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ON A FREE BOUNDARY PROBLEM FOR THE NAVIER-STOKES EQUATIONS

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Abstract. We consider a free boundary problem for the Navier-Stokes equation in $\mathbb{R}^n$ ($n \geqq 2$). We prove a local in time unique existence theorem for any initial data and a global in time unique existence theorem for some small initial data. The problem we consider in this paper was already treated by V. Solonnikov [15]. But, recently in [10] we proved an L_p-L_q maximal regularity theorem for the Stokes equation with Neumann boundary condition which is a linearized version of the free boundary problem for the Navier-Stokes equation treated in this paper. Our proof is based on this theorem. Therefore our solution is obtained in the space $W^{2,1}_{q,p}$ ($2 < p < \infty$ and $n < q < \infty$) while a solution in [15] is in $W^{2,1}_q = W^{2,1}_{q,q}$ ($n < q < \infty$). Moreover, our proof of the global in time existence theorem is much simpler than [15], because in [10] we established a maximal regularity theorem on the whole time interval $(0, \infty)$ with exponential stability. The results obtained in this paper were already announced in Shibata-Shimizu [11].

1. Introduction

This paper is concerned with a time-dependent problem with free surface for the Navier-Stokes equations which describes the motion of an isolated finite volume of viscous incompressible fluid without taking surface tension into account. The region $\Omega_t \subset \mathbb{R}^n$, $n \geqq 2$, occupied by the fluid is given only

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on the initial time $t = 0$, while for $t > 0$ it is to be determined. The velocity vector field $v(x,t) = (v_1, \dots, v_n)^*$ [3] and the pressure $\theta(x,t)$ for $x \in \Omega_t$ satisfy the Navier-Stokes equations:

$$\begin{aligned} &v_t + (v \cdot \nabla)v - \operatorname{Div} S(v,\theta) = f(x,t) && \text{in } \Omega_t,\ t>0 \\ &\operatorname{div} v = 0 && \text{in } \Omega_t,\ t>0 \\ &S(v,\theta)\nu_t + \theta_0(x,t)\nu_t = 0 && \text{in } \Gamma_t,\ t>0 \\ &v|_{t=0} = v_0 && \text{on } \Omega. \end{aligned} \tag{1.1}$$

Here, Γ_t denotes the boundary of Ω_t, $\nu_t(x)$ is the unit outward normal to Γ_t at the point $x \in \Gamma_t$, $\nabla = (\partial_1, \dots, \partial_n)$ with $\partial_i = \partial/\partial x_i$, and $S(v,\theta)$ is the stress tensor defined by the formula: $S(v,\theta) = D(v) - \theta I$, where $D(v)$ is the deformation tensor of the velocities with elements $D_{ij}(v) = \partial_i v_j + \partial_j v_i$ and I is the $n \times n$ identity matrix. For the $n \times n$ matrix of functions $S = (S_{ij})$ we set

$$\operatorname{Div} S = (\sum_{j=1}^n \partial_j S_{1j}, \dots, \sum_{j=1}^n \partial_j S_{nj})^*.$$

The external force $f(x,t)$ and the pressure $\theta_0(x,t)$ are functions defined on the whole space. In what follows, we may always assume that $\theta_0(x,t) = 0$, since we can arrive at this case by replacing $\theta(x,t)$ by $\theta + \theta_0$.

Aside from the dynamical boundary condition, a further kinematic condition for Γ_t is satisfied which gives Γ_t as a set of points $x = x(\xi,t)$, $\xi \in \Gamma_0$, where $x(\xi,t)$ is the solution of the Cauchy problem:

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi. \tag{1.2}$$

This expresses the fact that the free surface Γ_t consists for all $t > 0$ of the same fluid particles, which do not leave it and are not incident on it from inside Ω_t.

From now on, we write $\Omega = \Omega_0$ and $\Gamma = \Gamma_0$ and we assume that Γ is a $C^{2,1}$ compact hypersurface. The problem (1.1) can therefore be written as an initial boundary-value problem in the given region Ω if we go over the Euler coordinates $x \in \Omega_t$ to the Lagrange coordinates $\xi \in \Omega$ connected with x by (1.2). If a velocity vector field $u(\xi,t) = (u_1, \dots, u_n)^*$ is known as a function of the Lagrange coordinates ξ, then this connection can be written in the form:

$$x = \xi + \int_0^t u(\xi,\tau)\, d\tau \equiv X_u(\xi,t). \tag{1.3}$$

[3] M^* denotes the transpose of M.

Passing to the Lagrange coordinates in (1.1) and setting $\theta(X_u(\xi,t),t) = \pi(\xi,t)$, we obtain [4]

$$
\begin{aligned}
&u_t - \mathrm{Div}\,[S(u,\pi) + U(u,\pi)] = f(X_u(\xi,t),t) && \text{in } \Omega\times(0,T)\\
&\mathrm{div}\,u + E(u) = \mathrm{div}\,[u + \tilde{E}(u)] = 0 && \text{in } \Omega\times(0,T)\\
&[S(u,\pi) + U(u,\pi)]\nu = 0 && \text{on } \Gamma\times(0,T)\\
&u|_{t=0} = u_0 && \text{in } \Omega,
\end{aligned}
\tag{1.4}
$$

where $u_0(\xi) = v_0(x)$. Here and hereafter, ν denotes the unit outward normal to Γ, and $U(u,\pi)$, $E(u)$ and $\tilde{E}(u)$ are nonlinear terms of the following forms:

$$
U(u,\pi) = V_1(\int_0^t \nabla u\,d\tau)\nabla u + V_2(\int_0^t \nabla u\,d\tau)\pi,
$$

$$
E(u) = V_3(\int_0^t \nabla u\,d\tau)\nabla u, \quad \tilde{E}(u) = V_4(\int_0^t \nabla u\,d\tau)u,
$$

with some polynomials $V_j(\cdot)$ of $\int_0^t \nabla u\,d\tau$, $j = 1, 2, 3, 4$, such as $V_j(0) = 0$. Throughout the paper Ω is assumed to be a bounded domain in $\mathbb{R}^n$.

The main purpose of this paper is to show a local in time unique existence theorem for any initial data and right member f and a global in time unique existence theorem for some small initial data which are orthogonal to the rigid space in the case where $f = 0$.

In order to state our main results precisely, first of all we introduce the function spaces and some symbols which will be used throughout the paper. For any domain D in $\mathbb{R}^n$, integer m and $1 \leqq q \leqq \infty$, $L_q(D)$ and $W_q^m(D)$ denote the usual Lebesgue space and Sobolev space of functions defined on D with norms: $\|\cdot\|_{L_q(D)}$ and $\|\cdot\|_{W_q^m(D)}$, respectively. And also, for any Banach space X, interval I, integer ℓ and $1 \leqq p \leqq \infty$, $L_p(I,X)$ and $W_p^\ell(I,X)$ denote the usual Lebesgue space and Sobolev space of X-valued functions defined on I with norms $\|\cdot\|_{L_p(I,X)}$ and $\|\cdot\|_{W_p^\ell(I,X)}$, respectively. Set

$$
\begin{aligned}
&W_{q,p}^{m,\ell}(D\times I) = L_p(I, W_q^m(D)) \cap W_p^\ell(I, L_q(D)),\\
&\|u\|_{W_{q,p}^{m,\ell}(D\times I)} = \|u\|_{L_p(I,W_q^m(D))} + \|u\|_{W_p^\ell(I,L_q(D))},\\
&W_{p,0}^\ell((0,T),X) = \{u \in W_p^\ell((-\infty,T),X) : u = 0 \text{ for } t < 0\},\\
&W_q^0(D) = L_q(D), \;\; W_p^0(I,X) = L_p(I,X),\\
&W_{p,0}^0((0,T),X) = L_{p,0}((0,T),X).
\end{aligned}
$$

[4]The derivation of (1.4) is discussed in the appendix.

Given $\alpha \geqq 0$, we set

$$
\begin{aligned}
&< D_t >^\alpha u(t) = \mathcal{F}^{-1}[(1+s^2)^{\alpha/2}\mathcal{F}u(s)](t),\\
&H_p^\alpha(\mathbb{R}, X) = \{u \in L_p(\mathbb{R}, X) : < D_t >^\alpha u \in L_p(\mathbb{R}, X)\},\\
&\|u\|_{H_p^\alpha(\mathbb{R},X)} = \| < D_t >^\alpha u\|_{L_p(\mathbb{R},X)} + \|u\|_{L_p(\mathbb{R},X)}.
\end{aligned}
$$

Here and hereafter, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse formula, respectively. Set

$$
\begin{aligned}
&H^{1,1/2}_{q,p}(D \times \mathbb{R}) = H^{1/2}_p(\mathbb{R}, L_q(D)) \cap L_p(\mathbb{R}, W^1_q(D)),\\
&H^{1,1/2}_{q,p,0}(D \times \mathbb{R}_+) = \{u \in H^{1,1/2}_{q,p}(D \times \mathbb{R}) : u = 0 \text{ for } t < 0\},\\
&\|u\|_{H^{1,1/2}_{q,p}(D\times\mathbb{R})} = \|u\|_{H^{1/2}_p(\mathbb{R},L_q(D))} + \|u\|_{L_p(\mathbb{R},W^1_q(D))}.
\end{aligned}
$$

Finally, given $0 < T < \infty$ we set

$$
\begin{aligned}
H^{1,1/2}_{q,p,0}(D \times (0,T)) &= \{u : \exists\, v \in H^{1,1/2}_{q,p,0}(D \times \mathbb{R}_+),\ \ u = v \text{ on } D \times (0,T)\},\\
\|u\|_{H^{1,1/2}_{q,p,0}(D\times(0,T))} &= \inf\{\|v\|_{H^{1,1/2}_{q,p}(D\times\mathbb{R})} :\\
&\qquad \forall\, v \in H^{1,1/2}_{q,p,0}(D \times \mathbb{R}_+),\ \ v = u \text{ on } D \times (0,T)\}.
\end{aligned}
$$

Given a vector or matrix M, M^* denotes the transpose of M. $M = (M_{ij})$ denotes the $n \times n$ matrix. Given a Banach space X with norm $\|\cdot\|_X$, we set

$$
\begin{aligned}
&X^n = \{v = (v_1, \ldots, v_n)^* : v_j \in X\}, \quad \|v\|_X = \sum_{j=1}^n \|v_j\|_X,\\
&X^{n\times n} = \{M = (M_{ij}) : M_{ij} \in X\}, \qquad \|M\|_X = \sum_{i,j=1}^n \|M_{ij}\|_X.
\end{aligned}
$$

The dot $\cdot$ denotes the inner-product of $\mathbb{R}^n$. $F = (F_{ij})$ means an $n \times n$ matrix whose i-th column and j-th row component is F_{ij}. For the differentiation of an $n \times n$ matrix of functions $F = (F_{ij})$, an n-vector of functions $u = (u_1, \ldots, u_n)^*$ and a scalar function θ, we use the following symbols: $\theta_t = \partial_t\theta = \partial\theta/\partial t$, $\partial_j\theta = \partial\theta/\partial x_j$,

$$
\begin{gathered}
\nabla\theta = (\partial_1\theta, \ldots, \partial_n\theta)^*,\ \ \nabla^k\theta = (\partial_x^\alpha\theta : |\alpha| = k),\\
u_t = \partial_t u = (\partial_t u_1, \ldots, \partial_t u_n),\ \ \nabla u = (\partial_i u_j),\\
\nabla^k u = (\partial_x^\alpha u_i, |\alpha| = k, i = 1, \ldots, n),\ \ \mathrm{div}\, u = \sum_{j=1}^n \partial_j u_j,
\end{gathered}
$$

$$\operatorname{Div} F = (\sum_{j=1}^n \partial_j F_{1j}, \ldots, \sum_{j=1}^n \partial_j F_{nj})^*.$$

The inner products $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ are defined by

$$(u, v)_\Omega = \int_\Omega u(x)\cdot v(x)\,dx, \quad (u, v)_\Gamma = \int_\Gamma u(x)\cdot v(x)\,d\sigma, \tag{1.5}$$

where $d\sigma$ denotes the surface element of Γ. By C we denote a generic constant and $C_{a,b,\cdots}$ denotes the constant depending on the quantities $a, b, \ldots$. The constants C and $C_{a,b,\ldots}$ may change from line to line.

Second of all, in order to introduce our class of initial data for (1.4), we discuss an analytic semigroup approach to the initial boundary-value problem:

$$\begin{aligned} u_t - \operatorname{Div} S(u,\pi) = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega\times(0,\infty)\\ S(u,\pi)\nu|_\Gamma = 0, \quad u|_{t=0} = u_0. \end{aligned} \tag{1.6}$$

Since the time derivative of π is missing in (1.6), to obtain the evolution equation for u we have to eliminate π from (1.6). To do this, for a while instead of (1.6) we shall consider the resolvent problem:

$$\lambda u - \operatorname{Div} S(u,\pi) = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega\times(0,\infty), \quad S(u,\pi)\nu|_\Gamma = 0, \tag{1.7}$$

and we shall discuss how to eliminate π from (1.7). We introduce the second Helmholtz decomposition corresponding to (1.6). Set

$$\begin{aligned} J_q(\Omega) &= \{u = (u_1,\ldots,u_n)^* \in L_q(\Omega)^n : \operatorname{div} u = 0 \quad \text{in } \Omega\},\\ G_q(\Omega) &= \{\nabla\pi : \pi \in W^1_q(\Omega),\ \ \pi|_\Gamma = 0\}. \end{aligned}$$

Then, by Grubb and Solonnikov [6] (cf. also Shibata and Shimizu [10]) we know that $L_q(\Omega)^n = J_q(\Omega)\oplus G_q(\Omega)$ for $1 < q < \infty$, where $\oplus$ denotes the direct sum. Let P_q be the solenoidal projection: $L_q(\Omega)^n \to J_q(\Omega)$ along $G_q(\Omega)$. Then, substituting the 2nd Helmholtz decomposition of f: $f = P_q f + \nabla\theta$, where $\theta\in W^1_q(\Omega)$ and $\theta|_\Gamma = 0$, into (1.7), we have

$$\lambda u - \operatorname{Div} S(u, \pi-\theta) = P_q f,\ \ \operatorname{div} u = 0 \quad \text{in } \Omega, \quad S(u,\pi-\theta)\nu|_\Gamma = 0. \tag{1.8}$$

Denoting $\pi-\theta$ by π in (1.8) again, from now on we consider (1.7) under the condition that $\operatorname{div} f = 0$. Then, applying the divergence to (1.7) and multiplying the boundary condition by ν, we have

$$\Delta\theta = 0 \quad \text{in } \Omega, \quad \theta|_\Gamma = [D(u)\nu]\cdot\nu - \operatorname{div} u|_\Gamma, \tag{1.9}$$

where we have used the facts that $\operatorname{div} u = 0$ in Ω and $\nu \cdot \nu = 1$ on Γ. We know that given $u \in W_q^2(\Omega)^n$ there exists a unique $\theta \in W_q^1(\Omega)$ which solves (1.9) and enjoys the estimate:

$$\|\theta\|_{W_q^1(\Omega)} \leqq C\|u\|_{W_q^2(\Omega)}.$$

From this point of view, let us define the map $K : W_q^2(\Omega)^n \to W_q^1(\Omega)$ by $\theta = K(u)$ for $u \in W_q^2(\Omega)^n$. By using this symbol, the equation (1.7) is rewritten in the form:

$$\lambda u - \operatorname{Div} S(u, K(u)) = f \quad \text{in } \Omega, \quad S(u, K(u))\nu|_\Gamma = 0 \tag{1.10}$$

for $f \in J_q(\Omega)$. We set

$$A_q u = -\operatorname{Div} S(u, K(u)) \quad \text{for } u \in \mathcal{D}(A_q),$$
$$\mathcal{D}(A_q) = \{u \in J_q(\Omega) \cap W_q^2(\Omega)^n : S(u, K(u))\nu|_\Gamma = 0\}.$$

From Grubb and Solonnikov [6] and Shibata and Shimizu [9], we know the following theorem.

Theorem 1.1. *Let $1 < q < \infty$. Then, A_q generates an analytic semigroup $\{e^{-A_q t}\}_{t \geqq 0}$ on $J_q(\Omega)$.*

Remark 1.2. The function $e^{-A_q t}u_0$ is an initial flow for (1.4).

Now, we shall state our main results. For the initial data, we introduce the space: $\mathcal{D}_{q,p}(\Omega) = [J_q(\Omega), \mathcal{D}(A_q)]_{1-1/p,p}$. Here and hereafter, $[\cdot,\cdot]_{\theta,p}$ denotes the real interpolation functor. The first theorem is a local in time unique existence theorem of (1.4).

Theorem 1.3. *Let $2 < p < \infty$ and $n < q < \infty$. Then, for any $R > 0$ and $R' > 0$ there exists a time $T > 0$ depending on R and R' such that the equation (1.4) admits a unique solution*

$$(u, \pi) \in W_{q,p}^{2,1}(\Omega \times (0,T))^n \times L_p((0,T), W_q^1(\Omega)),$$

which satisfies the estimate:

$$\|u\|_{W_{q,p}^{2,1}(\Omega\times(0,T))} + \|\pi\|_{L_p((0,T),W_q^1(\Omega))} \leqq CR$$

for some constant C depending essentially only on p and q provided that $u_0 \in \mathcal{D}_{q,p}(\Omega)$, $f \in L_p(\mathbb{R}_+, L_q(\mathbb{R}^n))$, $\nabla f \in L_\infty(\mathbb{R}^n \times \mathbb{R}_+)$ and

$$\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} + \|f\|_{L_p(\mathbb{R}_+,L_q(\mathbb{R}^n))} \leqq R, \quad \|\nabla f\|_{L_\infty(\mathbb{R}^n\times\mathbb{R}_+)} \leqq R'.$$

In order to state a global in time unique existence theorem for (1.4), we introduce the rigid space $\mathcal{R}_d$ which is defined by the relation:

$$\mathcal{R}_d = \{Ax + b : A : \ n \times n \text{ anti-symmetric matrix}, \, b \in \mathbb{R}^n\}. \tag{1.11}$$

In what follows, we denote the basis of $\mathcal{R}_d$ by $\{p_\ell\}_{\ell=1}^M$, which are normalized by $(p_\ell, p_m)_\Omega = \delta_{\ell m}$ $(\ell, m = 1, \ldots, M)$, where $\delta_{\ell m}$ are Kronecker's delta symbols.

Theorem 1.4. *Let $2 < p < \infty$ and $n < q < \infty$. We consider the case where $T = \infty$ and $f = 0$ in (1.4). Then, there exist positive numbers ϵ and γ such that if $u_0 \in \mathcal{D}_{q,p}(\Omega)$, $\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} \leqq \epsilon$ and $(u_0, p_\ell)_\Omega = 0$ for $\ell = 1, \ldots, M$, then the equation (1.4) with $T = \infty$ and $f = 0$ admits a unique solution*

$$(u, \pi) \in W^{2,1}_{q,p}(\Omega \times (0, \infty))^n \times L_p((0, \infty), W^1_q(\Omega)),$$

which satisfies the estimate:

$$\|e^{\gamma t} u\|_{W^{2,1}_{q,p}(\Omega \times (0,\infty))} + \|e^{\gamma t} \pi\|_{L_p((0,\infty), W^1_q(\Omega))} \leqq C\epsilon$$

for some $\gamma > 0$ and the condition:

$$(u(\cdot, t), p_\ell)_\Omega = 0 \quad \textit{for } \ell = 1, \ldots, M \textit{ and } t \geqq 0.$$

Remark 1.5. Let us define the Besov space $B^{2(1-1/p)}_{q,p}(\Omega)$ by the real interpolation:

$$B^{2(1-1/p)}_{q,p}(\Omega) = [L_q(\Omega), W^2_q(\Omega)]_{1-1/p,p},$$

and set

$$JB^{2(1-1/p)}_{q,p}(\Omega) = \{u \in B^{2(1-1/p)}_{q,p}(\Omega)^n : \operatorname{div} u = 0 \text{ in } \Omega\}.$$

Then, we see that $\mathcal{D}_{q,p}(\Omega) \subset JB^{2(1-1/p)}_{q,p}(\Omega)$. Moreover, from Proposition 2.13 combined with Remarks 2.7 (c) in Steiger [16] (cf. also Triebel [17]) it follows that

$$\mathcal{D}_{q,p}(\Omega) = \begin{cases} \{v \in JB^{2(1-1/p)}_{q,p}(\Omega) : S(v, K(v))\nu|_\Gamma = 0\} & 2(1 - \frac{1}{p}) > 1 + \frac{1}{q} \\ JB^{2(1-1/p)}_{q,p}(\Omega) & 2(1 - \frac{1}{p}) < 1 + \frac{1}{q}. \end{cases}$$

To prove Theorems 1.3 and 1.4, we will use the usual fixed-point theorem based on the following two theorems obtained by Shibata and Shimizu [10] concerning the L_p-L_q maximal regularity of solutions to the following Stokes equation with Neumann boundary condition:

$$\begin{aligned} &u_t - \operatorname{Div} S(u, \pi) = f && \text{in } \Omega \times (0, T) \\ &\operatorname{div} u = g = \operatorname{div} \tilde{g} && \text{in } \Omega \times (0, T) \end{aligned}$$

$$S(u,\pi)\nu|_\Gamma = h, \quad u|_{t=0} = u_0. \tag{1.12}$$

Theorem 1.6. *Let $1 < p, q < \infty$ and $0 < T < \infty$. If u_0, f, g, $\tilde g$ and h satisfy the condition*

$$u_0 \in \mathcal{D}_{q,p}(\Omega), \quad f \in L_p((0,T), L_q(\Omega))^n, \quad g \in L_{p,0}((0,T), W^1_q(\Omega))$$
$$\tilde g \in W^1_{p,0}((0,T), L_q(\Omega))^n, \quad h \in H^{1,1/2}_{q,p,0}(\Omega \times (0,T))^n,$$

then the equation (1.12) *admits a unique solution*

$$(u,\pi) \in W^{2,1}_{q,p}(\Omega \times (0,T))^n \times L_p((0,T), W^1_q(\Omega))$$

which satisfies the estimate:

$$\begin{aligned}\|u\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\pi\|_{L_p((0,T),W^1_q(\Omega))} &\leqq C(1+T)\{\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} + \|f\|_{L_p((0,T),L_q(\Omega))} \\ &+ \|g\|_{L_p((0,T),W^1_q(\Omega))} + \|\tilde g\|_{W^1_p((0,T),L_q(\Omega))} + \|h\|_{H^{1,1/2}_{q,p,0}(\Omega\times(0,T))}\},\end{aligned}$$

where the constant C is independent of T, u, π, u_0, f, g, $\tilde g$ and h.

Theorem 1.7. *Let $1 < p, q < \infty$. Then, there exists a $\gamma_0 > 0$ such that if u_0, f, g, $\tilde g$ and h satisfy the following conditions:*

$$u_0 \in \mathcal{D}_{q,p}(\Omega),\ e^{\gamma t} f \in L_p((0,\infty), L_q(\Omega))^n, \quad e^{\gamma t} g \in L_{p,0}((0,\infty), W^1_q(\Omega))$$
$$e^{\gamma t}\tilde g \in W^1_{p,0}((0,\infty), L_q(\Omega))^n, \quad e^{\gamma t} h \in H^{1,1/2}_{q,p,0}(\Omega \times (0,\infty))^n$$

for some $\gamma \in [0,\gamma_0]$, and

$$(u_0, p_\ell)_\Omega = 0, \quad (f(\cdot,t), p_\ell)_\Omega + (h(\cdot,t), g_\ell)_\Gamma = 0$$

for $t \geqq 0$ and $\ell = 1, \ldots, M$, then the equation (1.12) *with $T = \infty$ admits a unique solution*

$$(u,\pi) \in W^{2,1}_{q,p}(\Omega \times (0,\infty))^n \times L_p((0,\infty), W^1_q(\Omega))$$

which satisfies the estimates:

$$\begin{aligned}&\|e^{\gamma t} u\|_{W^{2,1}_{q,p}(\Omega\times(0,\infty))} + \|e^{\gamma t}\pi\|_{L_p((0,\infty),W^1_q(\Omega))} \\ &\quad\leqq C\{\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} + \|e^{\gamma t} f\|_{L_p((0,\infty),L_q(\Omega))} \\ &\quad+ \|e^{\gamma t} g\|_{L_p(\mathbb{R}_+,W^1_q(\Omega))} + \|e^{\gamma t}\tilde g\|_{W^1_p(\mathbb{R}_+,L_q(\Omega))} + \|e^{\gamma t} h\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}\},\end{aligned}$$

and the condition: $(u(\cdot,t), p_\ell)_\Omega = 0$ for $t \geqq 0$ and $\ell = 1, \ldots, M$.

Concerning the free boundary problem for the Navier-Stokes equations, the pioneering work was done by Solonnikov [14] in 1977. In fact, he proved the local in time unique solvability of (1.4) in the framework of Hölder spaces $C^{2+\alpha,1+\frac{\alpha}{2}}$ with $\alpha \in (\frac{1}{2}, 1)$ by using the results of the linearized problem [13].Later on, Solonnikov [15] and Mucha and Zajączkowski [7, 8] proved the local in time unique solvability of (1.4) for arbitrary initial data and external force f, and Solonnikov [15] also proved the global in time unique solvability of (1.4) for $f = 0$ and sufficiently small initial data in the class of isotropic Sobolev spaces $W_q^{2,1}$ with $n < q < \infty$ when $n = 2, 3$.

In this paper, we extend the Solonnikov results in [15] in the class of anisotropic Sobolev spaces $W_{q,p}^{2,1}$ with $n < q < \infty$ and $2 < p < \infty$. Formally, the novelty of our results consists of two things. First, we extend the results of Solonnikov by unbinding the exponents of integrability with respect to the space and time variables. Second, we allow the exponent of integrability with respect to time p to go down to 2 providing a weaker setting than the one that was allowed by Solonnikov whose case was $p = q > n$.

But, the principle significance of this paper is the methodology of investigation of the linearized problem which in some sense is alternative to the approach of Solonnikov and his followers. In fact, Solonnikov proved the local in time L_p maximal regularity theorem of the linearized problem (a theorem corresponding to Theorem 1.6 with $p = q$) by utilization of the techniques of the hydrodynamical potentials with the careful estimates of the kernels of the corresponding singular integrals, and then by the contraction mapping principle he proved a local in time existence theorem for the nonlinear problem (1.4). But, Solonnikov did not prove the global in time L_p maximal regularity theorem of the linearized problem (a theorem corresponding to Theorem 1.7 with $p = q$), so that in order to prove the global in time unique existence of solutions to the nonlinear problem (1.4), he extended the local in time solutions to the infinite time interval by showing their *a priori* estimates whose constants are independent of time. To prove such estimates, he started with the L_2 decay estimate by using the usual energy method, and then he obtained the required *a priori* estimates in the L_p framework by using a rather complicated boot-strap argument.

On the other hand, we develop the semigroup approach which consists of the following. As stated in Theorem 1.1, the linear operator A_q obtained under the linearization of the free boundary problem generates an analytic semigroup $\{e^{-A_q t}\}_{t \geqq 0}$ on the functional space $J_q(\Omega)$ of q-integrable divergence free functions. The core of our approach is to show the global in time

L_p-L_q maximal regularity of the linearized problem on the whole time interval $(0,\infty)$ with exponential decay, which is stated in Theorem 1.7. To prove Theorem 1.7, first of all we show the exponential stability of the semigroup $\{e^{-A_q t}\}_{t\geqq 0}$ on the quotient space $J_q(\Omega)\setminus\mathcal{R}_d$, where $\mathcal{R}_d$ is the rigid space defined by (1.11), which is simply implied by the fact that 0 is contained in the resolvent set of A_q on $J_q(\Omega)\setminus\mathcal{R}_d$. As a result, the $W^1_q(\Omega)$ norm of the solution

$$u(t)=\int_0^t e^{-A_q(t-s)}f(s)\,ds$$

to the non-homogeneous problem

$$u_t+A_q u=f \quad \text{on } J_q(\Omega)\setminus\mathcal{R}_d$$

with zero initial condition is estimated by

$$C\int_0^t e^{-\gamma(t-s)}(t-s)^{-1/2}\|f(s)\|_{L_q(\Omega)}\,ds$$

with some constants C and $\gamma>0$. Therefore, we have the estimate of $\|e^{\gamma t}u\|_{L_p((0,\infty),W^1_q(\Omega))}$. To estimate u_t and the spatial derivatives of u of second order, after reducing the problem to the model problem in the whole space and half-space by the localization procedure, we applied the operator-valued Fourier multiplier theorem by showing the $\mathcal{R}$-boundedness of the solution operator to the model problem, which was recently developed by Weis [19], Denk, Hieber and Prüss [4] and Amann [1]. The perturbation terms in the localization procedure are estimated by $\|e^{\gamma t}u\|_{L_p((0,\infty),W^1_q(\Omega))}$. The combination of the exponential stability of the semigroup and $\mathcal{R}$-boundedness of the solution operator to the model problem as stated above can be applied to a wide class of initial boundary-value problems for parabolic equations to obtain the Sobolev maximal regularity. We believe that methodologically, this approach is simpler and more demonstrative than the estimates of kernels of singular integrals in the anisotropic Sobolev spaces.

Thanks to the global in time L_p-L_q maximal regularity of the linearized problem on the whole time interval $(0,\infty)$ with exponential decay (Theorem 1.7), our proof of the global in time existence theorem for the nonlinear problem (1.4) (Theorem 1.4) is much simpler than Solonnikov's proof [15]. In fact, we can show Theorem 1.4 simply by the contraction mapping principle.

2. A proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. We are looking for a solution to (1.4) of the form: $u=Z+v$ and $\pi=\Pi+\theta$, where (Z,Π) solves the linear equation:

$$\begin{aligned} Z_t - \operatorname{Div} S(Z, \Pi) = 0, \quad \operatorname{div} Z = 0 \quad \text{in } \Omega \times (0,2) \\ S(Z,\Pi)\nu|_{\Gamma} = 0, \quad Z|_{t=0} = u_0. \end{aligned} \tag{2.1}$$

By Theorem 1.6 with $f = 0$, $g = 0$, $\tilde g = 0$ and $h = 0$ we see that $(Z, \Pi) \in W^{2,1}_{q,p}(\Omega \times (0,2))^n \times L_p((0,2), W^1_q(\Omega))$ and

$$\|Z\|_{W^{2,1}_{q,p}(\Omega\times(0,2))} + \|\Pi\|_{L_p((0,2),W^1_q(\Omega))} \leqq C\|u_0\|_{\mathcal{D}_{q,p}(\Omega)}. \tag{2.2}$$

By the boundary condition of (2.1), we have

$$\Pi = [D(Z)\nu]\cdot\nu \quad \text{on } \Gamma \times (0,T). \tag{2.3}$$

If (u,π) solves (1.4), then (v,θ) enjoys the equation:

$$\begin{aligned} & v_t - \operatorname{Div}[S(v,\theta) + U(v+Z, \theta+\Pi)] \\ & \qquad = f(\xi + \int_0^t (v+Z)\,d\tau, t) && \text{in } \Omega\times(0,T) \\ & \operatorname{div} v + E(v+Z) = \operatorname{div}(v + \tilde E(v+Z)) && \text{in } \Omega\times(0,T) \\ & [S(v,\theta) + U(v+Z,\theta+\Pi)]\nu = 0 && \text{on } \Gamma\times(0,T) \\ & v(\xi,0) = 0 && \text{in } \Omega. \end{aligned} \tag{2.4}$$

In what follows, we shall solve (2.4) by the contraction mapping principle. To do this, we define the underlying space $\mathcal{I}_{R,T}$ by

$$\begin{aligned} \mathcal{I}_{R,T} = \{ & (v,\theta,\tilde\theta) \in W^{2,1}_{q,p}(\Omega\times(0,T))^n \times L_p((0,T), W^1_q(\Omega)) \times H^{1,\frac12}_{q,p,0}(\Omega\times\mathbb{R}_+) : \\ & \theta = \tilde\theta \ \text{ on } \Gamma\times(0,T), \quad v(\xi,0) = 0 \ \text{ on } \Omega, && \text{(As.1)} \\ & \|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\theta\|_{L_p((0,T),W^1_q(\Omega))} + \|\tilde\theta\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq R\}. && \text{(As.2)} \end{aligned}$$

Here, T is a positive number determined later and R is a large number such that $\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} \leqq R$. In particular, by (2.2) we see that

$$\|Z\|_{W^{2,1}_{q,p}(\Omega\times(0,2))} + \|\Pi\|_{L_p((0,2),W^1_q(\Omega))} \leqq CR. \tag{2.5}$$

The assumption $\theta = \tilde\theta$ on Γ in (As.1) holds in the sense of the trace operator for functions belonging to $W^1_q(\Omega)$ for almost all $t \in (0,T)$ [5]. Since we consider

[5]A useful characterization of the trace operator of functions belonging to Sobolev spaces of mixed order with respect to time and space derivatives is obtained by Denk, Hieber and Prüss [5] and Weidemaier [18].

the local in time solvability of (1.4), we may assume that $0 < T \leqq 1$ in the course of our proof of Theorem 1.3.

Given $(w, \eta, \tilde\eta) \in \mathcal{I}_{R,T}$, let v and θ be solutions to the linear equation:

$$
\begin{aligned}
&v_t - \operatorname{Div} S(v,\theta) = f(\xi + \int_0^t (w(\xi,\tau) + Z(\xi,\tau))\,d\tau, t) \\
&\qquad\qquad + \operatorname{Div} U(w + Z, \eta + \Pi) && \text{in } \Omega \times (0,T) \\
&\operatorname{div} v = -E(w+Z) = -\operatorname{div} \tilde E(w+Z) && \text{in } \Omega \times (0,T) \\
&S(v,\theta)\nu|_\Gamma = -U(w+Z, \tilde\eta + [D(Z)\nu]\cdot\nu)\nu|_\Gamma, \quad v|_{t=0} = 0, && (2.6)
\end{aligned}
$$

where we have used (2.3). Here and hereafter, ν is suitably extended to all of $\mathbb{R}^n$ as a vector of $C^{2,1}$ functions and such an extension is also denoted by ν.

To apply Theorem 1.6 to (2.6) we shall check the condition on the right members in (2.6). All the constants independent of R and T are denoted by C in what follows. To estimate the L_∞ norm of functions and the multiplication of several functions we use the following facts:

$$
\begin{aligned}
&W^1_q(\Omega) \subset L_\infty(\Omega), \quad \|f\|_{L_\infty(\Omega)} \leqq C\|f\|_{W^1_q(\Omega)} \quad \forall\, f \in W^1_q(\Omega), \\
&\|f\prod_{j=1}^N g_j\|_{L_q(\Omega)} \leqq C\|f\|_{L_q(\Omega)} \prod_{j=1}^N \|g_j\|_{W^1_q(\Omega)} \\
&\qquad\qquad \forall\, f \in L_q(\Omega),\ g_j \in W^1_q(\Omega)\ (j=1,\ldots,N), \\
&\|\prod_{j=1}^N f_j\|_{W^1_q(\Omega)} \leqq C\prod_{j=1}^N \|f_j\|_{W^1_q(\Omega)} \quad \forall\, f_j \in W^1_q(\Omega)\ (j=1,\ldots,N), \qquad (2.7)
\end{aligned}
$$

which follows from the Sobolev imbedding theorem and the assumption $n < q < \infty$.

By (As.2), (2.5) and (2.7), we have

$$
\begin{aligned}
&\Big\|\int_0^t (w(\cdot,\tau) + Z(\cdot,\tau))\,d\tau\Big\|_{W^1_\infty(\Omega)} \\
&\leqq C\int_0^T (\|w(\cdot,\tau)\|_{W^2_q(\Omega)} + \|Z(\cdot,\tau)\|_{W^2_q(\Omega)})\,d\tau \leqq CT^{1/p'}R. \qquad (2.8)
\end{aligned}
$$

Here and hereafter, p' denotes the dual exponent of p; that is, $p' = p/(p-1)$. In view of (2.8), we choose $T > 0$ so small that the map

$$\xi \mapsto \xi + \int_0^t (w(\xi,\tau) + Z(\xi,\tau))\,d\tau$$

is a C^1 diffeomorphism and

$$\Big\|\det\Big[I + \int_0^T ((\nabla w)(\cdot,\tau) + (\nabla Z)(\cdot,\tau))\,d\tau\Big]\Big\|_{L_\infty(\Omega)} \geqq c$$

with some positive constant c, and then we have

$$\begin{aligned} &\|f(\cdot + \int_0^t (w(\cdot,\tau) + Z(\cdot,\tau))\,d\tau, t)\|_{L_p((0,T),L_q(\Omega))} \\ &\leqq \Big\|\det\Big[I + \int_0^T ((\nabla w)(\cdot,\tau) + (\nabla Z)(\cdot,\tau))\,d\tau\Big]\Big\|_{L_\infty(\Omega)}^{-1} \|f\|_{L_p((0,\infty),L_q(\mathbb{R}^n))} \\ &\leqq c^{-1}\|f\|_{L_p((0,\infty),L_q(\mathbb{R}^n))} \end{aligned} \tag{2.9}$$

for any $(w, \eta, \tilde\eta) \in \mathcal{I}_{R,T}$ provided that $CT^{1/p'}R \leqq \epsilon_0$ for some small $\epsilon_0 > 0$.

To estimate $U(w+Z, \eta+\Pi)$, recalling that

$$\begin{aligned} &U(w+Z,\eta+\Pi) \\ &\qquad = V_1(\int_0^t \nabla(w+Z)\,d\tau)\nabla(w+Z) + V_2(\int_0^t \nabla(w+Z)\,d\tau)(\eta+\Pi), \end{aligned}$$

by (2.7) we have

$$\begin{aligned} &\|\mathrm{Div}\, U(w+Z,\eta+\Pi)\|_{L_q(\Omega)} \\ &\qquad \leqq C\{\|V_1(\int_0^t \nabla(w+Z)\,d\tau)\|_{W_q^1(\Omega)}\|\nabla(w+Z)\|_{W_q^1(\Omega)} \\ &\qquad\quad + \|V_2(\int_0^t \nabla(w+Z)\,d\tau)\|_{W_q^1(\Omega)}\|\eta+\Pi\|_{W_q^1(\Omega)}\}. \end{aligned} \tag{2.10}$$

To estimate the nonlinear terms V_j, we observe that

$$\begin{aligned} &\Big\|\int_0^t \nabla(w+Z)\,d\tau\Big\|_{W_q^1(\Omega)} \leqq \int_0^t \|w+Z\|_{W_q^2(\Omega)}\,dt \\ &\quad \leqq \Big(\int_0^t dt\Big)^{1/p'}(\|w\|_{L_p((0,T),W_q^2(\Omega))} + \|Z\|_{L_p((0,T),W_q^2(\Omega))}) \leqq CT^{1/p'}R, \end{aligned} \tag{2.11}$$

$0 < t \leqq T$, where we have used (As.2) and (2.5). In particular, we choose $T > 0$ so small that

$$\Big\|\int_0^t \nabla(w+Z)\,d\tau\Big\|_{W_q^1(\Omega)} \leqq 1, \qquad 0 \leqq t \leqq T. \tag{2.12}$$

Noting that V_j is a polynomial such as $V_j(0) = 0$, by (2.7), (2.11) and (2.12) we have

$$\Big\|V_j(\int_0^t \nabla(w+Z)\,d\tau)\Big\|_{L_\infty((0,T),W_q^1(\Omega))} \leqq CT^{1/p'}R, \tag{2.13}$$

which combined with (As.2), (2.5) and (2.10) implies that

$$\|\mathrm{Div}\,U(w+Z,\eta+\Pi)\|_{L_p((0,T),L_q(\Omega))} \leqq CT^{1/p'}R^2. \tag{2.14}$$

Now, we shall estimate the terms:

$$E(w+Z) = V_3(\int_0^t \nabla(w+Z)\,d\tau)\nabla(w+Z),$$

$$\tilde{E}(w+Z) = V_4(\int_0^t \nabla(w+Z)\,d\tau)(w+Z).$$

Employing the same argument as in the proof of (2.14), we see that

$$\begin{aligned} &\|E(w+Z)\|_{L_p((0,T),W_q^1(\Omega))} \leqq CT^{1/p'}R^2, \\ &\|\tilde{E}(w+Z)\|_{L_p((0,T),L_q(\Omega))} \leqq CT^{1/p'}R^2. \end{aligned} \tag{2.15}$$

To estimate $\|\partial_t\tilde{E}(w+Z)\|_{L_p((0,T),L_q(\Omega))}$, we shall use the following lemma.

Lemma 2.1. *Let $1 < p, q < \infty$ and I be a time interval. Set*

$$\hat{W}^{1,1}_{q,p}(\Omega\times I) = \{f \in W^{1,1}_{q,\infty}(\Omega\times I) : \partial_t f \in L_p(I, W_q^1(\Omega))\}.$$

Then, we have

$$\begin{aligned} \|\partial_t(fg)\|_{L_p((0,T),L_q(\Omega))} \leqq C\{&\|f\|_{L_\infty((0,T),W_q^1(\Omega))}\|g_t\|_{L_p((0,T),L_q(\Omega))} \\ &+ T^{(q-n)/(pq)}(\|f_t\|_{L_\infty((0,T),L_q(\Omega))}\|g\|_{L_\infty((0,T),L_q(\Omega))})^{(1-n/(2q))} \\ &\times (\|f_t\|_{L_p((0,T),W_q^1(\Omega))}\|g\|_{L_p((0,T),W_q^1(\Omega))})^{n/(2q)}\}, \end{aligned}$$

provided $f \in \hat{W}^{1,1}_{q,p}(\Omega\times(0,T))$ and $g \in L_\infty((0,T),L_q(\Omega)) \cap W^{1,1}_{q,p}(\Omega\times(0,T))$.

Proof. By (2.7) we have

$$\|fg_t\|_{L_p((0,T),L_q(\Omega))} \leqq C\|f\|_{L_\infty((0,T),W^1_q(\Omega))}\|g_t\|_{L_p((0,T),L_q(\Omega))}.$$

On the other hand, by Hölder's inequality and Sobolev's inequality we have

$$\|f_tg\|_{L_q(\Omega)} \leqq \|f_t\|_{L_{2q}(\Omega)}\|g\|_{L_{2q}(\Omega)} \leqq C_q\|f_t\|_{H^{n/(2q)}_q(\Omega)}\|g\|_{H^{n/(2q)}_q(\Omega)},$$

where $H^s_q(\Omega)$ ($s \in \mathbb{R}$, $1 < q < \infty$) denotes the Bessel potential space due to Calderón [3] and we have used the fact that $n(1/q - 1/(2q)) = n/(2q)$. Since

$$\|v\|_{H^{n/(2q)}_q(\Omega)} \leqq C_q\|v\|^{n/(2q)}_{W^1_q(\Omega)}\|v\|^{(1-n/(2q))}_{L_q(\Omega)}$$

as follows from the assumption $n < q < \infty$, we have

$$\|f_tg\|_{L_q(\Omega)} \leqq C_q[\|f_t\|_{W^1_q(\Omega)}\|g\|_{W^1_q(\Omega)}]^{n/(2q)}[\|f_t\|_{L_q(\Omega)}\|g\|_{L_q(\Omega)}]^{(1-n/(2q))},$$

which implies that

$$\|f_tg\|_{L_p((0,T),L_q(\Omega))} \leqq C_q\Big[\|f_t\|_{L_\infty((0,T),L_q(\Omega))}\|g\|_{L_\infty((0,T),L_q(\Omega))}\Big]^{(1-n/(2q))} \times \Big[\int_0^T \|f_t(\cdot,t)\|^{np/(2q)}_{W^1_q(\Omega)}\|g(\cdot,t)\|^{np/(2q)}_{W^1_q(\Omega)}\,dt\Big]^{1/p}.$$

Choosing r in such a way that $n/q + 1/r = 1$, by Hölder's inequality we have

$$\begin{aligned}
&\Big[\int_0^T \|f_t(\cdot,t)\|^{np/(2q)}_{W^1_q(\Omega)}\|g(\cdot,t)\|^{np/(2q)}_{W^1_q(\Omega)}\,dt\Big]^{1/p}\\
&\quad\leqq \Big[\int_0^T \|f_t(\cdot,t)\|^{np/q}_{W^1_q(\Omega)}\,dt\Big]^{1/(2p)}\Big[\int_0^T \|g(\cdot,t)\|^{np/q}_{W^1_q(\Omega)}\,dt\Big]^{1/(2p)}\\
&\quad\leqq \Big[\int_0^T dt\Big]^{1/(rp)}\Big[\int_0^T \|f_t(\cdot,t)\|^{p}_{W^1_q(\Omega)}\,dt\Big]^{n/(2pq)}\Big[\int_0^T \|g(\cdot,t)\|^{p}_{W^1_q(\Omega)}\,dt\Big]^{n/(2pq)}\\
&\quad\leqq T^{(q-n)/(pq)}(\|f_t\|_{L_p((0,T),W^1_q(\Omega))}\|g\|_{L_p((0,T),W^1_q(\Omega))})^{n/(2q)}.
\end{aligned}$$

Combining these inequalities, we have

$$\begin{aligned}
&\|f_tg\|_{L_p((0,T),L_q(\Omega))}\\
&\quad\leqq C_{p,q}T^{(q-n)/(pq)}[\|f_t\|_{L_\infty((0,T),L_q(\Omega))}\|g\|_{L_\infty((0,T),L_q(\Omega))}]^{(1-n/(2q))}\\
&\qquad\times[\|f_t\|_{L_p((0,T),W^1_q(\Omega))}\|g\|_{L_p((0,T),W^1_q(\Omega))}]^{n/(2q)}.
\end{aligned}$$

This completes the proof of the lemma. □

Setting $f = V_4(\int_0^t \nabla(w+Z)\,d\tau)$ and $g = w + Z$, we apply Lemma 2.1 and then we have

$$\begin{aligned}&\|\partial_t\{V_4(\int_0^t \nabla(w+Z)\,d\tau)(w+Z)\}\|_{L_p((0,T),L_q(\Omega))}\\ &\quad\leqq C\{\|V_4\|_{L_\infty((0,T),W^1_q(\Omega))}\|w_t+Z_t\|_{L_p((0,T),L_q(\Omega))}\\ &\qquad + T^{(q-n)/(pq)}(\|\partial_t V_4\|_{L_\infty((0,T),L_q(\Omega))}\|w+Z\|_{L_\infty((0,T),L_q(\Omega))})^{(1-n/(2q))}\\ &\qquad\quad \times (\|\partial_t V_4\|_{L_p((0,T),W^1_q(\Omega))}\|w+Z\|_{L_p((0,T),W^1_q(\Omega))})^{n/(2q)}\}. \end{aligned} \tag{2.16}$$

By (2.12), (2.7), (2.5) and (As.2) we have

$$\begin{aligned}&\Big\|\partial_t(V_4(\int_0^t \nabla(w+Z)\,d\tau))\Big\|_{L_\infty((0,T),L_q(\Omega))} \leqq C\|w+Z\|_{L_\infty((0,T),W^1_q(\Omega))},\\ &\Big\|\partial_t(V_4(\int_0^t \nabla(w+Z)\,d\tau))\Big\|_{L_p((0,T),W^1_q(\Omega))} \leqq C\|w+Z\|_{L_p((0,T),W^2_q(\Omega))} \leqq CR. \end{aligned} \tag{2.17}$$

To estimate $\|w+Z\|_{L_\infty((0,T),W^1_q(\Omega))}$, we use the embedding relation:

$$L_p(J,E_1)\cap W^1_p(J,E_0) \subset BUC(J,[E_0,E_1]_{1-1/p,p}) \tag{2.18}$$

for any two Banach spaces E_0 and E_1 such that E_1 is dense in E_0, $1 < p < \infty$ and subinterval J of $[0,\infty)$ (cf. [1]) and the following extension results.

Lemma 2.2. (1) *Let* $0 < T \leqq 1$ *and set*

$$W^{2,1}_{q,p,0}(\Omega\times(0,T)) = \{v \in W^{2,1}_{q,p}(\Omega\times(0,T)) : v(\xi,0) = 0\}.$$

Then, there exists a bounded linear operator $\mathbb{E}_1 : W^{2,1}_{q,p,0}(\Omega\times(0,T)) \to W^{2,1}_{q,p}(\Omega\times\mathbb{R})$ *such that* $\mathbb{E}_1 v = v$ *on* $\Omega\times(0,T)$, $\mathbb{E}_1 v = 0$ *for* $t \notin [0,2T]$ *and*

$$\|\mathbb{E}_1 v\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R})} \leqq C\|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))}$$

for any $v \in W^{2,1}_{q,p,0}(\Omega\times(0,T))$, *where* C *is independent of* T.
(2) *There exists a bounded linear operator* $\mathbb{E}_2 : W^{2,1}_{q,p}(\Omega\times(0,2)) \to W^{2,1}_{q,p}(\Omega\times\mathbb{R})$ *such that* $\mathbb{E}_2 v = v$ *on* $\Omega\times(0,1)$, $\mathbb{E}_2 v = 0$ *for* $t \notin [-2,2]$ *and*

$$\|\mathbb{E}_2 v\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R})} \leqq C\|v\|_{W^{2,1}_{q,p}(\Omega\times(0,2))}$$

for any $v \in W^{2,1}_{q,p}(\Omega\times(0,2))$.

(3) *Let* $0 < \gamma \leqq 1$ *and set*

$$W^{2,1}_{q,p,\gamma}(\Omega \times I) = \{v \in W^{2,1}_{q,p}(\Omega \times I) : \|e^{\gamma t} v\|_{W^{2,1}_{q,p}(\Omega\times I)} < \infty\}$$

for $I = \mathbb{R}_+$ *and* $\mathbb{R}$. *Then, there exists a bounded linear operator* $\mathbb{E}_3 : W^{2,1}_{q,p,\gamma}(\Omega \times \mathbb{R}_+) \to W^{2,1}_{q,p,\gamma}(\Omega \times \mathbb{R})$ *such that* $\mathbb{E}_3 v = v$ *on* $\Omega \times \mathbb{R}_+$, $\mathbb{E}_3 v = 0$ *for* $t \leqq -2$ *and*

$$\|e^{\gamma t}\,\mathbb{E}_3 v\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R})} \leqq C\|e^{\gamma t}\, v\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)}$$

for any $v \in W^{2,1}_{q,p,\gamma}(\Omega \times \mathbb{R}_+)$.

Remark 2.3. Although $\mathbb{E}_3$ will be used in Section 3 to prove Theorem 1.4, we state the assertion (3), because the idea of the proof of (3) is the same as those of (1) and (2).

Postponing a proof of Lemma 2.2, we shall finish estimating the norm $\|w + Z\|_{L_\infty((0,T),W^1_q(\Omega))}$. Since $w + Z = \mathbb{E}_1 w + \mathbb{E}_2 Z$ on $(0,T) \times \Omega$, as follows from the assumption $0 < T \leqq 1$, and since $1 < 2(1 - 1/p)$, as follows from the assumption $2 < p < \infty$, we have

$$\begin{aligned}\|w + Z\|_{L_\infty((0,T),W^1_q(\Omega))} &\leqq \|\mathbb{E}_1 w + \mathbb{E}_2 Z\|_{L_\infty((0,2),W^1_q(\Omega))} \\ &\leqq C\|\mathbb{E}_1 w + \mathbb{E}_2 Z\|_{L_\infty((0,2),B^{2(1-1/p)}_{q,p}(\Omega))}.\end{aligned}$$

Therefore, by (2.18) with $E_0 = L_q(\Omega)$, $E_1 = W^2_q(\Omega)$ and $J = (0,2)$, Lemma 2.2, (As.2) and (2.5) we have

$$\|w + Z\|_{L_\infty((0,T),W^1_q(\Omega))} \leqq C\|\mathbb{E}_1 w + \mathbb{E}_2 Z\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R})} \leqq CR,$$

where C is independent of T, which combined with (2.16), (2.13), (2.17), (As.2) and (2.5) gives us

$$\|\partial_t \tilde{E}(w + Z)\|_{L_p((0,T),L_q(\Omega))} \leqq CR^2(T^{1/p'} + T^{(q-n)/(pq)}). \tag{2.19}$$

A proof of Lemma 2.2. (1) Set

$$v_1(\xi,t) = \begin{cases} v(\xi,t) & 0 \leqq t \leqq T \\ 0 & t < 0, \end{cases}$$

$$\mathbb{E}_1 v(\xi,t) = \begin{cases} v_1(\xi,t) & t \leqq T \\ v_1(\xi, 2T - t) & t > T. \end{cases}$$

Then, we see easily that $\mathbb{E}_1$ has the required property.

(2) Let $\rho(t)$ be a function in $C_0^\infty(\mathbb{R})$ such that $\rho(t) = 1$ for $|t| \leqq 1$ and $\rho(t) = 0$ for $|t| \geqq 2$. Set

$$\mathbb{E}_2 v(\xi, t) = \begin{cases} (\rho v)(\xi, t) & t \geqq 0 \\ (\rho v)(\xi, -t) & t < 0. \end{cases}$$

Then, we see easily that $\mathbb{E}_2$ has the required properties.
(3) Let $\psi(t)$ be a function in $C^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $t \geqq -1$ and $\psi(t) = 0$ for $t \leqq -2$. Set

$$\mathbb{E}_3 v(\xi, t) = \begin{cases} v(\xi, t) & t \geqq 0 \\ \psi(t) v(\xi, -t) & t < 0. \end{cases}$$

Then, we see easily that $\mathbb{E}_3$ has the required properties. □

Finally, we shall estimate $\|U(w+Z, \tilde\eta + [D(Z)\nu]\cdot\nu)\nu\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))}$. From the definition of $H^{1,1/2}_{q,p}(\Omega \times (0, T))$, we have to extend not only w and Z but also $\int_0^t (\nabla w + \nabla Z)\, ds$ to the whole time $\mathbb{R}$. To do this, we use Lemma 2.2 and the following lemma.

Lemma 2.4. *Let $2 < p < \infty$ and $n < q < \infty$.*
(1) Let $\hat W^{1,1}_{q,p}(\Omega\times I)$ and $W^{2,1}_{q,p,0}(\Omega\times(0,T))$ be the spaces defined in Lemmas 2.1 and 2.2, respectively. Let $0 < T \leqq 1$. Then, there exist linear operators $\mathbb{F}_1 : W^{2,1}_{q,p,0}(\Omega \times (0,T)) \to \hat W^{1,1}_{q,p}(\Omega \times \mathbb{R})$ and $\mathbb{F}_2 : W^{2,1}_{q,p}(\Omega \times (0, 2)) \to \hat W^{1,1}_{q,p}(\Omega \times \mathbb{R})$ such that

$$\mathbb{F}_j v(\xi, t) = \int_0^t \nabla v(\xi, \tau)\, d\tau, \quad 0 \leqq t \leqq T, \tag{2.20}$$

$$\mathbb{F}_j v = 0 \quad \text{for } t \notin [0, 2T], \tag{2.21}$$

$$\|\mathbb{F}_j v\|_{L_\infty(\mathbb{R}, W^1_q(\Omega))} \leqq CT^{1/p'} \|v\|_{W^{2,1}_{q,p}(\Omega\times I_j)}, \tag{2.22}$$

$$\|\partial_t(\mathbb{F}_j v)\|_{L_\infty(\mathbb{R}, L_q(\Omega))} \leqq C\|v\|_{W^{2,1}_{q,p}(\Omega\times I_j)}, \tag{2.23}$$

$$\|\partial_t(\mathbb{F}_j v)\|_{L_p(\mathbb{R}, W^1_q(\Omega))} \leqq C\|v\|_{W^{2,1}_{q,p}(\Omega\times I_j)} \tag{2.24}$$

for $j = 1, 2$, where $I_1 = (0, T)$, $I_2 = (0, 2)$ and C is independent of v and T.
(2) Let $\gamma > 0$ and $W^{2,1}_{q,p,\gamma}(\Omega \times \mathbb{R}_+)$ be the same space as in Lemma 2.2. Then, there exists a linear operator $\mathbb{F}_3 : W^{2,1}_{q,p,\gamma}(\Omega \times \mathbb{R}_+) \to W^{1,1}_{q,\infty}(\Omega \times \mathbb{R})$ such that

$$\mathbb{F}_3 v(\xi, t) = \int_0^t \nabla v(\xi, \tau)\, d\tau, \quad t \geqq 0, \tag{2.25}$$

$$\mathbb{F}_3 v = 0 \quad \text{for } t \leqq 0, \tag{2.26}$$

$$\|\mathbb{F}_3 v\|_{L_\infty(\mathbb{R}, W^1_q(\Omega))} \leqq C_\gamma \|e^{\gamma t} v\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)}, \tag{2.27}$$

$$\|\partial_t(\mathbb{F}_3 v)\|_{L_\infty(\mathbb{R}, L_q(\Omega))} \leqq C_\gamma \|e^{\gamma t} v\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)}, \tag{2.28}$$

where C_γ is independent of v.

Remark 2.5. Although $\mathbb{F}_3$ will be used in Section 3 to prove Theorem 1.4, we state the assertion (3), because the idea of the proof of (3) is the same as that of (1) and (2).

Proof. (1) Set

$$w_j(\xi,t) = \begin{cases} \displaystyle\int_0^t \nabla \mathbb{E}_j v(\xi,\tau)\,d\tau & 0 \leqq t \leqq T \\ \quad 0 & t < 0, \end{cases}$$

$$\mathbb{F}_j v(\xi,t) = \begin{cases} w_j(\xi,t) & t \leqq T \\ w_j(\xi, 2T-t) & t \geqq T \end{cases}$$

for $j = 1, 2$. Then, we have (2.20) and (2.21) immediately.

To show (2.22) we use the estimates:

$$\begin{aligned} \|\mathbb{F}_j v(\cdot,t)\|_{W^1_q(\Omega)} &\leqq \int_0^t \|\mathbb{E}_j v(\cdot,\tau)\|_{W^2_q(\Omega)}\,d\tau \\ &\leqq CT^{1/p'}\|v\|_{L_p((0,T),W^2_q(\Omega))}, \qquad 0 \leqq t \leqq T, \\ \|\mathbb{F}_j v(\cdot,t)\|_{W^1_q(\Omega)} &\leqq \int_0^{2T-t} \|\mathbb{E}_j v(\cdot,\tau)\|_{W^2_q(\Omega)}\,d\tau \\ &\leqq CT^{1/p'}\|v\|_{L_p((0,T),W^2_q(\Omega))}, \qquad T \leqq t \leqq 2T, \end{aligned}$$

which, combined with (2.21) and Lemma 2.2 (1) and (2), implies (2.22).

Since

$$\partial_t \mathbb{F}_j v(\xi,t) = \begin{cases} \nabla \mathbb{E}_j v(\xi,t) & 0 \leqq t \leqq T \\ -\nabla \mathbb{E}_j v(\xi, 2T-t) & T \leqq t \leqq 2T \\ \quad 0 & t \notin [0, 2T], \end{cases} \tag{2.29}$$

noting that $1 < 2(1-1/p)$ and $0 < T \leqq 1$, we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|\partial_t \mathbb{F}_j v(\cdot,t)\|_{L_q(\Omega)} &\leqq C \sup_{0\leqq t\leqq T} \|\mathbb{E}_j v(\cdot,t)\|_{W^1_q(\Omega)} \\ &\leqq C \sup_{0\leqq t\leqq 2} \|\mathbb{E}_j v(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)}. \end{aligned}$$

Therefore, by (2.29) and (2.18) with $E_0 = L_q(\Omega)$, $E_1 = W_q^2(\Omega)$ and $J = (0,2)$ we have $\|\partial_t \mathbb{F}_j v\|_{L_\infty(\mathbb{R}, L_q(\Omega))} \leqq C\|\mathbb{E}_j v\|_{W_{q,p}^{2,1}(\Omega\times(0,2))}$, which, combined with Lemma 2.2 (1) and (2), implies (2.23). By (2.29) we have $\|\partial_t \mathbb{F}_j v\|_{L_p(\mathbb{R}, W_q^1(\Omega))} \leqq C\|\mathbb{E}_j v\|_{L_p(\mathbb{R}, W_q^2(\Omega))}$, which, combined with Lemma 2.2 (1) and (2), implies (2.24). This completes the proof of the assertion (1).

(2) Set

$$\mathbb{F}_3 v(\xi, t) = \begin{cases} \displaystyle\int_0^t \nabla v(\xi,\tau)\,d\tau & t \geqq 0 \\ 0 & t < 0, \end{cases} \tag{2.30}$$

and then obviously we have (2.25) and (2.26). When $t \geqq 0$, we have

$$\begin{aligned} &\|\mathbb{F}_3 v(\cdot,t)\|_{W_q^1(\Omega)} \leqq \int_0^t \|v(\cdot,\tau)\|_{W_q^2(\Omega)}\,d\tau \\ &\leqq \Big(\int_0^\infty e^{-p'\gamma t}\,dt\Big)^{1/p'} \Big(\int_0^\infty (e^{\gamma t}\|v(\cdot,t)\|_{W_q^2(\Omega)})^p\,dt\Big)^{1/p} \leqq C_\gamma \|e^{\gamma t} v\|_{W_{q,p}^{2,1}(\Omega\times\mathbb{R}_+)}, \end{aligned}$$

which, combined with (2.26), implies (2.27).

Since $(\partial_t \mathbb{F}_3 v)(\xi,t) = \nabla v(\xi,\tau)$ for $t \geqq 0$ and $(\partial_t \mathbb{F}_3 v)(\xi,t) = 0$ for $t < 0$ as follows from (2.30), noting that $1 < 2(1-1/p)$, we have

$$\sup_{t\geqq 0} \|\partial_t \mathbb{F}_3 v(\cdot,t)\|_{L_q(\Omega)} \leqq C \sup_{t\geqq 0} \|v(\cdot,t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)},$$

which, combined with (2.18) with $E_0 = L_q(\Omega)$ and $E_1 = W_q^2(\Omega)$, implies (2.28), because $\gamma > 0$. This completes the proof. □

Using Lemmas 2.2 and 2.4, we define $H(x,t)$ by the formula:

$$\begin{aligned} H = \{&V_1(\mathbb{F}_1 w + \mathbb{F}_2 Z)\nabla(\mathbb{E}_1 w + \mathbb{E}_2 Z) \\ &+ V_2(\mathbb{F}_1 w + \mathbb{F}_2 Z)(\tilde\eta + [D(\mathbb{E}_2 Z)\nu]\cdot\nu)\}\nu. \end{aligned}$$

Then, we have

$$U(w + Z, \tilde\eta + [D(Z)\nu]\cdot\nu)\nu = H \quad \text{on } \Omega\times(0,T). \tag{2.31}$$

To estimate H, we use the following lemma.

Lemma 2.6. *Let $1 < p < \infty$, $n < q < \infty$ and $0 < T \leqq 1$. Let $\hat W_{q,p}^{1,1}(\Omega\times\mathbb{R})$ be the same space as in Lemma 2.1. If $f \in \hat W_{q,p}^{1,1}(\Omega\times\mathbb{R})$, $g \in H_{q,p}^{1,1/2}(\Omega\times\mathbb{R})$ and f vanishes when $t \notin [0,2T]$, then we have*

$$\|fg\|_{H_{q,p}^{1,1/2}(\Omega\times\mathbb{R})} \leqq C_{p,q}[\|f\|_{L_\infty(\mathbb{R}, W_q^1(\Omega))}$$

$$+ T^{(q-n)/(pq)}\|f_t\|^{(1-n/(2q))}_{L_\infty(\mathbb{R},L_q(\Omega))}\|f_t\|^{n/(2q)}_{L_p(\mathbb{R},W^1_q(\Omega))}]\|g\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}.$$

To prove Lemma 2.6, first we shall show the following lemma.

Lemma 2.7. *Let $1 < p < \infty$, $n < q < \infty$ and $0 < T \leqq 1$. Let $f \in \hat{W}^{1,1}_{q,p}(\Omega\times\mathbb{R})$ and $g \in H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R})$. If f vanishes when $t \notin [0, 2T]$, then we have*

$$\begin{aligned}\|fg\|_{H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R})} &\leqq C_{p,q}[\|f\|_{L_\infty(\mathbb{R},W^1_q(\Omega))}\\ &\quad + T^{(q-n)/(pq)}\|f_t\|^{(1-n/(2q))}_{L_\infty(\mathbb{R},L_q(\Omega))}\|f_t\|^{n/(2q)}_{L_p(\mathbb{R},W^1_q(\Omega))}]\|g\|_{H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R})}.\end{aligned}$$

Here, we have set

$$H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R}) = H^{1/2}_p(\mathbb{R}, L_q(\Omega)) \cap L_p(\mathbb{R}, H^{1/2}_q(\Omega)),$$
$$H^\theta_q(\Omega) = [L_q(\Omega), W^1_q(\Omega)]_\theta, \quad 0 < \theta < 1,$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor.

Proof. To show the lemma, we shall use the complex interpolation relation:

$$H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R}) = [L_p(\mathbb{R}, L_q(\Omega)), W^{1,1}_{q,p}(\Omega\times\mathbb{R})]_{1/2}. \tag{2.32}$$

First, assuming that $g \in W^{1,1}_{q,p}(\Omega\times\mathbb{R})$, we estimate $\|fg\|_{W^{1,1}_{q,p}(\Omega\times\mathbb{R})}$. Employing the same argument as in the proof of Lemma 2.1 and noting that $f = 0$ for $t \notin [0, 2T]$, we have

$$\begin{aligned}&\|\partial_t(fg)\|_{L_p(\mathbb{R},L_q(\Omega))} = \|\partial_t(fg)\|_{L_p((0,2T),L_q(\Omega))}\\ &\leqq C[\|f\|_{L_\infty((0,2T),W^1_q(\Omega))}\|g_t\|_{L_p((0,2T),L_q(\Omega))} + (2T)^{(q-n)/(pq)}\|f_t\|^{(1-(n/(2q))}_{L_\infty((0,2T),L_q(\Omega))}\\ &\quad \times \|f_t\|^{n/(2q)}_{L_p((0,2T),W^1_q(\Omega))}\|g\|^{(1-(n/(2q))}_{L_\infty((0,2T),L_q(\Omega))}\|g\|^{n/(2q)}_{L_p((0,2T),W^1_q(\Omega))}].\end{aligned} \tag{2.33}$$

Noting that $0 < T \leqq 1$ and $B^{1-1/p}_{q,p}(\Omega) \subset L_q(\Omega)$, by (2.18) with $E_0 = L_q(\Omega)$ and $E_1 = W^1_q(\Omega)$ we have

$$\begin{aligned}\|g\|_{L_\infty((0,2T),L_q(\Omega))} &\leqq \|g\|_{L_\infty((0,2),B^{1-1/p}_{q,p}(\Omega))}\\ &\leqq C\|g\|_{W^{1,1}_{q,p}(\Omega\times(0,2))} \leqq C\|g\|_{W^{1,1}_{q,p}(\Omega\times\mathbb{R})},\end{aligned}$$

which, combined with (2.33), implies that

$$\begin{aligned}\|\partial_t(fg)\|_{L_p(\mathbb{R},L_q(\Omega))} &\leqq C[\|f\|_{L_\infty(\mathbb{R},W^1_q(\Omega))}\\ &\quad + T^{(q-n)/(pq)}\|f_t\|^{(1-(n/(2q))}_{L_\infty(\mathbb{R},L_q(\Omega))}\|f_t\|^{n/(2q)}_{L_p(\mathbb{R},W^1_q(\Omega))}]\|g\|_{W^{1,1}_{q,p}(\Omega\times\mathbb{R})}.\end{aligned} \tag{2.34}$$

On the other hand, by (2.7) we have

$$\|fg\|_{L_p(\mathbb{R},L_q(\Omega))} \leqq C_{p,q}\|f\|_{L_\infty(\mathbb{R},W_q^1(\Omega))}\|g\|_{L_p(\mathbb{R},L_q(\Omega))}, \tag{2.35}$$

$$\|fg\|_{L_p(\mathbb{R},W_q^1(\Omega))} \leqq C_{p,q}\|f\|_{L_\infty(\mathbb{R},W_q^1(\Omega))}\|g\|_{L_p(\mathbb{R},W_q^1(\Omega))}. \tag{2.36}$$

In particular, combining (2.34), (2.35) and (2.36), we have

$$\begin{aligned}\|fg\|_{W_{q,p}^{1,1}(\Omega\times\mathbb{R})} \leqq C[\|f\|_{L_\infty(\mathbb{R},W_q^1(\Omega))} \\ + T^{(q-n)/(pq)}\|f_t\|_{L_\infty(\mathbb{R},L_q(\Omega))}^{(1-(n/(2q))}\|f_t\|_{L_p(\mathbb{R},W_q^1(\Omega))}^{n/(2q)}]\|g\|_{W_{q,p}^{1,1}(\Omega\times\mathbb{R})}.\end{aligned} \tag{2.37}$$

On the other hand, when $g \in L_p(\mathbb{R}, L_q(\Omega))$ we know that (2.35) holds, and therefore combining (2.32), (2.35) and (2.37) implies Lemma 2.7. □

A proof of Lemma 2.6. Setting

$$M = C_{p,q}[\|f\|_{L_\infty(\mathbb{R},W_q^1(\Omega))} + T^{(q-n)/(pq)}\|f_t\|_{L_\infty(\mathbb{R},L_q(\Omega))}^{(1-n/(2q))}\|f_t\|_{L_p(\mathbb{R},W_q^1(\Omega))}^{n/(2q)}],$$

by Lemma 2.7 we have

$$\begin{aligned}\|fg\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} &\leqq \|fg\|_{H_{q,p}^{1/2,1/2}(\Omega\times\mathbb{R})} \\ &\leqq M\|g\|_{H_{q,p}^{1/2,1/2}(\Omega\times\mathbb{R})} \leqq M\|g\|_{H_{q,p}^{1,1/2}(\Omega\times\mathbb{R})},\end{aligned}$$

which, combined with (2.36), implies Lemma 2.6. □

Applying Lemma 2.6, we have

$$\begin{aligned}\|H\|_{H_{q,p}^{1,1/2}(\Omega\times\mathbb{R})} \leqq C\{\sum_{j=1}^{2}(\|V_j(\mathbb{F}_1 w + \mathbb{F}_2 Z)\|_{L_\infty(\mathbb{R},W_q^1(\Omega))} \\ + T^{(q-n)/(pq)}\|\partial_t V_j(\mathbb{F}_1 w + \mathbb{F}_2 Z)\|_{L_\infty(\mathbb{R},L_q(\Omega))}^{(1-n/(2q))}\|\partial_t V_j(\mathbb{F}_1 w + \mathbb{F}_2 Z)\|_{L_p(\mathbb{R},W_q^1(\Omega))}^{n/(2q)})\} \\ \times \|(\nabla(\mathbb{E}_1 w + \mathbb{E}_2 Z), \tilde{\eta} + [D(\mathbb{E}_2 Z)\nu]\cdot\nu)\|_{H_{q,p}^{1,1/2}(\Omega\times\mathbb{R})}.\end{aligned} \tag{2.38}$$

By Lemma 2.4, (As.2) and (2.5) we have

$$\|\mathbb{F}_1 w + \mathbb{F}_2 Z\|_{L_\infty(\mathbb{R},W_q^1(\Omega))} \leqq CT^{1/p'}R, \tag{2.39}$$

and therefore we choose $T > 0$ so small that

$$\|\mathbb{F}_1 w + \mathbb{F}_2 Z\|_{L_\infty(\mathbb{R},W_q^1(\Omega))} \leqq 1. \tag{2.40}$$

By (2.7), (2.39), (2.40), Lemma 2.2, Lemma 2.4, (As.2) and (2.5), we have

$$\|V_j(\mathbb{F}_1 w + \mathbb{F}_2 Z)\|_{L_\infty(\mathbb{R},W_q^1(\Omega))} \leqq CT^{1/p'}R,$$

$$\|\partial_t V_j(\mathbb{F}_1 w + \mathbb{F}_2 Z)\|_{L_\infty(\mathbb{R}, L_q(\Omega))} \leqq CR,$$
$$\|\partial_t V_j(\mathbb{F}_1 w + \mathbb{F}_2 Z)\|_{L_p(\mathbb{R}, W_q^1(\Omega))} \leqq CR \tag{2.41}$$

for $j = 1, 2$. On the other hand, since we know that

$$\|f\|_{H_p^{1/2}(\mathbb{R}, W_q^1(\Omega))} \leqq C\|f\|_{W_{q,p}^{2,1}(\Omega\times\mathbb{R})} \tag{2.42}$$

as follows from the mixed derivative theorem (cf. Sobolevskii [12], Denk-Hieber-Prüss [5], Shibata-Shimizu [10]), by Lemma 2.2, (As.2) and (2.5) we have

$$\|(\nabla(\mathbb{E}_1 w + \mathbb{E}_2 Z), \tilde{\eta} + [D(\mathbb{E}_2 Z)\nu] \cdot \nu)\|_{H_{q,p}^{1,1/2}(\Omega\times\mathbb{R})} \leqq CR,$$

which, combined with (2.38) and (2.41), implies that

$$H \in H_{q,p,0}^{1,1/2}(\Omega \times \mathbb{R}_+)^{n\times n},$$
$$\|H\|_{H_{q,p}^{1,1/2}(\Omega\times\mathbb{R})} \leqq CR^2(T^{1/p'} + T^{(q-n)/(pq)}), \tag{2.43}$$

because by (2.21) we see that $H = 0$ when $t < 0$. Therefore, from (2.31), the definition of $H_{q,p,0}^{1,1/2}(\Omega \times (0,T))$ and (2.43) we see that

$$U(w + Z, \tilde{\eta} + [D(Z)\nu] \cdot \nu)\nu \in H_{q,p,0}^{1,1/2}(\Omega \times (0,T))^{n\times n},$$
$$\|U(w + Z, \tilde{\eta} + [D(Z)\nu] \cdot \nu)\nu\|_{H_{q,p,0}^{1,1/2}(\Omega\times(0,T))} \leqq CR^2(T^{1/p'} + T^{(q-n)/(pq)}). \tag{2.44}$$

Applying Theorem 1.6 to (2.6) and using (2.9), (2.14), (2.15), (2.19) and (2.44) we see that the equation (2.6) admits a unique solution

$$(v, \theta) \in W_{q,p}^{2,1}(\Omega \times (0,T))^n \times L_p((0,T), W_q^1(\Omega))$$

such that

$$\|v\|_{W_{q,p}^{2,1}(\Omega\times(0,T))} + \|\theta\|_{L_p((0,T), W_q^1(\Omega))}$$
$$\leqq C\|f\|_{L_p(\mathbb{R}_+, L_q(\mathbb{R}^n))} + C(T^{1/p'} + T^{(q-n)/(pq)})R^2, \tag{2.45}$$

provided that (As.2) holds.

By the boundary condition of (2.6) we have

$$\theta = [D(v)\nu + U(w + Z, \tilde{\eta} + [D(Z)\nu] \cdot \nu)\nu] \cdot \nu \tag{2.46}$$

on $\Gamma \times (0,T)$. In view of (2.31), we set

$$\tilde{\theta} = [D(\mathbb{E}_1 v)\nu + H] \cdot \nu. \tag{2.47}$$

By Lemma 2.2 and (2.42) we see that

$$[D(\mathbb{E}_1 v)\nu]\cdot\nu = [D(v)\nu]\cdot\nu \quad \text{on } \Omega\times(0,T),$$
$$[D(\mathbb{E}_1 v)\nu]\cdot\nu \in H^{1,1/2}_{q,p,0}(\Omega\times\mathbb{R}_+),$$
$$\|[D(\mathbb{E}_1 v)\nu]\cdot\nu\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq C\|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))}. \tag{2.48}$$

By (2.31), (2.43), (2.46), (2.47) and (2.48) we have

$$\theta = \tilde\theta \quad \text{on } \Gamma\times(0,T), \quad \tilde\theta\in H^{1,1/2}_{q,p,0}(\Omega\times\mathbb{R}_+),$$
$$\|\tilde\theta\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq C\|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + C(T^{1/p'}+T^{(q-n)/(pq)})R^2. \tag{2.49}$$

Combining (2.45) and (2.49), we have

$$\|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\theta\|_{L_p((0,T),W^1_q(\Omega))} + \|\tilde\theta\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}$$
$$\leqq C\|f\|_{L_p(\mathbb{R}_+,L_q(\mathbb{R}^n))} + C(T^{1/p'}+T^{(q-n)/(pq)})R^2. \tag{2.50}$$

Choosing $R>0$ and $T>0$ in such a way that

$$C\|f\|_{L_p(\mathbb{R}_+,L_q(\mathbb{R}^n))} \leqq R/2, \quad C(T^{1/p'}+T^{(q-n)/(pq)})R \leqq 1/2,$$

by (2.50) we have

$$\|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\theta\|_{L_p((0,T),W^1_q(\Omega))} + \|\tilde\theta\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq R.$$

Therefore, if we define the map Φ by $\Phi(w,\eta,\tilde\eta) = (v,\theta,\tilde\theta)$, then Φ maps $\mathcal{I}_{R,T}$ into itself.

Now, we shall show that Φ is a contraction map. To do this, given $(w_i,\eta_i,\tilde\eta_i)\in\mathcal{I}_{R,T}$, we set $(v_i,\theta_i,\tilde\theta_i) = \Phi(w_i,\eta_i,\tilde\eta_i)$ for $i=1,2$. From (2.6) we see that v_1-v_2 and $\theta_1-\theta_2$ satisfy the linear equation:

$$\begin{aligned}
&(v_1-v_2)_t - \operatorname{Div} S(v_1-v_2,\theta_1-\theta_2) = f(\xi+\int_0^t (w_1(\xi,\tau)+Z(\xi,\tau))\,d\tau, t)\\
&\quad - f(\xi+\int_0^t (w_2(\xi,\tau)+Z(\xi,\tau))\,d\tau, t)\\
&\quad + \operatorname{Div} U(w_1+Z,\eta_1+\Pi) - \operatorname{Div} U(w_2+Z,\eta_2+\Pi) \qquad \text{in } \Omega\times(0,T)\\
&\operatorname{div}(v_1-v_2) = -(E(w_1+Z)-E(w_2+Z))\\
&\qquad = -\operatorname{div}(\tilde E(w_1+Z)-\tilde E(w_2+Z)) \qquad \text{in } \Omega\times(0,T)\\
&S(v_1-v_2,\theta_1-\theta_2)\nu = -[U(w_1+Z,\tilde\eta_1+[D(Z)\nu]\cdot\nu)\\
&\qquad - U(w_2+Z,\tilde\eta_2+[D(Z)\nu]\cdot\nu)]\nu \qquad \text{on } \Gamma\times(0,T)
\end{aligned}$$

$$(v_1 - v_2)(\xi, 0) = 0 \qquad \text{in } \Omega.$$

Applying the same argument as in the proof of (2.50), we can show that

$$\begin{aligned}
&\|v_1 - v_2\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\theta_1 - \theta_2\|_{L_p((0,T),W^1_q(\Omega))} + \|\tilde\theta_1 - \tilde\theta_2\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \\
&\leqq \|\nabla f\|_{L_\infty(\mathbb{R}^n\times\mathbb{R}_+)} T\|w_1 - w_2\|_{L_p((0,T),L_q(\Omega))} + C(T^{1/p'} + T^{(q-n)/(pq)})R \\
&\times [\|w_1 - w_2\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\eta_1 - \eta_2\|_{L_p((0,T),W^1_q(\Omega))} + \|\tilde\eta_1 - \tilde\eta_2\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}].
\end{aligned} \tag{2.51}$$

Choosing $T > 0$ so small that

$$\|\nabla f\|_{L_\infty(\mathbb{R}^n\times\mathbb{R}_+)} T \leqq 1/3, \quad C(T^{1/p'} + T^{(q-n)/(pq)})R \leqq 1/3$$

in (2.51), we see that Φ is a contraction map on $\mathcal{I}_{R,T}$. Therefore, Φ has the fixed point $(v, \theta, \tilde\theta) \in \mathcal{I}_{R,T}$, which solves (2.4). Combining this with (2.1), we see that for any $R > 0$ and $R' > 0$ there exists a time $T > 0$ depending on R and R' such that the equation (1.4) admits a solution $u = Z + v$ and $\pi = \Pi + \theta$ which satisfy the estimate:

$$\|u\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\pi\|_{L_p((0,T),L_q(\Omega))} \leqq CR$$

with some $C > 0$ depending essentially only on p and q provided that $\|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|f\|_{L_p(\mathbb{R}_+,L_q(\Omega))} \leqq R$ and $\|\nabla f\|_{L_\infty(\mathbb{R}^n\times\mathbb{R})} \leqq R'$.

Since we construct a solution by the contraction mapping principle, choosing $T > 0$ smaller if necessary, we can easily prove the uniqueness of solutions $u = Z + v$ and $\pi = \Pi + \theta$ with $(v, \theta, \tilde\theta) \in \mathcal{I}_{R,T}$, where $\tilde\theta$ is defined by the formula:

$$\begin{aligned}
\tilde\theta = \{&D(\mathbb{E}_1 v)\nu - V_1(\mathbb{F}_1 v + \mathbb{F}_2 Z)\nabla(\mathbb{E}_1 v + \mathbb{E}_2 Z) \\
&- V_2(\mathbb{F}_1 v + \mathbb{F}_2 Z)(\tilde\eta + [D(\mathbb{E}_2 Z)\nu]\cdot\nu)\}\cdot\nu
\end{aligned}$$

(cf. (2.47)). This completes the proof of Theorem 1.3.

3. A proof of Theorem 1.4

In order to prove Theorem 1.4 by the contraction mapping principle, we define the underlying space $\mathcal{I}_{\gamma,\epsilon}$ by

$$\begin{aligned}
\mathcal{I}_{\gamma,\epsilon} =\{&(v, \theta, \tilde\theta) \in W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)^n \times L_p(\mathbb{R}_+, W^1_q(\Omega)) \times H^{1,1/2}_{q,p}(\Omega\times\mathbb{R}) : \\
&\theta = \tilde\theta \quad \text{on } \Gamma\times\mathbb{R}_+,
\end{aligned} \tag{As.3}$$

$$\|e^{\gamma t}\, v\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)} + \|e^{\gamma t}\, \theta\|_{L_p(\mathbb{R}_+,W^1_q(\Omega))} + \|e^{\gamma t}\, \tilde\theta\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq \epsilon\}, \tag{As.4}$$

where $\mathbb{R}_+ = (0,\infty)$, γ is a positive constant in Theorem 1.6 and ϵ is some constant less than 1 which will be determined later. Given $(v,\theta,\tilde\theta) \in \mathcal{I}_{\gamma,\epsilon}$, we set

$$\begin{aligned} &G = V_3(\mathbb{F}_3 v)\nabla\mathbb{E}_3 v, \quad \tilde G = V_4(\mathbb{F}_3 v)\mathbb{E}_3 v,\\ &H = V_1(\mathbb{F}_3 v)\nabla\mathbb{E}_3 v + V_2(\mathbb{F}_3 v)\tilde\theta. \end{aligned} \tag{3.1}$$

Here, $\mathbb{E}_3$ and $\mathbb{F}_3$ are linear operators defined in Lemmas 2.2 and 2.4, respectively. By Lemmas 2.2 and 2.4, $E(v) = G$, $\tilde E(v) = \tilde G$ and $U(v,\tilde\theta) = H$ on $\Omega\times\mathbb{R}_+$. Let u and π be solutions to the equation:

$$\begin{aligned} &u_t - \mathrm{Div}\, S(u,\pi) = \mathrm{Div}\, U(v,\theta) && \text{in } \Omega\times\mathbb{R}_+\\ &\mathrm{div}\, u = -G = -\mathrm{div}\,\tilde G && \text{in } \Omega\times\mathbb{R}_+\\ &S(u,\pi)\nu = -H\nu && \text{on } \Gamma\times\mathbb{R}_+\\ &u(\xi,0) = u_0(\xi) && \text{in } \Omega. \end{aligned} \tag{3.2}$$

Since $\theta = \tilde\theta$ on $\Gamma\times\mathbb{R}_+$, $U(v,\theta) = H$ on $\Gamma\times\mathbb{R}_+$, and therefore the following orthogonal condition holds: $(\mathrm{Div}\, U(v,\theta), p_\ell)_\Omega + (-H\nu, p_\ell)_\Gamma = 0$ and $(u_0, p_\ell)_\Omega = 0$ for all $\ell = 1,2,\ldots,M$ and $t\in\mathbb{R}_+$.

We shall estimate the right members of (3.2). In what follows, all the constants independent of ϵ will be denoted by C. First we consider $U(v,\theta)$. Recalling that

$$U(v,\theta) = V_1(\int_0^t \nabla v\, d\tau)\nabla v + V_2(\int_0^t \nabla v\, d\tau)\theta,$$

by (2.7) we have

$$\begin{aligned} &\|e^{\gamma t}\,\mathrm{Div}\, U(v,\theta)\|_{L_p(\mathbb{R}_+,L_q(\Omega))}\\ &\qquad \leqq C(\sum_{j=1}^2 \|V_j(\int_0^t \nabla v\, d\tau)\|_{L_\infty(\mathbb{R}_+,W^1_q(\Omega))})\|e^{\gamma t}\,(\nabla v,\theta)\|_{L_p(\mathbb{R}_+,L_q(\Omega))}. \end{aligned} \tag{3.3}$$

By (As.4) we have

$$\begin{aligned} \Big\|\int_0^t \nabla v\, d\tau\Big\|_{W^1_q(\Omega)} &\leqq \int_0^t \|v(\cdot,\tau)\|_{W^2_q(\Omega)}\, d\tau\\ &\leqq \Big(\int_0^\infty e^{-\gamma p' t}\, dt\Big)^{1/p'} \|e^{\gamma t}\, v\|_{L_p(\mathbb{R}_+,W^2_q(\Omega))} \leqq C\epsilon. \end{aligned} \tag{3.4}$$

In particular, we choose $\epsilon > 0$ so small that

$$\Big\|\int_0^t \nabla v\, d\tau\Big\|_{W^1_q(\Omega)} \leqq 1 \quad \text{for all } t \geqq 0. \tag{3.5}$$

Since V_j is a polynomial such as $V_j(0) = 0$, by (2.7), (3.4) and (3.5) we have

$$\Big\|V_j(\int_0^t \nabla v\, d\tau)\Big\|_{L_\infty(\mathbb{R}_+, W^1_q(\Omega))} \leqq C\epsilon, \tag{3.6}$$

which, combined with (3.3) and (As.4), implies that

$$e^{\gamma t}\operatorname{Div} U(v,\theta) \in L_p(\mathbb{R}_+, L_q(\Omega))^n, \quad \|e^{\gamma t}\operatorname{Div} U(v,\theta)\|_{L_p(\mathbb{R}_+, L_q(\Omega))} \leqq C\epsilon^2. \tag{3.7}$$

Next, we consider the term $\tilde G = V_4(\mathbb{F}_3 v)\mathbb{E}_3 v$. By (2.26) we see that $\tilde G = 0$ for $t < 0$. By (2.35), (2.25) and (2.26) we have

$$\begin{aligned}\|e^{\gamma t}\tilde G\|_{W^1_p(\mathbb{R}_+, L_q(\Omega))} \leqq C\{&\|\partial_t(V_4(\mathbb{F}_3 v))\|_{L_\infty(\mathbb{R}_+, L_q(\Omega))}\|e^{\gamma t} v\|_{L_p(\mathbb{R}_+, W^1_q(\Omega))} \\ &+ \|V_4(\mathbb{F}_3 v)\|_{L_\infty(\mathbb{R}_+, W^1_q(\Omega))}\|e^{\gamma t} v\|_{W^1_p(\mathbb{R}_+, L_q(\Omega))}\}.\end{aligned} \tag{3.8}$$

By (2.27) and (As.4) we have

$$\|\mathbb{F}_3 v\|_{L_\infty(\mathbb{R}_+, W^1_q(\Omega))} \leqq C\epsilon. \tag{3.9}$$

Choosing ϵ small enough, in view of (3.9) we may assume that

$$\|\mathbb{F}_3 v\|_{L_\infty(\mathbb{R}_+, W^1_q(\Omega))} \leqq 1. \tag{3.10}$$

By (2.7), (3.9), (3.10), (2.28) and (As.4) we have

$$\begin{aligned}\|\partial_t V_j(\mathbb{F}_3 v)\|_{L_\infty(\mathbb{R}_+, L_q(\Omega))} &\leqq C\|\partial_t(\mathbb{F}_3 v)\|_{L_\infty(\mathbb{R}_+, L_q(\Omega))} \leqq C\epsilon, \\ \|V_j(\mathbb{F}_3 v)\|_{L_\infty(\mathbb{R}_+, W^1_q(\Omega))} &\leqq C\|\mathbb{F}_3 v\|_{L_\infty(\mathbb{R}_+, W^1_q(\Omega))} \leqq C\epsilon.\end{aligned} \tag{3.11}$$

Therefore, by (3.8), (3.11) and (As.4) we have

$$e^{\gamma t}\tilde G \in W^1_{p,0}(\mathbb{R}_+, L_q(\Omega))^n, \quad \|e^{\gamma t}\tilde G\|_{W^1_p(\mathbb{R}_+, L_q(\Omega))} \leqq C\epsilon^2. \tag{3.12}$$

Now, we consider the term $G = V_3(\mathbb{F}_3 v)\nabla\mathbb{E}_3 v$. By (2.26) we see that $G = 0$ for $t < 0$. By (2.7) we have

$$\|e^{\gamma t} G\|_{L_p(\mathbb{R}_+, W^1_q(\Omega))} \leqq C\|V_3(\mathbb{F}_3 v)\|_{L_\infty(\mathbb{R}_+, W^1_q(\Omega))}\|e^{\gamma t}\mathbb{E}_3 v\|_{L_p(\mathbb{R}_+, W^2_q(\Omega))},$$

which, combined with Lemma 2.2, (As.4) and (3.11), implies that

$$e^{\gamma t} G \in L_{p,0}(\mathbb{R}_+, W^1_q(\Omega)), \quad \|e^{\gamma t} G\|_{L_p(\mathbb{R}_+, W^1_q(\Omega))} \leqq C\epsilon^2. \tag{3.13}$$

Finally, we shall estimate $\|e^{\gamma t} H\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}$. To do this, we need the following lemma.

Lemma 3.1. *Let $1 < p < \infty$ and $n < q < \infty$. If $f \in W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})$ and $g \in H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})$, then $fg \in H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})$ and*

$$\|fg\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq C\|f\|_{W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})}\|g\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}.$$

To prove Lemma 3.1, first we shall show the following lemma.

Lemma 3.2. *Let $1 < p < \infty$ and $n < q < \infty$. If $f \in W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})$ and $g \in H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R})$, then $fg \in H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R})$ and*

$$\|fg\|_{H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq C\|f\|_{W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})}\|g\|_{H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R})}.$$

Proof. To show Lemma 3.2, we use (2.32). Therefore, first we consider the case where $g \in W^{1,1}_{q,p}(\Omega\times\mathbb{R})$. By (2.7) we have

$$\begin{aligned}\|\partial_t(fg)\|_{L_q(\Omega)} &\leqq \|f_t g\|_{L_q(\Omega)} + \|f g_t\|_{L_q(\Omega)}\\ &\leqq C_q(\|f_t\|_{L_q(\Omega)}\|g\|_{W^1_q(\Omega)} + \|f\|_{W^1_q(\Omega)}\|g_t\|_{L_q(\Omega)}),\end{aligned}$$

and therefore we have

$$\|\partial_t(fg)\|_{L_p(\mathbb{R},L_q(\Omega))} \leqq C_q\|f\|_{W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})}\|g\|_{W^{1,1}_{q,p}(\Omega\times\mathbb{R})}, \tag{3.14}$$

which, combined with (2.35) and (2.36), implies that

$$\|fg\|_{W^{1,1}_{q,p}(\Omega\times\mathbb{R})} \leqq C_q\|f\|_{W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})}\|g\|_{W^{1,1}_{q,p}(\Omega\times\mathbb{R})}. \tag{3.15}$$

On the other hand, by (2.35) we have

$$\begin{aligned}\|fg\|_{L_p(\mathbb{R},L_q(\Omega))} &\leqq C\|f\|_{L_\infty(\mathbb{R},W^1_q(\Omega))}\|g\|_{L_p(\mathbb{R},L_q(\Omega))}\\ &\leqq C\|f\|_{W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})}\|g\|_{L_p(\mathbb{R},L_q(\Omega))}\end{aligned}$$

for $g \in L_p(\mathbb{R}, L_q(\Omega))$, which, combined with (3.15) and (2.32), implies Lemma 3.2. □

Now, we shall prove Lemma 3.1. By Lemma 3.2 we have

$$\begin{aligned}\|fg\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} \leqq \|fg\|_{H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R})} &\leqq C_{q,p}\|f\|_{W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})}\|g\|_{H^{1/2,1/2}_{q,p}(\Omega\times\mathbb{R})}\\ &\leqq C_{q,p}\|f\|_{W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})}\|g\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})},\end{aligned}$$

which, combined with (2.36), implies Lemma 3.1. □

Recall that $H = V_1(\mathbb{F}_3 v)\nabla \mathbb{E}_3 v + V_2(\mathbb{F}_3 v)\tilde{\theta}$. By (2.26) we see that $H = 0$ for $t < 0$. By Lemma 3.1 we have

$$\|e^{\gamma t} H\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq C\Big(\sum_{j=1}^{2} \|V_j(\mathbb{F}_3 v)\|_{W^{1,1}_{q,\infty}(\Omega\times\mathbb{R})}\Big)\|e^{\gamma t}(\nabla \mathbb{E}_3 v, \tilde{\theta})\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})},$$

which, combined with (3.11), (2.42), Lemma 2.2 and (As.4), implies that

$$e^{\gamma t} H \in H^{1,1/2}_{q,p,0}(\Omega\times\mathbb{R}_+)^{n\times n}, \quad \|e^{\gamma t} H\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq C\epsilon^2. \tag{3.16}$$

Applying Theorem 1.7 to (3.2) and using (3.7), (3.12), (3.13) and (3.16), we see that the equation (3.2) admits a unique solution:

$$(u,\pi) \in W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)^n \times L_p(\mathbb{R}_+, W^1_q(\Omega)),$$

which satisfies the estimate

$$\|e^{\gamma t} u\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)} + \|e^{\gamma t}\pi\|_{L_p(\mathbb{R}_+, W^1_q(\Omega))} \leqq C\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} + C\epsilon^2. \tag{3.17}$$

By the boundary condition of (3.2), we have

$$\pi = [D(u)\nu + H\nu]\cdot\nu \quad \text{on } \Gamma\times\mathbb{R}_+. \tag{3.18}$$

Set

$$\tilde{\pi} = [D(\mathbb{E}_3 u)\nu + H\nu]\cdot\nu. \tag{3.19}$$

By Lemma 2.2 (3) and (2.42), we see that

$$\begin{gathered} [D(\mathbb{E}_3 u)\nu]\cdot\nu = [D(u)\nu]\cdot\nu \quad \text{on } \Omega\times\mathbb{R}_+, \\ e^{\gamma t}[D(\mathbb{E}_3 u)\nu]\cdot\nu \in H^{1,1/2}_{q,p}(\Omega\times\mathbb{R}), \\ \|e^{\gamma t}[D(\mathbb{E}_3 u)\nu]\cdot\nu\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq C\|e^{\gamma t} u\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)}. \end{gathered} \tag{3.20}$$

By (3.16), (3.18), (3.19) and (3.20) we have

$$\begin{gathered} \pi = \tilde{\pi} \quad \text{on } \Gamma\times\mathbb{R}_+, \quad e^{\gamma t}\tilde{\pi} \in H^{1,1/2}_{q,p}(\Omega\times\mathbb{R}), \\ \|e^{\gamma t}\tilde{\pi}\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq C\|e^{\gamma t} u\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)} + C\epsilon^2. \end{gathered} \tag{3.21}$$

Combining (3.17) and (3.21), we have

$$\begin{aligned} \|e^{\gamma t} u\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)} + \|e^{\gamma t}\pi\|_{L_p(\mathbb{R}_+, W^1_q(\Omega))} + \|e^{\gamma t}\tilde{\pi}\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} & \\ \leqq C\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} + C\epsilon^2. & \end{aligned} \tag{3.22}$$

Choosing u_0 and $\epsilon > 0$ in such a way that $C\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} \leqq \epsilon/2$, $C\epsilon \leqq 1/2$ in (3.22), we have

$$\|e^{\gamma t}\, u\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)} + \|e^{\gamma t}\,\pi\|_{L_p(\mathbb{R}_+, W^1_q(\Omega))} + \|e^{\gamma t}\,\tilde\pi\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq \epsilon. \tag{3.23}$$

If we define the map Φ by $\Phi(v,\theta,\tilde\theta) = (u,\pi,\tilde\pi)$, then it follows from (3.23) that Φ maps $\mathcal{I}_{\gamma,\epsilon}$ into itself.

In the same manner, we can show that given $(v_j,\theta_j,\tilde\theta_j) \in \mathcal{I}_{\gamma,\epsilon}$, $j = 1,2$, we have

$$\begin{aligned}
&\|e^{\gamma t}(u_1-u_2)\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)} + \|e^{\gamma t}\,(\pi_1-\pi_2)\|_{L_p(\mathbb{R}_+, W^1_q(\Omega))} \\
&\quad + \|e^{\gamma t}\,(\tilde\pi_1-\tilde\pi_2)\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leqq C\epsilon[\|e^{\gamma t}(v_1-v_2)\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R}_+)} \\
&\qquad + \|e^{\gamma t}\,(\theta_1-\theta_2)\|_{L_p(\mathbb{R}_+, W^1_q(\Omega))} + \|e^{\gamma t}\,(\tilde\theta_1-\tilde\theta_2)\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}],
\end{aligned} \tag{3.24}$$

where we have set $(u_j,\pi_j,\tilde\pi_j) = \Phi(v_j,\theta_j,\tilde\theta_j)$, $j = 1,2$. If we choose $\epsilon > 0$ so small that $C\epsilon < 1$ in (3.24), then we see that Φ is a contraction map on $\mathcal{I}_{\gamma,\epsilon}$, and therefore there exists a fixed point $(u,\pi,\tilde\pi)$ of Φ in $\mathcal{I}_{\gamma,\epsilon}$ which solves the equation

$$\begin{aligned}
&u_t - \mathrm{Div}\,[S(u,\pi) + U(u,\pi)] = 0 && \text{in } \Omega\times\mathbb{R}_+ \\
&\mathrm{div}\,u + E(u) = \mathrm{div}\,[u + \tilde E(u)] = 0 && \text{in } \Omega\times\mathbb{R}_+ \\
&[S(u,\pi) + U(u,\pi)]\nu|_\Gamma = 0, \quad u|_{t=0} = u_0.
\end{aligned} \tag{3.25}$$

Since we construct a solution (u,π) of (3.25) by the contraction mapping principle, the uniqueness of solutions belonging to $\mathcal{I}_{\gamma,\epsilon}$ follows immediately, which completes the proof of Theorem 1.4.

APPENDIX A. DERIVATION OF (1.4)

In this appendix, we shall derive (1.4) from (1.1). Since $u(\xi,t) = v(x,t)$ and since x and ξ are related by the formula (1.3) we have

$$\frac{\partial}{\partial t}u(\xi,t) = v_t + \sum_{j=1}^n \frac{\partial v}{\partial x_j}\frac{dx_j}{dt} = v_t + \sum_{j=1}^n \frac{\partial v}{\partial x_j}u_j = v_t + (v\cdot\nabla)v. \tag{A.1}$$

Set

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{jk} = \frac{\partial x_j}{\partial \xi_k} = \delta_{jk} + \int_0^t \frac{\partial u_j}{\partial \xi_k}(\xi,\tau)\,d\tau. \tag{A.2}$$

Since $\delta_{ij} = \frac{\partial \xi_i}{\partial \xi_j} = \sum_{k=1}^n \frac{\partial \xi_i}{\partial x_k} \frac{\partial x_k}{\partial \xi_j}$, we have

$$A^{-1} = \begin{pmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} & \cdots & \frac{\partial \xi_1}{\partial x_n} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} & \cdots & \frac{\partial \xi_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \xi_n}{\partial x_1} & \frac{\partial \xi_n}{\partial x_2} & \cdots & \frac{\partial \xi_n}{\partial x_n} \end{pmatrix}. \tag{A.3}$$

Let A_{ij} be the (i,j) cofactor of A; then from (A.3) we have

$$(\det A) \frac{\partial \xi_i}{\partial x_j} = A_{ij}. \tag{A.4}$$

We see that $\det A = 1$. In fact, from (A.2) and (A.4) we have

$$\begin{aligned} &\frac{\partial}{\partial t}(\det A) \\ &= \begin{bmatrix} \frac{\partial u_1}{\partial \xi_1} & \int_0^t \frac{\partial u_1}{\partial \xi_2}\, d\tau & \cdots & \int_0^t \frac{\partial u_1}{\partial \xi_n}\, d\tau \\ \frac{\partial u_2}{\partial \xi_1} & 1 + \int_0^t \frac{\partial u_2}{\partial \xi_2}\, d\tau & \cdots & \int_0^t \frac{\partial u_2}{\partial \xi_n}\, d\tau \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial u_n}{\partial \xi_1} & \int_0^t \frac{\partial u_n}{\partial \xi_2}\, d\tau & \cdots & 1 + \int_0^t \frac{\partial u_n}{\partial \xi_n} \end{bmatrix} + \cdots = \sum_{k=1}^n \Big(\sum_{j=1}^n \frac{\partial u_j}{\partial \xi_k} A_{kj} \Big) \\ &= \sum_{j=1}^n \Big(\sum_{k=1}^n \frac{\partial u_j}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_j} \Big) \det A = (\det A) \sum_{j=1}^n \frac{\partial v_j}{\partial x_j} = (\det A) \operatorname{div} v = 0, \end{aligned}$$

and therefore $\det A = \det A|_{t=0} = \det(\delta_{ij}) = 1$. By (A.4), we have

$$\frac{\partial \xi_i}{\partial x_j} = A_{ij} = (-1)^{i+j} \det \left(\begin{array}{cc|c|cc} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \hline a_{j1} & \cdots & a_{ji} & \cdots & a_{jn} \\ \hline \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{array} \right) \tag{A.5}$$

$$= \delta_{ij} + B_{ij} \Big(\int_0^t \nabla_i' u_1\, d\tau, \ldots, \int_0^t \nabla_i' u_{j-1}\, d\tau, \int_0^t \nabla_i' u_{j+1}\, d\tau, \ldots, \int_0^t \nabla_i' u_n\, d\tau \Big)$$

with some polynomial B_{ij} such as $B_{ij}(0, \ldots, 0) = 0$, where we have set

$$\nabla_i' v = (\partial v / \partial \xi_1, \ldots, \partial v / \partial \xi_{i-1}, \partial v / \partial \xi_{i+1}, \ldots, \partial v / \partial \xi_n).$$

To derive (1.4), we also use the fact that

$$\sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\Big(\frac{\partial \xi_i}{\partial x_j}\Big) = 0. \tag{A.6}$$

To show (A.6), we set $\mathbf{e}_k = (a_{1k}, \dots, a_{j-1\,k}, a_{j+1\,k}, \dots, a_{nk})^*$, where M^* denotes the transpose of M. Note that

$$\begin{aligned}\frac{\partial}{\partial \xi_\ell}\mathbf{e}_k = \Big(\int_0^t \frac{\partial^2 u_1}{\partial \xi_\ell \partial \xi_k}\, d\tau, \dots, \int_0^t \frac{\partial^2 u_{j-1}}{\partial \xi_\ell \partial \xi_k}\, d\tau, \int_0^t \frac{\partial^2 u_{j+1}}{\partial \xi_\ell \partial \xi_k}\, d\tau,\\ \dots, \int_0^t \frac{\partial^2 u_n}{\partial \xi_\ell \partial \xi_k}\, d\tau\Big)^* = \frac{\partial}{\partial \xi_k}\mathbf{e}_\ell. \end{aligned}\tag{A.7}$$

Therefore, by (A.7) we have

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\Big(\frac{\partial \xi_i}{\partial x_j}\Big) &= \sum_{i=1}^{n}(-1)^{i+j}\frac{\partial}{\partial \xi_i}\det(\mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_n)\\ = (-1)^{j+1}\{&\det(\partial_1\mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_n) + \det(\mathbf{e}_2, \partial_1\mathbf{e}_3, \dots, \mathbf{e}_n)\\ &+ \cdots + \det(\mathbf{e}_2, \mathbf{e}_3, \dots, \partial_1\mathbf{e}_n)\\ &- \det(\partial_2\mathbf{e}_1, \mathbf{e}_3, \dots, \mathbf{e}_n) - \det(\mathbf{e}_1, \partial_2\mathbf{e}_3, \dots, \mathbf{e}_n)\\ &- \cdots - \det(\mathbf{e}_1, \mathbf{e}_3, \dots, \partial_2\mathbf{e}_n)\\ &+ \det(\partial_3\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \dots, \mathbf{e}_n) + \det(\mathbf{e}_1, \partial_3\mathbf{e}_2, \mathbf{e}_4, \dots, \mathbf{e}_n)\\ &+ \cdots + \det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \dots, \partial_3\mathbf{e}_n)\\ &+ \cdots\cdots + (-1)^{n-1}\det(\partial_n\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n-1})\\ &+ \cdots + (-1)^{n-1}\det(\mathbf{e}_1, \mathbf{e}_2, \dots, \partial_n\mathbf{e}_{n-1})\} = 0.\end{aligned}$$

Given a vector of functions $f(x) = (f_1(x), \dots, f_n(x))^*$, we set $g(\xi) = f(x)$, and then by (A.6) we have

$$\operatorname{div} f = \sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j} = \sum_{j,k=1}^{n}\frac{\partial \xi_k}{\partial x_j}\frac{\partial g_j}{\partial \xi_k} = \sum_{k=1}^{n}\frac{\partial}{\partial \xi_k}\Big(\sum_{j=1}^{n}\frac{\partial \xi_k}{\partial x_j} g_j\Big). \tag{A.8}$$

Concerning the boundary condition, we may assume that Γ is defined locally by $F(\xi) = 0$ with some function $F \in C^{2,1}$. In particular, $\nu = (\nu_1, \dots, \nu_n)^*$ denoting the unit outward normal to Γ, we have

$$\nu \parallel (\frac{\partial F}{\partial \xi_1}, \dots, \frac{\partial F}{\partial \xi_n})^*, \tag{A.9}$$

where $a \parallel b$ means that the vectors a and b are parallel. Let $\xi = \xi(x,t)$ be the inverse of the map $x = X_u(\xi,t)$, and then $F(\xi(x,t)) = 0$ for $x \in \Gamma_t$, from which it follows that

$$dF(\xi(x,t)) = \sum_{k=1}^n \Big(\sum_{j=1}^n \frac{\partial F}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_k}\Big)\,dx_k.$$

Therefore, denoting the unit outward normal to Γ_t by ν_t, we have

$$\nu_t \parallel \Big(\sum_{j=1}^n \frac{\partial F}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_1}, \ldots, \sum_{j=1}^n \frac{\partial F}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_n}\Big)^*. \tag{A.10}$$

By (A.9) and (A.10), we have

$$\begin{aligned} f\cdot\nu_t &= c\sum_{j=1}^n g_j\Big(\sum_{k=1}^n \frac{\partial F}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j}\Big) \\ &= c\sum_{k=1}^n\Big(\sum_{j=1}^n \frac{\partial \xi_k}{\partial x_j}g_j\Big)\frac{\partial F}{\partial \xi_k} = c'\sum_{k=1}^n\Big(\sum_{j=1}^n \frac{\partial \xi_k}{\partial x_j}g_j\Big)\nu_k, \end{aligned}$$

where c and c' are some scalar functions. In particular, if $f\cdot\nu_t = 0$ on Γ_t then

$$\sum_{k=1}^n\Big(\sum_{j=1}^n \frac{\partial \xi_k}{\partial x_j}g_j\Big)\nu_k = 0 \quad \text{on } \Gamma. \tag{A.11}$$

By (A.5) we have

$$\begin{aligned} D(v)_{ij} = \partial_i v_j + \partial_j v_i &= \sum_{k=1}^n\Big(\frac{\partial \xi_k}{\partial x_i}\frac{\partial u_j}{\partial \xi_k} + \frac{\partial \xi_k}{\partial x_j}\frac{\partial u_i}{\partial \xi_k}\Big) \\ &= \sum_{k=1}^n\{(\delta_{ki}+B_{ki})\frac{\partial u_j}{\partial \xi_k} + (\delta_{kj}+B_{kj})\frac{\partial u_i}{\partial \xi_k}\} \\ &= D(u)_{ij} + \sum_{k=1}^n\Big(B_{ki}\frac{\partial u_j}{\partial \xi_k} + B_{kj}\frac{\partial u_i}{\partial \xi_k}\Big). \end{aligned} \tag{A.12}$$

In view of (A.8) and (A.12), we set

$$\begin{aligned} \tilde S_{ik}(u,\pi) &= \sum_{j=1}^n \frac{\partial \xi_k}{\partial x_j}\Big(D(u)_{ij} + \sum_{\ell=1}^n\Big(B_{\ell i}\frac{\partial u_j}{\partial \xi_\ell} + B_{\ell j}\frac{\partial u_i}{\partial \xi_\ell}\Big) - \delta_{ij}\pi\Big) \\ &= \sum_{j=1}^n(\delta_{kj}+B_{kj})\Big\{D(u)_{ij} + \sum_{\ell=1}^n\Big(B_{\ell i}\frac{\partial u_j}{\partial \xi_\ell} + B_{\ell j}\frac{\partial u_i}{\partial \xi_\ell}\Big) - \delta_{ij}\pi\Big\} \end{aligned}$$

$$
\begin{aligned}
&= D(u)_{ik} - \delta_{ik}\pi + \sum_{\ell=1}^{n}\Big(B_{\ell i}\frac{\partial u_k}{\partial \xi_\ell} + B_{\ell k}\frac{\partial u_i}{\partial \xi_\ell}\Big) + \sum_{j=1}^{n} B_{kj} D(u)_{ij} \\
&\qquad + \sum_{j,\ell=1}^{n}\Big(B_{kj}B_{\ell i}\frac{\partial u_j}{\partial \xi_\ell} + B_{kj}B_{\ell j}\frac{\partial u_i}{\partial \xi_\ell}\Big) - B_{ki}\pi \\
&= S_{ik}(u,\pi) + U_{ik}\Big(\int_0^t \nabla u\,d\tau\Big)\nabla u + V_{ik}\Big(\int_0^t \nabla u\,d\tau\Big)\pi,
\end{aligned}
$$

where we have set

$$
\begin{aligned}
U_{ik}\Big(\int_0^t \nabla u\,d\tau\Big)\nabla u &= \sum_{\ell=1}^{n}\Big\{(B_{\ell k} + B_{k\ell})\frac{\partial u_i}{\partial \xi_\ell} + B_{\ell i}\frac{\partial u_k}{\partial \xi_\ell} + B_{k\ell}\frac{\partial u_\ell}{\partial \xi_i}\Big\} \\
&\qquad + \sum_{j,\ell=1}^{n}\Big(B_{kj}B_{\ell i}\frac{\partial u_j}{\partial \xi_\ell} + B_{kj}B_{\ell j}\frac{\partial u_i}{\partial \xi_\ell}\Big), \\
V_{ik}\Big(\int_0^t \nabla u\,d\tau\Big) &= -B_{ki}\Big(\int_0^t \nabla u\,d\tau\Big).
\end{aligned}
$$

By (1.3), (A.1) and (A.8), the equation of motion becomes

$$
\partial_t u_i - \sum_{j=1}^{n}\frac{\partial}{\partial \xi_j}\tilde{S}_{ij}(u,\pi) = f_i(X_u(\xi,t),t), \quad \xi \in \Omega, \ \ i = 1,\dots,n.
$$

By (A.11), the boundary condition $S(v,\theta)\nu_t = 0$ becomes

$$
\sum_{j=1}^{n}\tilde{S}_{ij}(u,\pi)\nu_j = 0 \quad \text{on } \Gamma, \ \ i = 1,\dots,n.
$$

Concerning the divergence condition, by (A.5) and (A.6) we have

$$
\operatorname{div} v = \sum_{j,k=1}^{n}\frac{\partial \xi_k}{\partial x_j}\frac{\partial u_j}{\partial \xi_k} = \sum_{j,k=1}^{n}(\delta_{kj} + B_{kj})\frac{\partial u_j}{\partial \xi_k} = \operatorname{div} u + \sum_{j,k=1}^{n} B_{kj}\frac{\partial u_j}{\partial \xi_k}, \tag{A.13}
$$

$$
\begin{aligned}
\operatorname{div} v &= \sum_{j,k=1}^{n}\frac{\partial}{\partial \xi_k}\Big(\frac{\partial \xi_k}{\partial x_j}u_j\Big) = \sum_{j,k=1}^{n}\frac{\partial}{\partial \xi_k}\{(\delta_{kj} + B_{kj})u_j\} \\
&= \operatorname{div} u + \sum_{k=1}^{n}\frac{\partial}{\partial \xi_k}\Big(\sum_{j=1}^{n} B_{kj}u_j\Big).
\end{aligned} \tag{A.14}
$$

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