} f_0(X(0,t,x,v), V(0,t,x,v))dv$$
$$\leq \int_{v_N}^{\infty} 2^{1-N} F_0 I_{|X(0,t,x,v)| \leq C} dv$$
$$\leq 2^{1-N} F_0 \int I_{|X(0,t,x,v)| \leq C} \; \frac{(-\partial_V X(0,t,x,v))}{t} \, dv$$
$$= 2^{1-N} F_0 t^{-1} \int I_{|z|<C} dz = C2^{-N} t^{-1}.$$

Combining this with (3.3) yields

$$\int_{v_0}^{\infty} f(t,x,v)dv \leq Ct^{-1}\left(Nt^{-1} + 2^{-N}\right).$$

For $t \geq 3$ there exists an integer N with

$$\frac{\ln t}{\ln 2} < N < \left(1 + \frac{1}{\ln 2}\right) \ln t$$

and

$$\int_{v_0}^{\infty} f(t,x,v)dv \leq Ct^{-2} \ln t$$

follows.

We may show

$$\int_{-\infty}^{v_0} f(t,x,v)dv \leq Ct^{-2}\ln t$$

in a similar manner and the proof is complete.

Lemma 3.2. *Assume that the support of f_0 is a slanted set. There exists a positive constant, C, such that*

$$\int_{x_0}^{\infty}\int_{-\infty}^{v_0} f(t,x,v)dv\,dx + \int_{-\infty}^{x_0}\int_{v_0}^{\infty} f(t,x,v)dv\,dx \leq Ct^{-2}$$

for all (t,x_0,v_0) with $t>0$ and $f(t,x_0,v_0)>0$.

Proof. Define

$$x_1 = \inf\{x : f(t,x,v_0) > 0\}, \quad x_2 = \sup\{x : f(t,x,v_0) > 0\},$$
$$\mathcal{S}_0 = \text{support } f_0, \quad S = \text{closure } \{(x,v) : x_1 \leq x \leq x_2 \text{ and } f(t,x,v) > 0\},$$
$$\mathcal{M} = \int_{x_1}^{x_2}\int f(t,x,v)dv\,dx,$$

and for $0 \leq s \leq t$

$$\mathcal{S}(s) = \{(X(s,0,x,v), V(s,0,x,v)) : (x,v) \in \mathcal{S}_0\},$$
$$S(s) = \{(X(s,t,x,v), V(s,t,x,v)) : (x,v) \in S\}.$$

We claim that for every $(y,w) \in S(s)$,

$$X(s,t,x_1,v_0) \leq y \leq X(s,t,x_2,v_0) \tag{3.4}$$

holds. To prove this assume that $y > X(s,t,x_2,v_0)$ holds. We will derive a contradiction. Let

$$(x,v) = (X(t,s,y,w), V(t,s,y,w)).$$

First consider $w \geq V(s,t,x_2,v_0)$. By Lemma 2.1 we have $x = X(t,s,y,w) > x_2$. But $(y,w) \in S(s)$ so $(x,v) \in S$ and hence $x \leq x_2$, which is a contradiction.

Next consider $w < V(s,t,x_2,v_0)$. Then

$$(y,w) \leq (X(s,t,x_2,v_0), V(s,t,x_2,v_0)).$$

By Lemma 2.2 $\mathcal{S}(s)$ is a slanted set. Since $(y,w) \in S(s) \subset \mathcal{S}(s)$ and $(X(s,t,x_2,v_0), V(s,t,x_2,v_0)) \in S(s) \subset \mathcal{S}(s)$, it follows that

$$[X(s,t,x_2,v_0), y] \times [w, V(s,t,x_2,v_0)] \subset \mathcal{S}(s). \tag{3.5}$$

To contradict this, note that for $\tilde{x} > x_2$ we have $(x_2, v_0) \geq (\tilde{x}, v_0)$ and hence by Lemma 2.1

$$(X(s,t,\tilde{x},v_0), V(s,t,\tilde{x},v_0)) \leq (X(s,t,x_2,v_0), V(s,t,x_2,v_0)).$$

It follows that for $\tilde{x}$ sufficiently near x_2 (but greater than x_2) we have

$$(X(s,t,\tilde{x},v_0), V(s,t,\tilde{x},v_0)) \in [X(s,t,x_2,v_0), y] \times [w, V(s,t,x_2,v_0)],$$

and hence by (3.5) that $(X(s,t,\tilde{x},v_0), V(s,t,\tilde{x},v_0)) \in \mathcal{S}(s)$ and $(\tilde{x}, v_0) \in \mathcal{S}(t)$. But for $\tilde{x} > x_2$, $(\tilde{x}, v_0) \notin \mathcal{S}(t)$, which is a contradiction.

We have shown that $y \leq X(s,t,x_2,v_0)$. We may show $X(s,t,x_1,v_0) \leq y$ similarly so the claim, (3.4), is established. Let $Y(s) = X(s,t,x_2,v_0) - X(s,t,x_1,v_0)$. If follows from Lemma 2.1 that $Y(s) \geq 0$ for all $s \in [0,t]$. Note that $Y(t) = x_2 - x_1 > 0$, $\dot{Y}(t) = v_0 - v_0 = 0$, and by (3.4)

$$\begin{aligned}\ddot{Y}(s) &= \int_{X(s,t,x_1,v_0)}^{X(s,t,x_2,v_0)} \int f(s,y,w)\,dw\,dy \\ &\geq \iint_{S(s)} f(s,y,w)\,dw\,dy = \iint_{S(t)} f(t,x,v)\,dv\,dx = \mathcal{M}.\end{aligned}$$

It follows that

$$Y(s) \geq \frac{1}{2}\mathcal{M}(t-s)^2 \quad \text{for } 0 \leq s \leq t.$$

But since $(X(0,t,x_2,v_0), V(0,t,x_2,v_0))$ and $(X(0,t,x_1,v_0), V(0,t,x_1,v_0))$ are in the support of f_0, we have $C \geq Y(0) \geq \frac{1}{2}\mathcal{M}t^2$ and hence $Ct^{-2} \geq \mathcal{M}$. Consider (x,v) with $x \leq x_0$ and $v \geq v_0$. Suppose $f(t,x,v) > 0$. Then $(x,v) \geq (x_0,v_0)$ and $(x,v) \in \mathcal{S}(t)$ and $(x_0,v_0) \in \mathcal{S}(t)$. Since $\mathcal{S}(t)$ is a slanted set it follows that $[x,x_0] \times [v_0,v] \subset \mathcal{S}(t)$ and in particular that $(x,v_0) \in \mathcal{S}(t)$. Hence $x \geq x_1$. It follows that

$$\int_{-\infty}^{x_0} \int_{v_0}^{\infty} f(t,x,v)\,dv\,dx \leq \int_{x_1}^{x_0} \int f(t,x,v)\,dv\,dx \leq \mathcal{M} \leq Ct^{-2}.$$

It may be shown in a similar fashion that

$$\int_{x_0}^{\infty} \int_{-\infty}^{v_0} f(t,x,v)\,dv\,dx \leq Ct^{-2}$$

which completes the proof. □

Proof of Theorem 1.2. Consider a characteristic $(X(s,t,x,v), V(s,t,x,v))$ along which f is positive and write $(X(s), V(s)) = (X(s,t,x,v), V(s,t,x,v))$.

Let

$$\ell(s) = \int_{-\infty}^{X(s)} \int_{-\infty}^{V(s)} f(s,y,w)dw\,dy,$$
$$r(s) = \int_{X(s)}^{\infty} \int_{V(s)}^{\infty} f(s,y,w)dw\,dy,$$
$$d(s) = \int_{X(s)}^{\infty} \int_{-\infty}^{V(s)} f(s,y,w)dwdy + \int_{-\infty}^{X(s)} \int_{V(s)}^{\infty} f(s,y,w)dw\,dy,$$

then $\mathring{m} = \ell(s) + r(s) + d(s)$. It follows from Lemma 2.1 that $\ell(s)$ and $d(s)$ are nondecreasing and hence $\ell(s) \to L$ and $r(s) \to R$ as $s \to \infty$ for some $L \in [0, \mathring{m}]$ and $R \in [0, \mathring{m}]$. By Lemma 3.2 $0 \le d(s) \le Cs^{-2}$ and therefore $L + R = \mathring{m}$ and $L - \ell(s) + R - r(s) = d(s) \le Cs^{-2}$. Since $L - \ell(s) \ge 0$ and $R - r(s) \ge 0$, it follows that

$$\Big|E(s,X(s)) - (L - \frac{\mathring{m}}{2})\Big| = \Big|\int_{-\infty}^{X(s)} \int f(s,y,w)dw\,dy - L\Big|$$
$$= \Big|\int_{-\infty}^{X(s)} \int_{V(s)}^{\infty} f(s,y,w)dw\,dy + \ell(s) - L\Big| \le d(s) + L - \ell(s) \le Cs^{-2}.$$

Let $\mathcal{E} = L - \frac{\mathring{m}}{2}$ and note that $|\mathcal{E}| \le \frac{1}{2}\mathring{m}$ and $|E(s,X(s))| \le \frac{1}{2}\mathring{m}$, hence

$$|E(s,X(s)) - \mathcal{E}| \le \min\{\mathring{m}, Cs^{-2}\}. \tag{3.6}$$

Hence,

$$|V(s) - V(0) - \mathcal{E}s| = \Big|\int_0^s (E(\tau, X(\tau)) - \mathcal{E})d\tau\Big| \le C$$

and

$$\Big|X(s) - X(0) - V(0)s - \frac{1}{2}\mathcal{E}s^2\Big| = \Big|\int_0^s (V(\tau) - V(0) - \mathcal{E}\tau)d\tau\Big| \le Cs. \tag{3.7}$$

Suppose $s \ge 1$, then (3.6) and (3.7) yield

$$\big|2s^{-2}X(s) - E(s,X(s))\big| = \Big|2s^{-2}(X(s) - X(0) - V(0)s - \frac{1}{2}\mathcal{E}s^2)$$
$$+2s^{-2}X(0) + 2s^{-1}V(0) + (\mathcal{E} - E(s,X(s)))\big|$$
$$\le 2s^{-2}(Cs) + 2s^{-2}|X(0)| + 2s^{-1}|V(0)| + Cs^{-2} \le Cs^{-1}. \tag{3.8}$$

If $|X(s)| \le \frac{1}{4}\mathring{m}s^2$ then (3.8) becomes

$$|E_\infty(s,X(s)) - E(s,X(s))| = \big|2s^{-2}X(s) - E(s,X(s))\big| \le Cs^{-1}.$$

If $X(s) > \frac{1}{4}\mathring{m}s^2$, then (3.8) yields

$$\begin{aligned}|E_\infty(s, X(s)) - E(s, X(s))| &= \frac{1}{2}\mathring{m} - E(s, X(s)) \\ &< 2s^{-2}X(s) - E(s, X(s)) \leq Cs^{-1}.\end{aligned}$$

If $X(s) < -\frac{1}{4}\mathring{m}s^2$ then (3.8) yields

$$\begin{aligned}|E_\infty(s, X(s)) - E(s, X(s))| &= \left| -\frac{1}{2}\mathring{m} - E(s, X(s))\right| = \frac{1}{2}\mathring{m} + E(s, X(s)) \\ &< -2s^{-2}X(s) + E(s, X(s)) \leq Cs^{-1}.\end{aligned}$$

It follows that

$$|E_\infty(t, x) - E(t, x)| \leq Ct^{-1} \tag{3.9}$$

for $t \geq 1$ and for all x for which (x, v) is in the support of $f(t)$ for some v.

Let $\mathcal{S}(t)$ be the support of $f(t)$. Suppose $t \geq 1$ and $x \in \mathbb{R}$ are such that $(x, v) \notin \mathcal{S}(t)$ for all v. Assume that $\mathcal{S}(t) \cap ([x, \infty) \times \mathbb{R})) \neq \emptyset$ and define

$$\begin{aligned}x_2 &= \min\{\tilde{x} > x : \text{ there exists } \tilde{v} \text{ with } (\tilde{x}, \tilde{v}) \in \mathcal{S}(t)\}, \\ v_2 &= \min\{\tilde{v} : (x_2, \tilde{v}) \in \mathcal{S}(t)\},\end{aligned}$$

and $(y_2, w_2) = (X(0, t, x_2, v_2), V(0, t, x_2, v_2))$. If $\mathcal{S}(t) \cap ((-\infty, x] \times \mathbb{R}) = \emptyset$, then by Lemma 2.4 it follows that

$$\begin{aligned}E(s, X(s, t, x_2, v_2)) &= -\frac{1}{2}\mathring{m}, \\ X(s, t, x_2, v_2) &= y_2 + sw_2 - \frac{1}{4}\mathring{m}s^2, \\ x_2 = y_2 + tw_2 - \frac{1}{4}\mathring{m}t^2 &\leq Ct - \frac{1}{2}\mathring{m}t^2\end{aligned}$$

and hence

$$\begin{aligned}-\frac{1}{2}\mathring{m} \leq E_\infty(t, x) &\leq E_\infty(t, x_2) \leq E_\infty(t, Ct - \frac{1}{2}\mathring{m}t^2) \\ &\leq 2t^{-2}(Ct - \frac{1}{4}\mathring{m}t^2) = 2Ct^{-1} - \frac{1}{2}\mathring{m}\end{aligned}$$

and

$$|E_\infty(t, x_2) - E_\infty(t, x)| \leq Ct^{-1}. \tag{3.10}$$

Consider $\mathcal{S}(t) \cap ((-\infty, x] \times \mathbb{R})) \neq \emptyset$ now and define

$$\begin{aligned}x_1 &= \max\{\tilde{x} < x : \text{ there exists } \tilde{v} \text{ with } (\tilde{x}, \tilde{v}) \in \mathcal{S}(t)\}, \\ v_1 &= \max\{\tilde{v} : (x_1, \tilde{v}) \in \mathcal{S}(t)\},\end{aligned}$$

and $(y_1, w_1) = (X(0,t,x_1,v_1), V(0,t,x_1,v_1))$. Then by Lemma 2.4 if follows that $x_2 - x_1 = (y_2 + w_2 t) - (y_1 + w_1 t) \le Ct$ so

$$|E_\infty(t,x_2) - E_\infty(t,x)| \le Ct^{-2}(x_2 - x) \le Ct^{-2}(x_2 - x_1) \le Ct^{-1}. \quad (3.11)$$

Now by (3.9), (3.10), and (3.11) it follows (for $\mathcal{S}(t) \cap ((-\infty, x] \times \mathbb{R})$ empty or nonempty) that

$$\begin{aligned} |E_\infty(t,x) - E(t,x)| = \; & |E_\infty(t,x) - E(t,x_2)| \\ \le \; & |E_\infty(t,x) - E_\infty(t,x_2)| + |E_\infty(t,x_2) - E(t,x_2)| \\ \le \; & Ct^{-1} + Ct^{-1} = Ct^{-1}. \end{aligned} \quad (3.12)$$

The assumption $\mathcal{S}(t) \cap ([x,\infty) \times \mathbb{R})) \neq \emptyset$ was used to establish (3.12). A similar argument may be used if $\mathcal{S}(t) \cap ((-\infty, x] \times \mathbb{R}) \neq \emptyset$ is assumed. If $\mathcal{S}(t) \cap ((-\infty, x] \times \mathbb{R}) = \mathcal{S}(t) \cap ([x,\infty) \times \mathbb{R}) = \emptyset$, then $f = 0$ at all points and there is nothing to prove, hence the proof is complete.

References

[1] J. Batt and G. Rein, *Global classical solutions of the periodic Vlasov-Poisson system in three dimensions,* C. R. Acad. Sci. Paris, 313 (1991), 411–416.

[2] J. Batt, M. Kunze, and G. Rein, *On the asymptotic behavior of a one-dimensional, monocharged plasma and a rescaling method,* Advances in Differential Equations, 3 (1998), 271–292.

[3] J. R. Burgan, M. R. Feix, E. Fijalkow, and A. Munier, *Self-similar and asymptotic solutions for a one-dimensional Vlasov beam,* J. Plasma Physics, 29 (1983), 139–142.

[4] R. Glassey and W. Strauss, *Remarks on collisionless plasmas,* Contemporary Mathematics, 28 (1984), 269–279.

[5] E., Horst, *Symmetric plasmas and their decay,* Commun. Math. Phys., 126 (1990), 613–633.

[6] E. Horst, *On the asymptotic growth of the solutions of the Vlasov-Poisson system,* Math. Meth. in the Appl. Sci., 16 (1993), 79–85.

[7] R. Illner and G. Rein, *Time decay of the solutions of the Vlasov-Poisson system in the plasma physical case,* Math. Meth. in the Appl. Sci., 19 (1996), 1409–1413.

[8] P.-L. Lions and B. Perthame, *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system,* Invent. Math., 105 (1991), 415–430.

[9] B. Perthame, *Time decay, propagation of low moments and dispersive effects for kinetic equations,* Commun. Part. Diff. Eqns., 21 (1996), 659–686.

[10] K. Pfaffelmoser, *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data,* J. Diff. Eqns., 95 (1992), 281–303.

[11] G. Rein, *Growth estimates for the solutions of the Vlasov-Poisson system in the plasma physics case,* Math. Nachrichten, 191 (1998), 269–278.

[12] G. Rein and A. D. Rendall, *Global existence of classical solutions to the Vlasov-Poisson system in a three-dimensional, cosmological setting,* Arch. Rational Mech. Anal, 126 (1994), 183–201

[13] J. Schaeffer, *Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions,* Commun. Part. Diff Eqns., 16 (1991), 1313–1335.