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HÖLDER CONTINUITY IN A SHAPE-OPTIMIZATION PROBLEM WITH PERIMETER

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Abstract. Here, we prove Hölder continuity of the state function in a shape-optimization problem arising, for instance, in the electromagnetic shaping of a liquid metal. The equilibrium state is obtained as minimizing the total energy, which is given in the form

$$\mathcal{E}_\lambda(\Omega) = J(\Omega) + P(\Omega) + \lambda||\Omega| - m|,$$

where Ω is the domain occupied by the liquid, $P(\Omega)$ and $|\Omega|$ being respectively the perimeter and the Lebesgue measure of Ω. We prove that the state function associated with the optimal shape is $\frac{1}{2}$-Hölder continuous, the main difficulty coming from the fact that the state function is not assumed to be non-negative.

Introduction

The goal of this note is to give a proof of the Hölder continuity of the state function in a shape-optimization problem. This problem arises, for instance, when modelling the shape of a liquid metal confined by an electromagnetic field (see [6], [9] and the references in [7]). The approach is variational: the equilibrium shape is defined as minimizing the total energy of the considered system. This naturally leads to a shape-optimization problem with a volume constraint that takes into account the fact that the quantity of liquid metal is given. The total energy is composed of two terms: One is the surface tension energy, which, as usual, is proportional to the perimeter of the domain occupied by the liquid; the second term is the electromagnetic energy, which is described via the electromagnetic potential, solution of a Dirichlet problem. Actually, this kind of energy functional appears generically in many applications (see [7]). Here, we concentrate on a penalized version of the problem where the volume constraint is relaxed. As explained below, this is essentially equivalent to the constrained situation.

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The difficulty here comes from the fact that the state function may change sign. A proof of its regularity has been given in [8] for the non-negative case. The situation is quite a bit more complicated in general, and its analysis requires, in particular, the use of the famous monotonicity lemma of [1] and [4]. The proof of the Hölder continuity is the main contribution of the paper. Here, we adapt the method developed in [3], in which the authors study the regularity of the state function for the same kind of functional but without the perimeter.

To be more precise, let us recall the shape-optimization problem. Let D be an open subset of $\mathbf{R}^d$ and $f \in L^\infty(D)$. For any measurable subset Ω of D with finite Lebesgue measure $|\Omega|$, we denote by $P(\Omega)$ its perimeter and by $J(\Omega)$ its "Dirichlet energy," defined as follows:

$$J(\Omega) = \inf\{G(v) : v \in H_0^1(\Omega)\}, \tag{0.1}$$

where $G(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - f \cdot v$. In the case where Ω is open, it is well-known that $J(\Omega) = G(u_\Omega)$, where u_Ω is the solution to the Dirichlet problem

$$u_\Omega \in H_0^1(\Omega),\ -\Delta u_\Omega = f \text{ in } \Omega. \tag{0.2}$$

The definition of $H_0^1(\Omega)$ may be extended to any measurable set Ω (see e.g. [7]) and u_Ω may then be defined as the unique function $u_\Omega \in H_0^1(\Omega)$ such that $J(\Omega) = G(u_\Omega)$.

Now we fix $m \in (0, |D|)$, and for $\lambda > 0$, we consider the penalized energy

$$\mathcal{E}_\lambda(\Omega) = J(\Omega) + P(\Omega) + \lambda||\Omega| - m|. \tag{0.3}$$

Our goal is to study the regularity of optimal shapes Ω_λ^* solutions of

$$\mathcal{E}_\lambda(\Omega_\lambda^*) = \min\{\mathcal{E}_\lambda(\Omega) : \Omega \text{ measurable},\ \Omega \subset D,\ |\Omega| \text{ finite}\}. \tag{0.4}$$

In this article, we prove the following result:

Theorem 1. *The state function $u_{\Omega_\lambda^*}$ associated to an optimal shape Ω_λ^* of the problem* (0.4) *is locally $\frac{1}{2}$-Hölder continuous.*

Remark 1. When the function f is non-negative, the same conclusion may be reached in a much easier way, and also with the weaker assumption $f \in L^q$, for $q > N$ (see [8]). One can also prove higher regularity for the state function (Lipschitz continuity) and even conclude a rather precise regularity result for the boundary of the optimal shape (see [8]). Here, where the state function may change sign, we can only reach $\frac{1}{2}$-Hölder continuity. If we could prove Lipschitz continuity, or even $(\frac{1}{2} + \varepsilon)$-Hölder continuity with $\varepsilon > 0$, then we would reach the same regularity result for the boundary as in [8], using the regularity theory for quasi-minimizers.

Remark 2. As proved in [8], in many situations the constrained problem $|\Omega^*| = m$, $J(\Omega^*)+P(\Omega^*) = \min\{J(\Omega)+P(\Omega) : |\Omega| = m,\ \Omega$ measurable, $\Omega \subset D\}$ is equivalent to (0.4). Therefore, the Hölder continuity proved here carries over to this problem.

The note is organized as follows:

- In the first section, we prove some elementary results concerning this problem.
- In the second section, we prove the continuity of the state function, this property being needed as a preliminary step for the final proof.
- Finally, in the third section, we give the *key estimate* in Lemma 3.1 whence we can prove the $\frac{1}{2}$-Hölder continuity.
- In the appendix, we recall some results and formulas related to the Laplacian.

1. First properties of the optimal shape

First of all, let us mention that optimal shapes of the problem (0.4) exist ([8]). Here, and until the end of this paper, we will consider an optimal shape called Ω. We will call u the state function associated to Ω. As a first property of Ω, obtained using the well-known inequality $P(\Omega \cup B) + P(\Omega \cap B) \leq P(\Omega) + P(B)$, we have

Lemma 1.1. *Let B be an open ball included in D and let v belong to $H_0^1(B \cup \Omega)$. Then*

$$J(\Omega) \leq G(v) + P(B) - P(\Omega \cap B) + \lambda|\Omega^c \cap B|.$$

As a first corollary, we have

Lemma 1.2. *There exists a constant $C(\lambda) > 0$ such that, for any ball $B \subset D$ and any v a solution of*

$$-\Delta v = f \ \text{in } B,\ v - u \in H_0^1(B),$$

one has

$$\int_B |\nabla(u - v)|^2 \leq C(\lambda)(P(B) + |B|).$$

Proof. We only have to apply Lemma 1.1 with v equal to u outside B and defined as in the lemma on B. □

As a second corollary to Lemma 1.1, we have

Lemma 1.3. *Let B be an open ball included in D and $\varphi \in \mathcal{C}_0^\infty(B)$. Then*

$$|\langle \Delta u + f, \varphi \rangle| \leq 2\Big(\int |\nabla \varphi|^2\Big)^{\frac{1}{2}} (P(B) - P(B \cap \Omega) + \lambda |B \cap \Omega^c|)^{\frac{1}{2}}.$$

Proof. Let $t \in (0, +\infty)$. From Lemma 1.1 applied with $v = u + t\varphi$, we have

$$G(u) \leq G(u + t\varphi) + P(B) - P(B \cap \Omega) + \lambda |B \cap \Omega^c|.$$

Expanding and dividing by t leads to

$$\langle \Delta u + f, \varphi \rangle \leq t \int |\nabla \varphi|^2 + \frac{1}{t}(P(B) - P(B \cap \Omega) + \lambda |B \cap \Omega^c|),$$

and finally, minimizing in t, we obtain the result. □

Now we can prove the regularity of u in the "interior" of Ω in the following sense:

Lemma 1.4. *Let B be a ball included in D such that $|B \cap \Omega^c| = 0$. Then u is a solution of*

$$-\Delta u = f \text{ in } B \text{ in the sense of distributions.}$$

Proof. Since $|B \cap \Omega^c| = 0$, we have $P(\Omega \cap B) = P(B)$. We then apply Lemma 1.3. □

Whereas the preceding lemma gives Δu in the interior of Ω, the following lemma gives some *key* information about the nature of Δu across the boundary of Ω.

Lemma 1.5. *There exist two positive measures μ^+ and μ^- such that*

$$\Delta u^+ + f \mathbf{1}_{[u>0]} = \mu^+, \tag{1.1}$$

$$\Delta u^- - f \mathbf{1}_{[u<0]} = \mu^-. \tag{1.2}$$

Moreover, when u is continuous, we have

$$\mu^+([u \neq 0]) = \mu^-([u \neq 0]) = 0. \tag{1.3}$$

Proof. Let us take some $\varphi \in \mathcal{C}_c^\infty(D, \mathbf{R}^+)$ and some $t > 0$. Since the function $v_t = (u - t\varphi)^+ - u^-$ belongs to $H_0^1(\Omega)$, we have $J(u) \leq J(v_t)$; that is to say,

$$\frac{1}{2}\int |\nabla u^+|^2 - \int f \cdot u^+ \leq \frac{1}{2}\int |\nabla (u - t\varphi)^+|^2 - \int f \cdot (u - t\varphi)^+.$$

Since $(u^+ - t\varphi) = (u - t\varphi)^+ - (u^+ - t\varphi)^-$, we have

$$\int |\nabla (u - t\varphi)^+|^2 \leq \int |\nabla (u^+ - t\varphi)|^2,$$

and so

$$-\int \nabla u^+ \cdot \nabla\varphi + O(t) \geq \frac{1}{t}\Big(\int f\cdot(u-t\varphi)^+ - \int f\cdot u^+\Big).$$

But now, this last term has a limit, which is $-\int_{[u^+>0]} f\cdot\varphi$, when t goes to 0. So one finds that

$$\big\langle \Delta u^+ + f\mathbf{1}_{[u>0]}, \varphi\big\rangle \geq 0;$$

that is to say, $\Delta u^+ + f\mathbf{1}_{[u>0]}$ is a positive distribution, and so a positive Radon measure, which we denote μ^+.

If we take now test functions of the form $w_t = u^+ - (u+t\varphi)^-$, we get the equality (1.2).

Now suppose u is continuous. Since the set $[u>0]$ is open, we can apply Lemma 1.4, and we get that $\Delta u + f = 0$ upon the set $[u>0]$. But since we also have that $\Delta u^+ + f\mathbf{1}_{[u>0]} = 0$ upon the open set $[u<0]$, we get that $\Delta u^+ + f\mathbf{1}_{[u>0]} = 0$ upon the set $[u\neq 0]$. Also, $\mu^+ = \Delta u^+ + f\mathbf{1}_{[u>0]}$ being a positive Radon measure, this last equality implies that $\mu^+([u\neq 0]) = 0$. We prove in the same way that $\mu^-([u\neq 0]) = 0$. □

Now, from the preceding lemma, we have $-\Delta|u| \leq |f|$, whence

Lemma 1.6. *The function u belongs to $L^\infty(D)$, and*

$$\|u\|_{L^\infty(D)} \leq C(d, |D|, \|f\|_{L^\infty(D)}).$$

Proof. It is an application of Theorem 8.16 from [5]. □

Remark 3. Notice that we can apply the last part of Lemma A.1 to $|u|$; i.e., since we have just proved that u is bounded, we have that Δu is a locally finite measure, and that $\Delta u^+ \geq -\|f\|_\infty$, $\Delta u^- \geq -||f||_\infty$, Lemma A.1 tells us that the following representation of u is well-defined everywhere ($\fint_E u$ denotes the average of u over E):

$$x \mapsto \lim_{r\to 0} \fint_{\partial B(x,r)} u.$$

In the following, we will suppose that this representation is chosen for u.

2. Continuity of the state function

In the general case that we consider here, we want to apply the monotonicity lemma to the function u in order to prove Hölder continuity. But to do so, one must have proven that the function u is continuous. This is the topic of this section.

Theorem 2. *The state function u is continuous in D.*

Proof. Let $(x_n)_n$ be a sequence in D converging to some x_∞ in D. Let $\delta_n = |x_n - x_\infty|$. We can suppose that the δ_n are non-increasing. If, for some n, $|B(x_\infty, \delta_n) \cap \Omega^c| = 0$, then we saw (see Lemma 1.4) that $-\Delta u = f$ in $B(x_\infty, \delta_n)$, and so u is continuous at the point x_∞.

Thus, we can suppose that for every n, $|B(x_\infty, \delta_n) \cap \Omega^c| \neq 0$. Let us take $u_n(\xi) = u(x_\infty + \delta_n \xi)$. Since these functions are uniformly bounded, one may suppose (up to extracting a subsequence) that they converge, at least in $L^\infty(\mathbf{R}^d)$ with its weak-$*$ topology, toward some function $u_\infty \in L^\infty(\mathbf{R}^d)$.

We want to prove that $u_\infty = 0$ and that the convergence is uniform in $B_1 = \{\xi \in \mathbf{R}^d : |\xi| \leq 1\}$.

For any $R \geq 1$, we introduce the function v_R a solution of

$$\begin{cases} -\Delta v_R = f \text{ in } B(x_\infty, \delta_n R), \\ v_R \in u + H_0^1(B(x_\infty, \delta_n R)). \end{cases}$$

Let us call $v_n(\xi) = v_R(x_\infty + \delta_n \xi)$. Then, thanks to Lemma 1.2,

$$\int_{B_R} |\nabla(u_n - v_n)|^2 \leq C(\lambda, R)\delta_n, \text{ and } -\Delta v_n = \delta_n^2 f(x_\infty + \delta_n \xi).$$

In particular, $v_n - u_n$ tends to 0 in $H_0^1(B_R)$. But since the functions v_n are bounded and Δv_n converges to 0 in L^∞, the convergence of the v_n is uniform upon each compact subset of B_R. The limit, which must be u_∞, is harmonic on B_R. But since R can be any number, we have that u_∞ is harmonic on $\mathbf{R}^d$. As u_∞ is in L^∞, it must be a constant. We also can suppose that the convergence of the u_n holds in $H^1_{loc}(\mathbf{R}^d)$.

Now, one wants to prove that $u_\infty = 0$. To do so, let us suppose that $u_\infty > 0$. Then, u_n^- tends to 0 in $H^1_{loc}(\mathbf{R}^d)$. But we have the inequality $-\Delta u_n^- \leq \delta_n^2 f(x_\infty + \delta_n \xi)$, and so the convergence is uniform on compact sets. For each n, we have $|B(x_\infty, \delta_n) \cap \Omega^c| \neq 0$, and so we can choose $y_n = x_\infty + \delta_n \xi_n$ for some ξ_n in B_1 with $u(y_n) = 0$. Let us call $B_s = B(y_n, s)$. We then apply Lemma 1.3 with a function

$$\varphi \in \mathcal{D}(B_{2s}),\ 0 \leq \varphi \leq 1,\ \varphi|_{B_s} = 1,\ \|\nabla \varphi\|_{L^\infty(B_{2s})} \leq C/s.$$

We find

$$\left| \langle \mu^+ - \mu^-, \varphi \rangle \right| \leq C s^{d-1-1/2}. \tag{2.1}$$

Then, μ^- and μ^+ being two nonnegative Radon measures,

$$\mu^+(B_s) \leq C s^{d-1-1/2} + \mu^-(B_{2s}),$$

and so

$$\Delta u^+(B_s) \le \Delta u^-(B_{2s}) + Cs^{d-1-1/2}.$$

Multiplying this last inequality by s^{1-d} and integrating between 0 and δ_n, we obtain (using the formulas of Lemma A.1)

$$⨍_{\partial B_{\delta_n}} u^+ \le ⨍_{\partial B_{2\delta_n}} u^- + C\int_0^{\delta_n} s^{-1/2}ds \le ⨍_{\partial B_{2\delta_n}} u^- + C\sqrt{\delta_n};$$

$$\text{i.e., } ⨍_{\partial B_1} u_n^+(\xi_n+\xi)d\xi \le ⨍_{\partial B_2} u_n^-(\xi_n+\xi)d\xi + C\sqrt{\delta_n}.$$

But the right part of this inequality goes to 0 as n goes to infinity, and the left part is non-negative. Then the left part also goes to 0. Furthermore, the sequence $(\xi_n)_n$ stays in B_1. We may thus suppose, up to extracting a subsequence, that this sequence converges to some $\xi_\infty \in \overline{B_1}$. And so, on one hand, the sequence $(u_n(\xi_n+\cdot))_n$ converges in $H^1_{loc}(\mathbf{R}^d)$ to $u_\infty(\xi_\infty+\cdot) = u_\infty > 0$, and on the other hand, $u_n(\xi_n+\cdot)|_{\partial B_1}$ converges in $L^2(\partial B_1)$ to 0. This is a contradiction. We can prove in a similar way that, supposing $u_\infty < 0$, we have a similar contradiction. And so we must have $u_\infty = 0$.

To conclude, let us notice that (from Lemma 1.5), $-\Delta|u| \le |f|$, and so, with a change of variables, $-\Delta|u_n| \le \delta_n^2|f(x_\infty+\delta_n\cdot)|$. Since f is in L^∞, $\delta_n^2|f(x_\infty+\delta_n\cdot)|$ goes to 0 in L^∞ and so *the u_n converge to 0 uniformly on the compact sets*, and this achieves the proof. □

3. Hölder continuity of the state function

3.1. **Hölder continuity.**

Theorem 3. *The state function u is locally $\frac{1}{2}$-Hölder continuous.*

This regularity result comes from the study of the measure $|\Delta|u||$ on the boundary of Ω. The following estimation is the key to the Hölder continuity:

Lemma 3.1 (Key Estimate)**.** *For any $\delta \in (0, 1/2)$, there exists a constant $C = C(d, f, \lambda, \delta)$ such that for any ball $B(x_0, \delta) \subset D$ with $u(x_0) = 0$, and for any $r \le \delta/4$, we have*

$$|\Delta|u||(\overline{B(x_0,r)}) \le Cr^{d-1-\frac{1}{2}}. \tag{3.1}$$

In what follows, $\partial\Omega$ will always denote the measure-theoretic boundary of Ω; i.e.,

$$\partial\Omega = \{x \in D : \forall r > 0,\ 0 < |B(x,r)\cap\Omega| < |B(x,r)|\}.$$

Moreover, let us define $d(x) = d(x, \partial\Omega)$.

First of all, let us recall that u is almost-everywhere zero outside Ω, by definition of $H_0^1(\Omega)$. Then, u being continuous, we have the following result:

Lemma 3.2. *Let us take x_0 in D such that $|B(x_0,r)\cap\Omega^c|>0$ for all $r>0$. Then $u(x_0)=0$.*

Assuming, for now, that Lemma 3.1 is valid, we can prove the main theorem:

Proof of the Hölder continuity. Let $\delta\in(0,\frac{1}{3})$. Let us call $D_\delta=\{x\in D : d(x,\partial D)\geq 6\delta\}$.

Lemma 3.3. *There exists $C_\delta>0$ such that for any $x_0\in D_\delta$, $|u(x_0)|\leq C_\delta d(x_0)^{\frac{1}{2}}$.*

Proof. Take x_0 in D_δ. First suppose $d(x_0)\geq\delta$. Then, since u is bounded,

$$|u(x_0)|\leq\frac{\|u\|_\infty}{\sqrt{\delta}}d(x_0)^{\frac{1}{2}}\leq C_\delta d(x_0)^{\frac{1}{2}}.$$

Now, suppose $r_0=d(x_0)<\delta$ and take $y_0\in\partial\Omega$ such that $r_0=d(x_0,y_0)$. First of all, we integrate the equation (3.1) and obtain that

$$\int_0^{2r_0}s^{1-d}\int_{B(z_0,s)}d(|\Delta|u||)ds\leq C\sqrt{r_0}.$$

Furthermore, we saw in Lemma 3.2 that $u(y_0)=0$, so that, applying Lemma A.1, we get

$$\begin{aligned}\fint_{\partial B(y_0,2r_0)}|u| &= \fint_{\partial B(y_0,2r_0)}|u|-|u(y_0)|\\ &= (d\omega_d)^{-1}\int_0^{2r_0}s^{1-d}\int_{B(z_0,s)}d(\Delta|u|)ds\leq C\sqrt{r_0}.\end{aligned}$$

Next, applying point (ii) of Lemma A.2, we get

$$\|u\|_{L^\infty(B(x_0,r_0))}\leq\|u\|_{L^\infty(B(y_0,2r_0))}\leq C(3r_0+\fint_{\partial B(y_0,2r_0)}|u|),$$

so that $|u(x_0)|\leq Cd(x_0)^{\frac{1}{2}}$. □

Lemma 3.4. *There exists C'_δ such that for any $x_0\in D_\delta$ with $d(x_0)>0$, we have*

$$\|\nabla u\|_{L^\infty(B(x_0,\frac{d(x_0)}{4}))}\leq C'_\delta\max\Big\{1,\frac{1}{d(x_0)^{\frac{1}{2}}}\Big\}.$$

Proof. First, suppose that $|B(x_0, d(x_0)) \cap \Omega| = 0$. Thanks to Lemma 3.2, we know that $u \equiv 0$ in $B(x_0, d(x_0))$. In particular, $|\nabla u(x_0)| = 0$. Now suppose that $|B(x_0, d(x_0)) \cap \Omega^c| = 0$. We have $-\Delta u = f$ in $B(x_0, d(x_0))$ (by Lemma 1.4) so that, applying point (i) of Lemma A.2,

$$\|\nabla u\|_{L^\infty(B(x_0, \frac{d(x_0)}{4}))} \leq C\Big[1 + \frac{1}{d(x_0)}\|u\|_{L^\infty\left(B\left(x_0, \frac{d(x_0)}{2}\right)\right)}\Big] \leq C\Big[1 + \frac{1}{d(x_0)^{\frac{1}{2}}}\Big].$$

□

To conclude the proof of the Hölder continuity, take x and y in D_δ. Denote $\Omega_{\text{int}} = \{x \in D : \exists r > 0,\ B(x,r) \subset D,\ |B(x,r) \cap \Omega^c| = 0\}$. Note that Ω_{int} is open. Suppose x or y belong to Ω_{int}^c. Then Lemmas 3.2 and 3.3 show that there exists a constant C such that

$$|u(x) - u(y)| \leq C d(x,y)^{\frac{1}{2}}.$$

Now suppose both x and y belong to Ω_{int}. First suppose $d(x,y) \leq d(x)/4$. Using that u is regular in $B(x, d(x))$, and the estimate on ∇u given by Lemma 3.4, we find that

$$\begin{aligned} |u(x) - u(y)| &\leq C'_\delta \max\Big\{1, \frac{1}{d(x)^{\frac{1}{2}}}\Big\} d(x,y) \\ &\leq C \max\{d(x,y), d(x,y)^{\frac{1}{2}}\} \leq C d(x,y)^{\frac{1}{2}}. \end{aligned}$$

If $d(x,y) \leq d(y)/4$, the result is the same by symmetry.

Now if $d(x,y) \geq \max\{d(y), d(x)\}/4$,

$$|u(x) - u(y)| \leq 2\max\{|u(x)|, |u(y)|\} \leq 2C \max\{d(x)^{\frac{1}{2}}, d(y)^{\frac{1}{2}}\} \leq C d(x,y)^{\frac{1}{2}},$$

and so there exists $C > 0$ such that for any x, y in D_δ, $|u(x) - u(y)| \leq C d(x,y)^{\frac{1}{2}}$. □

3.2. **Proof of the key lemma.** This proof will follow from the next result:

Lemma 3.5 (monotonicity lemma, [1], [4])**.** *Let $U \in H^1(B_{r_0})$, U continuous in $\overline{B_{r_0}}$, with $U(0) = 0$ and such that, for some $a \geq 0$, we have*

$$\Delta U^+ \geq -a,\ \Delta U^- \geq -a \ \ \text{in } B_{r_0}.$$

We then define

$$\Phi(r) = \Big(\frac{1}{r^2}\int_{B_r} \frac{|\nabla U^+|^2}{|x|^{d-2}}\Big)\Big(\frac{1}{r^2}\int_{B_r} \frac{|\nabla U^-|^2}{|x|^{d-2}}\Big).$$

Then, if $a = 0$, the function Φ is non-increasing on $(0, r_0)$. In general, there exists some constant C such that

$$\forall r \in (0, r_0/2),\ \Phi(r) \leq C\Big[1 + \int_{B_{r_0}} U^2\Big].$$

We may apply this lemma to u because we have already proved that it is continuous.

Proof of Lemma 3.1. Note first that the proof is easy in the case $u \geq 0$. Indeed, we then have $\mu^- = 0$ and the estimate is directly given by (2.1). In the general case, we must exploit the connection between μ^+ and μ^- to derive (3.1) from (2.1). This is done through the monotonicity lemma. We will use the same approach as in [3]. Let us consider some open ball $B(x_0, \delta) \subset D$ with $u(x_0) = 0$. We can apply the monotonicity lemma to the function $U(\cdot) = u(x_0 + \cdot)$ in B_δ (because f is bounded). The function u being bounded in D, we have that $\Phi(r)$ is bounded for $r \in (0, \delta/2)$ by some constant $C(\delta)$. Translating, we have that, for $r \in (0, \delta/2)$,

$$r^{-2d}\Big(\int_{B(x_0,r)} |\nabla u^+|^2\Big)\Big(\int_{B(x_0,r)} |\nabla u^-|^2\Big) \leq \Phi(r) \leq C(\delta). \tag{3.2}$$

For any r, we introduce the functions $v^r = v^r_+ - v^r_-$ and $w^r = w^r_+ - w^r_-$ where v^r_+, v^r_-, w^r_+ and w^r_- are the solutions of

$$-\Delta v^r_+ = f^+,\ -\Delta v^r_- = f^- \text{ in } B(x_0, r),\ \ v^r_+ - u^+, v^r_- - u_- \in H^1_0(B(x_0, r)),$$

$$-\Delta w^r_+ = f^+,\ -\Delta w^r_- = f^- \text{ in } B(x_0, r),\ w^r_+, w^r_- \in H^1_0(B(x_0, r)).$$

Since, for $i \in \{+, -\}$, $v^r_i - w^r_i$ is harmonic in $B(x_0, r)$ and is equal to u^i on $\partial B(x_0, r)$, we have

$$\int_{B(x_0,r)} |\nabla(v^r_i - w^r_i)|^2 \leq \int_{B(x_0,r)} |\nabla u^i|^2,$$

and also (since $v^r_i - w^r_i$ is harmonic and $u_i - v^r_i + w^r_i = 0$ on $\partial B(x_0, r)$),

$$\int_{B(x_0,r)} |\nabla(u^i - v^r_i + w^r_i)|^2 = \int_{B(x_0,r)} \nabla(u^i - v^r_i + w^r_i)\nabla u^i \leq 2\int_{B(x_0,r)} |\nabla u^i|^2.$$

Then, using the inequality (3.2), we get

$$\Big(r^{-d}\int_{B(x_0,r)} |\nabla(u^+ - v^r_+ - w^r_+)|^2\Big)\Big(r^{-d}\int_{B(x_0,r)} |\nabla(u^- - v^r_- - w^r_-)|^2\Big) \leq C(\delta). \tag{3.3}$$

We also have

$$\int_{B(x_0,r)} |\nabla(u - v^r + w^r)|^2 \leq C \int_{B(x_0,r)} |\nabla(v^r - u)|^2 + |\nabla w^r|^2.$$

We now try to obtain upper bounds on the two terms of the right part of this inequality.

(i) As $v^r \in u + H_0^1(B(x_0,r))$, and using Lemma 1.1, we have

$$G(u) \leq G(v^r) + Cr^{d-1};$$

i.e., noticing that $G(u) - G(v^r) = \frac{1}{2}\int |\nabla(u - v^r)|^2$ $(-\Delta v^r = f$ in $B(x_0,r))$, we have

$$\int |\nabla(u - v^r)|^2 \leq Cr^{d-1}.$$

(ii) Now we bound the second term. Now consider, for $\xi \in B(0,1)$, the function defined by $W^r(\xi) = w^r(x_0 + r\xi)$. Then W^r is the solution of the following problem:

$$W^r \in H_0^1(B(0,1)), \quad -\Delta W^r(\xi) = r^2 f(x_0 + r\xi),$$

so that $\|W^r\|_{L^\infty} \leq Cr^2\|f\|_{L^\infty}$, and one computes

$$\begin{aligned} r^{-d} \int_{B(x_0,r)} |\nabla w^r(x)|^2 dx &= r^{-2} \int_{B(0,1)} |\nabla W^r(\xi)|^2 d\xi \\ = r^{-2} \int_{B(0,1)} r^2 f(x_0 + r\xi) W^r(x_0 + r\xi) d\xi &\leq Cr^2 \leq C. \end{aligned}$$

Then, from (i) and (ii), one has

$$\int_{B(x_0,r)} |\nabla(u - v^r + w^r)|^2 \leq Cr^{d-1}.$$

However, as

$$\begin{aligned} &\int_{B(x_0,r)} |\nabla(u - v^r + w^r)|^2 \\ &= \int_{B(x_0,r)} |\nabla(u^+ - v_+^r + w_+^r)|^2 + \int_{B(x_0,r)} |\nabla(u^- - v_-^r + w_-^r)|^2 \\ &+ 2\int_{B(x_0,r)} \nabla(u^+ - v_+^r + w_+^r)\nabla(u^- - v_-^r + w_-^r), \end{aligned}$$

we have (thanks to the Cauchy-Schwarz inequality)

$$\int_{B(x_0,r)} |\nabla(u^+ - v_+^r + w_+^r)|^2 + \int_{B(x_0,r)} |\nabla(u^- - v_-^r + w_-^r)|^2$$

$$\leq Cr^{d-1} + 2\|\nabla(u^+ - v_+^r + w_+^r)\|_{L^2(B(x_0,r))}\|\nabla(u^- - v_-^r + w_-^r)\|_{L^2(B(x_0,r))}.$$

In addition, using the estimation given by the monotonicity lemma, i.e., the inequality (3.3), we deduce

$$\int_{B(x_0,r)} |\nabla(u^+ - v_+^r + w_+^r)|^2 + \int_{B(x_0,r)} |\nabla(u^- - v_-^r + w_-^r)|^2 \leq Cr^{d-1}.$$

In the end, the expression $r^{-d}\int_{B(x_0,r)} |\nabla w_i^r|^2$ being bounded, we have

$$\int_{B(x_0,r)} |\nabla(u^+ - v_+^r)|^2 + \int_{B(x_0,r)} |\nabla(u^- - v_-^r)|^2 \leq Cr^{d-1}. \tag{3.4}$$

We now come back to the definitions of v_+^r and v_-^r. We have

$$\Delta(u^+ - v_+^r) = \mu^+ + f^+ - f\mathbf{1}_{[u>0]} \geq \mu^+,$$
$$\Delta(u^- - v_-^r) = \mu^- - f^- + f\mathbf{1}_{[u<0]} \geq \mu^-.$$

Using the Stokes formula and having noticed that $u^+ - v_+^r \in H_0^1(B(x_0,r))$, we have

$$\begin{aligned} \int_{B(x_0,r)} |\nabla(u^+ - v_+^r)|^2 &= \int_{B(x_0,r)} (v_+^r - u^+) d(\Delta(u^+ - v_+^r) \\ &\geq \int_{B(x_0,r)} (v_+^r - u^+) d(\mu^+). \end{aligned}$$

Moreover, $\mu^+([u \neq 0]) = 0$, and so we have

$$\int_{B(x_0,r)} |\nabla(u^+ - v_+^r)|^2 \geq \int_{B(x_0,r)} v_+^r d(\mu^+).$$

We have a similar result for the functions u^-, v_-^r and μ^-:

$$\int_{B(x_0,r)} |\nabla(u^- - v_-^r)|^2 \geq \int_{B(x_0,r)} v_-^r d(\mu^-).$$

Finally, equation (3.4) becomes

$$\int_{B(x_0,r)} v_+^r d\mu^+ + \int_{B(x_0,r)} v_-^r d\mu^- \leq Cr^{d-1}. \tag{3.5}$$

We now apply the formula of Lemma A.1 to the function $U = u^+ - v_+^r$ (so that $\Delta U \geq \mu^+$ and $U \leq 0$). One may then write that for any $z \in B(x_0, r/4)$ (so that $B(z, 3r/4) \subset B(x_0,r)$),

$$v_+^r(z) \geq \fint_{\partial B(z,3r/4)} U - U(z) \geq C(d) \int_0^{3r/4} s^{1-d} \int_{B(z,s)} d\mu^+. \tag{3.6}$$

Now let us notice that

$$\int_0^{3r/4} s^{1-d} \int_{B(z,s)} d\mu^+ \geq (3r/4)^{1-d} \int_{r/2}^{3r/4} ds\mu^+(B(z,s)) \geq Cr^{2-d}\mu^+(B(x_0, r/4)).$$

Plugging in (3.6) and integrating upon $B(x_0, r/4)$, together with (3.5), leads to

$$Cr^{d-1} \geq Cr^{2-d} \left(\mu^+(B(x_0, r/4))\right)^2,$$

that is to say,

$$\mu^+(B(x_0, r/4)) \leq Cr^{d-1-1/2}.$$

We get the same result for μ^-, and so finally for $|\Delta|u||$. □

Appendix A. Technical lemmas

The proofs of the two following more or less classical lemmas may be found in [3]. We denote by $⨍_E$ the average over the set E.

Lemma A.1. *Let $B(x_0, r_0)$ be an open ball and $U \in \mathcal{C}^2(B(x_0, r_0))$. Then, for all $r \in (0, r_0)$,*

$$⨍_{\partial B(x_0,r)} U - U(x_0) = (d\omega_d)^{-1} \int_0^r ds\, s^{1-d} \int_{B(x_0,s)} d(\Delta U).$$

This remains valid for all $U \in H^1(B(x_0, r_0))$ such that ΔU is a measure satisfying

$$\int_0^r ds\, s^{1-d} \int_{B(x_0,s)} d|\Delta U| < \infty, \tag{A.1}$$

and such that

$$U(x_0) = \lim_{r\to 0} ⨍_{\partial B(x_0,r)} U. \tag{A.2}$$

Moreover, (A.1) *is satisfied if $U \in L^\infty(B(x_0, r_0))$ and there exists $g \in L^q(B(x_0, r_0))$ with $q > d/2$ such that $\Delta U^+ \geq -g$ and $\Delta U^- \geq -g$.*

Remark 4. The proof shows, furthermore, that the condition (A.1) implies the existence of the limit in (A.2) for any x_0, so that we can take the representative of U defined thanks to (A.2).

Lemma A.2. *Let $B(x_0, r_0)$ be an open ball, $r_0 \leq 1$, $F \in L^q(B(x_0, r_0))$, $q > d$. Then, there exists some constant $C = C(\|F\|_{L^q(B(x_0,r_0))}, q, d)$ such that, for $r \in (0, r_0)$,*

(i) *if* $\Delta U = F$ *on* $B(x_0, r_0)$*, then*

$$\|\nabla U\|_{L^\infty(B(x_0,r/2))} \leq C[1 + r^{-1}\|U\|_{L^\infty(B(x_0,r))}], \tag{A.3}$$

(ii) *if* $\Delta U \geq F$ *and* $U \geq 0$ *on* $B(x_0, r_0)$*, then*

$$\|U\|_{L^\infty(B(x_0,2r/3))} \leq C\Big[r + \fint_{\partial B(x_0,r)} U\Big]. \tag{A.4}$$

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