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# SECOND-ORDER INTEGRODIFFERENTIAL EQUATION WITH NONAUTONOMOUS OPERATORS

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**Abstract.** In this paper, we establish some results to show the existence and uniqueness of classical solution of the non-autonomous second-order integrodifferential equation of the type:

$$\begin{aligned} u''(t) &= A(t)\,[u(t)+\int_0^t k_1(t,s)u(s)ds]' + B(t)\,[u(t)+\int_0^t k_2(t,s)u(s)ds] \\ &\quad +f(t),\ 0\le t\le T, \\ u(0) &= u_0\in E, \quad u'(0)=u_1\in E, \end{aligned}$$

on a Banach space $E$, by means of the matrix operator method.

## 1. Introduction

Let us consider the following equations:

$$u'(t)=A(t)u(t), \quad u(0)=u_0,\ t\in[0,T], \tag{1.1}$$

$$u'(t)=A(t)u(t)+f(t), \quad u(0)=u_0,\ t\in[0,T], \tag{1.2}$$

where $A(t):D(A(t)):=D\subset E\to E$ is a linear operator and $f:[0,T]\to E$, $u_0\in E$ are given. (1.1) or (1.2) is studied using the semigroup approach if $A(t)$ is independent of $t$ and if, for each fixed $t\in[0,T]$, $A(t)$ generates a strongly continuous semigroup, then (1.2) is studied in Kato [1], Pazy [2] and Tanabe [3] using the evolution operators. But some applications in partial differential equations involve non-densely defined operators $A(\cdot)$, for example, when seeking pointwise estimates of solutions and their derivatives in the space of continuous functions. Thus, Da Prato and Sinestrari [5, 4] extended the results of Kato [1], Pazy [2], and Tanabe [3] to (1.2) with non-densely defined operators $A(\cdot)$, and proved the existence and uniqueness of solutions of (1.2) under some given conditions.

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In this paper, we study the non-autonomous second-order Volterra integrodifferential problem of the following type,

$$\begin{aligned} u''(t) = A(t)[u(t) + \int_0^t k_1(t,s)u(s)ds]' + B(t)[u(t) + \int_0^t k_2(t,s)u(s)ds] \\ + f(t), \quad 0 \le t \le T, \\ u(0) = u_0 \in E, \quad u'(0) = u_1 \in E, \end{aligned} \tag{1.3}$$

in a Banach space $E$, where $\{A(t)\}_{0\le t\le T}$ and $\{B(t)\}_{0\le t\le T}$ are linear operators on $E$, with $D(A(t)) := D \subset E$, $D(B(t)) \supseteq D$ and $f$ a continuous function from $[0,T]$ to $E$.

The purpose of this paper is to show the existence and uniqueness of solutions of (1.3) by using the results of Oka and Tanaka [7] and the operator matrix method. We use an operator matrix method [6] to reduce (1.3) into a first-order integrodifferential equation, then conclusions of Oka and Tanaka [7] concerning the non-autonomous evolution operator of hyperbolic type for the integrodifferential equations of the type

$$u'(t) = A(t)u(t) + \int_0^t K(t,s)u(s)ds + f(t),\ u(0) = u_0,\ t \in [0,T],$$

are applied to establish the existence and uniqueness of a solution of (1.3).

In Section 2, we recall some preliminary results useful for our analysis. In the next section we prove the existence and uniqueness of a solution of (1.3). These results can be applied to a class of integrodifferential equations under certain assumptions on the kernel. This fact is illustrated by the example given at the end of this paper.

## 2. Preliminaries

In this section, we give some basic definitions and notation. Let $E$ be a Banach space with its norm denoted by $\|\cdot\|$. We will use in this paper the following Banach spaces of functions (endowed with their usual norms):

- $C([t_0,T];E)$: the space of all continuous functions $u : [t_0,T] \to E$.
- $C^n([t_0,T];E)$: the space of all $n$ times continuously differentiable functions $u : [t_0,T] \to E$.
- $L^p([t_0,T];E)$: the space of all measurable functions $u : [t_0,T] \to E$. such that $\|u(\cdot)\| \in L^p([t_0,T]); 1 \le p < \infty$.
- $W^{1,p}([t_0,T];E)$: the space of all functions $u : [t_0,T] \to E$ such that there exist $v \in L^p([t_0,T]); 1 \le p < \infty$, such that

$$u(t) = u(t_0) + \int_{t_0}^{t} v(s)ds, \quad t \in [t_0, T].$$

In the following, for a linear operator $A$ on a Banach space $E$, we denote the resolvent set of an operator $A(t)$ by $\rho(A(t))$ and the resolvent $(\lambda - A)^{-1}$ by $R(\lambda, A)$. By $L(E, F)$, we denote the set of all linear operators from $E$ to $F$. By $L(E)$, we denote the set of all linear operators from $E$ to $E$ itself. In the whole work, we suppose that $T > 0$ is a real number. Let $T_1 \in \mathbb{R}$ with $T_1 < T$, we use $\triangle(T_1, T)$ to denote the set $\{(s,t) : s, t \in \mathbb{R},\ T_1 \leq t \leq s \leq T\}$. Finally for short, we write the family of $A(t),\ 0 \leq t \leq T$, by $\{A(t)\}_{0 \leq t \leq T}$.

**Definition 2.1.** A function $u : [0, T] \to E$ is said to be a classical solution of (1.3) if it is twice continuously differentiable on $[0, T]$, $u(t), u'(t) \in D$, and (1.3) is satisfied.

We first recall the fundamental result obtained by Tanaka [8].

**Theorem 2.2.** (see [8], Theorem 1.8) *A family of operators $\{A(t)\}_{0 \leq t \leq T}$ satisfies the hyperbolic conditions if*

(H1) $A(t) : D := D(A(t)) \subset E \to E$, *is a linear operator from the Banach space $(D, \|\cdot\|_D)$ over $E$. Here $D$ may not be dense in $E$. Moreover, there exist $c_0 > 0$ such that $\forall t \in [0, T],\ x \in D$,*

$$c_0^{-1}\|u\|_D \leq \|u\| + \|A(t)u\| \leq c_0\|u\|_D.$$

(H2) *The family $\{A(t)\}_{0 \leq t \leq T}$ is stable; that is, there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A(t)),\ t \in [0, T]$ and*

$$\|R(\lambda, t_1, t_2, \ldots, t_n)\| \leq \frac{M}{(\lambda - \omega)^n},\ 0 \leq t_n \leq t_{n-1} \cdots \leq t_1,\ \lambda > \omega;$$

*here* $R(\lambda, t_1, t_2, \ldots, t_n) := [(\lambda - A(t_1))^{-1}(\lambda - A(t_1))^{-1} \cdots (\lambda - A(t_n))^{-1}]$.

(H3) *The mapping $t \to A(t)y$ is continuously differentiable in $E$ $\forall\, y \in D$.*

In general, however, it is not always easy to decide whether or not a given family of operators is stable. The following lemmas are useful criteria for the same.

**Lemma 2.3.** (see [2], Theorem 5.2.3) *Let $\{A(t)\}_{0 \leq t \leq T}$ be stable and suppose $\{B(t)\}_{0 \leq t \leq T}$ is a family of uniformly bounded operators. Then the family $\{A(t) + B(t)\}_{0 \leq t \leq T}$ with $D(A(t) + B(t)) := D(A(t))$ is stable.*

**Lemma 2.4.** (see [1], Proposition 4.4) *Let* $\{A(t)\}_{0\leq t\leq T}$ *be a stable family and* $\{S(t)\}_{0\leq t\leq T}$ *be a family of isomorphisms,* $S(t)\in L(E)$*, which is strongly continuously differentiable. Then the family*

$$\{\tilde{A}(t)\}_{0\leq t\leq T}=\{S(t)A(t)S^{-1}(t)\}_{0\leq t\leq T}$$

*with* $D(\tilde{A}(t))=\{u:S^{-1}(t)u\in D(A(t))\}$ *is stable.*

For other information on evolution equations, evolution families and the theory of operator matrices, we refer to, for instance, [10], [2], and [8].

## 3. Main results

The method we use to study (1.3) is as follows: For our purpose, we assume that there exists an invertible operator $A$ from $D\subset E$ onto $E$ with $A^{-1}\in L(E,D)$. We introduce new variables by defining

$$v_0:=Au,\ v_1:=u'. \tag{3.1}$$

Then we have

$$\begin{aligned} v_0' &= Au'=Av_1,\\ v_1' &= u''(t). \end{aligned}$$

Using the above notation (1.3) can be rewritten as

$$\begin{aligned} u''(t) &= A(t)[u(t)+\int_0^t k_1(t,s)u(s)ds]'+B(t)[u(t)+\int_0^t k_2(t,s)u(s)ds]\\ &\quad +f(t),\quad 0\leq t\leq T,\\ &= A(t)u'(t)+A(t)A^{-1}\int_0^t k_{1t}(t,s)Au(s)ds+k_1(t,t)A(t)u(t)+B(t)u(t)\\ &\quad +B(t)A^{-1}\int_0^t k_2(t,s)Au(s)ds+f(t),\quad 0\leq t\leq T. \end{aligned}$$

In view of the closedness of $A$ the second step follows. Then using the notation given in (3.1) we can rewrite (1.3) as an integrodifferential equation in $V:=(v_0,v_1)^T$ in the Banach space $E\times E$ as follows:

$$\begin{aligned} V'(t) &= \tilde{A}(t)V(t)+\int_0^t K(t,s)V(s)ds+\tilde{f}(t),\quad 0\leq t\leq T,\\ V(0) &= (Au_0,u_1)^T, \end{aligned} \tag{3.2}$$

where

$$\tilde{f}(t)=\begin{pmatrix} 0\\ f(t)\end{pmatrix},$$

$$K(t,s) = \begin{pmatrix} 0 & 0 \\ k_{1t}(t,s)I + k_2(t,s)B(t)A^{-1} & 0 \end{pmatrix} \quad \text{and}$$
$$\tilde{A}(t) = \begin{pmatrix} 0 & A \\ B(t)A^{-1} + k_1(t,t)A(t)A^{-1} & A(t) \end{pmatrix}$$

with $D(\tilde{A}(t)) := E \times D$. We easily see that if $\{A(t)\}_{0\le t\le T}$ satisfies the hyperbolic conditions, we can choose $A := A(0)$.

For our purpose we have the following lemma:

**Lemma 3.1.** *Let $u_i \in D$ $(i = 0, 1)$; the following statements hold true.*

(1) *If $u(t)$ is a solution of* (1.3), *then $u(t) \in D$, $u'(t) \in D$ and $(Au(t), u'(t))^T$ is a solution of* (3.2).

(2) *Conversely, if $(v_0(t), v_1(t))^T$ is a solution of* (3.2), *then*

$$u(t) := \int_0^t v_1(s)ds + u_0$$

*is a solution of* (1.3).

**Proof.** (1) $\Rightarrow$ (2). Let $u(t)$ be a solution of (1.3) with $u_0 \in D$ and $u_1 \in D$. In view of the closedness of $A$ we have

$$\int_0^t Au'(s)ds = A\int_0^t u'(s)ds = A[u(t) - u_0]. \tag{3.3}$$

Since $u_0 \in D$, it follows that $u(t) \in D$; hence, the function $t \to Au(t)$ is continuously differentiable and $\frac{d}{dt}Au(t) = Au'(t)$. Therefore $(v_0(t), v_1(t))^T := (Au(t), u'(t))^T$ is continuously differentiable and satisfies (3.2), and thus is a solution of (3.2).

(2) $\Rightarrow$ (1). Conversely, suppose that $(v_0(t), v_1(t))^T$ is a solution of (3.2). We define the function $u$ by

$$u(t) := \int_0^t v_1(s)ds + u_0; \tag{3.4}$$

then $u(t)$ is twice continuously differentiable. Furthermore, from (3.2) we have

$$v_0(t) = \int_0^t Av_1(s)ds + v_0(0) = A\Big(\int_0^t v_1(s)ds + A^{-1}v_0(0)\Big) = Au(t). \tag{3.5}$$

Thus,

$$u''(t) = v_1'(t)$$

$$= A(t)v_1(t) + A(t)A^{-1}\int_0^t k_{1t}(t,s)v_0(s)ds + k_1(t,t)A(t)A^{-1}v_0(t)$$

$$+ B(t)A^{-1}v_0(t) + B(t)A^{-1}\int_0^t k_2(t,s)v_0(s)ds + f(t)$$

$$= A(t)u'(t) + A(t)A^{-1}\int_0^t k_{1t}(t,s)Au(s)ds + k_1(t,t)A(t)u(t) + B(t)u(t)$$

$$+ B(t)A^{-1}\int_0^t k_2(t,s)Au(s)ds + f(t)$$

$$= A(t)[u(t) + \int_0^t k_1(t,s)u(s)ds]' + B(t)[u(t) + \int_0^t k_2(t,s)u(s)ds] + f(t).$$

Now, we will prove the following results concerning the leading operators in (3.2).

**Theorem 3.2.** *For the second-order integrodifferential equation* (3.2)*, we assume the following:*

(1) $\{A(t)\}_{0\le t\le T}$ *satisfies the hyperbolic conditions;*
(2) $\{B(t)\}_{0\le t\le T}$ *is a family of linear operators with* $D(B(t)) \supseteq D$ *such that* $B(t) \in L(D,E)$ *and* $t \to B(t)u$ *is continuously differentiable for each* $u \in D$;
(3) $\{k_i(t,s)\}_{0\le t\le T}, i = 1,2$ *are defined as* $k_1(t,s) \in C^2(\triangle(0,T))$ *and* $k_2(t,s) \in C^1(\triangle(0,T))$*, where* $\triangle(0,T) := \{(s,t) : 0 \le s \le t \le T\}$.

*Then*

(D1) *The family* $\{\tilde{A}(t)\}_{0\le t\le T}$ *satisfies the hyperbolic conditions on the domain* $E \times D$.

(D2) $\{K(t,s)\}_{0\le t\le T}$ *is a family of bounded linear operators from* $E \times D$ *to* $E \times E$ *such that for every* $y \in E \times D$, $K(t,s)y$ *is continuous on the set* $\triangle(0,T)$ *and continuously differentiable with respect to* $t$.

**Proof.** It is easy to see that $\{\tilde{A}(t)\}_{0\le t\le T}$ satisfies (H1) of the hyperbolic conditions. So we have to prove (H2) and (H3) only. For this we assume, without loss of generality, that $\omega < 0$; then $A(t)$ is invertible for every $t \in [0,T]$. To prove the stability of $\tilde{A}(t)$ we will use Lemmas 2.3 and 2.4. For that, we first prove the uniform boundedness of the families $B(t)A^{-1}$ and $k_1(t,t)A(t)A^{-1}$.

Since the operators $B(t)A^{-1}$ are everywhere defined in $E$ and closed, they are bounded. Since $B(t) \in L(D,E)$, $B(t)A^{-1}$ is bounded in $E$. Now using the strongly continuous differentiability of $\{B(t)A^{-1}\}$ and the principle of uniform boundedness, it follows that the family $\{B(t)A^{-1}\}$ is uniformly bounded.

Similar results hold for $A(t)A^{-1}$ also; that is, the family $A(t)A^{-1}$ is uniformly bounded and so is $k_1(t,t)A(t)A^{-1}$.

Now, if $A(t)$ is strongly continuously differentiable, then $A^{-1}(t)$ is also. The family $AA^{-1}(t)$ is strongly continuously differentiable. This can be achieved by the fact that the family $A(t)A^{-1}$ is strongly continuously differentiable and the equality $AA^{-1}(t) = (A(t)A^{-1})^{-1}$. We have

$$\begin{aligned}
&\begin{pmatrix} 0 & A \\ B(t)A^{-1} + k_1(t,t)A(t)A^{-1} & A(t) \end{pmatrix} \\
&= \begin{pmatrix} I & AA^{-1}(t) \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & A(t) \end{pmatrix} \begin{pmatrix} I & -AA^{-1}(t) \\ 0 & I \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 \\ B(t)A^{-1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_1(t,t)A(t)A^{-1} & 0 \end{pmatrix}.
\end{aligned}$$

The stability of the family $\tilde{P}(t) := \{\begin{pmatrix} 0 & 0 \\ 0 & A(t) \end{pmatrix}\}$ follows from the stability of the family $\{A(t)\}_{0\le t\le T}$. Thus the family $\{\tilde{P}(t)\}_{0\le t\le T}$, with $D(\tilde{P}(t)) := E \times D$ is stable in $E \times E$.

Now from the observation, the family of isomorphisms $\begin{pmatrix} I & AA^{-1}(t) \\ 0 & I \end{pmatrix}$ and of their inverses, $\begin{pmatrix} I & -AA^{-1}(t) \\ 0 & I \end{pmatrix}$, are strongly continuously differentiable. Using Lemmas 2.3 and 2.4, we conclude that the family $\{\tilde{A}(t)\}_{0\le t\le T}$ with $D(\tilde{A}(t)) := E \times D$ is stable in $E \times E$.

Finally, for (H3), it is easy to see that $\{\tilde{A}(t)\}$ is strongly continuously differentiable. Thus, the hyperbolic conditions are satisfied for the family $\{\tilde{A}(t)\}_{0\le t\le T}$.

(D2) Here $K(t,s) = \begin{pmatrix} 0 & 0 \\ k_{1t}(t,s)I + k_2(t,s)B(t)A^{-1} & 0 \end{pmatrix}$ can be written as

$$\begin{aligned}
K(t,s) &= \begin{pmatrix} 0 & 0 \\ k_{1t}(t,s)I + k_2(t,s)B(t)A^{-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ k_{1t}(t,s)I & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_2(t,s)B(t)A^{-1} & 0 \end{pmatrix} := K_1(t,s) + K_2(t,s),
\end{aligned}$$

and we can easily check that each of $K_i(t,s)$, $(i=1,2)$ is a family of bounded linear operators from $E\times D$ to $E\times E$ and for every $y\in E\times D$, $K_i(t,s)y$, $i=1,2$ is continuous on the set $\triangle(0,T)$ and continuously differentiable with respect to $t$. Using these arguments, the proof of (D2) follows.

**Remark.** In the proof of Theorem 3.2, we assumed $\omega<0$; that is, $A(t)$ is invertible. This assumption can be removed. In fact, if we take $\lambda\in\rho(A(t))$ such that $\lambda>\omega$, the family $\{A(t)\}$ satisfies the hyperbolic conditions, and if $A\in L(D,E)$, then by the identity

$$\begin{pmatrix} 0 & A \\ 0 & A(t) \end{pmatrix} =$$

$$\begin{pmatrix} I & -AR(\lambda,A(t)) \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & A(t)-\lambda \end{pmatrix}\begin{pmatrix} I & AR(\lambda,A(t)) \\ 0 & I \end{pmatrix},$$

we see that $\begin{pmatrix} 0 & A \\ 0 & A(t) \end{pmatrix}$ is stable.

Now, we recall the following result for the first-order Volterra integrodifferential equations given by Oka and Tanaka [7].

**Theorem 3.3.** *Consider the first-order Volterra integrodifferential equations of the type*

$$u'(t) = P(t)u(t) + \int_0^t Q(t,s)u(s)ds + f(t),\ 0\le t\le T, \tag{3.6}$$

$$u(0) = u_0. \tag{3.7}$$

*If the following conditions hold,*

(P1) *The family $\{P(t)\}_{0\le t\le T}$ satisfies the hyperbolic conditions with constant domain $D$, which is not necessarily dense in $E$;*

(P2) *$\{Q(t,s)\}_{0\le t\le T}$ is a family of bounded linear operators from $D$ to $E$ such that for every $y\in D$, $Q(t,s)y$, is continuous on the set $\triangle(0,T):=\{(s,t):0\le s\le t\le T\}$ and continuously differentiable with respect to $t$;*

*then* (3.6)–(3.7) *has a unique solution for every initial value $u_0\in D$, and $f\in W^{1,1}(\mathbb{R}_+,E)$ such that $P(0)u_0+f(0)\in\bar{D}$.*

Combining Theorem 3.3 and Lemma 3.1, we obtain the existence and uniqueness of the solution of (1.3). More precisely, we have the following theorem.

**Theorem 3.4.** *Consider the non-autonomous second-order Volterra integrodifferential equation* (1.3)

$$\begin{aligned} u''(t) &= A(t)[u(t) + \int_0^t k_1(t,s)u(s)ds]' + B(t)[u(t) + \int_0^t k_2(t,s)u(s)ds] \\ &\quad + f(t), \quad 0 \le t \le T, \\ u(0) &= u_0 \in E, \quad u'(0) = u_1 \in E, \end{aligned}$$

*where the families* $\{A(t)\}_{0\le t\le T}$ *and* $\{B(t)\}_{0\le t\le T}$ *have the following properties:*

(i) *The family* $\{A(t)\}_{0\le t\le T}$ *satisfies the hyperbolic conditions given in Theorem* 2.2 *and* $\{B(t)\}_{0\le t\le T}$ *is a family of linear operators with* $D(B(t)) \supseteq D$ *such that* $B(t) \in L(D,E)$ *and* $t \to B(t)u$ *is continuously differentiable for each* $u \in D$.

(ii) $\{k_i(t,s)\}_{0\le t\le T}$, $i = 1, 2$ *are defined as* $k_1(t,s) \in C^2(\triangle(0,T))$ *and* $k_2(t,s) \in C^1(\triangle(0,T))$, *where* $\triangle(0,T)) := \{(x,y) : 0 \le s \le t \le T\}$.

*Then, a unique solution of* (1.3) *exists for every initial value* $u_0 \in D$ *and* $f \in W^{1,1}([0,T],E)$, *such that* $Au'(0) \in E$ *and* $B(0)u(0) + k_1(0,0)A(0)u(0) + A(0)u'(0) + f(0) \in \bar{D}$ .

**Proof.** To find the existence and uniqueness conditions we convert the given second-order Volterra integrodifferential equation using the notation of (3.1) to the first-order Volterra integrodifferential equation on $E \times D$ as follows:

$$\begin{aligned} V'(t) &= \tilde{A}(t)V(t) + \int_0^t K(t,s)V(s)ds + \tilde{f}(t), 0 \le t \le T, \\ V(0) &= (Ax_0, x_1)^T, \qquad (3.8) \\ \tilde{f}(t) &= \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \; K(t,s) = \begin{pmatrix} 0 & 0 \\ k_{1t}(t,s)I + k_2(t,s)B(t)A^{-1} & 0 \end{pmatrix} \\ \text{and} \quad \tilde{A}(t) &= \begin{pmatrix} 0 & A \\ B(t)A^{-1} + k_1(t,t)A(t)A^{-1} & A(t) \end{pmatrix}. \end{aligned}$$

Now, we can prove easily by using Theorem 3.2 that the families $\{\tilde{A}(t)\}_{0\le t\le T}$ and $\{K(t,s)\}_{0\le t\le T}$ satisfy the conditions $P1$ and $P2$ (see Theorem 3.3) and we can easily check for the compatibility condition; i.e., $f \in W^{1,1}(\mathbb{R}_+,E)$, $Au'(0) \in E$ and $B(0)u(0) + k_1(0,0)A(0)u(0) + A(0)u'(0) + f(0) \in \bar{D}$. Now, using the result of Theorem 3.3, we obtain the existences and uniqueness of the solutions of (3.8) and then the existence and uniqueness of the solutions of (1.3) follow from Lemma 3.1.

## 4. Applications

**1)** If we choose the functions $k_i(t,s) :\equiv 0$ in the respective domains of their definitions then our problem reduces to the problem solved by Nguyen [6]. In this way our results generalize the results given in Nguyen [6].

**2)** Consider the following hyperbolic integrodifferential equation:

$$\begin{aligned} &u_{tt} + u_{tx}(t,x) + \int_0^t k(t-s)u_x(s,x)ds + \psi_x(t,x) = f(t,x), \quad \text{with} \\ &\psi(t,x) = E(t)u(t,x) + \int_0^t b(t,s)u(s,x)ds,\ (t,x) \in [0,T]\times[0,1], \qquad (4.1) \\ &u(t,0) = u(t,1),\ t\in[0,T],\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ x\in[0,1]. \end{aligned}$$

Here, in this example, we have $E(t) > 0, \forall\, t \in [0,T]$, and $k(0) = 0$. We may rewrite (4.1) as follows:

$$\begin{aligned} u_{tt} + u_{tx}(t,x) + \int_0^t k(t-s)u_x(s,x)ds + E(t)u_x(t,x) + \int_0^t b(t,s)u_x(s,x)ds \\ = f(t,x), (t,x) \in [0,T]\times[0,1], \qquad (4.2) \\ u(t,0) = u(t,1),\ t\in[0,T],\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ x\in[0,1]. \end{aligned}$$

Let $k_1$ be such that $k_1 \in C^2\triangle(0,T)$ with $\frac{\partial}{\partial t}\ k_1(t,s) = k(t-s),\ k_1(t,t) = k(0) = 0$.

With this setting of functions and notations, we can rewrite (4.2) as

$$\begin{aligned} &u_{tt} + u_{tx}(t,x) + \int_0^t \frac{\partial}{\partial t}k_1(t,s)u_x(s,x)ds + E(t)u_x(t,x) + k_1(t,t)u_x(t,x) \\ &\quad + E(t)\int_0^t \frac{b(t,s)}{E(t)}u_x(s,x)ds = f(t,x),\ (t,x)\in[0,T]\times[0,1], \qquad (4.3) \\ &u(t,0) = u(t,1),\ t\in[0,T],\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ x\in[0,1]. \end{aligned}$$

Finally, let $k_2(t,s) = \frac{b(t,s)}{E(t)} \in C^1(\triangle[0,T])$; then we write (4.3) as

$$\begin{aligned} &u_{tt} + u_{tx}(t,x) + \int_0^t \frac{\partial}{\partial t}k_1(t,s)u_x(s,x)ds + E(t)u_x(t,x) + k_1(t,t)u_x(t,x) \\ &\quad + E(t)\int_0^t k_2(t,s)u_x(s,x)ds = f(t,x),\ (t,x)\in[0,T]\times[0,1], \qquad (4.4) \\ &u(t,0) = u(t,1),\ t\in[0,T],\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ x\in[0,1]. \end{aligned}$$

Here, we have $E$, the Banach space $C[0,1]$ with $\|u\|_\infty = \sup\{|u(x)| : x \in [0,1]\}$ and $D$, the Banach space $\{u \in C^1[0,1] : u(0) = u(1)\}$, endowed with the graph norm $\|u\|_\infty + \|u_x\|_\infty$. Define two families $\{A(t) : t \in [0,1]\}$ and $\{B(t) : t \in [0,1]\}$ of linear operators in $E$ as follows:

$$\begin{aligned}(D(A(t)) &= D,\\ (A(t)u)(x) &= -u_x(x) \text{ for } u \in D,\\ (D(B(t)) &= C^1[0,1],\\ (B(t)u)(x) &= -E(t)u_x(x) \text{ for } \ u \in C^1[0,1],\end{aligned}$$

respectively. Then using the definitions of $A(t)$ and $B(t)$, (4.4) is reduced to (1.3). The family $\{A(t) : t \in [0,1]\}$ of closed linear operators satisfies conditions (H1–H3) (see [4], Theorem 6.1). Also it is easy to see that if $E$ is of class $C^1$, then $\{B(t)\}_{0\le t\le T}$ is a family of linear operators with $D(B(t)) \supseteq D$ such that $B(t) \in L(D, E)$ and $t \to B(t)u$ is continuously differentiable for each $u \in D$, since $\bar{D} = \{u \in C[0,1] : u(0) = u(1)\}$. Theorem (3.4) asserts that if $f \in W^{1,1}([0,T] : C[0,1])$, $u_{1x} \in C[0,1]$ and the compatibility conditions that

$$u_{1x}(0) + E(0)u_{0x}(0) + f(0,0) = u_{1x}(1) + E(0)u_{0x}(1) + f(0,1),$$

is satisfied then the problem considered has a unique solution $u \in C^2([0,T] : C[0,1]) \cap C([0,T] : C^1[0,1])$.

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