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APPROXIMATION OF SOLUTIONS TO NON-LINEAR INTEGRODIFFERENTIAL PARABOLIC EQUATIONS IN L^p-SPACES

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Abstract. This paper is concerned with non-linear parabolic integrodifferential equations arising in continuum mechanics, phase-field models, and elsewhere where memory terms are important. More precisely, we will consider a non-linear initial-value problem depending on a small parameter and related to a uniformly elliptic second-order differential operator A. The integrodifferential character of our problem is expressed by a convolution term multiplying A, the scalar kernel approximating a delta-type function. Consequently, the limit problem reduces to a non-linear parabolic differential problem involving a multiple of operator A. Two different non-linearities, which are locally Lipschitz-continuous in suitable metrics, are considered. The basic aim of the paper consists in determining the rate of convergence of the approximating solutions to the exact one. The results proved in the linear case (cf. [10]) are used here as starting points to solve our problems.

1. Introduction

In this paper, we will consider the following non-linear initial-value problem depending on the (small) positive parameter ε in $L^\alpha(\Omega)$, $\Omega \subset \mathbf{R}^n$ being a bounded domain with a C^2-boundary,

$$(P_\varepsilon)\quad \begin{cases} u_\varepsilon'(t) - \chi(\varepsilon)Au(t) - A((k_\varepsilon + \varphi(\varepsilon)h_\varepsilon) * u_\varepsilon)(t) \\ \qquad\qquad\qquad = f_\varepsilon(t) + Nu_\varepsilon(t), \quad \forall t \in (0,T], \\ u_\varepsilon(0) = u_{0,\varepsilon}. \end{cases} \tag{1.1}$$

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In problem (P_ε) we use the following notation:

$$l_\varepsilon(t) = \varepsilon^{-1} l(t/\varepsilon), \quad l_\varepsilon * u_\varepsilon(t) = \int_0^t l_\varepsilon(t-s) u_\varepsilon(s)\, ds. \tag{1.2}$$

Moreover, A denotes the second-order differential operator in divergence form:

$$A := A(x, D_x) = -\sum_{i,j=1}^{n} D_{x_i}[a_{i,j}(x) D_{x_j}] + a_0(x), \tag{1.3}$$

endowed with Dirichlet boundary conditions, where

$$a_{i,j} \in C^1(\overline{\Omega}), \quad a_0 \in C(\overline{\Omega}), \quad a_{i,j} = a_{j,i}, \quad i, j = 1, \dots, n, \tag{1.4}$$

$$c_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq c_1|\xi|^2, \quad \forall x \in \overline{\Omega},\ \forall \xi \in \mathbf{R}^n, \quad a_0(x) \geq \gamma,\ \forall \xi \in \mathbf{R}^n,$$

c_0, c_1 and γ being three positive constants.

We will also assume, as in [10], that the kernels k and h have particular series representation (see formula (2.1) in Section 2).

Moreover, χ and φ are *positive scalar* functions defined in $(0, \varepsilon_0]$ such that

$$\chi(\varepsilon) \to \chi_0 \geq 0 \quad \text{and} \quad \varphi(\varepsilon) \to 0 \text{ as } \varepsilon \to 0+. \tag{1.5}$$

Finally, N is a *non-linear* operator admitting either of the following representations:

(B)

$$N(u)(t, x) = \int_\Omega l(t, x, y)\Psi(u)(y) dy, \quad u \in L^\alpha(\Omega), \tag{1.6}$$

where the kernel l is a smooth function from $[0, T] \times \Omega \times \Omega$ into $\mathbf{R}$ and Ψ is a locally Lipschitz non-linear operator from $L^\alpha(\Omega)$ into $L^{\alpha_0}(\Omega)$, with $\alpha > \alpha_0$;

(C)

$$Nu(x) = \psi(u(x)), \tag{1.7}$$

where ψ is a smooth function from $\mathbf{R}$ into $\mathbf{R}$.

We stress that the non-linear operator N is, in both cases, *assumed to be not monotone* at all. For instance, in the latter case, a typical example is $Nu = u \sin(u^\gamma)$, with $\gamma > 0$.

Our main task will consist in showing that, under suitable assumptions on operators A and N, functions f_ε and the initial data $u_{0,\varepsilon}$, the solution u_ε to problem (1.1) converges in $L^p((0, T); L^\alpha(\Omega))$-norm, for suitable $p \in [1, +\infty)$

and s varying in a subinterval in $[1,+\infty)$ containing $\alpha = 2$, to the solution to the *limit problem*

$$(P_0)\ \begin{cases} u'(t) - (1+\chi_0)Au(t) = f(t) + Nu(t), & \forall t \in (0,T], \\ u(0) = u_0. \end{cases} \tag{1.8}$$

We note that the present paper is an extension to the non-linear case of our previous paper [10], concerned with the linear case and L^p convergence.

Our first task consists in proving also the existence of the solutions u_ε and u to problems (1.1) and (1.8) since, in our assumptions, the operator N will be non-globally Lipschitz-continuous.

Due to the lack of monotonicity of operator N, in both cases we are forced to deal with non-linear operators with a *sublinear growth*.

We hope that this paper may be useful for a possible approximation, in $L^p((0,T);L^\alpha(\Omega))$ spaces, of solutions to phase-field problems, usually related to Hilbert spaces. Concerning this subject, we quote, e.g., the papers [1], [2], [3], [4], [5], [8], and [9].

Our main results are stated in Section 2. Section 3 is devoted to showing some basic properties of the kernels k and h, and of the non-linear operator N. The proofs of our main theorems are given in Section 5. They are strictly related to some fundamental auxiliary results (cf. Section 4)—in a general Banach-space framework—which are concerned with existence and uniqueness of solutions to problems related to more general kernels k and h, and operators A and N. Such auxiliary results will be proved in Section 6.

Finally, we observe that, if the non-linear operator N has the representation (B), the operator A and the functions χ and φ have the same properties as in the linear case (cf. [10]). Instead, if N has the representation (C), an additional assumption on function χ is required, i.e., χ should converge, as $\varepsilon \to 0+$, to a *strictly positive* limit χ_0.

2. Main results

In this section, we give the particular form of the pair of kernels k and h we will consider. They admit the representations

$$k(t) = \sum_{n=0}^{+\infty} k_n \Gamma(\alpha_n)^{-1} t^{\alpha_n - 1} \mathrm{e}^{-\beta_n t}, \tag{2.1}$$

$$h(t) = \sum_{n=0}^{+\infty} h_n \Gamma(\gamma_n)^{-1} t^{\gamma_n - 1} \mathrm{e}^{-\delta_n t}, \qquad \forall t \in \mathbf{R}_+,$$

where $\{\alpha_n\}_{n=0}^{+\infty} \subset (0,1)$ and $\{\gamma_n\}_{n=0}^{+\infty} \subset (0,1)$ are two *non-decreasing* sequences converging to $\alpha_\infty \in (0,1)$ and $\gamma_\infty \in (0,1)$, respectively, while $\{\beta_n\}_{n=0}^{+\infty}$ and $\{\delta_n\}_{n=0}^{+\infty}$ are two sequences in $\mathbf{R}_+$, the first being bounded from below by a positive constant

$$\beta_n \geq \overline{\beta}_0, \qquad \forall n \in \mathbf{N}. \tag{2.2}$$

Finally, $\{k_n\}_{n=0}^{+\infty} \subset \mathbf{R}_+$ and $\{h_n\}_{n=0}^{+\infty} \subset \mathbf{R}_+$ are two sequences such that

$$\sum_{n=0}^{+\infty} k_n \beta_n^{-\alpha_n} = 1, \quad \sum_{n=0}^{+\infty} h_n \delta_n^{-\gamma_n} < +\infty, \quad \sum_{n=0}^{+\infty} h_n < +\infty. \tag{2.3}$$

Remark 2.1. The assumption concerning $\{\alpha_n\}$ and $\{\beta_n\}$ and the first equality in (2.3) imply $\sum_{n=0}^{+\infty} k_n < +\infty$. Likewise, the assumption $\sum_{n=0}^{+\infty} h_n$ in (2.3) can be gotten rid of if we assume that $\{\delta_n\}$ is bounded from below by a positive constant.

We list now the assumptions concerning operator N in cases (B) and (C) (cf. the Introduction), where assumption (C2) depends on the dimension n:

(B1) let $\alpha, \alpha_0 \in [1, +\infty)$, with $\alpha > \alpha_0$, $1/\alpha_0 + 1/\alpha_0' = 1$, and let $\Psi : L^\alpha(\Omega) \to L^{\alpha_0}(\Omega)$ be an operator for which there exist a positive constant C_0 and a non-decreasing function $C : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\|\Psi(u)\|_{L^{\alpha_0}(\Omega)} \leq C_0(1 + \|u\|_{L^\alpha(\Omega)}), \quad \forall u \in L^\alpha(\Omega),$$

$$\|\Psi(u_2) - \Psi(u_1)\|_{L^{\alpha_0}(\Omega)} \leq C(M)\|u_2 - u_1\|_{L^\alpha(\Omega)}, \quad \forall u_1, u_2, \tag{2.4}$$

$\|u_1\|_{L^\alpha(\Omega)} \leq M, \|u_2\|_{L^\alpha(\Omega)} \leq M$;

(B2) let $l : [0,T] \times \Omega \times \Omega \to \mathbf{R}$ be a function satisfying the following property for some positive constant C_1:

$$\sup_{t \in [0,T]} \int_\Omega \Big(\int_\Omega |l(t,x,y)|^{\alpha_0'} \, dy \Big)^{\alpha/\alpha_0'} dx \leq C_1;$$

(C1) let $\psi \in C^1(\mathbf{R})$ satisfy the growth conditions

$$|\psi(z)| \leq C_1(1 + |z|), \quad |\psi'(z)| \leq C_2(1 + |z|^\gamma), \quad \forall z \in \mathbf{R}, \tag{2.5}$$

C_1, C_2 being positive constants and γ satisfying (C2);

(C2) (I1) if $n = 1,2,3$: $\gamma \in [0, +\infty)$, $\alpha \in ((1+\gamma)\max\{n, 1/\gamma\}, +\infty)$;

(I2) if $n = 1,2,3$: $\gamma \in [0, \frac{1}{n})$, $\alpha \in ((\gamma+1)/\gamma/[\gamma(1 - \min\{4\gamma, 1\})])$;

(II) if $n \geq 4$: $\gamma \in (\frac{1}{n}, 4/(n-4)]$, $\alpha \in [\max\{n\gamma, n/(n-1)\}, 4n/(n-4))$.

We give now an example of an operator Ψ satisfying (B1): let $\Psi(u)(x) = \psi(u(x))$ where $\psi : \mathbf{R} \to \mathbf{R}$, $|\psi(z)| \leq C_3(1 + |z|^\gamma)$, $|\psi'(z)| \leq C_4(1 + |z|^{\gamma-1})$ for some positive constant C_2, C_3, $\gamma > 1$ and all $z \in \mathbf{R}$. Consequently

$\Psi : L^\alpha(\Omega) \to L^{\alpha_0}(\Omega)$, for any $\alpha \in [1, +\infty)$ and $\alpha_0 = \alpha/\gamma$. Let meas$(\Omega) < +\infty$; then inequalities in (2.4) are satisfied by $C(M) = C_5\{\text{meas}(\Omega) + 2M^\alpha[\gamma^{(\alpha\alpha_0)/(\alpha-\alpha_0)}]^{-1}\}^{(\alpha-\alpha_0)/(\alpha\alpha_0)}$, where C_5 is some positive constant. Indeed,

$$|\psi(u_2)(x) - \psi(u_1)(x)| \le \int_0^1 |\psi'((1-s)u_2(x) + su_1(x))|\, ds |u_2(x) - u_1(x)|$$

$$\le C_4\Big(1 + \int_0^1 [(1-s)|u_2(x)| + s|u_1(x)|]^{\gamma-1}\, ds\Big)|u_2(x) - u_1(x)|$$

$$\le C_6\Big(1 + \int_0^1 [(1-s)^{\gamma-1}|u_2(x)|^{\gamma-1} + s^{\gamma-1}|u_1(x)|^{\gamma-1}]\, ds\Big)|u_2(x) - u_1(x)|$$

$$= C_7\Big[1 + \frac{1}{\gamma}\Big(|u_2(x)|^{\gamma-1} + |u_1(x)|^{\gamma-1}\Big)\Big]|u_2(x) - u_1(x)|.$$

Let now $\rho = (\alpha\alpha_0)/(\alpha - \alpha_0)$. Taking the $L^{\alpha_0}(\Omega)$-norms, we get

$$\|\psi(u_2) - \psi(u_1)\|_{L^{\alpha_0}(\Omega)}$$

$$\le C_7\Big\{\int_\Omega \Big[1 + \frac{1}{\gamma}\Big(|u_2(x)|^{\gamma-1} + |u_1(x)|^{\gamma-1}\Big)\Big]^\rho dx\Big\}^{1/\rho}\|u_2 - u_1\|_{L^\alpha(\Omega)}$$

$$\le C_8\Big\{\int_\Omega \Big[1 + \frac{1}{\gamma^q}\Big(|u_2(x)|^{(\gamma-1)\rho} + |u_1(x)|^{(\gamma-1)\rho}\Big)\Big] dx\Big\}^{1/\rho}\|u_2 - u_1\|_{L^\alpha(\Omega)}$$

$$\le C_8\Big[\text{meas}(\Omega) + \gamma^{-\rho}\Big(\|u_2\|^\alpha_{L^\alpha(\Omega)} + \|u_1\|^\alpha_{L^\alpha(\Omega)}\Big)\Big]^{1/\rho}\|u_2 - u_1\|_{L^\alpha(\Omega)}$$

$$\le C_8\Big[\text{meas}(\Omega) + 2\gamma^{-\rho}M^\alpha\Big]^{1/\rho}\|u_2 - u_1\|_{L^\alpha(\Omega)}.$$

Now we can state the theorems concerning existence of the solutions u_ε and u to problems (P_ε) and (P_0), respectively, as well as the convergence of u_ε to u in $L^p((0,T); L^\alpha(\Omega))$-norm, for suitable $p \in [1, +\infty)$ and $\alpha \in [1, +\infty)$.

Theorem 2.1. *Let A, k, h, and N be defined by formulae* (1.3), (2.1), (2.3), and (1.6) *respectively, let Ψ and l satisfy assumptions* (B1) *and* (B2), *and let χ and φ satisfy* (1.5) *and $\alpha \in (\alpha_1, \alpha_2)$, where* (cf. [10])

$$\alpha_1 =: \tfrac{1}{2}(4 + c^2)^{1/2}[(4 + c^2)^{1/2} - c], \quad \alpha_2 =: \tfrac{1}{2}(4 + c^2)^{1/2}[(4 + c^2)^{1/2} + c], \tag{2.6}$$

and $c = \text{cotg}\big(\pi \max\{\alpha_\infty, \gamma_\infty\}/2\big)$. Then problems (1.1) *and* (1.8) *admit a unique solution u and u_ε in $C^{1+\sigma}([0,T]; L^\alpha(\Omega)) \cap C^\sigma([0,T]; W^{2,\alpha}(\Omega) \cap W_0^{1,\alpha}(\Omega))$ for any given pair $(f_\varepsilon, u_{0,\varepsilon}) \in C^\sigma([0,T]; L^\alpha(\Omega)) \times W^{2,\alpha}(\Omega) \cap W_0^{1,\alpha}(\Omega)$, $\varepsilon \in [0, \varepsilon_0]$, $\sigma \in (0,1)$, such that $Au_{0,\varepsilon} + Nu_{0,\varepsilon} + f_\varepsilon(0) \in W^{2\sigma,\alpha}(\Omega) \cap W_0^{\sigma,\alpha}(\Omega))$ with the agreement that $(f_0, u_{0,0}) = (f, u_0)$.*

Theorem 2.2. *Let A, k, h, and N be defined by formulae* (1.3), (2.1), (2.3), and (1.7) *respectively, and let ψ, χ, and φ satisfy assumptions* (C1) *and* (1.5) *with $\chi_0 > 0$. Assume that α satisfies, in addition to* (C2), *the inequality $\alpha_1 < \alpha < \alpha_2$* (cf. (2.6)). *Then problems* (1.1) *and* (1.8) *admit a unique solution u and u_ε in $C^{1+\sigma}([0,T];L^\alpha(\Omega)) \cap C^\sigma([0,T];W^{2,\alpha}(\Omega)\cap W_0^{1,\alpha}(\Omega))$ for any given pair $(f_\varepsilon, u_{0,\varepsilon}) \in C^\sigma([0,T];L^s(\Omega)) \times W^{2,\alpha}(\Omega)\cap W_0^{1,\alpha}(\Omega)$, $\varepsilon \in [0,\varepsilon_0]$, $\sigma \in (0,1)$, such that $Au_{0,\varepsilon} + Nu_{0,\varepsilon} + f_\varepsilon(0) \in W^{2\sigma,\alpha}(\Omega) \cap W_0^{\sigma,\alpha}(\Omega))$ with the agreement that $(f_0, u_{0,0}) = (f, u_0)$.*

Theorem 2.3. *Let A, k, h, and N be defined by formulae* (1.3), (2.1), (2.3), and (1.6), *respectively, let Ψ and l satisfy assumptions* (B1) *and* (B2), *and let χ and φ satisfy* (1.5). *Let $q \in [1,+\infty)$, $p \in [1,+\infty)$, and $\alpha \in (\alpha_1, \alpha_2)$* (cf. (2.6)), *and let $f_\varepsilon, f \in L^q((0,T);L^\alpha(\Omega))$ and $u_{0,\varepsilon}, u_0 \in L^\alpha(\Omega)$ be such that*

$$\|f_\varepsilon - f\|_{L^q((0,T);L^\alpha(\Omega))} \to 0, \quad \|u_{0,\varepsilon} - u_0\|_{L^\alpha(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0+\,.$$

Then, the solutions u_ε to the approximating problem (1.1) *converge, as $\varepsilon \to 0+$, in $L^p((0,T);L^\alpha(\Omega))$ to the solution u to the limit problem* (1.8). *More exactly, there exists a positive constant c_2 such that*

$$\begin{aligned}\|u_\varepsilon - u\|_{L^p((0,T);L^\alpha(\Omega))} \le c_2\big\{&\|u_{0,\varepsilon} - u_0\|_{L^\alpha(\Omega)} + \|f_\varepsilon - f\|_{L^q((0,T);L^\alpha(\Omega))}\\ &+ [\varepsilon^{1/p} + \varepsilon + |\chi(\varepsilon) - \chi_0| + \varphi(\varepsilon)]\big[\|u_0\|_{L^\alpha(\Omega)} + \|f\|_{L^q((0,T);L^\alpha(\Omega))}\big]\big\}.\end{aligned}$$

Theorem 2.4. *Let A, k, h, and N be defined by formulae* (1.3), (2.1), (2.3), and (1.7), *respectively, let ψ satisfy assumptions* (C1) *and* (C2), *and let χ and φ satisfy* (1.5) *with $\chi_0 > 0$. Let $q \in [1,+\infty)$ and $p \in [1,+\infty)$. Assume that α satisfies, in addition to* (C2), *the inequality $\alpha_1 < \alpha < \alpha_2$* (cf. (2.6)). *Let $f_\varepsilon, f \in L^q((0,T);L^\alpha(\Omega))$ and $u_{0,\varepsilon}, u_0 \in L^\alpha(\Omega)$ be such that*

$$\|f_\varepsilon - f\|_{L^q((0,T);L^\alpha(\Omega))} \to 0, \quad \|u_{0,\varepsilon} - u_0\|_{L^\alpha(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0+\,.$$

Then, the solutions u_ε to the approximating problem (1.1) *converge, as $\varepsilon \to 0+$, in $L^p((0,T);L^\alpha(\Omega))$ to the solution u to the limit problem* (1.8). *More exactly, there exists a positive constant c_3 such that*

$$\begin{aligned}\|u_\varepsilon - u\|_{L^p((0,T);L^\alpha(\Omega))} \le c_3\big\{&\|u_{0,\varepsilon} - u_0\|_{L^\alpha(\Omega)} + \|f_\varepsilon - f\|_{L^q((0,T);L^\alpha(\Omega))}\\ &+ [\varepsilon^{1/p} + \varepsilon + |\chi(\varepsilon) - \chi_0| + \varphi(\varepsilon)]\big[\|u_0\|_{L^\alpha(\Omega)} + \|f\|_{L^q((0,T);L^\alpha(\Omega))}\big]\big\}.\end{aligned}$$

3. PROPERTIES OF KERNELS k AND h AND OPERATOR N

First, we list the following three lemmata proved in [10]. For this purpose, let $\mathbf{C}_+ = \{\lambda \in \mathbf{C} : \mathrm{Re}\lambda > 0\}$ and $\mathbf{C}_- = \{\lambda \in \mathbf{C} : \mathrm{Re}\lambda < 0\}$.

Lemma 3.1. *The following properties hold true:*

(i) $k, h \in L^1((0,+\infty);\mathbf{R})$ *and* $\int_0^{+\infty} k(t)\,dt = 1$;

(ii) *the Laplace transform* $\widetilde{k}$ *and* $\widetilde{h}$ *of* k *and* h *are given, for all* $\lambda \in \overline{\mathbf{C}_+}$, *by* $\widetilde{k}(\lambda) = \sum_{n=0}^{+\infty} k_n(\lambda+\beta_n)^{-\alpha_n}$, $\widetilde{h}(\lambda) = \sum_{n=0}^{+\infty} h_n(\lambda+\delta_n)^{-\gamma_n}$;

(iii) *the Laplace transforms* $\widetilde{k}$ *and* $\widetilde{h}$ *extend to the sector* $\Sigma_{\varphi_1} = \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg\lambda| < \varphi_1\}$, $\varphi_1 \in (\pi/2,\pi)$, *as nowhere-vanishing holomorphic functions.*

Lemma 3.2. *Let* φ_1, φ_2 *be a pair such that*

$$\varphi_2 \in \Big(\frac{\pi}{2}(1+\max\{\alpha_\infty,\gamma_\infty\}),\pi\Big), \quad \varphi_1 \in \Big(\frac{\pi}{2}, \frac{\varphi_2}{1+\max\{\alpha_\infty,\gamma_\infty\}}\Big). \tag{3.1}$$

Then the functions $\lambda \to \lambda^{-1}\widetilde{k}(\lambda)$ *and* $\lambda \to \lambda^{-1}\widetilde{h}(\lambda)$ *map* Σ_{φ_1} *into* Σ_{φ_2}. *In particular,* $\lambda \to \lambda^{-1}\widetilde{k}(\lambda)$ *and* $\lambda \to \lambda^{-1}\widetilde{h}(\lambda)$ *map* $\Sigma^+_{\varphi_1} = \Sigma_{\varphi_1} \cap \mathbf{C}_+$ *into* $\Sigma^-_{\varphi_2} = \Sigma_{\varphi_1} \cap \mathbf{C}_-$ *and* $\Sigma^-_{\varphi_1}$ *into* $\Sigma^+_{\varphi_2}$.

Lemma 3.3. *The following properties hold true:*

(iv) $|\widetilde{h}(se^{\pm i\eta})| \le C_9$, *for all* $s \in \mathbf{R}_+$ *and some* $\eta \in (\pi/2,\pi)$,

(v) $|\widetilde{h}(\delta e^{i\theta})| \le C_{10}$, *for all* $\theta \in [-\eta,\eta]$ *and for some positive* δ;

(vi) $|\widetilde{k}(se^{\pm i\eta})| \le C_{11}$, *for all* $s \in \mathbf{R}_+$;

(vii) $|\widetilde{k}(\lambda) - 1| \le C_{12}|\lambda|$, *for all* $\lambda \in \Sigma_{\varphi_1} \cap B(0,r_0)$, *some* $r_0 > \delta$.

Now, we state two lemmata related to some basic properties of the nonlinear operator N.

Lemma 3.4. *Let* N *be defined by formula* (1.6) *and let* Ψ *and* l *satisfy assumptions* (B1) *and* (B2). *Then, for all* $\alpha \in [1,+\infty)$,

(viii)

$$N : \begin{cases} C^\sigma([0,T];L^\alpha(\Omega)) \to C^\sigma([0,T];L^\alpha(\Omega)), & \forall\sigma \in (0,1), \\ L^\infty((0,T);L^\alpha(\Omega)) \to L^\infty((0,T);L^\alpha(\Omega)); \end{cases}$$

(ix) $\|Nu(t,\cdot)\|_{L^\alpha(\Omega)} \le C_{13}(1+\|u\|_{L^\alpha(\Omega)})$, *for all* $u \in L^\alpha(\Omega)$, *all* $t \in [0,T]$ *and some positive constant* C_{13};

(x) *there exists a function* $C_{14} : \mathbf{R}_+ \to \mathbf{R}_+$ *such that* $\|Nv_2(t,\cdot) - Nv_1(t,\cdot)\|_{L^\alpha(\Omega)} \le C_{14}(M)\|v_2 - v_1\|_{L^\alpha(\Omega)}$, *for all* $t \in [0,T]$, $v_1, v_2 \in L^\alpha(\Omega)$ *satisfying* $\|v_1\|_{L^\alpha(\Omega)} \le M, \|v_2\|_{L^\alpha(\Omega)} \le M$.

Lemma 3.5. *Let N be defined by formula* (1.7), *ψ satisfy assumptions* (C1) *and* (C2) *and let $\widetilde{\alpha}$ be correspondingly such that*

(I1) *if $n = 1, 2, 3$: $\widetilde{\alpha} \in (\max\{n, 1/\gamma\}, \alpha/(1+\gamma))$;*
(I2) *if $n = 1, 2, 3$: $\widetilde{\alpha} \in (\max\{1/(\gamma+1), n, \alpha/(1+\alpha\gamma)\}, \min\{4, 1/\gamma, \alpha/(\gamma+1)\})$;*
(II) *if $n \geq 4$: $1/\widetilde{\alpha} = 1/\alpha + 1/n$.*

Then

(xi) *$N : L^{\overline{\alpha}}(\Omega) \to L^{\overline{\alpha}}(\Omega)$, with both $\overline{\alpha} = \alpha$ and $\overline{\alpha} = \widetilde{\alpha}$, and $\|Nu\|_{L^{\overline{\alpha}}(\Omega)} \leq C_0(1 + \|u\|_{L^{\overline{\alpha}}(\Omega)})$, for all $u \in L^{\overline{\alpha}}(\Omega)$;*
(xii) *N satisfies the inequality*

$$\|Nv_2 - Nv_1\|_{L^{\widetilde{\alpha}}(\Omega)} \leq \rho\Big(\sum_{i=1}^{2} \|v_i\|_{L^{\alpha}(\Omega)}\Big)\|v_2 - v_1\|_{L^{\alpha}(\Omega)}, \quad \forall v_1, v_2 \in L^{\alpha}(\Omega).$$

where $\rho : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous function such that $\rho(s) \leq C_{15}(1 + s^{\gamma})$, for all $s \in \mathbf{R}_+$.

Proof of Lemma 3.4. From assumptions (B1) and (B2) easily follow

$$\|Nu(t,\cdot)\|_{L^{\alpha}(\Omega)} \leq C_1^{1/\alpha}\|\Psi(u)\|_{L^{\alpha_0}(\Omega)},$$
$$\|Nu(t,\cdot)\|_{L^{\alpha}(\Omega)} \leq C_0 C_1^{1/\alpha}(1 + \|u\|_{L^{\alpha}(\Omega)}),$$
$$\|Nu_1(t,\cdot) - Nu_2(t,\cdot)\|_{L^{\alpha}(\Omega)} \leq C_1^{1/\alpha}C(M)\|u_2 - u_1\|_{L^{\alpha}(\Omega)}.$$

Proof of Lemma 3.5. We observe that the first growth condition in (2.5) ensures that N maps $L^{\alpha}(\Omega)$ into itself, for all $\alpha \in [1, +\infty]$. However the latter condition in (2.5) does not guarantee, whatever α may be, that N is a Lipschitz-continuous map from $L^{\alpha}(\Omega)$ into itself.

We easily get that N satisfies property (xi). Indeed, from the first estimate in (2.5), for all $v \in L^{\overline{\alpha}}(\Omega)$, $\overline{\alpha} \in [1, +\infty)$, we deduce

$$\|Nv\|_{L^{\overline{\alpha}}(\Omega)} \leq C_{13}(1 + \|v\|_{L^{\overline{\alpha}}(\Omega)}).$$

Now we show that property (xii) holds true. For this purpose, for all $v_1, v_2 \in L^{\alpha}(\Omega)$, we estimate

$$\begin{aligned} |Nv_2(x) - Nv_1(x)| &= |\psi(v_2(x)) - \psi(v_1(x))| \\ &\leq \int_0^1 |\psi'((1-y)v_1(x) + yv_2(x))|\, dy\, |v_2(x) - v_1(x)| \\ &\leq C_{16}\Big\{1 + \int_0^1 [(1-y)|v_1(x)| + y|v_2(x)|]^{\gamma}\, dy\Big\}|v_2(x) - v_1(x)| \end{aligned} \tag{3.2}$$

$$\leq C_{17}\Big\{1 + \int_0^1 [(1-y)^\gamma |v_1(x)|^\gamma + y^\gamma |v_2(x)|^\gamma]\, dy\Big\} |v_2(x) - v_1(x)|$$
$$= C_{17}\{1 + (\gamma+1)^{-1}[|v_1(x)|^\gamma + |v_2(x)|^\gamma]\}\, |v_2(x) - v_1(x)|.$$

We now consider separately the three cases (I1), (I2), (II).

(II) Taking the $L^{\widetilde{\alpha}}(\Omega)$-norm in (3.2), using twice Hölder's inequality, and recalling that $1 < n\gamma < \alpha$, we obtain

$$\begin{aligned}
&\|Nv_2 - Nv_1\|_{L^{\widetilde{\alpha}}(\Omega)} \qquad (3.3)\\
&\leq C_{17}\Big\{\int_\Omega \Big[1 + (\gamma+1)^{-1}(|v_1(x)|^\gamma + |v_2(x)|^\gamma)\Big]^n dx\Big\}^{\frac{1}{n}} \|v_2 - v_1\|_{L^\alpha(\Omega)}\\
&\leq C_{18}\Big[\int_\Omega (1 + |v_1(x)|^{\gamma n} + |v_2(x)|^{\gamma n})\, dx\Big]^{1/n} \|v_2 - v_1\|_{L^\alpha(\Omega)}\\
&\leq C_{18}\{[\mathrm{meas}(\Omega)]^{1/n} + \|v_1\|^\gamma_{L^{\gamma n}(\Omega)} + \|v_2\|^\gamma_{L^{\gamma n}(\Omega)}\}\|v_2 - v_1\|_{L^\alpha(\Omega)}\\
&\leq C_{18}\{[\mathrm{meas}(\Omega)]^{1/n} + [\mathrm{meas}(\Omega)]^{(\alpha-\gamma n)/(\alpha\gamma n)}[\|v_1\|^\gamma_{L^\alpha(\Omega)}\\
&\qquad + \|v_2\|^\gamma_{L^\alpha(\Omega)}]\}\|v_2 - v_1\|_{L^\alpha(\Omega)}\\
&\leq C_{19}[1 + (\|v_1\|_{L^\alpha(\Omega)} + \|v_2\|_{L^\alpha(\Omega)})^\gamma]\|v_2 - v_1\|_{L^\alpha(\Omega)}.
\end{aligned}$$

(I1)–(I2) In these cases, first we observe that $1/\widetilde{\alpha} - 1/\alpha \in (0,1)$ and $1/\widetilde{\alpha} - 1/\alpha < \gamma < \alpha(1/\widetilde{\alpha} - 1/\alpha)$. Hence estimates (3.3) are to be modified changing n to s where $1/s = 1/\widetilde{\alpha} - 1/\alpha$.

Summing up, in all the previous cases function ρ in property (xii) is defined by $\rho(s) = C_{15}(1 + s^\gamma)$. □

4. An auxiliary problem in a Banach space

The proofs of our explicit existence Theorems 2.1–2.2 and approximation Theorems 2.3–2.4 will be deduced as particular cases of corresponding results for problems (P_ε) and (P_0) (cf. (1.1) and (1.8)) related to a general Banach space X and a linear closed operator $A : \mathcal{D}(A) \subset X \to X$. As far as the operator N is concerned, we will consider two different non-linearities. In both cases we will assume that N has a *sublinear growth*. In the first case, the non-linearity concerns an operator N which is *locally* Lipschitz continuous from the Banach space X into itself. On the contrary, in the latter case the operator N is locally Lipschitz continuous *from a Banach space X to a different Banach space* X_1, with $X \hookrightarrow X_1$.

Further, we will assume that the scalar functions χ, φ satisfy (1.5) and the kernels $k_\varepsilon, h_\varepsilon$ are still defined by notation (1.2). Further, the functions k

and h satisfy the properties listed in Lemmata 3.1, 3.2, 3.3, where condition (3.1) and property (vii) in Lemma 3.2 and 3.3 can be replaced, respectively, with the more general condition $\pi/2 < \varphi_1 \le \varphi_2 < \pi$ and assumption H3, listed below.

We state now our assumptions on the data and operators related to all our problems:

- H1 $A : D(A) \subset X \to X$ is a linear closed operator in X—with domain $D(A)$ being possibly non-dense in X—such that the resolvent set of A contains the open sector Σ_{φ_2};
- H2 there exists $M_0 > 0$ such that $\|(\lambda I - (1+\chi(\varepsilon))A)^{-1}\|_{\mathcal{L}(X)} \le M_0|\lambda|^{-1}$ for any $\lambda \in \Sigma_{\varphi_2}$;
- H3 $|\widetilde{k}(\lambda) - 1| \le C_4|\lambda|^{\theta_0}$ for all $\lambda \in \Sigma_{\varphi_1} \cap B(0, r_0)$, some $r_0 > \delta$ and $\theta_0 > 1/p$;
- H4 $f_\varepsilon \in L^q((0,T);X)$, $q \in [1,+\infty)$, $\|f_\varepsilon - f\|_{L^q((0,T);X)} \le r_0$ for some positive r_0, and f_ε converges to f in $L^q((0,T);X)$ as $\varepsilon \to 0+$;
- H5 $u_{0,\varepsilon}, u_0 \in D(A)$ (endowed with the graph-norm), $\|u_{0,\varepsilon} - u_0\|_X \le r_0$ and $u_{0,\varepsilon}$ converges to u_0 in X as $\varepsilon \to 0+$;
- H6 Let X_1 be a Banach space such that $X \hookrightarrow X_1$ and let the operators R, R_ε defined by (5.1) and (5.2) be such that $R, R_\varepsilon \in L^r((0,T); \mathcal{L}(X_1, X))$ for some $r \in [1,+\infty)$ and $\|R\varepsilon(t)\|_{\mathcal{L}(X_1,X)} \le Ct^{-1+\kappa}$, for all $t \in (0,T)$, for some non-negative constants C, κ such that $(1-\kappa)r < 1$;
- H7 $N : X_j \to X_j$, for $j = 0,1$, $X_0 = X$, and $\|Nu\|_{X_j} \le C_0(1 + \|u\|_{X_j})$, for all $u \in X_j$ and $j = 0,1$;
- H8 N satisfies the inequality

$$\|Nv_2 - Nv_1\|_{X_1} \le \rho\Big(\sum_{i=1}^{2} \|v_i\|_X\Big)\|v_2 - v_1\|_X, \quad \forall v_1, v_2 \in X.$$

  Where $\rho : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous function such that $\rho(s) \le C_{15}(1 + s^\gamma)$, for all $s \in \mathbf{R}_+$ and some positive γ.

Theorem 4.1. *Let N satisfy properties* (viii)–(x) *in Lemma* 3.4 *with $L^\alpha(\Omega)$) being replaced by the Banach space X. Under assumptions* H1–H3, *problems* (1.1) *and* (1.8) *admit a unique solution u and u_ε in $C^{1+\sigma}([0,T];X) \cap C^\sigma([0,T];\mathcal{D}(A))$ for any given pair $(f_\varepsilon, u_{0,\varepsilon}) \in C^\sigma([0,T];X) \times D(A)$, $\varepsilon \in [0,\varepsilon_0]$, $\sigma \in (0,1)$, such that $Au_{0,\varepsilon} + Nu_{0,\varepsilon} + f_\varepsilon(0) \in D_A(\sigma,\infty)$ with the agreement that $(f_0, u_{0,0}) = (f, u_0)$.*

Theorem 4.2. *Under assumptions* H1–H3 *and* H6–H8, *problems* (1.1) *and* (1.8) *admit a unique solution u and u_ε in $C^{1+\sigma}([0,T];X) \cap C^\sigma([0,T];\mathcal{D}(A))$*

for any given pair $(f_\varepsilon, u_{0,\varepsilon}) \in C^\sigma([0,T];X) \times D(A)$, $\varepsilon \in [0,\varepsilon_0]$, $\sigma \in (0,1)$, *such that* $Au_{0,\varepsilon} + Nu_{0,\varepsilon} + f_\varepsilon(0) \in D_A(\sigma,\infty)$ *with the agreement that* $(f_0, u_{0,0}) = (f, u_0)$.

Theorem 4.3. *Let* N *satisfy properties* (viii)–(x) *in Lemma* 3.4 *with* $L^\alpha(\Omega))$ *being replaced by the Banach space* X. *Under assumptions* H1–H5, *the solutions* u_ε *to the approximating problem* (1.1) *converges in* $L^p((0,T);X)$, $p \in [1,+\infty)$, *to the solution* u *to the limit problem* (1.8), *as* $\varepsilon \to 0+$. *More exactly, there exists a positive constant* c_4 *depending only on* T, M, η, δ, ε_0, θ_0, p, *and* r_0 *such that*

$$\begin{aligned}\|u_\varepsilon - u\|_{L^p((0,T);X)} &\le c_4\big\{\|u_{0,\varepsilon} - u_0\|_X + \|f_\varepsilon - f\|_{L^q((0,T);X)} \\ &\quad + [\varepsilon^{1/p} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0| + \varphi(\varepsilon)]\big[\|u_0\|_X + \|f\|_{L^q((0,T);X)}\big]\big\}.\end{aligned}$$

Theorem 4.4. *Let us assume that the quadruplet* $(\kappa,\gamma,p,\theta_0,r)$ *satisfies either of the following conditions:*

$(K1)$ $p \in [1,+\infty)$, $\gamma \in (0,p\kappa)$, $\theta_0 \in (1/p,+\infty)$, $r \in [p/(p-\gamma), 1/(1-\kappa)]$;

$(K2)$ $\gamma \in (0,+\infty)$, $p \in (\max\{1/\theta_0, \gamma/\kappa\},+\infty)$, $r \in [p/(p-\gamma), 1/(1-\kappa)]$.

Under assumptions H1–H8, *the solutions* u_ε *to the approximating problem in* $L^p((0,T);X)$ *converge to the solution* u *to the limit problem* (1.8), *as* $\varepsilon \to 0+$. *More exactly, there exists a positive constant* c_5 *depending only on* T, M, η, δ, ε_0, θ_0, p, *and* r_0 *such that*

$$\begin{aligned}\|u_\varepsilon - u\|_{L^p((0,T);X)} &\le c_5\big\{\|u_{0,\varepsilon} - u_0\|_X + \|f_\varepsilon - f\|_{L^q((0,T);X)} \\ &\quad + [\varepsilon^{1/p} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0| + \varphi(\varepsilon)]\big[\|u_0\|_X + \|f\|_{L^q((0,T);X)}\big]\big\}.\end{aligned}$$

Remark 4.1. Condition (K1) is related to the case when we are given the pair (κ, p); i.e., we are interested in a specific convergence exponent. On the contrary, condition (K2) states that we are given the non-globally Lipschitz non-linearity operator N, which, in addition, loses regularity, and we wonder whether some exponent p may exist to ensure the convergence.

Remark 4.2. The previous theorem improves the convergence in space, but, unfortunately, worsens the convergence in time.

5. Proofs of Theorems 2.1–2.4

Proofs of Theorems 2.1 and 2.3. They immediately follow from Lemmata 3.1–3.4 and "abstract" Theorems 4.1 and 4.3, since the operator A satisfies assumptions H1 and H2 (cf. [10]). □

In order to prove Theorems 2.2 and 2.4 we need to introduce the following families of operators:

$$R(t) = e^{tA} = \frac{1}{2\pi i}\int_{\gamma(\eta,\delta)} e^{t\lambda}(\lambda - (1+\chi_0)A)^{-1}d\lambda, \tag{5.1}$$

$$R_\varepsilon(t) = \frac{1}{2\pi i}\int_{\gamma(\eta,\delta)} e^{t\lambda}[\lambda - (\chi(\varepsilon) + \widetilde{k}(\varepsilon\lambda) + \varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A]^{-1}d\lambda, \tag{5.2}$$

and

$$\gamma(\eta,\delta) = \gamma_1(\eta,\delta) \cup \gamma_2(\eta,\delta) \cup \gamma_3(\eta,\delta), \tag{5.3}$$

with $\gamma_1(\eta,\delta) = \{re^{-i\eta} : r \geq \delta\}$, $\gamma_2(\eta,\delta) = \{\delta e^{i\theta} : |\theta| \leq \eta\}$, $\gamma_3(\eta,\delta) = \{re^{i\eta} : r \geq \delta\}$, η and δ being the constants entering the properties (iv)–(vii) in Lemma 3.3.

Proofs of Theorems 2.2 and 2.4. In order to apply "abstract" Theorems 4.2 and 4.4, first we observe that operator A satisfies assumptions H1 and H2 (cf. [10]). Then we note that assumption H3 with $\theta_0 = 1$, H7, and H8 immediately follow from Lemmata 3.1–3.3 and 3.5. It remains to show assumption H6. It will be a consequence of the following three lemmata:

Lemma 5.1. *Let $\overline{\alpha} \in [1, +\infty)$, A satisfy, for some $\widetilde{M_0} > 0$,*

$$\|\nabla(\mu I - A)^{-1}\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))} \leq \widetilde{M_0}|\mu|^{-1/2}, \forall \mu \in \Sigma_{\varphi_2}. \tag{5.4}$$

Let k, h satisfy properties in Lemma 3.2 and (i)–(vii) in Lemma 3.3, let χ, φ satisfy (1.5) and let

$$g(\varepsilon,\lambda) = \chi(\varepsilon) + \widetilde{k}(\varepsilon\lambda) + \varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda) \tag{5.5}$$

satisfy the inequalities

$$|g(\varepsilon,\lambda)| \geq \widetilde{C}, \quad \forall \lambda \in \Sigma_{\varphi_1},$$

for all $\varepsilon \in (0, \varepsilon_0)$ and some positive C. Then

$$\|\nabla R_\varepsilon(t)\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))} \leq Ct^{-1/2}, \quad \forall t \in \mathbf{R}_+. \tag{5.6}$$

Lemma 5.2. *Let $\overline{\alpha} \in [4/3, 4)$ and let A satisfy, for some $\widetilde{M_0} > 0$,*

$$\|\nabla(\mu I - A)^{-1}\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))} \leq \begin{cases} \widetilde{M_0}|\mu|^{-1/\overline{\alpha}}, & \forall \mu \in \Sigma_{\varphi_1} \cap \overline{B(0,\beta)} \\ \widetilde{M_0}|\mu|^{-(4-\overline{\alpha})/(2\overline{\alpha})}, & \forall \mu \in \Sigma_{\varphi_1} \cap \overline{B(0,\beta)}^{\,c}. \end{cases} \tag{5.7}$$

Let k, h satisfy properties in Lemma 3.2 and (i)–(vii) in Lemma 3.1 and 3.2, let χ, φ satisfy (1.5), $g(\varepsilon,\lambda)$ be defined by (5.5) and satisfy the inequality

$$|g(\varepsilon,\lambda)| \geq C, \quad \forall \lambda \in \Sigma_{\varphi_1},$$

for all $\varepsilon \in (0, \varepsilon_0)$ *and some positive* C*. Then*

$$\|\nabla R_\varepsilon(t)\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))} \le \overline{C}[t^{-1+1/\overline{\alpha}} + t^{-1+(4-\overline{\alpha})/(2\overline{\alpha})}], \quad \forall t \in \mathbf{R}_+. \tag{5.8}$$

Lemma 5.3. *Let* k, h *be defined by* (2.1) *with property* (2.3)*, let* φ, χ *satisfy* (1.5) *with* $\chi_0 > 0$*, respectively. Then function* g (cf. (5.5)) *satisfies the following inequality, for some positive constant* C*, independent of* ε*:*

$$|g(\varepsilon, \lambda)| \ge C, \quad \forall \lambda \in \Sigma_{\varphi_1}.$$

The previous lemmata will be proved at the end of this section.

Our next task consists in showing that the operator A defined by (1.3) and (1.4) satisfies assumptions in Lemmata 5.1–5.2. From Theorems 3.1 and 4.1 in [7], with $m \equiv 1$ and $\rho = 1$, and from the following interpolation inequality, holding for any $q \in (1, +\infty)$ and $\forall w \in L^{\overline{\alpha}}(\Omega)$,

$$\|\nabla(\mu I - A)^{-1} w\|_{L^{\overline{\alpha}}(\Omega)} \le \widetilde{M}_0 \|(\mu I - A)^{-1} w\|^{1/2}_{L^{\overline{\alpha}}(\Omega)} \|A(\mu I - A)^{-1} w\|^{1/2}_{L^{\overline{\alpha}}(\Omega)},$$

we deduce that (5.4) and (5.7) hold true respectively, for $\overline{\alpha} \in (1, 2)$ and $\overline{\alpha} \in [2, 4)$. We also note that (5.6) and (5.8) imply $\nabla R_\varepsilon \in L^1((0,T), \mathcal{L}(L^{\overline{\alpha}}(\Omega))$ for all $\overline{\alpha} \in (1, 4)$.

Now, we go on with our proof. Taking advantage of Lemma 5.3 we can prove that R_ε satisfies assumption H6. For this purpose we recall that the domain of the operator A is the Banach space $\mathcal{D}(A) = W^{2,\alpha}(\Omega) \cap W_0^{1,\alpha}(\Omega)$. We recall also that $g(\varepsilon, \lambda)$ is defined by (5.5). Consequently, from Lemmata 5.1 and 5.3, with $\overline{\alpha} = \widetilde{\alpha}$, we get that

$$\|\nabla R_\varepsilon(t)\|_{\mathcal{L}(L^{\widetilde{\alpha}}(\Omega))} \le \begin{cases} Ct^{-1/2}, & \forall t \in \mathbf{R}_+ \text{ and } \widetilde{\alpha} \in (1, 2), \\ Ct^{-1+\eta(\widetilde{\alpha})}, & \forall t \in \mathbf{R}_+ \text{ and } \widetilde{\alpha} \in [2, 4), \end{cases} \tag{5.9}$$

where $\eta(\widetilde{\alpha}) = (4 - \widetilde{\alpha})/(2\widetilde{\alpha})$. Hence, from (5.9), we get

$$\nabla R_\varepsilon \in L^{r(\widetilde{\alpha})}([0,T]; \mathcal{L}(L^{\widetilde{\alpha}}(\Omega))), \tag{5.10}$$

where $r(\widetilde{\alpha}) \in [1, 2)$ if $\widetilde{\alpha} \in [1, 2)$, and $r(\widetilde{\alpha}) \in [1, (1 - \eta(\widetilde{\alpha}))^{-1}]$ if $\widetilde{\alpha} \in [2, 4)$. Moreover, the property

$$R_\varepsilon \in L^\infty([0,T]; \mathcal{L}(L^{\widetilde{\alpha}}(\Omega))), \tag{5.11}$$

is a consequence of the following Lemma 5.4, proved in [10].

Lemma 5.4. *There exists a positive constant* $C(\eta, \delta)$ *such that*

$$\|R_\varepsilon(t)\|_{\mathcal{L}(X)} \le M_0 C(\eta, \delta), \quad \forall t \in \mathbf{R}_+.$$

Consequently, from (5.10) and (5.11), we get

$$R \in L^{r(\widetilde{\alpha})}((0,T); \mathcal{L}(L^{\widetilde{\alpha}}(\Omega), W^{1,\widetilde{\alpha}}(\Omega)).$$

According to assumptions (I1), (I2), and (II) in (C2) (cf. Section 2), we derive the following inclusions, via Sobolev embedding theorems:

(I1)–(I2) $W^{1,\widetilde{\alpha}}(\Omega) \hookrightarrow C(\overline{\Omega}) \hookrightarrow L^{\alpha}(\Omega)$, $\widetilde{\alpha} \in (n,4)$, $\alpha \in [1,+\infty)$, if $n = 1,2,3$;

(II) $W^{1,\widetilde{\alpha}}(\Omega) \hookrightarrow L^{\alpha}(\Omega)$, $\widetilde{\alpha} \in [1,4)$, $1/\alpha = 1/\widetilde{\alpha} - 1/n$, if $n \geq 4$.

Then

$$R_\varepsilon \in L^{r(\widetilde{\alpha})}((0,T);\mathcal{L}(L^{\widetilde{\alpha}}(\Omega), L^{\alpha}(\Omega))), \quad \text{if } n \geq 1. \tag{5.12}$$

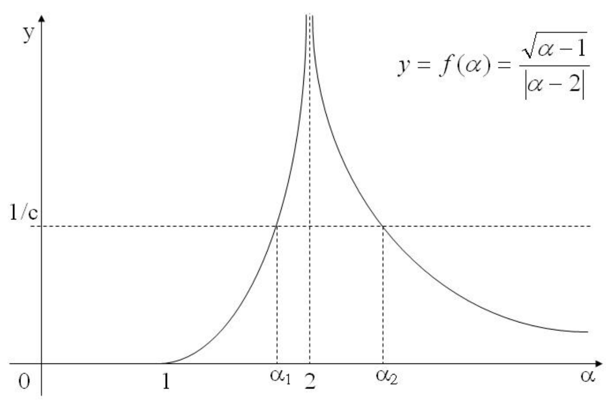
Consequently, from (5.12) assumption H6 is fulfilled with $X = L^{\alpha}(\Omega)$ and $X_1 = L^{\widetilde{\alpha}}(\Omega)$. Indeed, we have

$$\begin{aligned}\|R_\varepsilon(t)\|_{\mathcal{L}(L^{\widetilde{\alpha}}(\Omega),L^{\alpha}(\Omega))} &\leq c_0\big(\|R_\varepsilon(t)\|_{\mathcal{L}(L^{\widetilde{\alpha}}(\Omega))} + \|\nabla R_\varepsilon(t)\|_{\mathcal{L}(L^{\widetilde{\alpha}}(\Omega))}\big)\\ &\leq c_1(1+t^{-1+\kappa(\widetilde{\alpha})}) \leq c_2(T)t^{-1+\kappa(\widetilde{\alpha})}, \quad \forall t \in [0,T],\end{aligned}$$

where $\kappa(\widetilde{\alpha}) = 1/2$, if $\widetilde{\alpha} \in [1,2)$, $\kappa(\widetilde{\alpha}) = (4-\widetilde{\alpha})/(2\widetilde{\alpha})$, if $\widetilde{\alpha} \in [2,4)$. This concludes the proof of our theorem. □

It remains to explain why we have inserted the condition $\alpha \in (\alpha_1, \alpha_2)$ (cf. (2.6)) into the statement of our theorems. According to the results, related to operator A, proved in [10], in Theorems 2.1 and 2.3 the admissible spaces $L^{\alpha}(\Omega)$ are those corresponding to the α's satisfying the inequality

$$\frac{\sqrt{\alpha-1}}{|\alpha-2|} > \frac{1}{c}, \quad \text{where } c = \operatorname{cotg}\big(\pi \max\{\alpha_\infty, \gamma_\infty\}/2\big). \tag{5.13}$$



Instead, in Theorems 2.2 and 2.4, the admissible spaces $L^{\alpha}(\Omega)$ are those corresponding to the α's satisfying the inequality (5.13) as well as (C2). First we need to choose $\widetilde{\alpha}$ such that $\alpha \in (\alpha_1, \alpha_2)$. Hence, $\widetilde{\alpha}$ has to be chosen in the following way:

(I1) if $n = 1, 2, 3$, $\widetilde{\alpha} \in (\max\{n, 1/\gamma\}, \min\{4, \alpha_2(1+\gamma)\})$;

(I2) if $n = 1, 2, 3$,

either $\widetilde{\alpha} \in (\frac{\alpha_1}{\gamma+1}, \frac{\alpha_2}{\gamma+1}) \cap (\max\{n, \frac{1}{\gamma+1}\}, \min\{4, \frac{1}{\gamma}\})$

or $\widetilde{\alpha} \in (\frac{\alpha_1}{\gamma\alpha_1+1}, \frac{\alpha_2}{\gamma\alpha_2+1}) \cap (\max\{n, \frac{1}{\gamma+1}\}, \min\{4, \frac{1}{\gamma}\})$;

(II) if $n \geq 4$, $\widetilde{\alpha} \in (n\alpha_1/(n+\alpha_1), n\alpha_2/(n+\alpha_2)) \cap (\max\{1, n\gamma/(\gamma+1)\}, 4)$.

Finally, it remains to show that there exists $\alpha \in (\alpha_1, \alpha_2)$ satisfying assumption (C2). For this purpose, concerning the cases (I2) and (II), note the following:

(I2) the interval $(\alpha_1/(\gamma+1), \alpha_2/(\gamma+1)) \cap (\max\{n, 1/(\gamma+1)\}, \min\{4, 1/\gamma\})$ is not empty if

either $\max\{n, 1/(\gamma+1)\} < \alpha_1/(\gamma+1) < \min\{4, 1/\gamma\}$,

or $\max\{n, 1/(\gamma+1)\} < \alpha_2/(\gamma+1) < \min\{4, 1/\gamma\}$.

Likewise, the interval $(\alpha_1/(\gamma\alpha_1+1), \alpha_2/(\gamma\alpha_2+1)) \cap (\max\{n, 1/(\gamma+1)\}, \min\{4, 1/\gamma\})$ is not empty if

either $\max\{n, 1/(\gamma+1)\} < \alpha_1/(\gamma\alpha_1+1) < \min\{4, 1/\gamma\}$,

or $\max\{n, 1/(\gamma+1)\} < \alpha_2/(\gamma\alpha_2+1) < \min\{4, 1/\gamma\}$.

(II) the interval $(n\alpha_1/(n+\alpha_1), n\alpha_2/(n+\alpha_2)) \cap (\max\{1, n\gamma/(\gamma+1)\}, 4)$ is not empty if either of the following condition is satisfied:

$$\max\Big\{1, \frac{n\gamma}{\gamma+1}\Big\} < \frac{n\alpha_1}{n+\alpha_1} < 4, \qquad \max\Big\{1, \frac{n\gamma}{\gamma+1}\Big\} < \frac{n\alpha_2}{n+\alpha_2} < 4.$$

We now prove Lemmata 5.1–5.3.

Proof of Lemma 5.1. Observe that

$$\begin{aligned} &\|\nabla[\lambda I - (\chi(\varepsilon) + \widetilde{k}(\varepsilon\lambda) + \varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A]^{-1}\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))} \qquad (5.14)\\ &= \|\nabla[\lambda I - g(\varepsilon,\lambda)A]^{-1}\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))} = |g(\varepsilon,\lambda)|^{-1}\Big\|\nabla\Big(\frac{\lambda}{g(\varepsilon,\lambda)}I - A\Big)^{-1}\Big\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))}\\ &\leq \frac{\widetilde{M}_0}{|g(\varepsilon,\lambda)|^{1/2}|\lambda|^{1/2}} \leq \widetilde{M}_0 C^{-1/2}|\lambda|^{-1/2}, \quad \forall\lambda \in \Sigma_{\varphi_1}. \end{aligned}$$

Consequently, due to (5.14) and the analiticity of the integrand in formula (5.2), we can set $\delta = 0$ in formula (5.2) and can easily deduce the estimate (5.6). □

Proof of Lemma 5.2. First observe that

$$\begin{aligned}
&\|\nabla[\lambda I - (\chi(\varepsilon) + \widetilde{k}(\varepsilon\lambda) + \varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A]^{-1}\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))} \\
&= \|\nabla[\lambda I - g(\varepsilon,\lambda)A]^{-1}\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))} = |g(\varepsilon,\lambda)|^{-1}\Big\|\nabla\Big(\frac{\lambda}{g(\varepsilon,\lambda)}I - A\Big)^{-1}\Big\|_{\mathcal{L}(L^{\overline{\alpha}}(\Omega))} \\
&\qquad\qquad \le \begin{cases} \widetilde{M_0}C_{19}|\lambda|^{-1/\overline{\alpha}}, & \forall\lambda \in \Sigma_{\varphi_1} \cap \overline{B(0,\beta)} \\ \widetilde{M_0}C_{19}|\lambda|^{(\overline{\alpha}-4))/(2\overline{\alpha})}, & \forall\lambda \in \Sigma_{\varphi_1} \cap \overline{B(0,\beta)}^{\ c}. \end{cases}
\end{aligned}$$

Observe now that $-1/\overline{\alpha}$ and $(\overline{\alpha}-4))/(2\overline{\alpha})$ lie in interval $(0,1)$, since $\overline{\alpha} \in [4/3, 4)$. Therefore we can split the integral in formula (5.2) with $\delta = 0$ into two integrals $I_1(t)$ and $I_2(t)$, where the first is related to $\gamma(0,\eta) \cap B(0,\beta)$ and the latter to $\gamma(0,\eta) \cap B(0,\beta)^c$.

First, we estimate $I_1(t)$:

$$|I_1(t)| \le C_{20}\int_0^\beta e^{tr\cos\eta} r^{-1/\alpha}dr \le C_{20}\int_0^\beta r^{-1/\alpha}dr \le C_{21}, \quad t \in \mathbf{R}_+.$$

Then to estimate $I_2(t)$ we use the change of variable $r = s/t$:

$$\begin{aligned}
|I_1(t)| &\le C_{22}\int_\beta^{+\infty} e^{tr\cos\eta} r^{(\overline{\alpha}-4))/(2\overline{\alpha})}dr \\
&\le C_{22}t^{-1+(4-\overline{\alpha})/(2\overline{\alpha})}\int_0^{+\infty} e^{s\cos\eta} s^{(\overline{\alpha}-4)/(2\overline{\alpha})}ds. \qquad \square
\end{aligned}$$

Proof of Lemma 5.3. Let $\varepsilon_0 \in (0,1]$. First we note that the following estimate holds:

$$|z_1 + z_2| \ge \sqrt{2}\cos(\varphi/2)(|z_1|^2 + |z_2|^2)^{1/2},$$

$\forall z_1, z_2 \in \Sigma_\varphi^+ \quad (\text{or } z_1, z_2 \in \Sigma_\varphi^-)$, $\forall\varphi \in (\pi/2, \pi)$. Indeed,

$$\begin{aligned}
|z_1 + z_2|^2 &= |z_1|^2 + 2\mathrm{Re}\,|z_1||z_2|\exp[i(\arg z_1 - \arg z_2)] + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||z_2|\cos(\arg z_1 - \arg z_2) + |z_2|^2 \\
&\ge |z_1|^2 + 2|z_1||z_2|\cos\varphi + |z_2|^2 \ge 2\cos^2(\varphi/2)(|z_1|^2 + |z_2|^2).
\end{aligned}$$

It is now a simple exercise to show that

$$|z_1 + z_2 + z_3| \ge 2\cos^2(\varphi/2)(|z_1|^2 + |z_2|^2 + |z_3|^2)^{1/2}, \tag{5.15}$$

$\forall z_1, z_2, z_3 \in \Sigma_\varphi^+ \quad (\text{or } z_1, z_2, z_3 \in \Sigma_\varphi^-)$, $\forall\varphi \in (\pi/2, \pi)$. Note now that

$$|\lambda^{-1}\chi(\varepsilon)|, |\lambda^{-1}\widetilde{k}(\varepsilon\lambda)|, |\lambda^{-1}\widetilde{h}(\varepsilon\lambda)| \in \Sigma_{\varphi_2}^+, \qquad \forall\lambda \in \Sigma_{\varphi_1}^-,$$

$$|\lambda^{-1}\chi(\varepsilon)\lambda|, |\lambda^{-1}\widetilde{k}(\varepsilon\lambda)\lambda|, |\lambda^{-1}\widetilde{h}(\varepsilon\lambda)| \in \Sigma_{\varphi_2}^-, \qquad \forall\lambda \in \Sigma_{\varphi_1}^+.$$

Consequently, from (5.15) and assumption $\chi_0 > 0$, we easily obtain

$$\begin{aligned} |\lambda^{-1} g(\varepsilon, \lambda)| &= |\lambda^{-1}\chi(\varepsilon) + \lambda^{-1}\widetilde{k}(\varepsilon\lambda) + \lambda^{-1}\widetilde{h}(\varepsilon\lambda)| \\ &\geq 2\cos^2(\varphi_2/2)\big[|\lambda^{-1}\chi(\varepsilon)|^2 + |\lambda^{-1}\widetilde{k}(\varepsilon\lambda)|^2 + \big|\lambda^{-1}\widetilde{h}(\varepsilon\lambda)|^2\big]^{1/2} \\ &\geq 2\cos^2(\varphi_2/2)|\lambda^{-1}\chi(\varepsilon)|. \end{aligned} \tag{5.16}$$

6. Proofs of Theorems 4.1–4.5.

According to assumptions H1, H2, H4, and H5 the solutions of problems (1.1) and (1.8) are represented (cf. [6]) by

$$u_\varepsilon(t) = R_\varepsilon(t)u_{0,\varepsilon} + \int_0^t R_\varepsilon(t-s)f_\varepsilon(s)ds + \int_0^t R_\varepsilon(t-s)Nu_\varepsilon(s)ds, \tag{6.1}$$

$$u(t) = R(t)u_{0,\varepsilon} + \int_0^t R(t-s)f(s)ds + \int_0^t R(t-s)Nu(s)ds, \tag{6.2}$$

$\forall\varepsilon \in (0, \varepsilon_0]$, where R, R_ε are defined by (5.1)–(5.3).

To simplify our notation, from now as far as Lemma 6.5, *the subscript* $\varepsilon = 0$ *will refer to quantities related to the limit problem* (P_0).

Our first task consists in proving Theorems 4.1 and 4.2 concerning the existence of the solutions u_ε to the equation (6.1). For this purpose we need the following lemma, where we derive a pointwise estimate in X for the solutions u_ε.

Lemma 6.1. *Let N satisfy property* (ix) *in Lemma* 3.4, *where $L^\alpha(\Omega)$ is replaced with X. Let u_ε be a solution to equations* (6.1). *Then the following inequality holds:*

$$\|u_\varepsilon(t)\|_X^p \leq [g_\varepsilon(t)]^p + \int_0^t \widehat{l}_\varepsilon(t-s)\{1 + [g_\varepsilon(s)]^p\}\, ds, \quad \forall\varepsilon \in [0, \varepsilon_0], \tag{6.3}$$

where

$$\begin{aligned} [g_\varepsilon(t)]^p &= C_{22}\Big[\|u_{0,\varepsilon}\|_X^p + \|R_\varepsilon\|_{L^1((0,T),\mathcal{L}(X))}^{p/p'} \int_0^t \|R_\varepsilon(t-s)\|_{\mathcal{L}(X)} \|f_\varepsilon(s)\|_X^p\, ds\Big], \\ l_\varepsilon(t) &= C_{22}\|R_\varepsilon\|_{L^1((0,T),\mathcal{L}(X))}^{p/p'} \|R_\varepsilon(t)\|_{\mathcal{L}(X)}, \end{aligned} \tag{6.4}$$

for some positive constant C_{22}, $\widehat{l}_\varepsilon$ being the solution to the following integral equation:

$$\widehat{l}_\varepsilon(t) - l_\varepsilon(t) - l_\varepsilon * \widehat{l}_\varepsilon(t) = 0.$$

Proof. From equation (6.1), Lemma 5.1, and Lemma 5.4 in [11] and (ix), we get

$$\|u_\varepsilon(t)\|_X \le C_{23}\Big\{\|u_{0,\varepsilon}\|_X + \|R_\varepsilon\|^{1/p'}_{L^1((0,T),\mathcal{L}(X))}\Big[\Big(\int_0^t \|R_\varepsilon(t-s)\|_{\mathcal{L}(X)}\|f_\varepsilon(s)\|^p_X\,ds\Big)^{\frac{1}{p}} + \Big(\int_0^t \|R_\varepsilon(t-s)\|_{\mathcal{L}(X)}(1+\|u_\varepsilon(s)\|_X)^p\,ds\Big)^{1/p}\Big]\Big\}.$$

Consequently,

$$\begin{aligned}1+\|u_\varepsilon(t)\|^p_X &\le 1 + C_{22}\Big\{\|u_{0,\varepsilon}\|^p_X + \|R_\varepsilon\|^{p/p'}_{L^1((0,T),\mathcal{L}(X))}\Big[\int_0^t \|R_\varepsilon(t-s)\|_{\mathcal{L}(X)}\|f_\varepsilon(s)\|^p_X\,ds \\ &\quad + \int_0^t \|R_\varepsilon(t-s)\|_{\mathcal{L}(X)}(1+\|u_\varepsilon(s)\|_X)^p\,ds\Big]\Big\} \\ &\le 1+[g_\varepsilon(t)]^p + C_{22}\|R_\varepsilon\|^{p/p'}_{L^1((0,T),\mathcal{L}(X))}\int_0^t \|R_\varepsilon(t-s)\|_{\mathcal{L}(X)}(1+\|u_\varepsilon(s)\|_X)^p\,ds, \\ &\le 1+[g_\varepsilon(t)]^p + \int_0^t l_\varepsilon(t-s)(1+\|u_\varepsilon(s)\|_X)^p\,ds. \qquad (6.5)\end{aligned}$$

Let $\widehat{l}_\varepsilon$ be the corresponding solving kernel, i.e., that defined by equation (6.5). Now we get

$$1+\|u_\varepsilon(t)\|^p_X \le 1+[g_\varepsilon(t)]^p + \int_0^t \widehat{l}_\varepsilon(t-s)\{1+[g_\varepsilon(s)]^p\}\,ds;$$

i.e., u satisfies inequality (6.3). □

We can now define the function-space where we search for the solution to equation (6.2):

$$\mathcal{U}_\varepsilon := \Big\{u \in L^p((0,T);X) : \|u(t)\|^p_X \le [g_\varepsilon(t)]^p + \int_0^t \widehat{l}_\varepsilon(t-s)\{1+[g_\varepsilon(s)]^p\}\,ds, \quad \text{for a.e. } t\in[0,T]\Big\}. \qquad (6.6)$$

Lemma 6.2. *Let S_ε, $\varepsilon \in [0,\varepsilon_0]$, be the operator defined by*

$$S_\varepsilon u(t) = R_\varepsilon(t)u_{0,\varepsilon} + \int_0^t R_\varepsilon(t-s)f_\varepsilon(s)ds + \int_0^t R_\varepsilon(t-s)Nu_\varepsilon(s)ds, \qquad (6.7)$$

where N satisfies property (ix) in Lemma 3.4, *with $L^\alpha(\Omega)$ being replaced by X. Then S_ε maps the space $\mathcal{U}_\varepsilon$, defined by* (6.6), *into itself.*

Proof. Let $u \in \mathcal{U}_\varepsilon$; then

$$\|S_\varepsilon u(t)\|_X^p \le [g_\varepsilon(t)]^p + \int_0^t l_\varepsilon(t-s)(1+\|u_\varepsilon(s)\|_X)^p\,ds$$
$$\le [g_\varepsilon(t)]^p + \int_0^t l_\varepsilon(t-s)\Big\{1+[g_\varepsilon(s)]^p + \int_0^s \widehat{l}_\varepsilon(s-\sigma)\{1+[g_\varepsilon(\sigma)]^p\}\,d\sigma\Big\}ds$$
$$\le [g_\varepsilon(t)]^p + \int_0^t l_\varepsilon(t-s)\{1+[g_\varepsilon(s)]^p\}ds + \int_0^t \widehat{l}_\varepsilon * l_\varepsilon(t-s)\{1+[g_\varepsilon(s)]^p\}\,ds\Big\}$$
$$\le [g_\varepsilon(t)]^p + \int_0^t l_\varepsilon(t-s)\{1+[g_\varepsilon(s)]^p\}ds + \int_0^t [\widehat{l}_\varepsilon(t-s) - l_\varepsilon(t-s)]\{1+[g_\varepsilon(s)]^p\}\,ds$$
$$\le [g_\varepsilon(t)]^p + \int_0^t \widehat{l}_\varepsilon(t-s)\{1+[g_\varepsilon(s)]^p\}ds.$$

This proves our assertion. □

Proof of Theorem 4.1. Let $u_1, u_2 \in \mathcal{U}_\varepsilon$. Then, according to property (ix), we have

$$\begin{aligned}&\|S_\varepsilon u_1 - S_\varepsilon u_2\|_{L^p([0,\tau],X)} \\ &\le C_{24}\Big[\int_0^\tau \|R_\varepsilon(s)\|_{\mathcal{L}(X)}\|Nu_1 - Nu_2\|^p_{L^p([0,\tau-s],X)}\,ds\Big]^{1/p} \\ &\le C_{24}C_{13}(M_v e)\Big[\int_0^\tau \|R_\varepsilon(s)\|_{\mathcal{L}(X)}\|u_1-u_2\|^p_{L^p([0,\tau-s],X)}\,ds\Big]^{1/p},\end{aligned} \tag{6.8}$$

where

$$M_\varepsilon = \Big(\int_0^T \Big\{[g_\varepsilon(t)]^p + \int_0^t \widehat{l}_\varepsilon(t-s)\{1+[g_\varepsilon(s)]^p\}\,ds\Big\}dt\Big)^{1/p}.$$

We note now that estimate (6.8) implies that the iterates S_ε^m are a contracting mappings in $L^p([0,T],X)$ for large enough m.

We have so proved that equation (6.1) admits unique solutions $u_\varepsilon \in L^p([0,T],X)$, for all $\varepsilon \in [0,\varepsilon_0]$.

We have now to show the regularity of u_ε appearing in the statement of the theorem. For this purpose we recall that, from the definition of R_ε (cf. (5.2) and assumption (H2), we easily deduce the estimates

$$\|R'_\varepsilon(t)\|_{\mathcal{L}(X)} \le C_{25}(\varepsilon)t^{-1}, \quad \forall t \in \mathbf{R}_+, \forall\varepsilon \in [0,\varepsilon_0].$$

From the first inequality, we easily obtain

$$\|R_\varepsilon(t_2) - R_\varepsilon(t_1)\|_{\mathcal{L}(X)} \le \int_{t_1}^{t_2} \|R'_\varepsilon(t)\|_{\mathcal{L}(X)}\,dt \le C_{25}\int_{t_1}^{t_2} t^{-1}\,dt \tag{6.9}$$
$$\le C_{25}\log(t_2/t_1) \le C\log((t_2-t_1)/t_1+1) \le C_{26}(t_2-t_1)^\sigma t_1^{-\sigma}, \quad 0 < t_1 < t_2.$$

Then, we estimate the increments of the convolution $R * l$, where we have $l \in L^\infty([0,T];X)$. From (6.9), for $0 \le t_1 \le t_2 \le T$, we get

$$\begin{aligned}
&\Big\| \int_0^{t_2} R_\varepsilon(t_2-s)l(s)\,ds - \int_0^{t_1} R_\varepsilon(t_1-s)l(s)\,ds\Big\|_X \qquad (6.10)\\
&\le \Big\| \int_0^{t_1} [R_\varepsilon(t_2-s)-R(t_1-s)]l(s)\,ds\Big\|_X + \Big\| \int_{t_1}^{t_2} R_\varepsilon(t_2-s)l(s)\,ds\Big\|_X\\
&\le \Big[C_{26}(t_2-t_1)^\sigma \int_0^{t_1}(t_1-s)^{-\sigma}\,ds + C_{25}(t_2-t_1)\Big]\|l\|_{L^\infty([0,T];X)}\\
&\le C_{27}\|l\|_{L^\infty([0,T];X)}(t_2-t_1)^\sigma[t_1^{1-\sigma}+t_2^{1-\sigma}]\\
&\le 2C_{27}\|l\|_{L^\infty([0,T];X)}(t_2-t_1)^\sigma T^{1-\sigma}.
\end{aligned}$$

From estimate (6.10) and property (viii) in Lemma 3.4 we deduce that $R_\varepsilon * (f_\varepsilon + Nu_\varepsilon)$ belongs to $C^\sigma([0,T],X)$.

Now we estimate $R_\varepsilon(t)u_{0,\varepsilon}$, recalling that $u_{0,\varepsilon} \in \mathcal{D}(A)$. Observe that

$$\begin{aligned}
R'_\varepsilon(t)u_{0,\varepsilon} &= \frac{1}{2\pi i}\int_\gamma e^{t\lambda}\lambda(\lambda-(\chi(\varepsilon)+\widetilde{k}(\varepsilon\lambda)+\varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A)^{-1}u_{0,\varepsilon}\,d\lambda\\
&= \frac{1}{2\pi i}\int_\gamma e^{t\lambda}\lambda(\lambda-(\chi(\varepsilon)+\widetilde{k}(\varepsilon\lambda)+\varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A)^{-1}A^{-1}Au_0\,d\lambda\\
&= \frac{1}{2\pi i}\int_\gamma e^{t\lambda}(\lambda-(\chi(\varepsilon)+\widetilde{k}(\varepsilon\lambda)+\varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A + (\chi(\varepsilon)+\widetilde{k}(\varepsilon\lambda)+\varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A)\\
&\qquad \times (\lambda-(\chi(\varepsilon)+\widetilde{k}(\varepsilon\lambda)+\varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A)^{-1}A^{-1}Au_{0,\varepsilon}\,d\lambda\\
&= \frac{1}{2\pi i}\int_\gamma e^{t\lambda}[A^{-1}+(\chi(\varepsilon)+\widetilde{k}(\varepsilon\lambda)+\varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))AA^{-1}\\
&\qquad \times (\lambda-(\chi(\varepsilon)+\widetilde{k}(\varepsilon\lambda)+\varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A)^{-1}]Au_{0,\varepsilon}\,d\lambda\\
&= \frac{1}{2\pi i}\int_\gamma e^{t\lambda}(\chi(\varepsilon)+\widetilde{k}(\varepsilon\lambda)+\varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))\\
&\qquad \times (\lambda-(\chi(\varepsilon)+\widetilde{k}(\varepsilon\lambda)+\varphi(\varepsilon)\widetilde{h}(\varepsilon\lambda))A)^{-1}Au_{0,\varepsilon}\,d\lambda,
\end{aligned}$$

where we have used the following identity:

$$\int_\gamma e^{t\lambda}u_0\,d\lambda = 0.$$

Consequently, we get

$$\|R'_\varepsilon(t)u_{0,\varepsilon}\|_X \le C_{28}\|Au_{0,\varepsilon}\|_X \int_\gamma e^{t\mathrm{Re}\lambda}\frac{|d\lambda|}{|\lambda|} = C_{29}\|Au_{0,\varepsilon}\|_X.$$

From the previous estimate, we easily deduce

$$\|[R_\varepsilon(t_2)-R_\varepsilon(t_1)]u_{0,\varepsilon}\|_X \le \int_{t_1}^{t_2} \|R'_\varepsilon(t)u_{0,\varepsilon}\|_X\,dt \le C_{29}\|Au_{0,\varepsilon}\|_X(t_2-t_1).$$

Since $u_{0,\varepsilon}\in\mathcal{D}(A)$ and $f_\varepsilon+Nu_\varepsilon\in L^\infty((0,T),X)$, from the equation

$$u_\varepsilon(t)=R_\varepsilon(t)u_{0,\varepsilon}+R_\varepsilon*[f_\varepsilon+Nu_\varepsilon](t),$$

we deduce that u_ε actually belongs to the space $C^\sigma([0,T];D(A))$. Finally, from the assumption (viii) in Lemma 3.4 on N it follows $f_\varepsilon+Nu_\varepsilon\in C^\sigma([0,T];X)$. Since $Au_{0,\varepsilon}+Nu_{0,\varepsilon}+f_\varepsilon(0)\in D_A(\sigma,\infty)$, from the results in [6] and [12], we conclude that $u_\varepsilon\in C^{1+\sigma}([0,T];X)\cap C^\sigma([0,T];D(A))$. □

To prove Theorem 4.2 we need two lemmata, the first being a variant of the well-known Young's theorem on convolutions.

Lemma 6.3. *Let $(p,\widetilde{p},r)$ be a triplet such that*

$$1\le p\le+\infty,\quad 1\le r\le+\infty,\quad 1/\widetilde{p}=1/p-1/r+1.$$

Then, for any $k\in L^r((0,T))$ and $u\in L^{\widetilde{p}}([0,T])$, the following estimates hold for all $\tau\in(0,T]$:

$$\begin{aligned}\|k*u\|^p_{L^p((0,\tau))} &\le \|k\|^{p-r}_{L^r([0,\tau])}\|u\|^{p-\widetilde{p}}_{L^{\widetilde{p}}((0,\tau))}\int_0^\tau k(s)^r\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}((0,\tau-s))}\,ds\\ &\le \|k\|^p_{L^r([0,\tau])}\|u\|^p_{L^{\widetilde{p}}((0,\tau))}.\end{aligned}\tag{6.11}$$

Proof. Let $\mu,\nu\in[1,+\infty]$ be such that

$$1/\mu=1/r-1/p,\quad 1/\nu=1/\widetilde{p}-1/p;$$

consequently, $1/\mu+1/\nu+1/p=1$.

Consider now the following identity,

$$|k*u(t)|\le\int_0^t|k(t-s)|^{r/\mu}|u(s)|^{\widetilde{p}/\nu}\{|k(t-s)|^{r(1/r-1/\mu)}|u(s)|^{\widetilde{p}(1/\widetilde{p}-1/\nu)}\}\,ds,$$

and observe that

$$pr(1/r-1/\mu)=r,\quad p\widetilde{p}(1/\widetilde{p}-1/\nu)=\widetilde{p}.$$

Then, we apply the Hölder inequality with exponents μ, ν, and p to get

$$\begin{aligned}|k*u(t)| &\le\Big(\int_0^t|k(t-s)|^r\,ds\Big)^{\frac{1}{\mu}}\Big(\int_0^t|u(s)|^{\widetilde{p}}\,ds\Big)^{\frac{1}{\nu}}\Big(\int_0^t|k(t-s)|^r|u(s)|^{\widetilde{p}}\,ds\Big)^{\frac{1}{p}}\\ &\le\|k\|^{r/\mu}_{L^r([0,\tau])}\|u\|^{\widetilde{p}/\nu}_{L^{\widetilde{p}}([0,\tau])}\Big(\int_0^t|k(s)|\,|u(t-s)|^{\widetilde{p}}\,ds\Big)^{1/p},\quad\forall t\in(0,\tau].\end{aligned}$$

Taking the $L^p(0,\tau)$-norms, $\tau \in (0,T]$, we easily obtain (6.11). Indeed, we have

$$\|k * u\|^p_{L^p((0,\tau))} \le \|k\|^{pr/\mu}_{L^r([0,\tau])} \|u\|^{p\widetilde{p}/\nu}_{L^{\widetilde{p}}((0,\tau))} \int_0^\tau |k(s)|^r \, ds \int_s^\tau |u(t-s)|^{\widetilde{p}} \, ds.$$

Lemma 6.4. *Let $(p, \widetilde{p}, r)$ be a triplet such that*

$$1 \le p \le +\infty, \quad 1 \le r \le +\infty, \quad 1/\widetilde{p} = 1/p - 1/r + 1.$$

Then $R_\varepsilon * w \in L^p((0,T);X)$ for all $w \in L^{\widetilde{p}}((0,T);X_1)$, and the following estimates hold:

$$\begin{aligned} &\|R_\varepsilon * w\|^p_{L^p([0,\tau),X)} \\ &\le \ \|R_\varepsilon\|^{p-r}_{L^r((0,\tau),\mathcal{L}(X_1;X))} \|w\|^{p-\widetilde{p}}_{L^{\widetilde{p}}((0,\tau),X_1))} \int_0^\tau \|R_\varepsilon\|^r \|w\|^{\widetilde{p}}_{L^{\widetilde{p}}((0,\tau-s))} \, ds \\ &\le \|R_\varepsilon\|^p_{L^r((0,\tau);\mathcal{L}(X_1);X))} \|w\|^p_{L^{\widetilde{p}}((0,\tau),X_1)))}. \end{aligned} \tag{6.12}$$

Proof. It suffices to recall that

$$\int_0^t \|R_\varepsilon(t-s)w(s)\|_X \, ds \le \int_0^t \|R_\varepsilon(t-s)\|_{\mathcal{L}(X_1,X)} \|w(s)\|_{X_1} \, ds.$$

Consequently, we get

$$\|R_\varepsilon * w\|_{L^p((0,\tau),X)} \le \|k * u\|_{L^p((0,\tau))},$$

where $k(t) = \|R_\varepsilon(t)\|_{\mathcal{L}(X_1,X)}$ and $u(t) = \|w(t)\|_{X_1}$. Applying Lemma 6.3, we easily get inequalities (6.12). □

Proof of Theorem 4.2. Let $u_1, u_2 \in \mathcal{U}_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$. First, we estimate the difference $Nu_1 - Nu_2$ in $L^{\widetilde{p}}((0,\tau), X_1)$. From

$$\|Nu_1(t,\cdot) - Nu_2(t,\cdot)\|_{X_1} \le \rho\Big(\sum_{i=1}^2 \|u_i(t)\|_X\Big) \|u_2(t,\cdot) - u_1(t,\cdot)\|_X$$

(cf. assumption H8) and Hölder's inequality, recalling that $1 \le r' \le p/\gamma$, $1/r + 1/r' = 1$, we easily deduce the inequality

$$\|Nu_1 - Nu_2\|_{L^{\widetilde{p}}((0,\tau),X_1)} \le \|u_2 - u_1\|_{L^p([0,\tau],X)} \Big\{ \int_0^\tau \rho\Big(\sum_{i=1}^2 \|u_i(t)\|_X\Big)^{r'} dt \Big\}^{1/r'}$$

$$\le C_{15} \|u_2 - u_1\|_{L^p([0,\tau],X)} \Big\{ \int_0^\tau \Big[1 + \sum_{i=1}^2 \|u_i(t)\|_X^{\gamma r'}\Big] dt \Big\}^{1/r'}$$

$$\leq C_{16}\|u_2-u_1\|_{L^p([0,\tau],X)}\Big[\tau+\sum_{i=1}^{2}\|u_i\|^p_{L^p([0,\tau],X)}\Big]^{1/r'}. \tag{6.13}$$

Then, if $k_\varepsilon(t)=\|R_\varepsilon(t)\|_{\mathcal{L}(X_1,X)}$, from definition (6.7) and (6.13), we have

$$\begin{aligned}
&\|S_\varepsilon u_1-S_\varepsilon u_2\|^p_{L^p([0,\tau],X)}=\|R_\varepsilon * Nu_1-R_\varepsilon * Nu_2\|^p_{L^p([0,\tau],X)} \qquad (6.14)\\
&\leq \|k_\varepsilon\|^{p-r}_{L^r((0,T))}\|Nu_1-Nu_2\|^{p-\widetilde{p}}_{L^{\widetilde{p}}((0,\tau),X_1)}\\
&\qquad\times\int_0^\tau |k_\varepsilon(s)|^r\|Nu_1-Nu_2\|^{\widetilde{p}}_{L^{\widetilde{p}}((0,\tau-s),X_1)}\,ds\\
&\leq C_{16}^p\Big[\tau+\sum_{i=1}^{2}\|u_i\|^p_{L^p([0,\tau],X)}\Big]^{(p-\widetilde{p})/r'}\|k_\varepsilon\|^{p-r}_{L^r((0,T))}\|u_1-u_2\|^{p-\widetilde{p}}_{L^p((0,\tau),X)}\\
&\qquad\times\Big[\tau+\sum_{i=1}^{2}\|u_i\|^p_{L^p([0,\tau],X)}\Big]^{\widetilde{p}/r'}\int_0^\tau |k_\varepsilon(s)|^r\|u_1-u_2\|^{\widetilde{p}}_{L^p([0,\tau-s],X)}\,ds\\
&\leq C_{16}^p[\tau+2M_\varepsilon]^{p/r'}\|k_\varepsilon\|^{p-r}_{L^r((0,T))}\|u_1-u_2\|^{p-\widetilde{p}}_{L^p((0,\tau),X)}\\
&\qquad\times\int_0^\tau |k_\varepsilon(s)|^r\|u_1-u_2\|^{\widetilde{p}}_{L^p([0,\tau-s],X)}\,ds.
\end{aligned}$$

Since $R_\varepsilon\in\mathcal{L}(X_1,X)$ (cf. assumption H6), we have

$$k_\varepsilon(t) \;=\; \|R_\varepsilon(t)\|_{\mathcal{L}(X_1,X)}\leq Ct^{-1+\kappa},\quad t\in(0,+\infty). \tag{6.15}$$

As a consequence, from (6.14) and (6.15), provided $(1-\kappa)r<1$, we deduce

$$\begin{aligned}
&\|S_\varepsilon u_1-S_\varepsilon u_2\|^p_{L^p([0,\tau],X)}\\
&\leq C_{31}(\varepsilon)\|u_1-u_2\|^{p-\widetilde{p}}_{L^p((0,\tau),X)}\int_0^\tau s^{-(1-\kappa)r}\|u_1-u_2\|^{\widetilde{p}}_{L^p([0,\tau-s],X)}\,ds.\\
&\leq C_{32}(\varepsilon)\|u_1-u_2\|^p_{L^p((0,\tau),X)}\tau^{1-(1-\kappa)r}.
\end{aligned}$$

By induction, we can prove that the m-th iterate S_ε^m of S_ε satisfies

$$\|S^m_\varepsilon u_1-S^m_\varepsilon u_2\|^p_{L^p([0,\tau],X)} \;\leq\; C_{33}(m,\varepsilon)\|u_1-u_2\|^p_{L^p((0,\tau),X)}\tau^{m[1-(1-\kappa)r]},$$

where

$$C_{33}(m,\varepsilon)=\frac{[C_{32}(\varepsilon)\Gamma(1-(1-\kappa)r)]^m}{\Gamma(1+m[1-(1-\kappa)r])}.$$

Indeed, we have

$$\|S^{m+1}_\varepsilon u_1-S^{m+1}_\varepsilon u_2\|^p_{L^p([0,\tau],X)}$$

$$
\begin{aligned}
&\le C_{32}(\varepsilon)\|u_1-u_2\|^{p-\widetilde{p}}_{L^p((0,\tau),X)}\int_0^\tau s^{-(1-\kappa)r}\|S^m_\varepsilon u_1 - S^m_\varepsilon u_2\|^{\widetilde{p}}_{L^p([0,\tau-s],X)}\,ds\\
&\le C_{32}(\varepsilon)C_{33}(m,\varepsilon)\|u_1-u_2\|^{p-\widetilde{p}}_{L^p((0,\tau),X)}\\
&\qquad\times\int_0^\tau s^{-(1-\kappa)r}(\tau-s)^{m[1-(1-\kappa)r]}\|u_1-u_2\|^{\widetilde{p}}_{L^p([0,\tau-s],X))}\,ds\\
&\le C_{32}(\varepsilon)C_{33}(m,\varepsilon)\|u_1-u_2\|^{p}_{L^p((0,\tau),X)}\\
&\qquad\times\int_0^\tau s^{-(1-\kappa)r}(\tau-s)^{m[1-(1-\kappa)r]}\,ds\\
&= C_{32}(\varepsilon)C_{33}(m,\varepsilon)\|u_1-u_2\|^{p}_{L^p((0,\tau),X)}\tau^{(m+1)[1-(1-\kappa)r]}\\
&\qquad\times\int_0^1 s^{-(1-\kappa)r}(1-s)^{m[1-(1-\kappa)r]}\,ds\\
&= C_{32}(\varepsilon)C_{33}(m,\varepsilon)\frac{\Gamma(1-(1-\kappa)r)\Gamma(1+m[1-(1-\kappa)r])}{\Gamma(1+(m+1)[1-(1-\kappa)r])}\\
&\qquad\times\|u_1-u_2\|^{p}_{L^p((0,\tau),X)}\tau^{(m+1)[1-(1-\kappa)r]}\\
&= C_{33}(m+1,\varepsilon)\|u_1-u_2\|^{p}_{L^p((0,\tau),X)}\tau^{(m+1)[1-(1-\kappa)r]}.
\end{aligned}
$$

This proves that S^m_ε is a contracting mapping in $L^p([0,T],X)$ for large enough m.

We have thus proved that equation (6.1) admits unique solutions $u_\varepsilon \in L^p([0,T],X)$, for all $\varepsilon\in[0,\varepsilon_0]$.

To show the regularity for u_ε appearing in the statement of the theorem we can use the same arguments as in the proof of Theorem 2.1. □

To prove Theorems 4.3 and 4.4, first we show that the solution u actually belongs to $L^\infty([0,T],X)$. For this purpose we make use of the following Lemma 6.5.

Lemma 6.5. *Let u_ε be defined by* (6.1) *with n satisfying assumption* (ix) *in Lemma* 3.4 *with $L^\alpha(\Omega)$ replaced with X. Then*

$$\|u_\varepsilon\|_{L^\infty([0,T];X)}\le G(u_{0,\varepsilon},f_\varepsilon)\exp\left[M_0C(\eta,\delta)C_{12}T\right],\quad \forall\varepsilon\in[0,\varepsilon_0],\tag{6.16}$$

where

$$G(u_0,f)=M_0C(\eta,\delta)(\|u_0\|+\|f\|_{L^1((0,T);X)}+C_{12}T).$$

Proof. From relations (6.1) we easily get

$$
\begin{aligned}
\|u_\varepsilon(t)\|_X &\le \|R_\varepsilon(t)\|_{\mathcal{L}(X)}\|u_{0,\varepsilon}\|_X+\|R_\varepsilon(\cdot)\|_{\mathcal{L}(X)}*\|f_\varepsilon(\cdot)\|_X\\
&\quad+\|R_\varepsilon(\cdot)\|_{\mathcal{L}(X)}*\|Nu_\varepsilon(\cdot)\|_X.
\end{aligned}
$$

From assumptions H4, H5, Lemma 5.4, and (ix), we obtain

$$\begin{aligned}\|u_\varepsilon(t)\|_X &\le \|R_\varepsilon(t)\|_{\mathcal{L}(X)}\|u_{0,\varepsilon}\|_X + \|R_\varepsilon(\cdot)\|_{\mathcal{L}(X)} * \|f_\varepsilon(\cdot)\|_X \\ &\quad + C_{12}\|R_\varepsilon(\cdot)\|_{\mathcal{L}(X)} * (1+\|u_\varepsilon(\cdot)\|_X). \\ &\le M_0C(\eta,\delta)(\|u_{0,\varepsilon}\|_X + \|f_\varepsilon\|_{L^1((0,T);X)} + C_{12}T) \\ &\quad + M_0C(\eta,\delta)C_{12}\int_0^t \|u_\varepsilon(s)\|_X ds \\ &\le G(u_{0,\varepsilon}, f_\varepsilon) + M_0C(\eta,\delta)C_{12}\int_0^t \|u_\varepsilon(s)\|_X ds.\end{aligned}$$

Applying Gronwall's lemma, we get

$$\begin{aligned}\|u_\varepsilon(t)\|_X &\le G(u_{0,\varepsilon}, f_\varepsilon) + M_0C(\eta,\delta)C_{12}G(u_{0,\varepsilon}, f_\varepsilon) \\ &\qquad \times \int_0^t \exp\left[M_0C(\eta,\delta)C_{12}(t-s)\right] ds \\ &\le G(u_{0,\varepsilon}, f_\varepsilon)\exp\left[M_0C(\eta,\delta)C_{12}T\right]. \qquad \square\end{aligned}$$

The next Lemmas 6.6, 6.7, and 6.8 are needed to prove our approximation results in Theorems 4.3 and 4.4. The proofs of Lemmata 6.6–6.8 can be found in [10], where such lemmata are numbered as 5.2, 5.3, and 5.4.

Lemma 6.6. *Let* $p \in [1,+\infty)$ *and* $\theta_0 \in (1/p,+\infty)$. *Then*

$$\|R_\varepsilon - R\|_{L^p((0,T);\mathcal{L}(X))} \le C_{34}[\varepsilon^{1/p} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0| + \varphi(\varepsilon)], \tag{6.17}$$

for all $\varepsilon \in (0,\varepsilon_1)$, *with* $\varepsilon_1 = \min\{\varepsilon_0, r_0/\delta\}$, *and some positive constant* C_{34} *depending only on* T, M_0, η, δ, ε_0, θ_0, p, *and* r_0.

Lemma 6.7. *If* f_ε *and* f *satisfy* H4, *then*

$$\begin{aligned}&\|R_\varepsilon * f_\varepsilon - R * f\|_{L^r((0,T);X)} \\ &\le C_{35}\{\|f_\varepsilon - f\|_{L^q((0,T);X)} + [\varepsilon^{1/p} + \varepsilon^{\theta_0} + |\chi(\varepsilon)-\chi_0| + \varphi(\varepsilon)]\|f\|_{L^q((0,T);X)}\},\end{aligned}$$

where $R*f(t) = \int_0^t R(t-s)f(s)\,ds$, C_{35} *is a positive constant depending only on* T, M, η, δ, ε_0, θ_0, p, *and* r_0, *and*

$$1/r = \begin{cases} 1/p + 1/q - 1, & \text{if } 1/p + 1/q \ge 1, \\ 1/p, & \text{if } 1/p + 1/q < 1. \end{cases}$$

In particular, $R_\varepsilon * f_\varepsilon \to R * f$, *in* $L^r((0,T);X)$, *as* $\varepsilon \to 0^+$.

Lemma 6.8. *If* $u_{0,\varepsilon}$ *and* u_0 *satisfy* H5, *then, for any* $\varepsilon \in (0,\varepsilon_1]$,

$$\|R_\varepsilon(\cdot)u_{0,\varepsilon} - R(\cdot)u_0\|_{L^p((0,T);X)}$$

$$\le C_{36}\{\|u_{0,\varepsilon}-u_0\|_X+[\varepsilon^{1/p}+\varepsilon^{\theta_0}+|\chi(\varepsilon)-\chi_0|+\varphi(\varepsilon)]\|u_0\|_X,$$

where C_{36} *is a positive constant depending only on* T, M_0, η, δ, ε_0, θ_0, p, *and* r_0. *In particular,* $R_\varepsilon(\cdot)u_{0,\varepsilon}\to R(\cdot)u_0$, *in* $L^p((0,T);X)$, *as* $\varepsilon\to 0^+$,

Remark 6.2. Note that the function $G(u_{0,\varepsilon},f_\varepsilon)$ is uniformly bounded, according to assumptions H4 and H5.

To prove Theorems 4.3 and 4.4, recalling (6.1) and (6.2), we have to estimate in advance the following difference:

$$\begin{aligned}u_\varepsilon(t)-u(t)&=R_\varepsilon(t)(u_{0,\varepsilon}-u_0)+[R_\varepsilon(t)-R(t)]u_0\\&+\int_0^t R_\varepsilon(t-s)[f_\varepsilon(s)-f(s)]ds+\int_0^t[R_\varepsilon(t-s)-R(t-s)]f(s)ds\\&+\int_0^t R_\varepsilon(t-s)[Nu_\varepsilon(s)-Nu(s)]ds+\int_0^t[R_\varepsilon(t-s)-R(t-s)]Nu(s)ds.\end{aligned}\tag{6.18}$$

The estimates we will derive in the following hold when the operator N satisfies the assumptions listed in Theorem 4.3 or in Theorem 4.4.

Now we can note that the identity (6.18), the boundedness estimates (6.16), with $\varepsilon=0$, (6.17) and assumptions H4 and H5 imply

$$\begin{aligned}&\|u_\varepsilon-u\|_{L^\infty([0,T];X)}\le\|u_\varepsilon\|_{L^\infty([0,T];X)}+\|u\|_{L^\infty([0,T];X)}\\&\le\exp[M_0C(\eta,\delta)C_{12}T](G(u_{0,\varepsilon},f_\varepsilon)+G(u_0,f))\\&\le\exp[M_0C(\eta,\delta)C_{12}T]2M_0C(\eta,\delta)(\|u_0\|+\|f\|_{L^1((0,T);X)}+r+C_{12}T)=:M.\end{aligned}\tag{6.19}$$

Proof of Theorem 4.3. From inequality (6.19), we deduce, via the property (x) in Lemma 3.4, the integral inequality

$$\begin{aligned}\|u_\varepsilon(t)-u(t)\|_X&\le D_\varepsilon^{(1)}(t)+\|R_\varepsilon(\cdot)-R(\cdot)\|_{\mathcal{L}(X)}*\|Nu(\cdot)\|_X(t)\\&\quad+M_0C(\eta,\delta)\int_0^t\|Nu_\varepsilon(s)-Nu(s)\|_Xds\\&\le D_\varepsilon^{(1)}(t)+C_{12}\|R_\varepsilon(\cdot)-R(\cdot)\|_{\mathcal{L}(X)}*(1+\|u(\cdot)\|_X)(t)\\&\quad+M_0C(\eta,\delta)C_{13}(M)\int_0^t\|u_\varepsilon(s)-u(s)\|_Xds\\&=:D_\varepsilon^{(1)}(t)+D_\varepsilon^{(2)}(t)+M_0C(\eta,\delta)C_{13}(M)\int_0^t\|u_\varepsilon(s)-u(s)\|_Xds,\end{aligned}$$

where

$$D_\varepsilon^{(1)}(t)=\|R_\varepsilon(t)\|_{\mathcal{L}(X)}\|u_{0,\varepsilon}-u_0\|_X+\|R_\varepsilon(t)-R(t)]\|_{\mathcal{L}(X)}\|u_0\|_X$$

$$+\int_0^t \|R_\varepsilon(t-s)\|_{\mathcal{L}(X)}\|f_\varepsilon(s)-f(s)\|_X ds$$
$$+\int_0^t \|R_\varepsilon(t-s)-R(t-s)\|_{\mathcal{L}(X)}\|f(s)\|_X ds,$$
$$D_\varepsilon^{(2)}(t)=C_{12}\|R_\varepsilon(\cdot)-R(\cdot)\|_{\mathcal{L}(X)}*(1+\|u(\cdot)\|_X)(t).$$

Applying Gronwall's inequality , we obtain

$$\|u_\varepsilon(t)-u(t)\|_X \le D_\varepsilon^{(1)}(t)+D_\varepsilon^{(2)}(t)+M_0C(\eta,\delta)C(M) \times \int_0^t \exp[M_0C(\eta,\delta)C(M)(t-s)][D_\varepsilon^{(1)}(s)+D_\varepsilon^{(2)}(s)]ds.$$

Taking the $L^p((0,T);X)$-norm and using Young's inequality, we get

$$\|u_\varepsilon-u\|_{L^p((0,T);X)} \le [\|D_\varepsilon^{(1)}\|_{L^p[0,T]}+\|D_\varepsilon^{(2)}\|_{L^p[0,T]}]\exp[M_0C(\eta,\delta)C(M)T].$$

Applying Young's inequality again, from Lemmas 6.6, 6.7, and 6.8, we easily deduce

$$\|D_\varepsilon^{(1)}\|_{L^p[0,T]} \le C_{37}(T,M,\eta,\delta,\varepsilon_0,r_0)\big\{\|u_{0,\varepsilon}-u_0\|_X+\|f_\varepsilon-f\|_{L^q((0,T);X)} + [\varepsilon^{1/p}+\varepsilon^{\theta_0}+|\chi(\varepsilon)-\chi_0|+\varphi(\varepsilon)]\big[\|u_0\|_X+\|f\|_{L^q((0,T);X)}\big]\big\},$$
$$\|D_\varepsilon^{(2)}\|_{L^p[0,T]} \le C_{38}\|R_\varepsilon-R\|_{L^p((0,T);\mathcal{L}(X))}(T+\|u\|_{L^1((0,T);X)}) \le C_{38}\|R_\varepsilon-R\|_{L^p((0,T);\mathcal{L}(X))}[T+C_{39}\|G(u_0,f)\|_{L^1[0,T]}(1+C_{40})].$$

Consequently,

$$\|u_\varepsilon-u\|_{L^p((0,T);X)} \le \exp[M_0C(\eta,\delta)C(M)T]\Big\{C(T,M_0,\eta,\delta,\varepsilon_0,r_0)\big\{\|u_{0,\varepsilon}-u_0\|_X + \|f_\varepsilon-f\|_{L^q((0,T);X)}+[\varepsilon^{1/p}+\varepsilon^{\theta_0}+|\chi(\varepsilon)-\chi_0|+\varphi(\varepsilon)] \times\big[\|u_0\|_X+\|f\|_{L^q((0,T);X)}\big]\big\} + C_{38}\|R_\varepsilon-R\|_{L^p((0,T);\mathcal{L}(X))}[T+C_{39}\|G(u_0,f)\|_{L^1[0,T]}(1+C_{40})]\Big\}. \qquad \square$$

Proof of Theorem 4.4. Our task consists in estimating the convolution term $R_\varepsilon*(Nu_\varepsilon-Nu)$ in (6.18)—indeed, the other terms can be estimated as in the linear case (see [10]). First we note that estimate (6.19) holds also in this case. Consequently, from assumptions H7 and H8, we deduce

$$\|Nu_\varepsilon(t,\cdot)-Nu(t,\cdot)\|_{X_1} \le \rho(\|u_\varepsilon\|_X+\|u\|_X)\|u_\varepsilon(t,\cdot)-u(t,\cdot)\|_X \le \big[1+(\|u_\varepsilon\|_X+\|u\|_X)^\gamma\big]\|u_\varepsilon(t,\cdot)-u(t,\cdot)\|_X \tag{6.20}$$

$$\leq C_{14}\big[1+2^{\gamma}M^{\gamma}\big]\|u_{\varepsilon}(t,\cdot)-u(t,\cdot)\|_{X}.$$

Let now $\widetilde{p}\in[1,+\infty]$ be defined by $1/\widetilde{p}=1/p-1/r+1$. Taking the $L^{\widetilde{p}}([0,T])$-norms, in (6.20), we get

$$\begin{aligned}\|Nu_{\varepsilon}-Nu\|_{L^{\widetilde{p}}([0,T];X_1)} &\leq \|u_{\varepsilon}-u\|_{L^{p}((0,T);X)}C_{14}\big[1+2^{\gamma}M^{\gamma}\big]\Big(\int_0^T 1dt\Big)^{1/r}\\ &\leq C_{41}(M)\|u_{\varepsilon}-u\|_{L^{p}((0,T);X)}. \qquad (6.21)\end{aligned}$$

The statement of the theorem follows immediately from Lemma 6.4. □

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