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LOW REGULARITY GLOBAL WELL-POSEDNESS FOR THE KLEIN-GORDON-SCHRÖDINGER SYSTEM WITH THE HIGHER-ORDER YUKAWA COUPLING

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Abstract. In this paper, we consider the Klein-Gordon-Schrödinger system with the higher-order Yukawa coupling in $\mathbb{R}^{1+1}$, and prove the local and global well-posedness in $L^2 \times H^{1/2}$. The method to be used is adapted from the scheme originally by J. Colliander, J. Holmer, and N. Tzirakis [8] to use the available L^2 conservation law of u and control the growth of n via the estimates in the local theory.

1. Introduction

The Cauchy problem

$$\begin{cases} iu_t + \Delta u = - nu, \quad x \in \mathbb{R}^d, t \in \mathbb{R}; \\ n_{tt} + (1-\Delta)n = |u|^2, \quad x \in \mathbb{R}^d, t \in \mathbb{R}; \\ u(0) = u_0, \quad n(0) = n_0, \quad n_t(0) = n_1 \end{cases} \tag{1.1}$$

has been considered in [3], [10], [11], and [12]. Here $u : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{C}$ is the nucleon field and $n : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ is the meson field. H. Pecher [20] considered the system (1.1) in $\mathbb{R}^{3+1}$ by the Fourier truncation method [6]. N. Tzirakis [22] consider the same system in one, two, and three dimensions by the I-method [16]. They also obtained a polynomial-in-time bound for the growth of the norms. Recently, using the available L^2 conservation law of u and controlling the growth of n via the estimate in the local theory, J. Colliander et al. [8] obtained the optimal global well-posedness of (1.1) in $\mathbb{R}^{3+1}$. It is also applicable to 1D and 2D cases.

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Just as in [4] and [7], the system (1.1) is naturally generalized to the following system:

$$\begin{cases} iu_t + \Delta u = - nf(|u|^2)u; \\ n_{tt} + (1-\Delta)n = F(|u|^2), \quad F' = f, F(0) = f(0) = 0; \\ u(0) = u_0, \quad n(0) = n_0, \quad n_t(0) = n_1. \end{cases}$$

The restricted case $F(s) = s^m$ in GKLS will be called KLS_m. In this paper, we consider only the case $1 \leq m < 2$, $d = 1$, that is,

$$\begin{cases} iu_t + \partial_x^2 u = - mn|u|^{2(m-1)}u, \quad x \in \mathbb{R}, t \in \mathbb{R}; \\ n_{tt} + (1-\partial_x^2)n = |u|^{2m}, \quad x \in \mathbb{R}, t \in \mathbb{R}; \\ (u, n, \dot{n})(0) = (u_0, n_0, n_1). \end{cases} \tag{1.2}$$

The reason that the higher-order powers are introduced into the physically relevant dispersive PDEs is to adjust the strength of the non-linearity relative to the dispersion to work toward understanding the balance between the two effects. We give similar scaling analyses in next section.

It is well known that the following conservation laws hold for (1.2):

$$\begin{cases} M(u)(t) =: \|u(t)\|, \\ E(u,n)(t) =: \|\partial_x u(t)\|^2 + \dfrac{1}{2}\big(\|An(t)\|^2 + \|n_t(t)\|^2\big) - \displaystyle\int_{\mathbb{R}} |u(t)|^{2m} n(t) dx, \end{cases} \tag{1.3}$$

where $\|\cdot\|$ denotes the norm of $L^2(\mathbb{R})$ and A denotes $(I - \partial_x^2)^{\frac{1}{2}}$. Here, we use the method in [8] rather than the Fourier truncation method and the I-method to consider the low regularity. The idea is to use the available L^2 conservation law of u and control the growth of n via the estimates in the local theory.

Our main result is the following theorem

Theorem 1.1. *Let $1 \leq m < 2$; then the KLS_m (1.2) in dimension $d = 1$ is global well-posedness for $(u_0, n_0, n_1) \in L^2 \times H^{1/2} \times H^{-1/2}$. More precisely, the solution $(u, n) \in C(\mathbb{R}; L^2) \times C(\mathbb{R}; H^{\frac{1}{2}})$ satisfies, for $t \in \mathbb{R}$,*

$$\big\|u(t)\big\|_{L^2} = \big\|u_0\big\|_{L^2}$$

and

$$\begin{aligned} &\big\|n(t)\big\|_{H^{1/2}} + \big\|\partial_t n(t)\big\|_{H^{-1/2}} \\ &\lesssim \exp(c|t|\big\|u_0\big\|_{L^2}^{4m-2}) \max(\big\|n_0\big\|_{H^{1/2}} + \big\|n_1\big\|_{H^{-1/2}}, \big\|u_0\big\|_{L^2}^{2m}). \end{aligned}$$

Remark 1.1. In the same way, we can also get the global well-posedness of (1.2) in $L^2 \times H^{d/2} \times H^{d/2-1}$ for dimension $d = 2$ and $1 \le m < \frac{3}{2d} + \frac{1}{2}$.

Remark 1.2. Since the energy $E(u, n)$ plays no role in the proof, the same result applies to certain Hamiltonian generalizations of (1.2). This is analogous to Remark 1.2 in [8].

The paper is organized as follows. In Section 2, we first give some scaling analyses of the criticality, then give the linear and non-linear estimates along the lines of Ginibre, Tsutsumi and Velo [13] in the $X^{s,b}$ spaces, which were introduced by Bourgain [5]; Kenig, Ponce and Vega [17]; and Klainerman and Machedon [18], [19]. We also can refer to Foschi [9], Grünrock [14] and Selberg [21].

We have the free dispersive equation of the form

$$iu_t + \varphi(D_x)u = 0, \quad D_x = -i\partial_x, \tag{1.4}$$

where φ is a measurable function. For $\lambda \in \mathbb{R}$, the Japanese symbol $\langle\lambda\rangle$ denotes $\left(1 + |\lambda|^2\right)^{1/2}$. Let $X^{s,b}_\varphi$ be the completion of $\mathcal{S}(\mathbb{R})$ with respect to

$$\begin{aligned}\|f\|_{X^{s,b}_\varphi} :=& \|\langle\xi\rangle^s\langle\tau\rangle^b\mathcal{F}(e^{-it\varphi(D_x)}f(x,t))\|_{L^2_{\xi,\tau}}\\ =& \|\langle\xi\rangle^s\langle\tau-\varphi(\xi)\rangle^b\widehat{f}(\xi,\tau))\|_{L^2_{\xi,\tau}}.\end{aligned}$$

In general, we use the notation $X^{s,b}_\pm$ for $\varphi(\xi) = \pm\langle\xi\rangle$ and $X^{s,b}$ for $\varphi(\xi) = -|\xi|^2$ without confusion. For a given time interval I, we define

$$\|f\|_{X^{s,b}(I)} = \inf_{\widetilde{f}_{|I}=f} \|\widetilde{f}\|_{X^{s,b}} \quad \text{where} \quad \widetilde{f} \in X^{s,b};$$

$$\|f\|_{X^{s,b}_\pm(I)} = \inf_{\widetilde{f}_{|I}=f} \|\widetilde{f}\|_{X^{s,b}_\pm} \quad \text{where} \quad \widetilde{f} \in X^{s,b}_\pm.$$

In Section 3, we transform the KLS_m (1.2) into an equivalent system of first order in t in the usual way, then make use of Strichartz-type estimates to give the local well-posedness in the $X^{0,b}([0,\delta]) \times X^{1/2,b}_\pm([0,\delta])$ spaces for some $0 < b < 1/2$, which is useful for the iteration procedure. In general, we can obtain the local well-posedness for $b \ge \frac{1}{2}$, but in order to get the global well-posedness, we use $0 < b < 1/2$ to obtain some gains.

In Section 4, we show that the local result can be iterated to get a solution on any time interval $[0, T]$. We first can construct the solution step by step on some time intervals, which is dependent only on $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Then we can repeat this entire procedure to get the desired time

T, each time advancing a time of length $\sim 1/\|u_0\|_{L^2}^{4m-2}$ (independent of $\|(n(t),\partial_t n(t))\|_{H^{1/2},H^{-1/2}}$).

We use the following standard facts about the space $X^{s,b}_{\varphi}$ [14].

Let $\psi \in C_0^\infty(\mathbb{R})$ and satisfy $\operatorname{supp}\{\psi\} \subset (-2,2)$; $\psi|_{[-1,1]} = 1$; $\psi(t) = \psi(-t)$, $\psi \geq 0$. For $0 < \lambda \leq 1$, define $\psi_\lambda(t) = \psi(\frac{t}{\lambda})$.

For $s \in \mathbb{R}$ and $b \geq 0$, we have the following homogeneous estimate:

$$\|\psi_\delta\ e^{i\varphi(D_x)t} f(x)\|_{X^{s,b}_{\varphi}} \leq c\delta^{\frac{1}{2}-b}\|f\|_{H^s_x}; \tag{1.5}$$

$$\|e^{i\varphi(D_x)t} f(x)\|_{C(R,H^s_x)} = \|f\|_{H^s_x}. \tag{1.6}$$

For $b'+1 \geq b \geq 0 \geq b' > -\frac{1}{2}$, we have the following inhomogeneous estimates:

$$\left\|\psi_\delta \int_0^t e^{i(t-s)\varphi(D_x)} F(s)ds\right\|_{X^{s,b}_{\varphi}} \leq c\delta^{1+b'-b}\|F\|_{X^{s,b'}_{\varphi}}; \tag{1.7}$$

$$\left\|\int_0^t e^{i(t-s)\varphi(D_x)} F(s)ds\right\|_{C([0,\delta],H^s_x)} \leq c\delta^{\frac{1}{2}+b'}\|F\|_{X^{s,b'}_{\varphi}}. \tag{1.8}$$

For $1 < p \leq 2$ and $b \leq \frac{1}{2} - \frac{1}{p}$, we have the following Sobolev inequality:

$$\|f\|_{X^{s,b}_{\varphi}} \leq c\|f\|_{L^p_t(\mathbb{R},H^s_x(\mathbb{R}))}. \tag{1.9}$$

Finally, we introduce the following notation: $a+$ (respectively $a-$) denotes a number slightly larger (respectively smaller) than a.

2. Linear and Non-Linear Estimates

In this section, we first transform the Klein-Gordon-Schrödinger system into an equivalent system of first order in t in the usual way to discuss the notion of criticality for the system (1.2). Later, we give some useful linear and non-linear estimates.

First, for the notion of criticality, we define

$$n_\pm := \tfrac{1}{2}\big(n \pm \tfrac{1}{iA} n_t\big), \quad A = (I - \partial_x^2)^{\frac{1}{2}}.$$

Then we have $n = n_+ + n_-$, $n_t = iA(n_+ - n_-)$, $n_+ = \overline{n}_-$, and the equivalent system is

$$\begin{cases} iu_t + \partial_x^2 u = -\,m(n_+ + n_-)|u|^{2(m-1)}u \\ i\partial_t n_\pm \pm An_\pm = \pm\,\tfrac{1}{2}A^{-1}\big(|u|^{2m}\big) \\ u(0) = u_0 \in H^k_x, \quad n_\pm(0) = \tfrac{1}{2}\big(n_0 \pm \tfrac{1}{iA}n_1\big) \in H^l_x. \end{cases} \tag{2.1}$$

We follow [13] to discuss the criticality through scaling. Consider the following similar system:

$$\begin{cases} i\partial_t u + \partial_x^2 u = -\, m(n_+ + n_-)|u|^{2(m-1)}u \\ i\partial_t n_\pm \pm (-\partial_x^2)^{1/2} n_\pm = \pm \frac{1}{2}(-\partial_x^2)^{-1/2}\big(|u|^{2m}\big). \end{cases} \tag{2.2}$$

If there were not the term $\partial_x^2 u$ in the left-hand side of the first equation in (2.2), which means that the linear effect of the wave equation is stronger than that of the Schrödinger equation, then the system (2.2) would be invariant under the dilation

$$u \to u_\lambda = \lambda^{3/(4m-2)} u(\lambda t, \lambda x), \quad n \to n_\lambda = \lambda^{(2-m)/(2m-1)} n(\lambda t, \lambda x),$$

and the system (2.2) would be critical for $(u_0, n_\pm(0)) \in H_x^k \times H_x^l$ for $k = \frac{d}{2} - \frac{3}{4m-2}$, $l = \frac{d}{2} - \frac{2-m}{2m-1}$. Hence it is a $L_x^2 \times H_x^{1/2}$-subcritical case for $d = 1$, $1 \le m < 2$.

If there were not the term $i\partial_t u$ in the left-hand side of the first equation in (2.2), then the system (2.2) would be invariant under the dilation

$$u \to u_\lambda = \lambda^{2/(2m-1)} u(\lambda t, \lambda x), \quad n \to n_\lambda = \lambda^{2/(2m-1)} n(\lambda t, \lambda x),$$

and the system (2.2) would be critical for $(u_0, n_\pm(0)) \in H_x^k \times H_x^l$ for $k = \frac{d}{2} - \frac{2}{2m-1}$, $l = \frac{d}{2} - \frac{2}{2m-1}$. Hence, it is an $L_x^2 \times L_x^2$-subcritical case for $d = 1$, $1 \le m < \frac{5}{2}$.

If there were not the term $\pm(-\partial_x^2)^{1/2} n_\pm$ in the left-hand side of the second equation in (2.2), which means that the linear effect of the Schrödinger equation is stronger than that of the wave equation, then the system (2.2) would be invariant under the dilation

$$u \to u_\lambda = \lambda^{5/(4m-2)} u(\lambda^2 t, \lambda x), \quad n \to n_\lambda = \lambda^{(3-m)/(2m-1)} n(\lambda^2 t, \lambda x),$$

and the system (2.2) would be critical for $(u_0, n_\pm(0)) \in H_x^k \times H_x^l$ for $k = \frac{d}{2} - \frac{5}{4m-2}$, $l = \frac{d}{2} - \frac{3-m}{2m-1}$. Hence, it is an $L_x^2 \times H_x^{1/2}$-subcritical case for $d = 1$, $1 \le m < 3$.

If there were not the term $i\partial_t n_\pm$ in the left-hand side of the second equation in (2.2), then the system (2.2) would be invariant under the dilation

$$u \to u_\lambda = \lambda^{2/(2m-1)} u(\lambda^2 t, \lambda x), \quad n \to n_\lambda = \lambda^{2/(2m-1)} n(\lambda^2 t, \lambda x),$$

and the system (2.2) would be critical for $(u_0, n_\pm(0)) \in H_x^k \times H_x^l$ for $k = \frac{d}{2} - \frac{2}{2m-1}$, $l = \frac{d}{2} - \frac{2}{2m-1}$. Hence, it is an $L_x^2 \times L_x^2$-subcritical case for $d = 1$, $1 \le m < \frac{5}{2}$.

Because there is no strict scaling analysis about (1.2), (2.1) and (2.2), the discussion above is heuristic by dropping various terms. We conjecture that for the 1D case, the first analysis is the most relevant; while for the higher case, the third analysis is the most relevant. It has been shown for the 1D Zakaharov system in [15]. That is the reason why we here focus on the local and global well-posedness of the system (2.1) in $L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ for $1 \leq m < 2$. We will take the other cases into account in the forthcoming papers.

Second, we give some known linear estimates. For the Schrödinger equation, we have

Lemma 2.1 (Strichartz estimate [13], [14])**.** *Assume that* $4 \leq q \leq +\infty$, $2 \leq r \leq +\infty$, $0 \leq \frac{2}{q} \leq \frac{1}{2} - \frac{1}{r}$, *and* $s = \frac{1}{2} - \frac{1}{r} - \frac{2}{q}$. *Then we have*

$$\|u\|_{L^q_t L^r_x(\mathbb{R})} \leq c\|u\|_{X^{s,\frac{1}{2}+}}.$$

In particular, combining with the trivial equality $\|u\|_{L^2_{t,x}} = \|u\|_{X^{0,0}}$, we have

Lemma 2.2 ([13], [14])**.** *Assume that* $0 < \frac{1}{r} \leq \frac{1}{2}$, $\frac{1}{2} - \frac{1}{r} \leq \frac{2}{q} < \frac{1}{2} + \frac{1}{r}$ *and* $b > \frac{1}{2} - \frac{1}{q} + \frac{1}{2}(\frac{1}{2} - \frac{1}{r})$. *Then the estimate*

$$\|u\|_{L^q_t L^r_x(\mathbb{R})} \leq c\|u\|_{X^{0,b}}, \tag{2.3}$$

holds true for all $u \in X^{0,b}$.

For the Klein-Gordon equation, we will use the fact that

$$\|n_\pm\|_{L^p_t H^s_x(\mathbb{R})} \leq c\|n_\pm\|_{X^{s,b}}, \quad \text{for} \quad 2 < p < \infty, \quad b > \tfrac{1}{2} - \tfrac{1}{p}, \tag{2.4}$$

which can be obtained from the interpolation between

$$\|n_\pm\|_{L^\infty_t H^s_x} \leq c\|n_\pm\|_{X^{s,\frac{1}{2}+}_\pm}$$

and the trivial equality

$$\|n_\pm\|_{L^2_t H^s_x} = \|n_\pm\|_{X^{s,0}_\pm}.$$

Finally, we give some useful non-linear estimates, which are especially important to the iteration procedure.

Lemma 2.3 (non-linear estimates)**.** *Let* $1 \le m < 2$; *then there exists some* $0 < \epsilon < 1 - \frac{m}{2}$ *such that the estimates*

$$\big\| n_\pm |u|^{2(m-1)} u \big\|_{X^{0,b_1'}} \le c \big\| n_\pm \big\|_{X_\pm^{\frac{1}{2},b_2}} \big\| u \big\|_{X^{0,b_1}}^{2m-1},$$
$$\big\| |u|^{2m} \big\|_{X_\pm^{-\frac{1}{2},b_2'}} \le c \big\| u \big\|_{X^{0,b_1}}^{2m}$$

hold for any $n_\pm \in X_\pm^{\frac{1}{2},b_2}$ *and* $u \in X^{0,b_1}$, *where* $b_1 = b_2 = \frac{2m-1}{4m} + \epsilon$ *and* $b_1' = b_2' = -\frac{1}{2} + 2m\epsilon$.

Proof. By (1.9) and Hölder's inequality, we have

$$\begin{aligned} \big\| n_\pm |u|^{2(m-1)} u \big\|_{X^{0,b_1'}} &\le c \big\| n_\pm |u|^{2(m-1)} u \big\|_{L_t^{\frac{1}{1-2m\epsilon}} L_x^2} \\ &\le c \big\| n_\pm \big\|_{L_t^{4m} L_x^{\frac{1}{\theta}}} \big\| u \big\|_{L_t^q L_x^{\frac{4m-2}{1-2\theta}}}^{2m-1} \le c \big\| n_\pm \big\|_{L_t^{4m} H_x^{1/2}} \big\| u \big\|_{L_t^q L_x^{\frac{4m-2}{1-2\theta}}}^{2m-1}, \end{aligned}$$

where $\frac{1}{4m} + \frac{2m-1}{q} = 1 - 2m\epsilon$ and $0 < \theta \ll 1$.

From (2.4), we have

$$\big\| n_\pm \big\|_{L_t^{4m} H_x^{1/2}} \le c \big\| n_\pm \big\|_{X_\pm^{\frac{1}{2},b_2}}.$$

From Lemma 2.2, we have

$$\big\| u \big\|_{L_t^q L_x^{\frac{4m-2}{1-2\theta}}} \le c \big\| u \big\|_{X^{0,b_1}},$$

under the conditions

$$\begin{cases} \theta + 2\epsilon < 2 - m, \\ \theta - 4m\epsilon < m + \frac{1}{2m} - 2, \end{cases} \tag{2.5}$$

which can be satisfied for $1 \le m < 2$. Therefore, we obtain

$$\big\| n_\pm |u|^{2(m-1)} u \big\|_{X^{0,b_1'}} \le c \big\| n_\pm \big\|_{X_\pm^{\frac{1}{2},b_2}} \big\| u \big\|_{X^{0,b_1}}^{2m+1}.$$

In addition, by (1.9), Sobolev inequality and Lemma 2.2, we have

$$\begin{aligned} \big\| |u|^{2m} \big\|_{X_\pm^{-\frac{1}{2},b_2'}} &\le c \big\| |u|^{2m} \big\|_{L_t^{\frac{1}{1-2m\epsilon}} H_x^{-1/2}} \le c \big\| |u|^{2m} \big\|_{L_t^{\frac{1}{1-2m\epsilon}} L_x^1} \\ &\le c \big\| u \big\|_{L_t^{\frac{2m}{1-2m\epsilon}} L_x^{2m}}^{2m} \le c \big\| u \big\|_{X^{0,b_1}}^{2m}. \end{aligned}$$

The proof is completed.

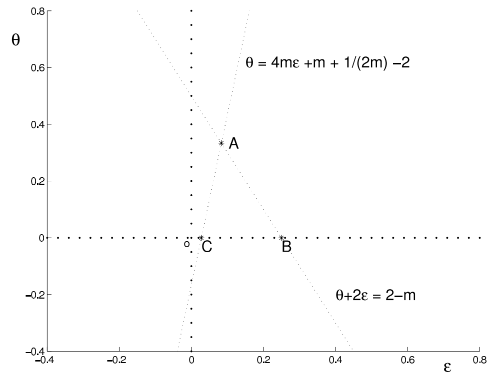

FIGURE 1. $\theta - \epsilon$ parameter picture for $1 \leq m \leq 1 + \frac{\sqrt{2}}{2}$.

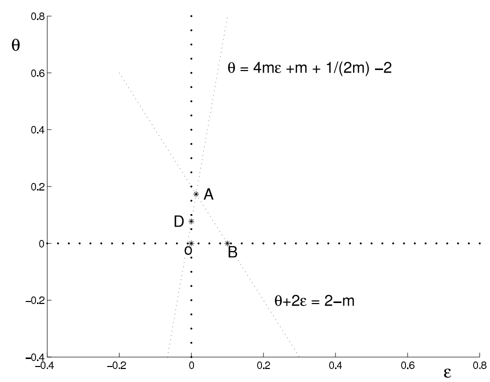

FIGURE 2. $\theta - \epsilon$ parameter picture for $1 + \frac{\sqrt{2}}{2} \leq m \leq 1 + \frac{\sqrt{3}}{2}$.

Remark 2.1. If $1 \leq m \leq 1 + \frac{\sqrt{2}}{2}$, we take the value of θ and ϵ in the region ABC of Figure 1. If $1 + \frac{\sqrt{2}}{2} \leq m \leq 1 + \frac{\sqrt{3}}{2}$, we take the value of θ and ϵ in the region $ABoD$ of Figure 2 (see next page). If $1 + \frac{\sqrt{3}}{2} \leq m < 2$, we take the value of θ and ϵ in the region ABo of Figure 3 (see next page). As we know, when $m = 2$, it is difficult to prove the non-linear estimates in the above lemma for some b_1, b_2, b_1', b_2' satisfying $2m + b_1' + b_2' = (4m-1)b_1 + b_2$. Hence, we cannot prove the global well-posedness in Theorem 1.1 for the endpoint case $m = 2$.

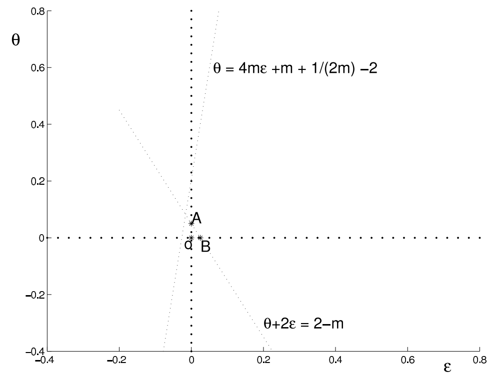

FIGURE 3. $\theta - \epsilon$ parameter picture for $1 + \frac{\sqrt{3}}{2} \leq m < 2$.

3. Local Well-Posedness

In this section, we construct a solution of (2.1) in some time interval $[0, \delta]$ using a fixed-point argument.

The KLS_m (2.1) has the following equivalent integral equation formulation:

$$u(t) = U(t)u_0 + imU_{*R}[(n_+ + n_-)|u|^{2(m-1)}u](t),$$
$$n_\pm = W_\pm n_\pm(0) \mp \frac{i}{2} W_{\pm *R}(A^{-1}|u|^{2m})(t),$$

where

$$U(t)u_0 = e^{it\partial_x^2}u_0, \qquad U_{*R}F(t) = \int_0^t U(t-s)F(s)ds;$$
$$W_\pm(t)n_\pm(0) = e^{\pm itA}n_\pm(0), \quad W_{\pm *R}G(t) = \int_0^t W(t-s)G(s)ds.$$

For $0 < \delta < 1$, we define a mapping $M = \big(\Lambda_S(u, n_\pm), \Lambda_{W_\pm}(u, n_\pm)\big)$ by

$$\begin{cases} \Lambda_S(u, n_\pm) = \psi_\delta U(t)u_0 + im\psi_\delta U_{*R}[(n_+ + n_-)|u|^{2(m-1)}u](t); \\ \Lambda_{W_\pm}(u, n_\pm) = \psi_\delta W_\pm n_\pm(0) \mp \frac{i}{2}\psi_\delta W_{\pm *R}(A^{-1}|u|^{2m})(t). \end{cases}$$

Proposition 3.1 (**local well-posedness**). *Let $1 \leq m < 2$ and $\epsilon > 0$ as in Lemma 2.3, $b_1 = b_2 = \frac{2m-1}{4m} + \epsilon$, and $b_1' = b_2' = -\frac{1}{2} + 2m\epsilon$. Assume that $u_0 \in L^2(\mathbb{R})$ and $n_\pm(0) \in H^{1/2}(\mathbb{R})$. Then there exists a positive number δ*

satisfying

$$\delta^{m+\frac{1}{2}+b_2'-(2m-1)b_1-b_2}\big\|u_0\big\|_{L^2}^{2m-1} \lesssim 1; \tag{3.1}$$

$$\delta^{m+\frac{1}{2}+b_1'-(2m-1)b_1-b_2}\big\|u_0\big\|_{L^2}^{2m-2}\big\|n_\pm(0)\big\|_{H^{1/2}} \lesssim 1; \tag{3.2}$$

$$\delta^{m+\frac{1}{2}+b_2'-2mb_1}\big\|u_0\big\|_{L^2}^{2m} \lesssim \big\|n_\pm(0)\big\|_{H^{1/2}}, \tag{3.3}$$

such that the above Cauchy problem (2.1) *has a unique solution* $u(t,x) \in C([0,\delta],L^2)$ *and* $n_\pm(t,x) \in C([0,\delta],H^{1/2})$ *with the property*

$$\big\|u\big\|_{X^{0,b_1}([0,\delta])} \lesssim \delta^{\frac{1}{2}-b_1}\big\|u_0\big\|_{L^2}, \quad \big\|n_\pm\big\|_{X_\pm^{\frac{1}{2},b_2}([0,\delta])} \lesssim \delta^{\frac{1}{2}-b_2}\big\|n_\pm(0)\big\|_{H^{1/2}}.$$

Remark 3.1. According to the value of b_1, b_2, b_1', b_2', we have

$$\begin{aligned} m+\tfrac{1}{2}+b_2'-(2m-1)b_1-b_2 &= m+\tfrac{1}{2}+b_1'-(2m-1)b_1-b_2 \\ &= m+\tfrac{1}{2}+b_2'-2mb_1 = \tfrac{1}{2}. \end{aligned}$$

Proof. Define the closed set Y as

$$\begin{aligned} Y = \Big\{ &\big\|u\big\|_{X^{0,b_1}([0,\delta])} \le 2c\delta^{\frac{1}{2}-b_1}\big\|u_0\big\|_{L^2}, \\ &\big\|n_\pm\big\|_{X_\pm^{\frac{1}{2},b_2}([0,\delta])} \le 2c\delta^{\frac{1}{2}-b_2}\big\|n_\pm(0)\big\|_{H^{1/2}}\Big\}. \end{aligned}$$

Define the metric in the set Y as

$$d\big((u_1,n_{1\pm}),(u_2,n_{2\pm})\big) = \big\|u_1-u_2\big\|_{X^{0,b_1}([0,\delta])} + \big\|n_{1\pm}-n_{2\pm}\big\|_{X_\pm^{\frac{1}{2},b_2}([0,\delta])}.$$

First, we prove that M maps Y into itself under some conditions on δ.

Now take any $(u,n_\pm) \in Y$. By (1.5), (1.7) and Lemma 2.3, we have

$$\begin{aligned} &\big\|\Lambda_S(u,n_\pm)\big\|_{X^{0,b_1}([0,\delta])} \le c\delta^{\frac{1}{2}-b_1}\big\|u_0\big\|_{L^2} + c\delta^{1+b_1'-b_1}\big\|n_\pm|u|^{2(m-1)}u\big\|_{X^{0,b_1'}} \\ &\le c\delta^{\frac{1}{2}-b_1}\big\|u_0\big\|_{L^2} + c\delta^{1+b_1'-b_1}\big\|n_\pm\big\|_{X_\pm^{\frac{1}{2},b_2}}\big\|u\big\|_{X^{0,b_1}}^{2m-1} \\ &\le c\delta^{\frac{1}{2}-b_1}\big\|u_0\big\|_{L^2} + c\delta^{1+b_1'-b_1}\delta^{\frac{1}{2}-b_2}\big\|n_\pm(0)\big\|_{H^{1/2}}\big(\delta^{\frac{1}{2}-b_1}\big\|u_0\big\|_{L^2}\big)^{2m-1} \\ &\le 2c\delta^{\frac{1}{2}-b_1}\big\|u_0\big\|_{L^2}, \end{aligned}$$

under the condition (3.2). In addition, we have

$$\begin{aligned} &\big\|\Lambda_{W_\pm}(u,n_\pm)\big\|_{X_\pm^{\frac{1}{2},b_2}([0,\delta])} \le c\delta^{\frac{1}{2}-b_2}\big\|n_\pm(0)\big\|_{H^{1/2}} + c\delta^{1+b_2'-b_2}\big\||u|^{2m}\big\|_{X_\pm^{-\frac{1}{2},b_2'}} \\ &\le c\delta^{\frac{1}{2}-b_2}\big\|n_\pm(0)\big\|_{H^{1/2}} + c\delta^{1+b_2'-b_2}\big\|u\big\|_{X^{0,b_1}}^{2m} \\ &\le c\delta^{\frac{1}{2}-b_2}\big\|n_\pm(0)\big\|_{H^{1/2}} + c\delta^{1+b_2'-b_2}\big(\delta^{\frac{1}{2}-b_1}\big\|u_0\big\|_{L^2}\big)^{2m} \le 2c\delta^{\frac{1}{2}-b_2}\big\|n_\pm(0)\big\|_{H^{1/2}}, \end{aligned}$$

under the condition (3.3). Therefore, we prove that M maps Y into itself.

Second, we can prove that M is a contraction map under other conditions on δ.

Take any $(u_1, n_{1\pm}), (u_2, n_{2\pm}) \in Y$; we have

$$\begin{aligned}
&\big\|\Lambda_S(u_1, n_{1\pm}) - \Lambda_S(u_2, n_{2\pm})\big\|_{X^{0,b_1}([0,\delta])} \\
\le& c\delta^{1+b_1'-b_1}\big\|n_{1\pm}|u_1|^{2(m-1)}u_1 - n_{2\pm}|u_2|^{2(m-1)}u_2\big\|_{X^{0,b_1'}} \\
\le& c\delta^{1+b_1'-b_1}\Big(\|n_{1\pm}\|_{X_\pm^{\frac{1}{2},b_2}}\big(\|u_1\|_{X^{0,b_1}}^{2(m-1)} + \|u_2\|_{X^{0,b_1}}^{2(m-1)}\big)\|u_1-u_2\|_{X^{0,b_1}} \\
&\qquad + \big(\|u_1\|_{X^{0,b_1}}^{2m-1} + \|u_2\|_{X^{0,b_1}}^{2m-1}\big)\|n_{1\pm}-n_{2\pm}\|_{X_\pm^{\frac{1}{2},b_2}}\Big) \\
\le& c\delta^{1+b_1'-b_1}\Big(\delta^{\frac{1}{2}-b_2}\|n_\pm(0)\|_{H^{1/2}}\delta^{(m-1)(1-2b_1)}\|u_0\|_{L^2}^{2(m-1)}\|u_1-u_2\|_{X^{0,b_1}} \\
&\qquad + \delta^{(2m-1)(\frac{1}{2}-b_1)}\|u_0\|_{L^2}^{2m-1}\|n_{1\pm}-n_{2\pm}\|_{X_\pm^{\frac{1}{2},b_2}}\Big) \\
\le& \frac{1}{4}\big(\|u_1-u_2\|_{X^{0,b_1}([0,\delta])} + \|n_{1\pm}-n_{2\pm}\|_{X_\pm^{\frac{1}{2},b_2}([0,\delta])}\big),
\end{aligned}$$

under the conditions (3.2) and

$$\delta^{m+\frac{1}{2}+b_1'-2mb_1}\|u_0\|_{L^2}^{2m-1} \lesssim 1,$$

which is equivalent to (3.1) for $b_1 = b_2$ and $b_1' = b_2'$.

In addition, we have

$$\begin{aligned}
&\big\|\Lambda_{W_\pm}(u_1, n_{1\pm}) - \Lambda_{W_\pm}(u_2, n_{2\pm})\big\|_{X_\pm^{\frac{1}{2},b_2}([0,\delta])} \\
&\le c\delta^{1+b_2'-b_2}\big\||u_1|^{2m} - |u_2|^{2m}\big\|_{X_\pm^{-\frac{1}{2},b_2'}} \\
&\le c\delta^{1+b_2'-b_2}\big(\|u_1\|_{X^{0,b_1}}^{2m-1} + \|u_2\|_{X^{0,b_1}}^{2m-1}\big)\|u_1-u_2\|_{X^{0,b_1}} \\
&\le c\delta^{1+b_2'-b_2}\big(\delta^{\frac{1}{2}-b_1}\|u_0\|_{L^2}\big)^{2m-1}\|u_1-u_2\|_{X^{0,b_1}} \le \tfrac{1}{4}\|u_1-u_2\|_{X^{0,b_1}},
\end{aligned}$$

under the condition (3.1).

The standard fixed-point arguments give a unique solution in the time interval $[0,\delta]$. According to (1.6) and (1.8), we can get that $u \in C([0,\delta], L^2)$ and $n_\pm \in C([0,\delta], H^{1/2})$. Summarizing, the proof is completed.

4. Global Well-Posedness

In this section, we show that the process can be iterated to get a solution on any time interval $[0,T]$. We first can construct the solution step by step on some time intervals, which is dependent only on $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Then we can repeat this entire procedure to get the desired time T.

According the mass conservation in (1.3), we conclude that $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$. In order to iterate the local result to obtain the global well-posedness, we are only concerned with the growth in $\|n_\pm(t)\|_{H^{1/2}}$ from one time step to the next step.

Suppose that after some number of iterations we reach a time where $\|n_\pm(t)\|_{H^{1/2}} \gg \|u(t)\|_{L^2}^{2m} = \|u_0\|_{L^2}^{2m}$. Take this time position as the initial time $t=0$ so that $\|u_0\|_{L^2}^{2m} \ll \|n_\pm(0)\|_{H^{1/2}}$. Then (3.3) is automatically satisfied, and by (3.2) we may select a time increment of size

$$\delta \sim \left(\|u_0\|_{L^2}^{2m-2}\|n_\pm(0)\|_{H^{1/2}}\right)^{-1/(m+\frac{1}{2}+b_1'-(2m-1)b_1-b_2)}. \tag{4.1}$$

Since

$$n_\pm(t) = W_\pm n_\pm(0) \mp \frac{i}{2} W_{\pm *R}(A^{-1}|u|^{2m})(t),$$

We can apply (1.6), (1.8) and Proposition 3.1 to obtain

$$\begin{aligned}\|n_\pm(\delta)\|_{H^{1/2}} \le& \|n_\pm(0)\|_{H^{1/2}} + c\delta^{\frac{1}{2}+b_2'}\left\||u|^{2m}\right\|_{X_\pm^{-\frac{1}{2},b_2'}}\\ \le& \|n_\pm(0)\|_{H^{1/2}} + c\delta^{\frac{1}{2}+b_2'}\|u\|_{X^{0,b_1}[0,\delta]}^{2m}\\ \le& \|n_\pm(0)\|_{H^{1/2}} + c\delta^{m+\frac{1}{2}+b_2'-2mb_1}\|u_0\|_{L^2}^{2m},\end{aligned}$$

where c is some fixed constant. From this we can see that we can carry out N iterations on time intervals each of length (4.1), where

$$N \sim \frac{\|n_\pm(0)\|_{H^{1/2}}}{\delta^{m+\frac{1}{2}+b_2'-2mb_1}\|u_0\|_{L^2}^{2m}}, \tag{4.2}$$

before the quantity $\|n_\pm(t)\|_{H^{1/2}}$ doubles. The total time we advance after these N iterations, by (4.1) and (4.2) and $2m+b_1'+b_2' = (4m-1)b_1+b_2$, is

$$\begin{aligned}N\delta \sim& \frac{\|n_\pm(0)\|_{H^{1/2}}}{\delta^{m-\frac{1}{2}+b_2'-2mb_1}\|u_0\|_{L^2}^{2m}} \sim \frac{1}{\delta^{2m+b_1'+b_2'-(4m-1)b_1-b_2}\|u_0\|_{L^2}^{4m-2}}\\ \sim& \frac{1}{\|u_0\|_{L^2}^{4m-2}},\end{aligned}$$

which is independent of $\|n_\pm(t)\|_{H^{1/2}}$.

We can now repeat this entire procedure, each time advancing a time of length $\sim 1/\|u_0\|_{L^2}^{4m-2}$. Upon each repetition, the size of $\|n_\pm(t)\|_{H^{1/2}}$ will at most double, giving the exponential-in-time upper bound stated in Theorem 1.1. This completes the proof of Theorem 1.1,

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