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MULTIPLE CONTINUATION BEYOND BLOW-UP

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Abstract. We construct solutions of a supercritical parabolic equation which blow up in finite time but possess multiple global continuations (as weak solutions).

1. Introduction

We consider the equation

$$u_t = \Delta u + u^p, \quad u > 0, \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where

$$p > p_S := \begin{cases} \infty & \text{if} \quad N \le 2, \\ \frac{N+2}{N-2} & \text{if} \quad N > 2. \end{cases} \tag{1.2}$$

Our main aim is to show that for every positive integer n there is a backward self-similar solution u of (1.1) which blows up at the time $t = T$, $T \in \mathbb{R}$, and has at least n different continuations for $t > T$ provided $N > 2$ and

$$p < p_{JL} := \begin{cases} \infty & \text{if} \quad N \le 10, \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{if} \quad N > 10. \end{cases} \tag{1.3}$$

The first construction of a peaking solution by continuing a blowing-up backward self-similar solution u_b by a forward self-similar solution u_f can be found in [11]. It was shown there that for the equation

$$u_t = \Delta u + e^u, \qquad x \in \mathbb{R}^N, \tag{1.4}$$

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where $3 \le N \le 9$, there are two self-similar solutions u_b, u_f defined on $(-\infty, T)$ and (T, ∞), respectively, $T \in \mathbb{R}$, such that u_b blows up at $t = T$, $|u_b(x,T)| < \infty$ for $x \neq 0$ and $u_b(x,T) = u_f(x,T)$. A similar result for equation (1.1) was established in [7] for

$$p_S < p < p_L := \begin{cases} \infty & \text{if} \quad N \le 10, \\ 1 + \frac{6}{N-10} & \text{if} \quad N > 10. \end{cases} \tag{1.5}$$

It was also proved, in [7], that this construction indeed yields a global weak solution. The question of uniqueness of the continuation was left open in [11] and [7], although a remark on possible non-uniqueness was made in [11].

More recently, blowing-up solutions which can be extended beyond blow-up were studied in [3]–[6] and [16]–[20]. In most cases, the weak solutions considered there are the minimal (often called proper) solutions which are uniquely determined.

Examples of non-uniqueness of weak solutions of the Cauchy problem

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbb{R}^N, \quad t > 0, \\ u(\cdot, 0) = u_0, & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

or of the problem

$$\begin{aligned} &u_t = \Delta u + f(u), && x \in B_R := \{x \in \mathbb{R}^N : |x| < R\}, \quad t > 0, \\ &u = 0, && x \in \partial B_R, \quad t > 0, \\ &u(\cdot, 0) = u_0, && x \in B_R, \end{aligned}$$

with $f(u) = |u|^{p-1}u$, $p > 1$, or $f(u) = e^u$ are well known (see [1], [7], [9], [15], [21], [22], [24] and [26]), but in these examples $\|u(\cdot,t)\|_{L^q} \to \infty$ as $t \to 0$ for some sufficiently large $q \le \infty$. This means that initial data are only taken in the L^q-sense with q small enough. Another difference between previous non-uniqueness results and our example is that in the former, non-uniqueness occurs at $t = 0$, while in the latter, the solution branches at the time $t = T > 0$ when it develops a singularity; see Theorem 3.2 below.

2. Self-similar solutions

2.1. **Backward self-similar solutions.** By a backward self-similar solution of the equation

$$u_t = u_{rr} + \frac{N-1}{r} u_r + |u|^{p-1}u, \qquad r > 0, \qquad p > 1, \tag{2.1}$$

we mean a solution of the form

$$u(r,t) = (T-t)^{-\frac{1}{p-1}}\psi(y), \qquad y = \frac{r}{\sqrt{T-t}}, \quad T\in\mathbb{R}, \quad t<T,$$

where ψ is a solution of the ODE

$$\psi_{yy} + \Big(\frac{N-1}{y} - \frac{y}{2}\Big)\psi_y + |\psi|^{p-1}\psi - \frac{1}{p-1}\psi = 0, \qquad y>0, \tag{2.2}$$

which can also be written as

$$\big(\rho(y)\psi_y\big)_y + \rho(y)\Big(|\psi|^{p-1}\psi - \frac{1}{p-1}\psi\Big) = 0, \qquad y>0, \tag{2.3}$$

where $\rho(y) := y^{N-1}\exp\big(-\frac{y^2}{4}\big)$. Bounded solutions of this equation satisfy the initial conditions

$$\psi(0) = a, \qquad \psi_y(0) = 0. \tag{2.4}$$

In the case $p \le p_S$, the only bounded solutions of (2.2) are the constants

$$\psi \equiv 0, \qquad \psi \equiv \pm\kappa, \quad \kappa := (p-1)^{-\frac{1}{p-1}};$$

see [8]. On the other hand, for $p_S < p < p_{JL}$ there exists an increasing sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \to \infty$, such that the solution ψ_n of (2.2), (2.4) with $a = a_n$ satisfies

$$\psi_n(y) > 0 > (\psi_n)_y(y) \quad \text{for} \quad y>0, \qquad \psi_n(y)\to 0 \quad \text{as} \quad y\to\infty;$$

moreover, ψ_n has $n+1$ intersections with the explicit singular solution of (2.2) given by

$$\varphi_\infty(y) := c_\infty y^{-m}, \quad m := \frac{2}{p-1}, \quad c_\infty := \{m\,(N-2-m)\}^{\frac{m}{2}}; \tag{2.5}$$

see [23], [2], and [12]. For $N > 10$ and

$$p_{JL} \le p < p_L,$$

there are positive solutions of (2.2), (2.4), all of them crossing φ_∞ at least twice; see [13].

Matos showed in [14] that for every solution of (2.2), (2.4) there is a constant $c > 0$ such that

$$\psi(y) = \pm c y^{-m}\big(1 - d_1 y^{-2} - d_2 y^{-4} + o(y^{-4})\big) \quad \text{as} \quad y\to\infty,$$

where

$$d_1 := c^{p-1} - m\,(N-2-m)\,, \qquad d_2 := -p d_1\Big(m + \frac{d_1}{2}\Big).$$

Lemma 2.1. *Assume that $N > 2$ and $p_S < p < p_{JL}$.*

(i) *There are* $C_1, C_2, y_0 > 0$ *such that for all* $n \in \mathbb{N}$ *the following holds:*

$$-(\psi_n)_y(y) \le C_1 y^{-m-1}, \qquad y \ge y_0\,, \tag{2.6}$$

$$|(\psi_n)_{yy}(y)| \le C_2 y^{-m-2}, \qquad y \ge y_0\,. \tag{2.7}$$

(ii) *For every* $\delta > 0$ *there is* $C_\delta > 0$ *such that*

$$\psi_n,\ |(\psi_n)_y|,\ |(\psi_n)_{yy}| \le C_\delta \quad on \quad [\delta,\infty) \quad for\ all \quad n \in \mathbb{N}.$$

Proof. (i) It follows from Lemma 2.2 in [17] that

$$\psi_n(y) \le M y^{-m}, \quad \text{for all} \quad n \in \mathbb{N}, \quad M := \{m\,(N-1-m)\}^{\frac{m}{2}}\,. \tag{2.8}$$

Integrating the inequality

$$(\rho(\psi_n)_y)_y \le \frac{1}{p-1}\rho\psi_n$$

between y and ∞ and using (2.8) we obtain

$$-(\psi_n)_y(y) \le \frac{M}{(p-1)\rho(y)} \int_y^\infty \rho(\eta)\eta^{-m}\, d\eta, \qquad y > 0, \tag{2.9}$$

which yields (2.6). Differentiating (2.2), we have

$$(\rho(y)\psi_{yy})_y = \rho(y)\left\{\left(\frac{1}{p-1} + \frac{1}{2}\right) + \frac{N-1}{y^2} - p\psi^{p-1}\right\}\psi_y.$$

Therefore, (2.6) and (2.8) imply (2.7).

It follows easily from (2.8) and (2.9) that (ii) holds. □

Lemma 2.2. *Assume that* $N > 2$ *and* $p_S < p < p_{JL}$. *Then* $\psi_n \to \varphi_\infty$ *in* $C^2_{loc}(0,\infty)$ *as* $n \to \infty$.

Proof. Putting $w_n(\xi) = a_n^{-1}\psi_n(a_n^{-\frac{p-1}{2}}\xi)$, $\{w_n\}$ converges in $C^2_{loc}[0,\infty)$ to a solution w of

$$w_{\xi\xi} + \frac{N-1}{\xi} w_\xi + w^p = 0 \quad \text{in } (0,\infty), \tag{2.10}$$

satisfying $w_\xi(0) = 0$ and $w(0) = 1$. Let $\{\eta_k\}$ be an increasing sequence, $\eta_k \to \infty$ as $k \to \infty$ such that $v(\eta) := \xi^m w(\xi)$ with $\eta = \log\xi$ has a local maximum at η_k for each k. Such a sequence exists because w intersects φ_∞ infinitely many times; see [10] and [25]. We show that

$$v(\eta_k) \to c_\infty \text{ as } k \to \infty. \tag{2.11}$$

Indeed, v satisfies

$$v_{\eta\eta} + (N-2-2m)\,v_\eta - m\,(N-2-m)\,v + v^p = 0, \tag{2.12}$$

in $\mathbb{R}$. Define

$$E[v](\eta) = \frac{1}{2} v_\eta^2(\eta) + F(v(\eta)),$$

with

$$F(x) = -\frac{m}{2}(N-2-m)\, x^2 + \frac{1}{p+1} x^{p+1}.$$

Multiplying (2.12) by v_η yields

$$\frac{d}{d\eta} E[v](\eta) = -(N-2-2m)\, v_\eta^2(\eta) \le 0. \tag{2.13}$$

It is immediate that there is $K_1 > 0$ such that

$$0 \le v + |v_\eta| + |v_{\eta\eta}| \le K_1 \quad \text{in } \mathbb{R}. \tag{2.14}$$

Suppose that $v(\eta_k) \not\to c_\infty$ as $k \to \infty$. Then we may assume without loss of generality that $v(\eta_k) \to c$ as $k \to \infty$ for some $c \neq c_\infty$ taking a subsequence if necessary. It follows from (2.13) that

$$E[v](\eta_k) \searrow F(c) > F(c_\infty),$$

as $k \to \infty$. Taking a sequence $\{\eta_k^*\}$ such that

$$v(\eta_k^*) = c_\infty, \qquad E[v](\eta_k^*) \ge E[v](\eta_k),$$

for all k, we see

$$v_\eta^2(\eta_k^*) > 2\{F(c) - F(c_\infty)\} \quad \text{for all } k.$$

From (2.14), there is $\delta > 0$ such that

$$v_\eta^2(\eta) \ge F(c) - F(c_\infty) \quad \text{in } [\eta_k^* - \delta, \eta_k^* + \delta] \text{ for all } k.$$

Therefore we have

$$\frac{E[v](0) + K_2}{N-2-2m} \ge \int_0^\infty v_\eta^2(\eta)\, d\eta \ge \sum_{k=1}^\infty \int_{\eta_k^*-\delta}^{\eta_k^*+\delta} v_\eta^2(\eta)\, d\eta = \infty,$$

since $|E[v](\eta)| \le K_2$ for some $K_2 > 0$ and all $\eta \in \mathbb{R}$. This contradiction implies that (2.11) holds.

Then, there are sequences $\{\varepsilon_k\}$, $\{\xi_k\}$, $\{n_k\}$ with $\varepsilon_k \to 0$ and $a_{n_k}^{-\frac{1}{m}} \xi_k \to 0$ as $k \to \infty$ such that

$$w_{n_k}(\xi_k) = (c_\infty + \varepsilon_k)\, \xi_k^{-m} \quad \text{and} \quad (\xi^m w_{n_k}(\xi))_\xi (\xi_k) = 0 \quad \text{for all } k.$$

Putting $y_{n_k} = a_{n_k}^{-\frac{1}{m}} \xi_k$, we see that $y_{n_k} \to 0$ as $k \to \infty$,

$$\psi_{n_k}(y_{n_k}) = (c_\infty + \varepsilon_k)\, y_{n_k}^{-m} \quad \text{and} \quad (y^m \psi_{n_k}(y))_y (y_{n_k}) = 0 \quad \text{for all } k.$$

Setting $\phi_n(\eta) = y^m \psi_n(y)$ and $\eta = \log y$, ϕ_n satisfies

$$\phi_{\eta\eta} + (N-2-2m)\,\phi_\eta - \frac{e^{2\eta}}{2}\phi_\eta - m\,(N-2-m)\,\phi + \phi^p = 0, \tag{2.15}$$

in $\mathbb{R}$. Putting $\eta_{n_k} = \log y_{n_k}$, it holds that

$$\phi_{n_k}(\eta_{n_k}) = c_\infty + \varepsilon_k \quad \text{and} \quad (\phi_{n_k})_\eta(\eta_{n_k}) = 0 \quad \text{for all } k.$$

Multiplying (2.15) with $\phi = \phi_{n_k}$ by $(\phi_{n_k})_\eta$ yields

$$\frac{d}{d\eta} E[\phi_{n_k}](\eta) = \left\{ \frac{e^{2\eta}}{2} - (N-2-2m) \right\} (\phi_{n_k})_\eta^2(\eta). \tag{2.16}$$

Hence,

$$\frac{d}{d\eta} E[\phi_{n_k}](\eta) \le 0 \quad \text{for } \eta \le \frac{1}{2}\log\{2\,(N-2-2m)\}. \tag{2.17}$$

It follows now from Lemma 2.1 (ii) that $\psi_{n_k} \to \psi$ in $C^1_{loc}(0,\infty)$ as $k \to \infty$ (through a subsequence), where ψ is either a regular positive solution of (2.2) or $\psi = \varphi_\infty$.

Now, suppose that $\psi \ne \varphi_\infty$. Putting $\phi(\eta) = y^m \psi(y)$ and $\eta = \log y$, there is η^* with $\eta^* < \frac{1}{2}\log\{2\,(N-2-2m)\}$ such that $E[\phi](\eta^*) > F(c_\infty)$. Then, it holds that $E[\phi_{n_k}](\eta^*) > E[\phi_{n_k}](\eta_{n_k})$ for sufficiently large k. Since $\eta_{n_k} < \eta^*$ for sufficiently large k, this contradicts (2.17). In fact, this argument shows that φ_∞ is the only possible limit (in $C^1_{loc}(0,\infty)$) for any subsequence of ψ_n. Hence, the proof is complete. □

Proposition 2.3. *Assume that $N > 2$ and $p_S < p < p_{JL}$. Let $c_n = \lim_{y\to\infty} y^m \psi_n(y)$. Then $c_n \to c_\infty$ as $n \to \infty$.*

Proof. Putting $\phi_n(\eta) = y^m \psi_n(y)$ and $\eta = \log y$, ϕ_n satisfies (2.15). Then, it follows from (2.6), (2.7) and (2.8) that

$$|(\phi_n)_{\eta\eta}(\eta)| \le C_3 \quad \text{for } \eta \ge \eta_0 := \log y_0 \text{ for all } n,$$

for some $C_3 > 0$, and hence

$$|(\phi_n)_\eta(\eta)| \le C_4 e^{-2\eta} \quad \text{for } \eta \ge \eta_0 \text{ for all } n, \tag{2.18}$$

with some $C_4 > 0$. The bound (2.18) together with (2.16) with ϕ_n, instead of ϕ_{n_k}, imply the existence of $C_5 > 0$ such that

$$E[\phi_n](\eta_2) - E[\phi_n](\eta_1) \le C_5 e^{-2\eta_1}, \tag{2.19}$$

for each $\eta_2 > \eta_1 \ge \eta_0$ and n. Fix sufficiently large $\widetilde{\eta}_1 > \eta_0$. By Lemma 2.2, $\phi_n(\widetilde{\eta}_1) \to c_\infty$ as $n \to \infty$. Supposing that $c_n \not\to c_\infty$ as $n \to \infty$, there is a subsequence $\{c_{n_k}\}$ such that $c_{n_k} \to c$ as $k \to \infty$ for some non-negative constant $c \neq c_\infty$. Then, we can take $\{\eta_k\}$ with $\eta_k \to \infty$ as $k \to \infty$ such that

$$E[\phi_{n_k}](\eta_k) - E[\phi_{n_k}](\widetilde{\eta}_1) > \frac{1}{2}\{F(c) - F(c_\infty)\},$$

for all k. This contradicts (2.19) with $n = n_k$, $\eta_2 = \eta_k$, and $\eta_1 = \widetilde{\eta}_1$, which completes the proof. □

2.2. **Forward self-similar solutions.** By a forward self-similar solution of the equation (2.1) we mean a solution of the form

$$u(r,t) = (t-T)^{-\frac{1}{p-1}}\theta(y), \qquad y = \frac{r}{\sqrt{t-T}}, \quad T \in \mathbb{R}, \quad t > T,$$

where θ is a solution of the ODE

$$\theta_{yy} + \left(\frac{N-1}{y} + \frac{y}{2}\right)\theta_y + |\theta|^{p-1}\theta + \frac{1}{p-1}\theta = 0, \qquad y > 0. \tag{2.20}$$

The equation (2.20) with the initial conditions

$$\theta(0) = \beta, \qquad \theta_y(0) = 0, \tag{2.21}$$

was first studied by Haraux and Weissler in [9]. It was shown there that if θ_β is the solution of (2.20), (2.21) then

$$L(\beta) := \lim_{y\to\infty} y^{\frac{2}{p-1}}\theta_\beta(y), \tag{2.22}$$

exists and is a locally Lipschitz-continuous function of $\beta \in \mathbb{R}$, if $p \leq 1 + \frac{2}{N}$ then there is no positive solution, and if $p \geq p_S$ and $\beta > 0$ then $\theta_\beta(y) > 0$ for $y > 0$ and $L(\beta) > 0$.

Souplet and Weissler showed in [22] that $L(\beta)$ oscillates around the constant c_∞ infinitely many times as $\beta \to \infty$ if $p_S < p < p_{JL}$.

3. MULTIPLE CONTINUATION

We begin this section with the definition of a global L^1-solution of problem (1.6).

Definition 3.1. *By a global L^1-solution of* (1.6) *we mean a function $u \in C([0,\infty); L^1_{loc}(\mathbb{R}^N))$ such that $f(u) \in L^1_{loc}(\mathbb{R}^N \times (0,\infty))$, and the equality*

$$\int_{\mathbb{R}^N} [u\Psi]_\tau^t \, dx - \int_\tau^t \int_{\mathbb{R}^N} u\Psi_t \, dx\, ds = \int_\tau^t \int_{\mathbb{R}^N} (u\Delta\Psi + f(u)\Psi)\, dx\, ds$$

holds for any $0 \leq \tau < t$ and $\Psi \in C^2(\mathbb{R}^N \times [0,\infty))$ with compact support.

We shall prove the following:

Theorem 3.2. *Assume that $f(u) = u^p$, $N > 2$ and $p_S < p < p_{JL}$. Then for every positive integer k there is a bounded positive function $u_0 \in C^\infty(\mathbb{R}^N)$ such that problem (1.6) has at least k different global L^1-solutions which are classical and coincide for $t \in [0, T)$, $T > 0$.*

Proof. Since the function L defined in (2.22) is continuous and oscillates around the constant c_∞ infinitely many times, Proposition 2.3 yields that for any $k \in \mathbb{N}$ the equation

$$L(\beta) = c_n,$$

has at least k roots $\beta_1, \ldots, \beta_k$ if n is large enough. Let ψ_n be the solution of (2.2) with

$$\lim_{y \to \infty} y^m \psi_n(y) = c_n,$$

and let θ_i be the solution of (2.20), (2.21) with $\beta = \beta_i$, $i \in \{1, \ldots, k\}$. Set for $T > 0$

$$u_b(x,t) = (T-t)^{-\frac{1}{p-1}} \psi_n\left(\frac{|x|}{\sqrt{T-t}}\right), \qquad x \in \mathbb{R}^N, \quad t < T,$$

$$u_f^i(x,t) = (t-T)^{-\frac{1}{p-1}} \theta_i\left(\frac{|x|}{\sqrt{t-T}}\right), \qquad x \in \mathbb{R}^N, \quad t > T,$$

and

$$u_i(x,t) = \begin{cases} u_b(x,t), & x \in \mathbb{R}^N, \quad t < T, \\ c_n|x|^{-m} = \lim_{t \nearrow T} u_b(x,t), & x \in \mathbb{R}^N \setminus \{0\}, \quad t = T, \\ u_f^i(x,t), & x \in \mathbb{R}^N, \quad t > T. \end{cases}$$

We shall verify that u_i is a global L^1-solution of problem (1.6) with $u_0(x) = u_b(x, 0)$.

It is easy to see that $u_i \in C([0,\infty); L^1_{loc}(\mathbb{R}^N))$ and $u_i^p \in L^1_{loc}(\mathbb{R}^N \times (0,\infty))$ because there is a constant $C > 0$ such that

$$u_i(x,t) \le C|x|^{-m}, \qquad x \in \mathbb{R}^N, \quad t \ge 0.$$

Now, we choose a function $\Psi \in C^2(\mathbb{R}^N \times [0,\infty))$ such that supp $\Psi = \overline{\Omega} \times [T_1, T_2]$ where $\Omega \subset \mathbb{R}^N$ is bounded, $0 \in \Omega$ and $0 \le T_1 < T < T_2$. We show that

$$\int_\Omega u_f^i(x,t)\Psi(x,t)\,dx - \int_\Omega u_b(x,\tau)\Psi(x,\tau)\,dx$$
$$= \int_\tau^T \int_\Omega \Big(u_b(\Psi_t + \Delta\Psi) + u_b^p \Psi\Big)\,dx\,ds$$

$$+\int_T^t \int_\Omega \Big(u_f^i(\Psi_t + \Delta\Psi) + (u_f^i)^p \Psi\Big)\, dx\, ds, \quad (3.1)$$

for any $\tau \in [T_1, T)$ and $t \in (T, T_2]$. To do this, we set $B_\varepsilon = \{x \in \mathbb{R}^N : |x| < \varepsilon\}$ where $\varepsilon > 0$ is such that $B_\varepsilon \subset \Omega$ and $\Omega_\varepsilon := \Omega \setminus B_\varepsilon$. Then we test the equations

$$(u_b)_t = \Delta u_b + u_b^p, \qquad x \in \mathbb{R}^N, \quad 0 \le t < T,$$
$$(u_f^i)_t = \Delta u_f^i + (u_f^i)^p, \qquad x \in \mathbb{R}^N, \quad t > T,$$

with $\Psi\chi_{\Omega_\varepsilon}$. This yields

$$\int_{\Omega_\varepsilon} u_b(x,T)\Psi(x,T)\, dx - \int_{\Omega_\varepsilon} u_b(x,\tau)\Psi(x,\tau)\, dx$$
$$= \int_\tau^T \int_{\Omega_\varepsilon} \Big(u_b(\Psi_t + \Delta\Psi) + u_b^p\Psi\Big)\, dx\, ds$$
$$+ \int_T^t \int_{\partial B_\varepsilon} \left(\frac{\partial u_b}{\partial\nu}\Psi - u_b\frac{\partial\Psi}{\partial\nu}\right) d\sigma\, ds,$$

and we have a similar identity for u_f^i. Since

$$-(\psi_n)_y(y) \le \tilde{C} y^{-m-1}, \qquad y > 0,$$

we can pass to the limit as $\varepsilon \to 0$, and the integral over ∂B_ε will tend to zero. We proceed analogously for u_f^i, and (3.1) then follows if we take into account that $u_b(x,T) = u_f^i(x,T)$. □

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