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GLOBAL WELL-POSEDNESS OF TWO INITIAL–BOUNDARY-VALUE PROBLEMS FOR THE KORTEWEG–DE VRIES EQUATION

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Abstract. Two initial–boundary-value problems for the Korteweg–de Vries equation in a half-strip with two boundary conditions and in a bounded rectangle are considered and results on local and global well-posedness of these problems are established in Sobolev spaces of various orders, including fractional. Initial and boundary data satisfy natural (or close to natural) conditions, originating from properties of solutions of a corresponding initial-value problem for a linearized KdV equation. An essential part of the study is the investigation of special solutions of a "boundary potential" type for this linearized KdV equation.

1. Introduction

The main goal of this paper is to study global well-posedness in fractional-order Sobolev spaces of initial–boundary-value problems for the Korteweg–de Vries equation (KdV)

$$u_t + u_{xxx} + au_x + uu_x = f(t,x) \tag{1.1}$$

($u = u(t,x)$, a is an arbitrary real constant) in two domains: a left half-strip $\Pi_T^- = (0,T)\times\mathbb{R}_-$ and a bounded rectangle $Q_T = (0,T)\times(0,1)$ (here $T>0$ is arbitrary, $\mathbb{R}_- = (-\infty,0)$). For each of these problems we set the initial condition

$$u(0,x) = u_0(x) \tag{1.2}$$

(for $x \le 0$ in the case of the first problem and for $x\in[0,1]$ in the case of the second one) and for $t\in[0,T]$ the following boundary conditions: for the problem in Π_T^- two conditions,

$$u(t,0) = u_2(t), \qquad u_x(t,0) = u_3(t), \tag{1.3}$$

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and for the problem in Q_T three,

$$u(t,0) = u_1(t), \qquad u(t,1) = u_2(t), \qquad u_x(t,1) = u_3(t). \tag{1.4}$$

The present paper continues the paper [16], where similar problems of global well-posedness were studied for an initial–boundary-value problem for the KdV equation (1.1) in a right half-strip $\Pi_T^+ = (0,T)\times\mathbb{R}_+$ ($\mathbb{R}_+ = (0,+\infty)$) under initial condition (1.2) for $x \ge 0$ and one boundary condition

$$u(t,0) = u_1(t), \qquad 0 \le t \le T. \tag{1.5}$$

In any theory of well-posedness the problem of optimality of conditions on initial and boundary data is very important. In order to try to answer somehow such a question in this case let us consider an initial-value problem for a linear equation,

$$v_t + v_{xxx} = 0, \tag{1.6}$$
$$v(0,x) = v_0(x), \qquad x \in \mathbb{R}.$$

If $v_0 \in H^s(\mathbb{R})$ for some $s \in \mathbb{R}$, then there exists a unique solution of this problem $v(t,x) \in C(\mathbb{R}^t; H^s(\mathbb{R}^x))$ and for every $x \in \mathbb{R}$

$$\|D_t^{1/3} v(\cdot,x)\|_{H^{s/3}(\mathbb{R}^t)} = \|v_x(\cdot,x)\|_{H^{s/3}(\mathbb{R}^t)} = c(s)\|v_0\|_{H^s(\mathbb{R})} \tag{1.7}$$

(see, for example, [21]). Therefore, one can assume, that conditions of the type $u_0 \in H^s$, $u_1, u_2 \in H^{(s+1)/3}$, $u_3 \in H^{s/3}$ are natural for the considered problems (1.1)–(1.3) and (1.1), (1.2), (1.4), at least if the smoothness properties of solutions of the (1.7) type are required. Of course, we do not assert that such conditions are necessary for well-posedness in other functional classes.

Previously problems of existence of solutions or well-posedness of initial–boundary-value problems for the KdV equation in the domains Π_T^- and Q_T were studied in [2]–[20]. Certain approaches to initial–boundary-value problems in half-strips for the KdV equation based on an inverse scattering technique can be found in [1]–[17].

The paper [2] was the first one in which certain results on local well-posedness of the problem (1.1)–(1.3) in Π_T^- were established under zero boundary data. In [23] first results on global well-posedness of the problems (1.1)–(1.3) in Π_T^- and (1.1), (1.2), (1.4) in Q_T were established also under zero boundary data. The papers [6]–[10] are concerned with initial–boundary-value problems for the KdV equation in Q_T under boundary conditions different from (1.4).

In [13]–[15] first results on global well-posedness of the problems (1.1)–(1.3) in Π_T^- and (1.1), (1.2), (1.4) in Q_T were established under non-homogeneous boundary data. Initial data was assumed to be in certain Sobolev spaces H^k of non-negative integer orders. The hypothesis on boundary data differed from natural ones in the aforementioned sense. In particular, for $u_0 \in L_2$ the conditions $u_1 \in (L_{6+\varepsilon} \cap H^{1/6})(0,T)$ for local and, in addition, $u_1 \in W_1^{1/3}(0,T)$ for global results, $u_2 \in W_1^{5/6+\varepsilon}(0,T)$, $u_3 \in L_2(0,T)$ were assumed. Note, that these requirements on u_1 (especially for local results) can't be treated as more restrictive than the natural one $u_1 \in H^{1/3}(0,T)$. The same assumptions on u_1 were used in [12] for the problem (1.1), (1.2), (1.5) in Π_T^+. In the recent paper [4] certain relaxation of conditions on u_1 (in comparison with natural ones) in results of local well-posedness for the same problem was achieved.

In [3] first results on local and global well-posedness in a scale of fractional order Sobolev spaces of the problem (1.1),(1.2),(1.4) in Q_T were presented for $u_0 \in H^s(0,1)$, $s \geq 0$. The local results were obtained there under natural assumptions on boundary data, the global ones — under natural assumptions in the case $s \geq 3$ and nonoptimal in the case $s < 3$. The constructed solutions, however, do not possess smoothness properties with respect to t of the (1.7) type.

In the paper [20] existence of local-in-time solutions of the three aforementioned initial–boundary-value problems was proved under natural assumptions on initial and boundary data for $-3/4 < s < 3/2$, $s \neq 1/2$ (the problems of uniqueness and continuous dependence were not discussed in that paper). Classes of constructed solutions provide the smoothness properties of the (1.7) type. The method of investigation extends the one from [9] and is based on an introduction of an analytic family of boundary forcing operators. Such a method differs essentially from [16] and the present paper. The author uses in his paper functional spaces of Bourgain type different from the ones in [16] and the present paper (see Remark 2.1 below). It seems that his choice makes it difficult to extend existence results to the greater values of s.

In [11] results on global well-posedness both for the problem in Π_T^- and in Q_T were established for $u_0 \in L_2$ under ε-close-to-natural conditions on boundary data, for $u_0 \in H^2(\mathbb{R}_-)$ under natural conditions for the problem in Π_T^- and for $u_0 \in H^3(0,1)$ also under natural conditions for the problem in Q_T. The Bourgain-type spaces were not used in [11].

In the present paper local and global well-posedness in Bourgain-type spaces are established for (1.1)–(1.3) and (1.1), (1.2), (1.4), if $u_0 \in H^s$ for

$s \geq 0$, $s \neq 3k + 1/2$, $s \neq 3k + 3/2$ ($k \geq 0$ an integer). Conditions on boundary data u_1, u_2, u_3 are natural in the aforementioned sense for the local well-posedness results and in the case $s > 0$ for the global results. In the case $s = 0$ global well-posedness is established under ε-close-to-natural assumptions on boundary data. The smoothness properties of the (1.7) type are satisfied for constructed solutions.

The main difficult part of the present study is to establish these results for small values of s ($0 \leq s \leq 2$ for the first problem and $0 \leq s \leq 3$ for the second one). However, certain additional efforts for the larger values of s are also needed.

The essential part of the paper is, in fact, devoted to the local results. We have to redevelop partially the theory from [20] (moreover, our approach seems to be simpler, at least, for non-negative values of s, which are considered here).

Local solutions are constructed on the base of the contraction principle as fixed points of mappings $v = \Lambda u$, where functions v are solutions to the corresponding initial–boundary-value problems for an equation

$$v_t + v_{xxx} + av_x = f - uu_x$$

(such an approach for problems around KdV, in fact, goes back to [21]).

It is clear that this method requires, first of all, the study of initial–boundary-value problems for the linear equation

$$v_t + v_{xxx} + av_x = f. \tag{1.8}$$

A similar method was previously used in [16] for the problem in Π_T^+. In that paper solutions of the corresponding linear problem were constructed with the use of a special solution J_a of a "boundary potential" type to the homogeneous equation (1.8). Note that such potentials were introduced for the first time in the paper [8] in the case $a = 0$, where it was shown that a function

$$J(t,x;\mu) \equiv \int_{-\infty}^{t} \frac{3}{t-\tau} A''\left(\frac{x}{(t-\tau)^{1/3}}\right) \mu(\tau)\, d\tau, \tag{1.9}$$

where

$$A(\theta) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(\xi^3 + \theta\xi)}\, d\xi \equiv \mathcal{F}^{-1}[e^{i\xi^3}](\theta) \tag{1.10}$$

is the well-known Airy function, under certain assumptions on the function μ satisfied the equation (1.6) for $x > 0$ and, in addition, $J|_{x=0+0} = \mu$. Later on, properties of the potentials J_a were considered also in [13], [12], [5] and [16]. In the present paper analogous boundary potentials Q_a and R_a for the problem in $\Pi_T^-$are constructed and studied. The construction of these boundary potentials is based on the Laplace transform as in [8] and [5].

The potentials J_a, Q_a and R_a are also used in the present paper for the study of an initial–boundary-value problem in Q_T for the equation (1.8). As in [20] construction of its solutions is based on the idea of conversion of a certain operator.

Global a priori estimates of the problems (1.1)–(1.3) and (1.1), (1.2), (1.4) (just as in all preceding papers) are based on conservation laws for the KdV equation. The use of properties of the boundary potentials gives us an opportunity to establish these estimates under natural or ε-close-to-natural assumptions on boundary data. Thus, boundary potentials turned out to be useful both for local and global theory. The global a priori estimates (and, consequently, global well-posedness) are established first in functional spaces of integer orders, then for small positive values of s and after that nonlinear interpolation is applied. The use of nonlinear interpolation results in the case of initial–boundary-value problems for the KdV equation for the first time was introduced in [5].

Consider first a conservation law in L_2. Let $u(t,x)$ be a sufficiently smooth solution of one of the considered problems (also tending to zero with its derivatives as $x \to -\infty$ for the problem in a half-strip). Let a symbol I denote either $\mathbb{R}_-$ or $(0,1)$ and a symbol ∂I denote either the point $x=0$ for $\mathbb{R}_-$ or the endpoints of the interval $(0,1)$. Multiplying (1.1) by $2u$ and integrating over I one easily obtains an equality

$$\frac{d}{dt}\int_I u^2\,dx + \Bigl(2uu_{xx} - u_x^2 + au^2 + \frac{2}{3}u^3\Bigr)\Big|_{\partial I} = 2\int_I fu\,dx. \tag{1.11}$$

Obviously, in the case $u|_{\partial I} \equiv 0$ an estimate of the solution u in $L_2(I)$ uniform with respect to $t \in [0,T]$ for any $T>0$ follows from (1.11). Otherwise, the presence of the term $uu_{xx}|_{\partial I}$ obstructs derivation of such an estimate directly from (1.11). Then it is quite natural to introduce an auxiliary function ϕ such that $\phi|_{\partial I} = u|_{\partial I}$, and define a new function $U(t,x) \equiv u(t,x) - \phi(t,x)$. The function U satisfies an equation

$$U_t + U_{xxx} + aU_x + UU_x + (\phi U)_x + \phi\phi_x = F \equiv f - \phi_t - \phi_{xxx} - a\phi_x, \tag{1.12}$$

so multiplying (1.12) by $2U$ and integrating over I, we find that

$$\frac{d}{dt}\int_I U^2\,dx - U_x^2\Big|_{\partial I} + \int_I \phi_x(U^2 + 2U\phi)\,dx = 2\int_I FU\,dx. \tag{1.13}$$

This approach implies that the function ϕ can be chosen such that its properties ensure a possibility of derivation of a relevant estimate of a solution in L_2 from (1.13). In particular, it seems natural that ϕ must satisfy the following condition:

$$\phi_x \in L_1(0,T;L_\infty(I)). \tag{1.14}$$

In order to provide this property in the present paper for both problems in Π_T^- and Q_T the functions ϕ are constructed on the base of the boundary potential J (see (1.9)). Just the necessity of realization of (1.14) caused the ε-worsening of assumptions on boundary data (in comparison with natural ones) for $s=0$.

As in [16] solutions of the considered problems are constructed in special spaces X_s of Bourgain type, introduced first in [7] for classes of global well-posedness for the initial-value problem for the KdV equation. In the paper [9] these spaces were modified for similar results for the initial–boundary-value problem in Π_T^+.

The paper is organized as follows. Section 2 contains the main notation, definitions and certain materials on functional spaces. The main results of the paper on global well-posedness of the considered problems are formulated also in this section. In Section 3 special solutions of potential type for the linear equation (1.8) are introduced and studied. Section 4 is devoted to initial–boundary-value problems in Π_T^- and Q_T for linearized KdV equations. The proof of main results, in particular, local well-posedness and global a priori estimates for the nonlinear problems, is contained in Section 5.

2. Notation. Statement of main results.

In what follows (if there are no other conditions) in introduced notation we use a symbol I for an arbitrary interval (bounded or unbounded) on the real axis, $k\geq 0$ for an integer, and $p\in[1,+\infty]$.

Let $\eta(\theta)$, $\theta\in\mathbb{R}$, be a certain "cut-off" function, namely, $\eta\in C^\infty(\mathbb{R})$, $\eta\geq 0$, $\eta'\geq 0$, $\eta(\theta)=0$ for $\theta\leq 0$, $\eta(\theta)=1$ for $\theta\geq 1$, $\eta(\theta)+\eta(1-\theta)\equiv 1$, $\eta'(\theta)>0$ for $\theta\in(0,1)$.

Define $\psi(\theta)\equiv\eta(2-|\theta|)$ and $\psi_\delta(\theta)\equiv\psi(\theta/\delta)$ for $\delta>0$.

Let $\chi_I(\theta)$ be a characteristic function of the interval I. Define $\chi(\theta)\equiv\chi_{(-1,1)}(\theta)$.

Let $\widehat{f}(\xi)\equiv\mathcal{F}[f](\xi)$ and $\mathcal{F}^{-1}[f](\xi)$ be respectively the direct and inverse Fourier transforms of a function f, considered as operations in $\mathcal{S}'(\mathbb{R}^n)$. In particular, for $f\in\mathcal{S}(\mathbb{R})$

$$\widehat{f}(\xi)=\int_{\mathbb{R}} e^{-i\xi x}f(x)\,dx, \qquad \mathcal{F}^{-1}[f](x)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}f(\xi)\,d\xi.$$

Let $C_b^k(\overline{I})$ be a space of functions with all derivatives up to the order k continuous and bounded in $\overline{I}$. Define $C_b(\overline{I}) = C_b^0(\overline{I})$. If I is a bounded interval, the index b is omitted. Similar notation is used for functions, defined on domains $\Omega \subset \mathbb{R}^2$.

Define by $W_p^k(I)$ the Sobolev space

$$W_p^k(I) = \Big\{ f : f^{(j)} \in L_p(I), j = 0, \dots, k \Big\}.$$

For $s \in \mathbb{R}$ define the fractional-order Sobolev space on the whole real axis

$$H^s = H^s(\mathbb{R}) = \big\{ f : \mathcal{F}^{-1}[(1 + |\xi|^s)\widehat{f}(\xi)] \in L_2(\mathbb{R}) \big\}$$

and let $H^s(I)$ be a space of restrictions on I of functions from H^s. Evidently, $H^s(I) \subset C_b^k(\overline{I})$ if $s > k + 1/2$.

Next, define

$$H_0^s(I) = \big\{ f \in H^s : \operatorname{supp} f \subset \overline{I} \big\}.$$

The spaces $H^s(I)$ and $H_0^s(I)$ form the real interpolation scales $(\cdot,\cdot)_{\theta,2}$. The space $C_0^\infty(I)$ is dense in $H_0^s(I)$. If ∂I is a finite part of the boundary of the interval I, then for $s \in (k + 1/2, k + 3/2)$

$$H_0^s(I) = \big\{ f \in H^s(I) : f^{(j)}\big|_{\partial I} = 0, j = 0, \dots, k \big\}.$$

Note that $H_0^s(I) = H^s(I)$ for $s \in [0, 1/2)$. The theory of these spaces can be found, for example, in [25] and [26].

For noninteger $s > 0$ the symbol $D^s f(x) = \mathcal{F}^{-1}[|\xi|^s \widehat{f}(\xi)](x)$ denotes the Riesz potential.

Further, if $I = \mathbb{R}$ the symbol $\mathbb{R}$ in the notation for functional spaces is omitted: $L_p = L_p(\mathbb{R})$, $H^s = H^s(\mathbb{R})$ and so on; if $I = \mathbb{R}_+$ or $I = \mathbb{R}_-$ the lower indices "+" and "−," respectively, are used, namely, $L_{p,+} = L_p(\mathbb{R}_+)$, $L_{p,-} = L_p(\mathbb{R}_-)$, $H_+^s = H^s(\mathbb{R}_+)$, $H_-^s = H^s(\mathbb{R}_-)$, $C_{b,+} = C_b(\mathbb{R}_+)$, $C_{0,+}^\infty = C_0^\infty(\mathbb{R}_+)$ and so on.

Define for $s_1, s_2 \geq 0$

$$H^{s_1,s_2} = H^{s_1,s_2}(\mathbb{R}^2) = \big\{ f(t,x) : \mathcal{F}^{-1}[(1 + |\lambda|^{s_1} + |\xi|^{s_2})\widehat{f}(\lambda,\xi)] \in L_2(\mathbb{R}^2) \big\}.$$

Let $H^{s_1,s_2}(\Omega)$ be a space of restrictions of functions from H^{s_1,s_2} on a domain $\Omega \subset \mathbb{R}^2$.

If X is a certain Banach (or full countable-normed) space, define by $C_b(\overline{I}; X)$ a space of continuous bounded mappings from $\overline{I}$ to X. Let

$$C_b^k(\overline{I}; X) = \big\{ f(t) : D_t^j f \in C_b(\overline{I}; X), 0 \leq j \leq k \big\}.$$

If I is a bounded interval, the index b is omitted. Define by $C_0^\infty(I; X)$ the space of functions from $C_b^\infty(\overline{I}; X)$ with compact support in I.

The symbol $L_p(I;X)$ is used in the usual sense for the space of Bochner-measurable mappings from I to X, summable with order p (essentially bounded if $p=+\infty$).

Now we as in [16] introduce special spaces of Bourgain type.

Definition 2.1. *For $s\geq 0$, $a\in\mathbb{R}$, $b\in(0,1/2)$, $\alpha\in(1/2,2/3)$ define*

$$X_{s,a,b,\alpha}=\Big\{f(t,x)\in\mathcal{S}'(\mathbb{R}^2):$$
$$\|f\|_{X_{s,a,b,\alpha}}=\Big(\iint_{\mathbb{R}^2}(1+|\lambda|^{1/3}+|\xi|)^{2s}\beta^2(\lambda,\xi)|\widehat{f}(\lambda,\xi)|^2\,d\xi\,d\lambda\Big)^{1/2}<\infty\Big\},$$

where

$$\beta(\lambda,\xi)=\beta_{a,b,\alpha}(\lambda,\xi)\equiv(1+|\lambda-\xi^3+a\xi|)^b+\chi(\xi)(1+|\lambda|)^\alpha,$$

$$Y_{s,a,b,\alpha}=\Big\{f(t,x)\in\mathcal{S}'(\mathbb{R}^2):$$
$$\|f\|_{Y_{s,a,b,\alpha}}=\Big(\iint_{\mathbb{R}^2}(1+|\lambda|^{1/3}+|\xi|)^{2s}\gamma^2(\lambda,\xi)|\widehat{f}(\lambda,\xi)|^2\,d\xi\,d\lambda\Big)^{1/2}<\infty\Big\},$$

where

$$\gamma(\lambda,\xi)=\gamma_{a,b,\alpha}(\lambda,\xi)\equiv\frac{1}{(1+|\lambda-\xi^3+a\xi|)^b}+\frac{\chi(\xi)}{(1+|\lambda|)^{1-\alpha}}.$$

If Ω is a domain in $\mathbb{R}^2$, then define by $X_{s,a,b,\alpha}(\Omega)$ and $Y_{s,a,b,\alpha}(\Omega)$ restrictions of $X_{s,a,b,\alpha}$ and $Y_{s,a,b,\alpha}$ on Ω, respectively, with natural restriction norms.

For further simplicity we often omit the indices a, b, and α and use the notation X_s, Y_s, $X_s(\Omega)$, and $Y_s(\Omega)$. Obviously, with respect to the parameter s these spaces form the real interpolation scales $(\cdot,\cdot)_{\theta,2}$. The space $\mathcal{S}(\mathbb{R}^2)$ is dense in X_s and Y_s. Evidently, $H^{s/3,s}\subset Y_s$.

Remark 2.1. In the well-posedness theory for the initial-value problem for the KdV equation, smoothness properties in Bourgain-type spaces (starting from the pioneer work [7]) were governed by Sobolev spatial weight $(1+|\xi|)^s$. In the first application of such spaces to the initial–boundary-value problem in the right half-strip in the paper [9] the authors preserved this weight, but established well-posedness only for $s<1/2$. In the paper [20] special efforts were needed to extend their result to the range $s<3/2$ in the same spaces. The use of joint spatial and temporal weight $(1+|\lambda|^{1/3}+|\xi|)^s$ gives us an opportunity to consider any non-negative value of s without additional troubles.

Next we state certain properties of these spaces. Their proofs can be found in [9] and [16].

1) if $T > 0$ and $\delta \in (0, T]$, then

$$\|\psi_\delta(t) f(t,x)\|_{X_{s,a,b,\alpha}} \le c(s,a,b,\alpha,T)\delta^{1/2-\alpha-s/3}\|f(t,x)\|_{X_{s,a,b,\alpha}}; \tag{2.1}$$

2) if $b \in [7/16, 1/2)$, $\alpha \in (1/2, 2/3)$, then

$$\big\|(uv)_x\big\|_{Y_{s,a,b,\alpha}} \le c(s,a,b,\alpha)\|u\|_{X_{s,a,b,\alpha}}\|v\|_{X_{s,a,b,\alpha}}; \tag{2.2}$$

3) if $f \in Y_s$, then $D_t^{m-1} f \in C_b(\mathbb{R}^t; H^{s-3m})$ for natural $m \le s/3$.

In this paper we also need certain refinement of the estimate (2.2) for small values of s. In the case $s = 0$ it was established in [16]. Note that with the help of the following inequality (2.3) we can also improve the result on global well-posedness for the problem in the right half-strip from [16] (see Remark 5.4 below).

Lemma 2.1. *If $b \in [7/16, 1/2)$, $T > 0$, then there exists $\alpha_0(b) \in (1/2, 2/3)$ such that for every $s \in [0, 3\alpha_0(b) - 3/2)$ and $\alpha \in (1/2, \alpha_0(b) - s/3)$ there exists $\varepsilon(b, s, \alpha) > 0$ such that for all functions $u, v \in X_{s,a,b,\alpha}$ and all $\delta \in (0, T]$*

$$\begin{aligned}&\big\|\psi_\delta^2(t)(uv)_x\big\|_{Y_{s,a,b,\alpha}} \\ &\le c(s,a,b,\alpha,T)\delta^{\varepsilon}\left(\|u\|_{X_{s,a,b,\alpha}}\|v\|_{X_{0,a,b,\alpha}} + \|u\|_{X_{0,a,b,\alpha}}\|v\|_{X_{s,a,b,\alpha}}\right).\end{aligned} \tag{2.3}$$

Proof. The scheme of the proof repeats that of Lemma 2.2 in [16]. Therefore, we present here only those details that are different from those in [16]. According to (2.1), it is sufficient to prove the inequality

$$\big\|(uv)_x\big\|_{Y_{s,a,b,\alpha}} \le c\delta^{\theta}\left(\|u\|_{X_{s,a,b,\alpha}}\|v\|_{X_{0,a,b,\alpha}} + \|u\|_{X_{0,a,b,\alpha}}\|v\|_{X_{s,a,b,\alpha}}\right) \tag{2.4}$$

for all functions $u, v \in X_{s,a,b,\alpha}$ such that $u = v \equiv 0$ when $|t| > \delta$, and for some θ such that $1 - 2(\alpha + s/3) + \theta > 0$. Arguing as in the proof of the inequality (2.2) in [9] one can obtain that the estimate (2.4) can be derived from the following inequality: for an arbitrary non-negative function $w \in L_2(\mathbb{R}^2)$

$$\begin{aligned}\iiiint\limits_{\substack{\xi=\xi_1+\xi_2,\\ \lambda=\lambda_1+\lambda_2}} |\xi| w(\lambda,\xi)\gamma(\lambda,\xi)\frac{U_s(\lambda_1,\xi_1)}{\beta(\lambda_1,\xi_1)}\cdot\frac{V_0(\lambda_2,\xi_2)}{\beta(\lambda_2,\xi_2)}\,d\lambda_1\,d\xi_1\,d\lambda_2\,d\xi_2 \\ \le c\delta^{\theta}\|w\|_{L_2(\mathbb{R}^2)}\|U_s\|_{L_2(\mathbb{R}^2)}\|V_0\|_{L_2(\mathbb{R}^2)},\end{aligned} \tag{2.5}$$

where

$$U_s(\lambda,\xi) \equiv (1 + |\lambda|^{1/3} + |\xi|)^s \beta(\lambda,\xi)|\widehat{u}(\lambda,\xi)|, \qquad V_0(\lambda,\xi) \equiv \beta(\lambda,\xi)|\widehat{v}(\lambda,\xi)|.$$

Let for $\rho > 0$

$$F_\rho(t,x) \equiv \mathcal{F}^{-1}\left[\frac{U_s(\lambda,\xi)}{(1+|\lambda-\xi^3+a\xi|)^\rho}\right](t,x)$$

and the same definition holds for G_ρ, where U_s is replaced by V_0. It is shown in [9], that the left part of (2.5) does not exceed

$$c\|w\|_{L_2(\mathbb{R}^2)}\Big[\|U_s\|_{L_2(\mathbb{R}^2)}\big(\|G_b\|_{L_4(\mathbb{R}^2)} + \|D_x^{1/2}G_b\|_{L_4(\mathbb{R}^x;L_2^t)}\big) + \|V_0\|_{L_2(\mathbb{R}^2)}\big(\|F_b\|_{L_4(\mathbb{R}^2)} + \|D_x^{1/2}F_b\|_{L_4(\mathbb{R}^x;L_2^t)}\big)\Big].$$

Therefore, it is sufficient to prove an inequality

$$\|F_b\|_{L_4(\mathbb{R}^2)} + \|D_x^{1/2}F_b\|_{L_4(\mathbb{R}^x;L_2^t)} \le c\delta^\theta\|U_s\|_{L_2(\mathbb{R}^2)} \tag{2.6}$$

(the analogous one for G_b was established in [16]). Choose $p \in \big(1,(1-(b-1/3)/(\alpha+s/3))^{-1}\big)$ and $b_1 \in (1/3,b)$ such that $\alpha+s/3 \le p(b-b_1)/(p-1)$. As in the proof of Lemma 2.2 in [16] we can write down that

$$\|F_b\|_{L_4(\mathbb{R}^2)} + \|D_x^{1/2}F_b\|_{L_4(\mathbb{R}^x;L_2^t)} \le c\big\|U_s(\lambda,\xi)(1+|\lambda-\xi^3+a\xi|)^{b_1-b}\big\|_{L_2(\mathbb{R}^2)}. \tag{2.7}$$

Note that

$$(1+|\lambda|^{1/3}+|\xi|)^s\beta(\lambda,\xi) \le c(1+|\xi|)^s(1+|\lambda-\xi^3+a\xi|)^{\alpha+s/3}.$$

Therefore, continuing the inequality (2.7), we obtain that

$$\|F_b\|_{L_4(\mathbb{R}^2)} + \|D_x^{1/2}F_b\|_{L_4(\mathbb{R}^x;L_2^t)} \le c\big\|U_s\big\|^{1/p}_{L_2(\mathbb{R}^2)}\big\|u\big\|^{(p-1)/p}_{L_2(\mathbb{R}^t;H^s)},$$

where similarly to the proof of Lemma 2.2 in [16] one can show that

$$\big\|u\big\|_{L_2(\mathbb{R}^t;H^s)} \le c\delta^b\|U_s\|_{L_2(\mathbb{R}^2)}$$

and thus derive (2.6) for $\theta = b(1-1/p)$. Let as in [16] $\alpha_0 = \alpha_0(b) \in (1/2,2/3)$ be the greater root of the quadratic equation $2\alpha^2-\alpha-b(b-1/3)=0$. Then if $\alpha+s/3 \in (1/2,\alpha_0)$, choosing p sufficiently close to $(1-(b-1/3)/(\alpha+s/3))^{-1}$ one can obtain that $1-2(\alpha+s/3)+\theta > 0$. □

Now we turn to the considered problems and give definitions of generalized solutions. Let $u_0 \in L_{2,-}$, $u_2,u_3 \in L_1(0,T)$, $f \in L_1(0,T;L_{2,-})$.

Definition 2.2. *A function $u(t,x) \in L_2(\Pi_T^-)$ is called a generalized solution of the problem (1.1)–(1.3) in Π_T^-, if for any function $\phi(t,x)$ such that $\phi \in L_\infty(0,T;H_-^3)$, $\phi_t \in L_\infty(0,T;L_{2,-})$, and $\phi\big|_{t=T}=0$, $\phi\big|_{x=0}=0$, the following equality is valid:*

$$\iint_{\Pi_T^-} \left[u(\phi_t + \phi_{xxx} + a\phi_x) + \frac{1}{2}u^2\phi_x + f\phi \right] dx\,dt \\ + \int_{\mathbb{R}_-} u_0\phi(0,x)\,dx + \int_0^T [u_3\phi_x(t,0) - u_2\phi_{xx}(t,0)]\,dt = 0. \quad (2.8)$$

Let $u_0 \in L_2(0,1)$, $u_1, u_2, u_3 \in L_1(0,T)$, $f \in L_1(0,T; L_2(0,1))$.

Definition 2.3. *A function $u(t,x) \in L_2(Q_T)$ is called a generalized solution of the problem* (1.1), (1.2), (1.4) *in Q_T, if for any function $\phi(t,x)$ such that $\phi \in L_\infty(0,T; H^3(0,1))$, $\phi_t \in L_\infty(0,T; L_2(0,1))$, and $\phi\big|_{t=T} = 0$, $\phi\big|_{x=0} = \phi_x\big|_{x=0} = \phi\big|_{x=1} = 0$, the following equality is valid:*

$$\iint_{Q_T} \left[u(\phi_t + \phi_{xxx} + a\phi_x) + \frac{1}{2}u^2\phi_x + f\phi \right] dx\,dt \quad (2.9)$$
$$+ \int_0^1 u_0\phi(0,x)\,dx + \int_0^T [u_1\phi_{xx}(t,0) - u_2\phi_{xx}(t,1) + u_3\phi_x(t,1)]\,dt = 0.$$

In fact, solutions of the considered problems are constructed in special functional spaces $Z_s(\Pi_T^-)$ and $Z_s(Q_T)$.

Definition 2.4. *For any $T > 0$ and $s \geq 0$, let $Z_s((0,T) \times I)$ be a space of functions $u(t,x)$ such that*

$$D_t^m u \in C([0,T]; H^{s-3m}(I)) \qquad \text{for any integer } m \in [0, s/3],$$
$$D_x^l u \in C_b(\overline{I}^x; H^{(s-l+1)/3}(0,T)) \qquad \text{for any integer } l \in [0, s+1],$$
$$u \in X_{s,a,b,\alpha}((0,T) \times I) \qquad \text{for some } b \in [7/16, 1/2) \text{ and } \alpha \in (1/2, 2/3).$$

In order to formulate compatibility conditions for the considered problems we now define certain special functions.

Definition 2.5. *Let $\Phi_0(x) \equiv u_0(x)$ and for any natural m*

$$\Phi_m(x) \equiv D_t^{m-1} f(0,x) - \Phi'''_{m-1}(x) - a\Phi'_{m-1}(x) \quad (2.10)$$
$$- \sum_{l=0}^{m-1} \binom{m-1}{l} \Phi_l(x)\Phi'_{m-l-1}(x).$$

Now we can state the main results of the paper.

Theorem 2.1. *Let $u_0 \in H_-^s$, $u_2 \in H^{(s+1)/3+\varepsilon}(0,T)$, $u_3 \in H^{s/3+\varepsilon}(0,T)$, and $f \in H^{s/3,s}(\Pi_T^-)$ for an arbitrary $T > 0$ and $s \geq 0$ be such that the numbers $(s/3 - 1/6)$ and $(s/3 - 1/2)$ are noninteger, where $\varepsilon > 0$ is arbitrarily small in the case $s = 0$ and $\varepsilon = 0$ in the case $s > 0$. Assume also that $u_2^{(m)}(0) =$*

$\Phi_m(0)$ for any integer $m \in [0, s/3 - 1/6)$, $u_3^{(m)}(0) = \Phi'_m(0)$ for any integer $m \in [0, s/3 - 1/2)$. Then there exists a unique generalized solution $u(t,x)$ of the problem (1.1)–(1.3) in the space $Z_s(\Pi_T^-)$. The mapping $(u_0, u_2, u_3, f) \mapsto u$ is Lipschitz continuous on any ball in the norm of the mapping $H_-^s \times H^{(s+1)/3+\varepsilon}(0,T) \times H^{s/3+\varepsilon}(0,T) \times H^{s/3,s}(\Pi_T^-) \to Z_s(\Pi_T^-)$.

Theorem 2.2. *Let $u_0 \in H^s(0,1)$, $u_1 \in H^{(s+1)/3+\varepsilon}(0,T)$, $u_2 \in H^{(s+1)/3+\varepsilon}(0,T)$, $u_3 \in H^{s/3+\varepsilon}(0,T)$, and $f \in H^{s/3,s}(Q_T)$ for an arbitrary $T > 0$ and $s \geq 0$ such that the numbers $(s/3 - 1/6)$ and $(s/3 - 1/2)$ are noninteger, where $\varepsilon > 0$ is arbitrarily small in the case $s = 0$ and $\varepsilon = 0$ in the case $s > 0$. Assume also that $u_1^{(m)}(0) = \Phi_m(0)$, and $u_2^{(m)}(0) = \Phi_m(1)$ for any integer $m \in [0, s/3 - 1/6)$, $u_3^{(m)}(0) = \Phi'_m(1)$ for any integer $m \in [0, s/3 - 1/2)$. Then there exists a unique generalized solution $u(t,x)$ of the problem (1.1), (1.2), (1.4) in the space $Z_s(Q_T)$. The mapping $(u_0, u_1, u_2, u_3, f) \mapsto u$ is Lipschitz continuous on any ball in the norm of the mapping $H^s(0,1) \times \big(H^{(s+1)/3+\varepsilon}(0,T)\big)^2 \times H^{s/3+\varepsilon}(0,T) \times H^{s/3,s}(Q_T) \to Z_s(Q_T)$.*

3. Potentials for a linearized KdV equation

First of all, consider potentials related to the initial-value problem for the equation (1.8). For $v_0 \in \mathcal{S}'$ and $f \in L_1^{loc}(\mathbb{R}^t; \mathcal{S}')$ define

$$S_a(t,x;v_0) \equiv \mathcal{F}_x^{-1}\Big[e^{it(\xi^3 - a\xi)}\widehat{v}_0(\xi)\Big](x), \tag{3.1}$$

$$K_a(t,x;f) \equiv \int_0^t S_a(t-\tau, x; f(\tau,\cdot))\,d\tau. \tag{3.2}$$

The following properties of these potentials succeed from the results of the papers [16], [9] and [22]:

1) for any $T > 0$ and $s \geq 0$

$$\|S_a(\cdot,\cdot;v_0)\|_{Z_s(\Pi_T)} + \|K_a(\cdot,\cdot;f)\|_{Z_s(\Pi_T)} \leq c(s,a,b,\alpha,T)(\|v_0\|_{H^s} + \|f\|_{Y_{s,a,b,\alpha}}), \tag{3.3}$$

where $\Pi_T = (0,T) \times \mathbb{R}$,

2) for any $T > 0$ and $t_0 \in (0,T]$

$$\|K_a(\cdot,\cdot;f)\|_{C([0,t_0];L_2)} + \|K_{a\,x}(\cdot,\cdot;f)\|_{C_b(\mathbb{R}^x;L_2(0,t_0))} \leq c(a,T)\|f\|_{L_1(0,t_0;L_2)}, \tag{3.4}$$

3) for any $T > 0$, $t_0 \in (0,T]$ and $s \in [-1,2]$

$$\|K_a(\cdot,\cdot;f)\|_{C_b(\mathbb{R}^x;H^{(s+1)/3}(0,t_0))} \leq c(a,T)t_0^{1/3-s/6}\|f\|_{L_2(0,t_0;H^s)}, \tag{3.5}$$

4) for any $T > 0$ and $t_0 \in (0,T]$

$$\|S_a(\cdot,\cdot;v_0)\|_{L_2(\mathbb{R}^x;C[0,t_0])} + \|K_a(\cdot,\cdot;f)\|_{L_2(\mathbb{R}^x;C[0,t_0])} \\ \leq c(a,T)\left(\|v_0\|_{H^1} + \|f\|_{L_1(0,t_0;H^1)}\right). \quad (3.6)$$

Further, we always assume that if the functions v_0 and f are originally defined in more restricted domains, for construction of corresponding potentials S_a and K_a they are extended to the whole real axis and a certain strip, respectively, with preservation of their properties.

Now in order to define boundary potentials we first introduce certain additional notation. Let

$$\varphi_a(\xi) \equiv \xi^3 - a\xi.$$

Properties of this function are obvious. For $a \leq 0$ it increases monotonically on the whole real line $\mathbb{R}$, while for $a > 0$, on the rays $(-\infty, -\sqrt{a/3})$ and $(\sqrt{a/3}, +\infty)$ (and decreases monotonically on the interval $(-\sqrt{a/3}, \sqrt{a/3})$), where $\varphi_a(\pm\sqrt{a/3}) = \mp 2(a/3)^{3/2}$, $\varphi_a(\pm 2\sqrt{a/3}) = \pm 2(a/3)^{3/2}$. Let $\kappa_a(\lambda)$ when $a \leq 0$ be an inverse function to $\varphi_a(\xi)$ (it is defined for all λ) and when $a > 0$ be an inverse function to $\varphi_a(\xi)$ only for $|\xi| \geq 2\sqrt{a/3}$ (it is defined for $|\lambda| \geq 2(a/3)^{3/2}$).

Consider an algebraic equation

$$z^3 + az + \varepsilon + i\lambda = 0. \quad (3.7)$$

If $\varepsilon > 0$, then this equation has one root z_0 such that $\Re z_0 < 0$, and two roots z_1 and z_2 such that $\Re z_1 > 0$ and $\Re z_2 > 0$. With the use of the Cardano formula it is easy to show that there exist continuous (with respect to λ) limits of these roots

$$r_j(\lambda) = r_j(\lambda; a) = \lim_{\varepsilon \to +0} z_j(\varepsilon + i\lambda), \qquad j = 0, 1, 2.$$

The values $r_j(\lambda)$ are roots of the equation

$$r^3 + ar + i\lambda = 0$$

and have the following structure. It is the most simple, of course, if $a = 0$, namely

$$r_0(\lambda) = -\frac{\sqrt{3}}{2}|\lambda|^{1/3} - \frac{i}{2}\lambda^{1/3}, \;\; r_1(\lambda) = \frac{\sqrt{3}}{2}|\lambda|^{1/3} - \frac{i}{2}\lambda^{1/3}, \;\; r_2(\lambda) = i\lambda^{1/3} \quad (3.8)$$

(for r_1 and r_2 here and further enumeration can be changed if necessary). If $a < 0$, then $r_j(-\lambda) = \overline{r_j(\lambda)}$, $j = 0, 1, 2$, and

$$r_0(\lambda) = -p(\lambda) + iq(\lambda), \quad r_1(\lambda) = p(\lambda) + iq(\lambda), \quad r_2(\lambda) = i\kappa_a(\lambda), \quad (3.9)$$

where $p, q \in \mathbb{R}$, and for a certain constant $c > 0$

$$p(\lambda) \geq c(|\lambda|^{1/3} + |a|^{1/2}) \qquad \forall \lambda \in \mathbb{R}. \quad (3.10)$$

If $a > 0$, then $r_0(-\lambda) = \overline{r_0(\lambda)}$, $r_2(-\lambda) = \overline{r_1(\lambda)}$; for $\lambda \leq -2(a/3)^{3/2}$ they can be represented in the form (3.9), while for $\lambda \geq 2(a/3)^{3/2}$ the roots r_1 and r_2 in (3.9) are to be interchanged. Here the function p satisfies inequalities

$$p(\lambda) \geq c\left(|\kappa_a(\lambda)| - 2\sqrt{a/3}\right), \qquad |\lambda| \geq 2(a/3)^{3/2}, \tag{3.11}$$

$$p(\lambda) \geq c\max(|\lambda|^{1/3}, |q(\lambda)|), \qquad |\lambda| \geq 18(a/3)^{3/2}. \tag{3.12}$$

For $|\lambda| < 2(a/3)^{3/2}$ (if $a > 0$, of course)

$$r_j(\lambda) = i\xi_j(\lambda), \qquad j = 0, 1, 2, \tag{3.13}$$

where $\xi_2 < \xi_0 < \xi_1$ are roots of the equation $\xi^3 - a\xi = \lambda$. It can be also easily shown, that for all $a \in \mathbb{R}$ the roots $r_j(\lambda)$, $j = 0, 1, 2$, satisfy inequalities

$$|r_j(\lambda)| \leq c(|\lambda|^{1/3} + |a|^{1/2}) \qquad \forall \lambda \in \mathbb{R}, \tag{3.14}$$

$$|r_1(\lambda) - r_2(\lambda)| \geq c(|\lambda|^{1/3} + |a|^{1/2}) \qquad \forall \lambda \in \mathbb{R}. \tag{3.15}$$

The root $r_0(\lambda)$ was used in the paper [16] for construction of a boundary potential J_a for the linear problem in Π_T^+, namely, for $x \geq 0$ and a certain function $\mu(t)$

$$J_a(t, x; \mu) \equiv \mathcal{F}_t^{-1}\left[e^{r_0(\lambda)x}\widehat{\mu}(\lambda)\right](t). \tag{3.16}$$

The aforementioned properties of the root $r_0(\lambda)$ provide that if, for example, $\mu \in L_2$, then the function J_a is infinitely differentiable for $t \in \mathbb{R}$, $x > 0$ and for any $x_0 > 0$ and non-negative integers m and l

$$\sup_{x \geq x_0} |D_t^m D_x^l J_a(t, x; \mu)| \leq c(a, m, l, x_0)\|\mu\|_{L_2}. \tag{3.17}$$

It was proved in [16] that if $\mu \in L_2$, $\mu(t) = 0$ for $t < 0$, then the function J_a is a generalized solution in Π_∞^+ of a linear problem

$$v_t + v_{xxx} + av_x = 0, \qquad v\big|_{t=0} = 0, \qquad v\big|_{x=0} = \mu. \tag{3.18}$$

Moreover, the following estimates were established in [16]:
1) for any $T > 0$, $s \geq 0$

$$\|J_a(\cdot,\cdot; v_0)\|_{Z_s(\Pi_T^+)} \leq c(s, a, b, \alpha, T)\|\mu\|_{H^{(s+1)/3}}, \tag{3.19}$$

2) for any $T > 0$ and integer $l \geq 0$

$$\|D_x^l J_a(\cdot,\cdot; \mu)\|_{L_2(\mathbb{R}_+^x; C[0,T])} \leq c(a, T, l)\|\mu\|_{H^{(l+1)/3}} \tag{3.20}$$

(in [16] it was always assumed that $\mu(t) = 0$ for $t < 0$, but for the proof of (3.19) and (3.20) it was not used). By direct computation it can be shown that for $\mu \in L_2$

$$J_0(t, x; \mu) \equiv J(t, x; \mu), \tag{3.21}$$

where the function J is defined by the formula (1.9) (see, for example, [11]).

Now we establish certain additional properties of the potential J_a.

Lemma 3.1. *Let $\mu \in H^{(2s-1)/6}$ for some $s \geq 0$. Then for any $T > 0$*

$$\|J_a(\cdot,\cdot;\mu)\|_{L_2(0,T;H^s_+)} \leq c(a,T,s)\|\mu\|_{H^{(2s-1)/6}}. \tag{3.22}$$

Proof. Because of interpolation results it is enough to consider the case $s = n \geq 0$, an integer. Let

$$a_0 = \max(1, 18(|a|/3)^{3/2}), \tag{3.23}$$

$$\mu_1(t) \equiv \mathcal{F}_t^{-1}\left[\chi_{(-a_0,a_0)}(\lambda)\widehat{\mu}(\lambda)\right](t), \qquad \mu_2(t) \equiv \mu(t) - \mu_1(t). \tag{3.24}$$

Then by virtue of (3.19)

$$\|J_a(\cdot,\cdot;\mu_1)\|_{L_2(0,T;H^n_+)} \leq c(a,T,n)\|\mu_1\|_{H^{(n+1)/3}} \leq c_1(a,T,n)\|\mu\|_{H^{-1/6}}. \tag{3.25}$$

Next we use an extension of $J_a(t,x;\mu_2)$ to the whole plane $\mathbb{R}^2$, constructed in the proof of Lemma 3.9 in [16]:

$$J_a(t,x;\mu_2) = \tag{3.26}$$

$$\mathcal{F}_{t,x}^{-1}\Big[\frac{c(n)p^{n+1}(\lambda)\widehat{\mu}_2(\lambda)}{(p^2(\lambda)+(\xi-q(\lambda))^2)(p(\lambda)+2i(\xi-q(\lambda)))\cdots(p(\lambda)+2^n i(\xi-q(\lambda)))}\Big](t,x).$$

The inequalities (3.10), (3.12) and (3.14) provide from (3.26) that

$$|\widehat{J}_a(\lambda,\xi;\mu_2)| \leq c(a,n)\frac{|\lambda|^{(n+1)/3}|\widehat{\mu}_2(\lambda)|}{(|\lambda|^{1/3}+|\xi|)^{n+2}}, \tag{3.27}$$

and so

$$\|J_a(\cdot,\cdot;\mu_2)\|^2_{H^{0,n}} \leq c(a,n)\|\mu_2\|^2_{H^{-1/3}} +$$

$$c(a,n)\int_{\mathbb{R}}|\lambda|^{2(n+1)/3}|\widehat{\mu}_2(\lambda)|^2\int_{|\xi|\geq 1}(|\lambda|^{1/3}+|\xi|)^{-4}\,d\xi\,d\lambda \leq c_1(a,n)\|\mu\|^2_{H^{(2n-1)/6}}.$$

□

The well-known embedding theorem provides the following corollary.

Corollary 3.1. *Let $\mu \in H^{l/3+\varepsilon}$ for certain $\varepsilon > 0$ and integer $l \geq 0$. Then for any $T > 0$*

$$\|D^l_x J_a(\cdot,\cdot;\mu)\|_{L_2(0,T;C_{b,+})} \leq c(a,T,l,\varepsilon)\|\mu\|_{H^{l/3+\varepsilon}}. \tag{3.28}$$

Remark 3.1. For $a = 0$, $s = 1$ the estimate (3.22) was established in [13]. In the case $a = 1$ this estimate was proved in [4] via a method different from the one in the present paper.

The estimate (3.28) in the case $l = 1$ ensures the property (1.14), if $J_a(t, x; u_1)$ is chosen as an auxiliary function ϕ. In the paper [4] it was observed for the first time, that (3.28) is a simple corollary of (3.22).

In [13] for $a = 0$ and in [16] in the general case a different approach was used to establish the property (1.14) for the potential J_a, namely, it was proved there that if $\mu(t) = 0$ for $t < 0$, then for any $T > 0$

$$\|J_{a\,x}(\cdot,\cdot;\mu)\|_{L_1(0,T;C_{b,+})} \le c(a,T)\|\mu\|_{W_1^{1/3}(0,T)},$$

where the symbol $W_p^s(I)$ for noninteger values of s denotes the Slobodetskii space. It is known that $W_2^s(I) = H^s(I)$ (see, for example, [26]), but unfortunately $W_2^s(I)$ is not embedded in $W_1^s(I)$ for noninteger s even in the case of a bounded interval I. Certainly, $W_2^{s+\varepsilon}(0,T) \subset W_1^s(0,T)$ for any $\varepsilon > 0$, and so in [16] the assumption $u_1 \in H^{1/3+\varepsilon}(0,T)$ was also used to provide the property (1.14) for J_a.

In what follows we also use a primitive of the boundary potential J_a in the case $a = 0$ for $x \ge 0$

$$J^*(t,x;\mu) \equiv \mathcal{F}_t^{-1}\big[e^{r_0(\lambda)x}\widehat{\mu}(\lambda)r_0^{-1}(\lambda)\big](t), \tag{3.29}$$

where $r_0(\lambda)$ is defined by the formula (3.8).

Lemma 3.2. *Let $\mu \in L_2$. Then $J^* \in C_b(\mathbb{R}^t; L_{2,+})$ and for any $t \in \mathbb{R}$*

$$\|J^*(t,\cdot;\mu)\|_{L_{2,+}} \le c\|\mu\|_{L_2}. \tag{3.30}$$

Proof. The proof is based on the following fundamental inequality from [5]: if a certain continuous function $\gamma(\xi)$ satisfies an inequality $\Re\gamma(\xi) \le -\varepsilon|\xi|$ for some $\varepsilon > 0$ and all $\xi \in \mathbb{R}$, then

$$\Big\|\int_{\mathbb{R}} e^{\gamma(\xi)x} f(\xi)\,d\xi\Big\|_{L^x_{2,+}} \le c(\varepsilon)\|f\|_{L_2}. \tag{3.31}$$

Changing variables $\xi = \lambda^{1/3}$, we find that

$$J^*(t,x;\mu) = \frac{3}{2\pi}\int_{\mathbb{R}} e^{i\xi^3 t}e^{r_0(\xi^3)x}\widehat{\mu}(\xi^3)r_0^{-1}(\xi^3)\xi^2\,d\xi,$$

and since $\Re r_0(\xi^3) = -\sqrt{3}|\xi|/2$

$$\|J^*\|_{L_{2,+}} \le c\|\xi\widehat{\mu}(\xi^3)\|_{L_2} \le c_1\|\mu\|_{L_2}.$$

$\square$

Lemma 3.3. *Let $\mu \in H^s$ for some $s \ge 0$ and $\mu(t) = 0$, if $|t| \ge T$ for some $T > 0$. Then*

$$\|J^*(\cdot,0;\mu)\|_{H^{s+1/3}} \le c(T)\|\mu\|_{H^s}. \tag{3.32}$$

Proof. According to (3.29) and (3.8)

$$\|J^*(\cdot,0;\mu)\|_{H^{s+1/3}} \le c\Big(\int_{|\lambda|>1}(1+|\lambda|^s)^2|\widehat{\mu}(\lambda)|^2\,d\lambda\Big)^{1/2} + c\sup_{|\lambda|<1}|\widehat{\mu}(\lambda)| \le c(T)\|\mu\|_{H^s}.$$

□

From (3.29) and (3.8), it also obviously follows that for natural m

$$\|D_t^m J^*(\cdot,0;\mu)\|_{L_2} \le c(m)\|\mu\|_{H^{m-1/3}}. \tag{3.33}$$

In the study of the problem in Π_T^- we also use for $x\le 0$ the following functions:

$$\widetilde{J}(t,x;\mu) \equiv J(-t,-x;\widetilde{\mu}) = \mathcal{F}_t^{-1}\Big[e^{-\overline{r_0(\lambda)}x}\widehat{\mu}(\lambda)\Big](t), \tag{3.34}$$

$$\widetilde{J}^*(t,x;\mu) \equiv -J^*(-t,-x;\widetilde{\mu}) = \mathcal{F}_t^{-1}\Big[-e^{-\overline{r_0(\lambda)}x}\widehat{\mu}(\lambda)(\overline{r_0(\lambda)})^{-1}\Big](t), \tag{3.35}$$

where $\widetilde{\mu}(t)\equiv\mu(-t)$.

It is obvious that the properties of the functions J and J^* are also valid for the functions $\widetilde{J}$ and $\widetilde{J}^*$ with the natural substitution of $\mathbb{R}_+$ by $\mathbb{R}_-$.

Now we turn to boundary potentials for the linear problem in Π_T^-.

Definition 3.1. *For $\mu,\nu\in\mathcal{S}'$ define for $x\le 0$*

$$Q_a(t,x;\mu) \equiv \mathcal{F}_t^{-1}\Big[\frac{r_1(\lambda)e^{r_2(\lambda)x}-r_2(\lambda)e^{r_1(\lambda)x}}{r_1(\lambda)-r_2(\lambda)}\widehat{\mu}(\lambda)\Big](t), \tag{3.36}$$

$$R_a(t,x;\nu) \equiv \mathcal{F}_t^{-1}\Big[\frac{e^{r_1(\lambda)x}-e^{r_2(\lambda)x}}{r_1(\lambda)-r_2(\lambda)}\widehat{\nu}(\lambda)\Big](t). \tag{3.37}$$

Lemma 3.4. *Let $\mu\in H^{(s+1)/3}$, $\nu\in H^{s/3}$ for some $s\ge 0$. Then $Q_a,R_a\in C_b(\mathbb{R}^t;H_-^s)$ and for every $t\in\mathbb{R}$*

$$\|Q_a(t,\cdot;\mu)\|_{H_-^s} \le c(a,s)\|\mu\|_{H^{(s+1)/3}}, \tag{3.38}$$

$$\|R_a(t,\cdot;\nu)\|_{H_-^s} \le c(a,s)\|\mu\|_{H^{s/3}}. \tag{3.39}$$

Proof. For integer values of s the estimates (3.38), (3.39) succeed from the formulae (3.36), (3.37), inequalities (3.14), (3.15) and the following auxiliary Lemma 3.5. For other values of s they can be obtained via interpolation. □

Lemma 3.5. *Let*

$$I_j(t,x) \equiv \int_{\mathbb{R}} e^{i\lambda t}e^{r_j(\lambda)x}w(\lambda)\,d\lambda, \tag{3.40}$$

where $j = 1$ or $j = 2$. Then there exists an absolute constant $c > 0$ such that for any $t \in \mathbb{R}$

$$\|I_j(t,\cdot)\|_{L_{2,-}} \le c\big\|(|\lambda|^{1/3} + |a|^{1/2})w(\lambda)\big\|_{L_2}. \tag{3.41}$$

Proof. As for Lemma 3.2 the proof is based on the inequality (3.31) with the natural substitution of positive x by negative ones.

Let at first $a \le 0$. Changing variables $\xi = \lambda^{1/3}$ we deduce from (3.31) (since by virtue of (3.8) and (3.10) $p(\xi^3) \ge c|\xi|$) that

$$\|I_1(t,\cdot)\|_{L_{2,-}} \le c\|e^{i\xi^3 t}\xi^2 w(\xi^3)\|_{L_2} \le c_1\|\lambda^{1/3}w(\lambda)\|_{L_2}. \tag{3.42}$$

Then changing variables $\xi = \kappa_a(\lambda)$ (here, of course, $\lambda \equiv \varphi_a(\xi)$) we find that

$$I_2(t,x) = S_a\big(t,x;\mathcal{F}^{-1}[w(\varphi_a(\xi))\varphi_a'(\xi)]\big) \tag{3.43}$$

and

$$\|I_2(t,\cdot)\|_{L_2} = \|w(\varphi_a(\xi))\varphi_a'(\xi)\|_{L_2} = \|w(\lambda)(3\kappa_a^2(\lambda) + |a|)^{1/2}\|_{L_2} \le c\|(|\lambda|^{1/3} + |a|^{1/2})w(\lambda)\|_{L_2}. \tag{3.44}$$

Next let $a > 0$. Divide the integral I_1 into three parts:

$$I_1(t,x) = \int_{\lambda<-2(\frac{a}{3})^{\frac{3}{2}}} e^{i\lambda t}e^{(p(\lambda)+iq(\lambda))x}w(\lambda)\,d\lambda + \int_{|\lambda|<2(\frac{a}{3})^{\frac{3}{2}}} e^{i\lambda t}e^{i\xi_1(\lambda)x}w(\lambda)\,d\lambda + \int_{\lambda>2(\frac{a}{3})^{\frac{3}{2}}} e^{i\lambda t}e^{i\kappa_a(\lambda)x}w(\lambda)\,d\lambda \equiv I_{11}(t,x) + I_{12}(t,x) + I_{13}(t,x). \tag{3.45}$$

For I_{11} changing variables $\xi = \kappa_a(\lambda)$ (then $\lambda = \varphi_a(\xi)$) we find, similarly to (3.42) (since according to (3.11) $p(\varphi_a(\xi)) \ge c(|\xi| - 2\sqrt{a/3})$), that

$$\|I_{11}(t,\cdot)\|_{L_{2,-}} \le c\|e^{i\varphi_a(\xi)t}w(\varphi_a(\xi))\varphi_a'(\xi)\chi_{(-\infty,-2\sqrt{a/3})}(\xi)\|_{L_2} \le c_1\|(|\lambda|^{1/3} + |a|^{1/2})w(\lambda)\|_{L_2}. \tag{3.46}$$

In the integral I_{12} changing variables $\xi = \xi_1(\lambda)$ (then also $\lambda = \varphi_a(\xi)$) we find, similarly to (3.43), (3.44), that

$$I_{12}(t,x) = S_a\big(t,x;\mathcal{F}^{-1}[w(\varphi_a(\xi))\varphi_a'(\xi)\chi_{(\sqrt{a/3},2\sqrt{a/3})}(\xi)]\big) \tag{3.47}$$

and

$$\|I_{12}(t,\cdot)\|_{L_2} \le c|a|^{1/2}\|w\|_{L_2}. \tag{3.48}$$

Finally, estimating I_{13} quite similarly to (3.44) we obtain the inequality (3.41) for I_1. The integral I_2 is considered in the same way as I_1. □

Now we return to properties of the functions Q_a and R_a.

Lemma 3.6. *Let* $\mu \in H^{n/3+s}$ *for some* $s \geq 0$ *and integer* $n \geq 0$. *Then* $D_x^n Q_a \in C_b(\overline{\mathbb{R}}_-^x; H^s)$ *and for every* $x \leq 0$

$$\|D_x^n Q_a(\cdot, x; \mu)\|_{H^s} \leq c(a,n)\|\mu\|_{H^{n/3+s}}; \tag{3.49}$$

in addition,

$$D_x^n Q_a(t, 0-0; \mu) = \mathcal{F}_t^{-1}\Big[\frac{r_1(\lambda)r_2^n(\lambda) - r_2(\lambda)r_1^n(\lambda)}{r_1(\lambda) - r_2(\lambda)}\widehat{\mu}(\lambda)\Big](t). \tag{3.50}$$

Proof. Since $\Re r_1(\lambda) \geq 0$ and $\Re r_2(\lambda) \geq 0$, then according to (3.36), (3.14) and (3.15) uniformly with respect to $x \leq 0$

$$\|D_x^n Q_a(\cdot, x; \mu)\|_{H^s} \leq \big\|(1+|\lambda|^s)(|\lambda|^{1/3} + |a|^{1/2})^n \widehat{\mu}(\lambda)\big\|_{L_2} \leq c(a,n)\|\mu\|_{H^{n/3+s}}.$$

□

Lemma 3.7. *Let* $\nu \in H^{(n-1)/3+s}$ *for some* $s \geq 0$ *and natural* n. *Then* $D_x^n R_a \in C_b(\overline{\mathbb{R}}_-^x; H^s)$ *and for every* $x \leq 0$

$$\|D_x^n R_a(\cdot, x; \nu)\|_{H^s} \leq c(a,n)\|\nu\|_{H^{(n-1)/3+s}}; \tag{3.51}$$

in addition,

$$D_x^n R_a(t, 0-0; \nu) = \mathcal{F}_t^{-1}\Big[\frac{r_1^n(\lambda) - r_2^n(\lambda)}{r_1(\lambda) - r_2(\lambda)}\widehat{\nu}(\lambda)\Big](t). \tag{3.52}$$

Proof. The proof is similar to the proof of Lemma 3.6. □

Lemma 3.8. *Let* $\nu \in H^s$ *for some* $s \geq 0$. *Then* $R_a \in C(\overline{\mathbb{R}}_-^x; H^s) \cap C_b(\overline{\mathbb{R}}_-^x; H^{s+1/3}(-T,T))$ *for any* $T > 0$ *and for every* $x \leq 0$

$$\|R_a(\cdot, x; \nu)\|_{H^s} \leq |x| \cdot \|\nu\|_{H^s}, \tag{3.53}$$

$$\|R_a(\cdot, x; \nu)\|_{H^{s+1/3}(-T,T)} \leq c(a,T)\|\nu\|_{H^s}. \tag{3.54}$$

Proof. The estimate (3.53) obviously follows from the formula (3.37).

Next, the estimate (3.51) for $n = 1$, $s = 0$, the estimate (3.39) for $s = 0$ and elementary interpolation inequalities provide that uniformly with respect to $x \leq 0$

$$\|R_a(\cdot, x; \nu)\|_{L_2(-T,T)} \leq c(a,T)\|\nu\|_{L_2}. \tag{3.55}$$

Similarly to (3.51)

$$\|D_t^{1/3} R_a(\cdot, x; \nu)\|_{H^s} \leq c(a)\|\nu\|_{H^s}. \tag{3.56}$$

Combining (3.55) and (3.56) we deduce (3.54). □

Lemma 3.9. *Let $\mu \in H^{(n+2)/3}$, $\nu \in H^{(n+1)/3}$ for some integer $n \geq 0$. Then for any $T > 0$*

$$\|D_x^n Q_a(\cdot,\cdot;\mu)\|_{L_2(\mathbb{R}_-^x;C[0,T])} \leq c(a,T,n)\|\mu\|_{H^{(n+2)/3}}, \tag{3.57}$$

$$\|D_x^n R_a(\cdot,\cdot;\nu)\|_{L_2(\mathbb{R}_-^x;C[0,T])} \leq c(a,T,n)\|\nu\|_{H^{(n+1)/3}}. \tag{3.58}$$

Proof. It is enough to prove that for $j = 1$ and $j = 2$

$$\|I_j\|_{L_2(\mathbb{R}_-^x;C[0,T])} \leq c(a,T)\|(|\lambda|^{1/3} + |a|^{1/2})(1 + |\lambda|^{1/3})w(\lambda)\|_{L_2}, \tag{3.59}$$

where the functions $I_j(t,x)$ are defined by the formula (3.40).

If $a \leq 0$, then since

$$|I_1(t,x)| \leq 3\int_{\mathbb{R}} e^{p(\xi^3)x}|w(\xi^3)|\xi^2\,d\xi,$$

similarly to (3.42) we find that

$$\|I_1\|_{L_2(\mathbb{R}_-^x;C_b(\mathbb{R}^t))} \leq c\|\lambda^{1/3}w(\lambda)\|_{L_2}. \tag{3.60}$$

Next, by virtue of (3.43) and (3.6)

$$\begin{aligned}\|I_2\|_{L_2(\mathbb{R}^x;C[0,T])} &\leq c(a,T)\|(1+|\xi|)w(\varphi_a(\xi))\varphi_a'(\xi)\|_{L_2}\\ &\leq c_1(a,T)\|(1+|\lambda|^{1/3})(|\lambda|^{1/3}+|a|^{1/2})w(\lambda)\|_{L_2}.\end{aligned} \tag{3.61}$$

In the case $a > 0$ it is enough to consider $I_1(t,x)$. We again use the representation (3.45). Similarly to (3.60) and (3.46)

$$\begin{aligned}\|I_{11}\|_{L_2(\mathbb{R}_-^x;C_b(\mathbb{R}^t))} &\leq c\|w(\varphi_a(\xi))\varphi_a'(\xi)\chi_{(-\infty,-2\sqrt{a/3})}(\xi)\|_{L_2}\\ &\leq c_1\|(|\lambda|^{1/3}+|a|^{1/2})w(\lambda)\|_{L_2}.\end{aligned}$$

Similarly to (3.48) with the use of (3.47)

$$\begin{aligned}\|I_{12}\|_{L_2(\mathbb{R}^x;C[0,T])} &\leq c(a,T)\|(1+|\xi|)w(\varphi_a(\xi))\varphi_a'(\xi)\chi_{(\sqrt{a/3},2\sqrt{a/3})}(\xi)\|_{L_2}\\ &\leq c_1(a,T)\|w\|_{L_2}\end{aligned}$$

and, finally, $\|I_{13}\|_{L_2(\mathbb{R}^x;C[0,T])}$ can be estimated just as in (3.61). □

Lemma 3.10. *Let $\mu \in H^{(s+1)/3}$, $\nu \in H^{s/3}$ for some $s \geq 0$. Then for any $T > 0$*

$$\|Q_a(\cdot,\cdot;\mu)\|_{X_{s,a,b,\alpha}(\Pi_T^-)} \leq c(s,a,b,\alpha,T)\|\mu\|_{H^{(s+1)/3}}, \tag{3.62}$$

$$\|R_a(\cdot,\cdot;\nu)\|_{X_{s,a,b,\alpha}(\Pi_T^-)} \leq c(s,a,b,\alpha,T)\|\nu\|_{H^{s/3}}. \tag{3.63}$$

Proof. We again use a_0, μ_1, μ_2 from (3.23), (3.24) and a similar partition of ν.

Then for any $T>0$ and integer $m\ge 0$ according to (3.38) and (3.39)

$$\begin{aligned}&\|Q_a(\cdot,\cdot;\mu_1)\|_{H^{m,3m}(\Pi_T^-)}+\|R_a(\cdot,\cdot;\nu_1)\|_{H^{m,3m}(\Pi_T^-)}\\&\qquad\le c(a,T,m)\left(\|\mu_1\|_{H^{m+1/3}}+\|\nu_1\|_{H^m}\right)\le c_1(a,T,m)\left(\|\mu\|_{L_2}+\|\nu\|_{L_2}\right)\end{aligned}$$

which yields that for any $s\ge 0$

$$\|Q_a(\cdot,\cdot;\mu_1)\|_{X_s(\Pi_T^-)}+\|R_a(\cdot,\cdot;\nu_1)\|_{X_s(\Pi_T^-)}\le c(s,a,T)\left(\|\mu\|_{L_2}+\|\nu\|_{L_2}\right).$$

In order to evaluate $Q_a(t,x;\mu_2)$ and $R_a(t,x;\nu_2)$ in a proper way we prove that if $w(\lambda)=0$ for $|\lambda|<a_0$, then

$$\|I_j\|_{X_{s,a,b,\alpha}(\Pi_T^-)}\le c(s,a,b,\alpha,T)\big\||\lambda|^{(s+1)/3}w(\lambda)\big\|_{L_2}, \tag{3.64}$$

where the functions $I_j(t,x)$, $j=1$ and $j=2$, are defined by the formula (3.40).

Let $a\le 0$. For I_1 we consider at first the case $s=n\ge 0$ an integer. Let $\sigma(x;\lambda)\in H^{n+1}$ be an extension to the whole real axis $\mathbb{R}$ of the function $e^{p(\lambda)x}$ from the half-axis $\mathbb{R}_-$ via the Hestense method (see the proof of Lemma 3.9 in [16], where a similar extension from $\mathbb{R}_+$ was constructed for $e^{-p(\lambda)x}$). Then we extend $I_1(t,x)$ to the whole plane $\mathbb{R}^2$ by the formula

$$I_1(t,x)=2\pi\mathcal{F}_t^{-1}\left[\sigma(x;\lambda)e^{iq(\lambda)x}w(\lambda)\right](t)$$

and as in the proof of Lemma 3.9 in [16] obtain for I_1 the representation of the (3.26) type, where $\widehat{\mu}_2(\lambda)$ must be replaced by $w(\lambda)$. Evaluating $|\widehat{I}_1(\lambda,\xi)|$ similarly to (3.27), we find that

$$\|I_1\|_{X_n}^2\le c(a,n)\iint_{\mathbb{R}^2}(1+|\lambda|^{1/3}+|\xi|)^{2n}\beta^2(\lambda,\xi)\frac{|\lambda|^{2(n+1)/3}|w(\lambda)|^2}{(|\lambda|^{1/3}+|\xi|)^{2(n+2)}}\,d\xi\,d\lambda. \tag{3.65}$$

The corresponding part of the integral from the right part of (3.65) over the domain $(|\lambda|\ge a_0,\ |\xi|\ge 1)$ does not exceed

$$\begin{aligned}c(a)\iint_{|\lambda|\ge a_0,|\xi|\ge 1}&(|\lambda|^{1/3}+|\xi|)^{6b-4}|\lambda|^{2(n+1)/3}|w(\lambda)|^2\,d\xi\,d\lambda\\&\le c_1(a)\big\||\lambda|^{(n+1)/3+(b-1/2)}w(\lambda)\big\|_{L_2}^2\end{aligned} \tag{3.66}$$

since $b < 1/2$, while the corresponding part of this integral over the domain $(|\lambda| \geq a_0,\ |\xi| \leq 1)$ does not exceed

$$c(a) \int_{|\lambda| \geq a_0} |\lambda|^{2(n-1)/3+2\alpha} |w(\lambda)|^2 \, d\lambda \leq c_1(a) \big\| |\lambda|^{(n+1)/3+(\alpha-2/3)} w(\lambda) \big\|^2_{L_2}. \tag{3.67}$$

Therefore, from (3.65) we obtain (3.64) for $j = 1$, $s = n$ and then by interpolation for all s.

For I_2 the estimate (3.64) follows from the representation (3.43) and the inequality (3.3).

In the case $a > 0$ it is sufficient to consider $I_1(t,x)$. We use the partition (3.45). Note that here $I_{12} \equiv 0$. The integral I_{11} can be handled as I_1 in the case $a \leq 0$ and the integral I_{13} as I_2. □

4. Linear initial–boundary-value problems

First of all we consider an initial–boundary-value problem in Π_T^-

$$v_t + v_{xxx} + a v_x = F(t,x), \tag{4.1}$$

$$v\big|_{t=0} = v_0(x), \qquad v\big|_{x=0} = v_2(t), \qquad v_x\big|_{x=0} = v_3(t). \tag{4.2}$$

A generalized solution of this problem is unique, for example, in $L_2(\Pi_T^-)$ (see [13]).

Lemma 4.1. *Let $\mu \in H^{1/3}$, $\nu \in L_2$ and $\mu(t) = \nu(t) = 0$ for $t < 0$. Then a function*

$$v(t,x) \equiv Q_a(t,x;\mu) + R_a(t,x;\nu) \tag{4.3}$$

is a unique generalized solution of the problem (4.1), (4.2) *for $v_0 \equiv 0$, $v_2 \equiv \mu$, $v_3 \equiv \nu$, $F \equiv 0$ in any half-strip Π_T^-.*

Proof. By virtue of (3.38) and (3.39), without loss of generality one can assume that $\mu, \nu \in C_{0,+}^\infty$. In this case according to [13] there exists a solution $v(t,x)$ of the considered problem and $v \in C^\infty([0,T]; H_-^\infty)$ for any $T > 0$. Moreover, if $\mu(t) = \nu(t) = 0$ for $t \geq T_0 > 0$, then similarly to (1.11) it is easy to show that for $t \geq T_0$ and every integer $k \geq 0$

$$\frac{d}{dt} \|D_t^k v\|_{L_{2,-}} = 0,$$

whence with the use of the corresponding homogeneous equation (4.1) itself one can obtain that $v \in C_b^\infty(\overline{\mathbb{R}}_+^t; H_-^\infty)$.

Therefore, for any $p = \varepsilon + i\lambda$, where $\varepsilon > 0$, we can define the Laplace transform

$$\widetilde{v}(p,x) \equiv \int_{\mathbb{R}_+} e^{-pt} v(t,x)\,dt.$$

The function $\widetilde{v}(p,x)$ solves the problem

$$p\widetilde{v}(p,x) + a\widetilde{v}_x(p,x) + \widetilde{v}_{xxx}(p,x) = 0, \qquad x \le 0,$$

$$\widetilde{v}(p,0) = \widetilde{\mu}(p) \equiv \int_{\mathbb{R}_+} e^{-pt}\mu(t)\,dt, \qquad \widetilde{v}_x(p,0) = \widetilde{\nu}(p) \equiv \int_{\mathbb{R}_+} e^{-pt}\nu(t)\,dt.$$

Since $\widetilde{v}(p,x) \to 0$ as $x \to -\infty$, it follows that

$$\widetilde{v}(p,x) = \frac{z_1(p)e^{z_2(p)x} - z_2(p)e^{z_1(p)x}}{z_1(p) - z_2(p)}\widetilde{\mu}(p) + \frac{e^{z_1(p)x} - e^{z_2(p)x}}{z_1(p) - z_2(p)}\widetilde{\nu}(p), \tag{4.4}$$

where the values $z_j(\varepsilon + i\lambda)$ are the roots of the equation (3.7) such that $\Re z_j > 0$. Using the formula of inversion of the Laplace transform and passing to the limit as $\varepsilon \to +0$ we derive (4.3) from (4.4). □

Definition 4.1. *Let* $\widetilde{\Phi}_0(x) \equiv v_0(x)$ *and for natural* m

$$\widetilde{\Phi}_m(x) \equiv D_t^{m-1}F(0,x) - \widetilde{\Phi}'''_{m-1}(x) - a\widetilde{\Phi}'_{m-1}(x)$$
$$= (-1)^m(D_x^3 + aD_x)^m v_0(x) + \sum_{l=0}^{m-1}(-1)^{m-l-1}(D_x^3 + aD_x)^{m-l-1}D_t^l F(0,x). \tag{4.5}$$

Lemma 4.2. *Let* $v_0 \in H_-^s$, $v_2 \in H^{(s+1)/3}(0,T)$, $v_3 \in H^{s/3}(0,T)$, $F \in Y_{s,a,b,\alpha}(\Pi_T^-)$ *for some* $T > 0$ *and* $s \ge 0$ *such that* $(s/3 - 1/6)$ *and* $(s/3 - 1/2)$ *are noninteger numbers. Assume also that* $v_2^{(m)}(0) = \widetilde{\Phi}_m(0)$ *for any integer* $m \in [0, s/3 - 1/6)$, *and* $v_3^{(m)}(0) = \widetilde{\Phi}'_m(0)$ *for any integer* $m \in [0, s/3 - 1/2)$. *Then in* Π_T^- *there exists a unique solution* $v(t,x)$ *of the problem* (4.1), (4.2) *in the space* $Z_s(\Pi_T^-)$, *which is expressed by the formula*

$$v(t,x) = S_a(t,x;v_0) + K_a(t,x;F) + Q_a(t,x;v_2 - S_a(\cdot,0;v_0) - K_a(\cdot,0;F))$$
$$+ R_a(t,x;v_3 - S_{a\,x}(\cdot,0;v_0) - K_{a\,x}(\cdot,0;F)), \tag{4.6}$$

where the functions $v_2 - S_a(\cdot,0;v_0) - K_a(\cdot,0;F)$ *and* $v_3 - S_{a\,x}(\cdot,0;v_0) - K_{a\,x}(\cdot,0;F)$ *are extended by zero for* $t < 0$; *moreover,*

$$\|v\|_{Z_s(\Pi_T^-)} \le c(s,a,b,\alpha,T)\big(\|v_0\|_{H_-^s} + \|v_2\|_{H^{(s+1)/3}(0,T)}$$
$$+ \|v_3\|_{H^{s/3}(0,T)} + \|F\|_{Y_{s,a,b,\alpha}(\Pi_T^-)}\big). \tag{4.7}$$

For any natural $m \le s/3$ *the function* $D_t^m v$ *is a solution in* Π_T^- *of the problem of the* (4.1), (4.2) *type, where* v_0, v_2, v_3, F *are replaced by* $\widetilde{\Phi}_m$, $v_2^{(m)}$, $v_3^{(m)}$, $D_t^m F$.

Proof. First of all, note that for $m \le s/3$

$$D_t^m S_a(t,x;v_0) + D_t^m K_a(t,x;F) = S_a(t,x;\widetilde{\Phi}_m) + K_a(t,x;D_t^m F). \tag{4.8}$$

In particular, the compatibility conditions yield that

$$D_t^m S_a(0,0;v_0) + D_t^m K_a(0,0;F) = v_2^{(m)}(0) \quad \text{if } s > 3m+1/2,$$
$$D_t^m S_{a\,x}(0,0;v_0) + D_t^m K_{a\,x}(0,0;F) = v_3^{(m)}(0) \quad \text{if } s > 3m+3/2.$$

Therefore,

$$\widetilde{v}_2(t) \equiv \big[v_2(t) - S_a(t,0;v_0) - K_a(t,0;F)\big]\psi_T(t) \in H_{0,+}^{(s+1)/3},$$
$$\widetilde{v}_3(t) \equiv \big[v_3(t) - S_{a\,x}(t,0;v_0) - K_{a\,x}(t,0;F)\big]\psi_T(t) \in H_{0,+}^{s/3},$$

and by virtue of (3.3)

$$\|\widetilde{v}_2\|_{H^{(s+1)/3}} + \|\widetilde{v}_3\|_{H^{s/3}} \le c(a,T)\big(\|v_0\|_{H_-^s} + \|v_2\|_{H^{(s+1)/3}(0,T)} + \|v_3\|_{H^{s/3}(0,T)} + \|f\|_{Y_s(\Pi_T^-)}\big). \tag{4.9}$$

Let

$$v(t,x) \equiv S_a(t,x;v_0) + K_a(t,x;f) + Q_a(t,x;\widetilde{v}_2) + R_a(t,x;\widetilde{v}_3). \tag{4.10}$$

According to Lemma 4.1, the function $v(t,x)$ is a generalized solution in Π_T^- of the problem (4.1),(4.2), and the right parts of (4.10) and (4.6) coincide in Π_T^-.

The estimate (4.7) follows from (3.3), (3.38), (3.39), (3.49), (3.51), (3.54), (3.62), (3.63) and (4.9). The properties of the derivative $D_t^m v$ succeed from (4.8), (3.36) and (3.37). □

Besides (4.7) we also need certain estimates without the use of the space Y_s.

Lemma 4.3. *Let* $v_0 \in L_{2,-}$, $v_2 \in H^{1/3}(0,T)$, $v_3 \in L_2(0,T)$, $F \in L_2(\Pi_T^-)$ *for some* $T > 0$. *Consider a (unique) solution* $v(t,x)$ *of the problem* (4.1), (4.2) *in the space* $Z_0(\Pi_T^-)$. *Then for any* $t \in (0,T]$

$$\|v\|_{C([0,t];L_{2,-})} + \|v\|_{C_b(\overline{\mathbb{R}}_-^x;H^{1/3}(0,t))} + \|v_x\|_{C_b(\overline{\mathbb{R}}_-^x;L_2(0,t))}$$
$$\le c(a,T)\left(\|v_0\|_{L_{2,-}} + \|v_2\|_{H^{1/3}(0,T)} + \|v_3\|_{L_2(0,T)} + t^{1/3}\|F\|_{L_2(\Pi_t^-)}\right). \tag{4.11}$$

If, in addition, $v_0 \in H^1_-$, $v_2 \in H^{2/3}(0,T)$, $v_3 \in H^{1/3}(0,T)$, $F \in L_2(0,T;H^1_-)$, $v_0(0) = v_2(0)$, *then*

$$\|v\|_{L_2(\mathbb{R}^x_-;C[0,t])} \leq \tag{4.12}$$
$$c(a,T)\left(\|v_0\|_{H^1_-} + \|v_2\|_{H^{2/3}(0,T)} + \|v_3\|_{H^{1/3}(0,T)} + t^{1/6}\|F\|_{L_2(0,t;H^1_-)}\right).$$

If, in addition, $v_0 \in H^2_-$, $v_2 \in H^1(0,T)$, $v_3 \in H^{2/3}(0,T)$, $F \in L_2(0,T;H^2_-)$, $v_0(0) = v_2(0)$, $v_0'(0) = v_3(0)$, *then*

$$\|v\|_{C([0,T];H^2_-)} + \|v_{xxx}\|_{C_b(\overline{\mathbb{R}}^x_-;L_2(0,T))}$$
$$\leq c(a,T)\left(\|v_0\|_{H^2_-} + \|v_2\|_{H^1(0,T)} + \|v_3\|_{H^{2/3}(0,T)} + \|F\|_{L_2(0,T;H^2_-)}\right). \tag{4.13}$$

Proof. This lemma by analogy with (4.7) follows from the equality (4.6) and the properties of the potentials S_a, K_a, Q_a and R_a, where instead of (3.3) for K_a one must use (3.4), (3.5) and also (3.6), (3.57), (3.58) for (4.12). $\square$

Next we establish certain integral inequalities on solutions of the problem (4.1),(4.2).

Lemma 4.4. *Let* $v_0 \in L_{2,-}$, $v_2 \equiv 0$, $v_3 \in L_2(0,T)$, $F \in L_2(\Pi^-_T)$ *for some* $T > 0$. *Consider a (unique) solution* $v(t,x)$ *of the problem* (4.1), (4.2) *in the space* $Z_0(\Pi^-_T)$. *Then for any* $t \in (0,T]$

$$\int_{\mathbb{R}_-} v^2(t,x)\,dx = \int_{\mathbb{R}_-} v_0^2\,dx + \int_0^t v_3^2(\tau)\,d\tau + 2\int_0^t\!\!\int_{\mathbb{R}_-} Fv\,dx\,d\tau. \tag{4.14}$$

Proof. The equality (4.14) for smooth solutions can be derived similarly to (1.11), while in the general case obtained via closure on the base of the estimate (4.11). $\square$

Lemma 4.5. *Let* $v_0 \in H^2_-$, $v_2 \in H^1(0,T)$, $v_3 \equiv 0$, $F \in L_2(0,T;H^2_-)$ *for some* $T > 0$, $v_0(0) = v_2(0)$, $v_0'(0) = 0$. *Consider a (unique) solution* $v(t,x)$ *of the problem* (4.1), (4.2) *such, that* $v \in C([0,T];H^2_-)$, $v_{xxx} \in C_b(\overline{\mathbb{R}}^x_-;L_2(0,T))$ *(existing due to Lemma* 4.3*). Let* $\rho(x) \equiv 1 + \eta((x+2)/3)$. *Then for any* $t \in (0,T]$

$$\int_{\mathbb{R}_-} v_{xx}^2(t,x)\rho(x)\,dx + 2\int_0^t \int_{\mathbb{R}_-} v_{xxx}^2\rho'\,dx\,d\tau \tag{4.15}$$
$$\leq \int_{\mathbb{R}_-} (v_0'')^2\rho\,dx + c\int_0^t\!\!\int_{\mathbb{R}_-} v_{xx}^2\rho\,dxd\tau + 2\int_0^t \int_{\mathbb{R}_-} F_{xx}v_{xx}\rho\,dx\,d\tau$$
$$+ c\int_0^t \left(F^2(\tau,0) + F_x^2(\tau,0)\right)\,d\tau + c\int_0^t (v_2')^2\,d\tau,$$

where the constant $c > 0$ depends on a and T;

$$\frac{5}{6}\int_{\mathbb{R}_-} v^2(t,x)v_{xx}(t,x)\rho(x)\,dx - 5\int_0^t\int_{\mathbb{R}_-} v_x v_{xx}^2\rho'\,dx\,d\tau \tag{4.16}$$
$$\leq \int_0^t\int_{\mathbb{R}_-} v_{xxx}^2\rho'\,dx\,d\tau + c\int_0^t\big(1+v_2^2(\tau)\big)\int_{\mathbb{R}_-} v_{xx}^2\rho\,dx\,d\tau$$
$$+ c\int_0^t\int_{\mathbb{R}_-} v^2\,dxd\tau + c\int_0^t \sup_{x\leq 0} F^2(\tau,x)\,d\tau + c\|v_0\|^2_{H^2_-} + c\|v_2\|^2_{H^1(0,T)},$$

where here the constant c depends also on $\|v\|_{C([0,T];L_{2,-})}$ and $\|v_2\|_{L_2(0,T)}$.

Proof. As in the previous lemma it is enough to consider the smooth solution. Multiplying (4.1) by $2v_{xxxx}(t,x)\rho(x)$ and integrating over $\mathbb{R}_-$, we derive an equality

$$\frac{d}{dt}\int_{\mathbb{R}_-} v_{xx}^2\rho\,dx + 3\int_{\mathbb{R}_-} v_{xxx}^2\rho'\,dx - \int_{\mathbb{R}_-} v_{xx}^2\rho'''\,dx - a\int_{\mathbb{R}_-} v_{xx}^2\rho'\,dx$$
$$+ 2\big(v_t v_{xxx}\rho - v_t v_{xx}\rho' + v_{xxx}^2\rho - 4v_{xxx}v_{xx}\rho' + v_{xx}^2\rho'' - a v_{xx}^2\rho\big)\big|_{x=0}$$
$$= 2\int_{\mathbb{R}_-} F_{xx}v_{xx}\rho\,dx + 2\big(Fv_{xxx}\rho - F_x v_{xx}\rho - Fv_{xx}\rho'\big)\big|_{x=0}. \tag{4.17}$$

Properties of the function ρ yield an inequality

$$\big|v_{xx}\big|_{x=0}\big| \leq c\Big(\int_{\mathbb{R}_-} v_{xxx}^2\rho'\,dx\Big)^{1/4}\Big(\int_{\mathbb{R}_-} v_{xx}^2\rho\,dx\Big)^{1/4} + c\Big(\int_{\mathbb{R}_-} v_{xx}^2\rho\,dx\Big)^{1/2} \tag{4.18}$$

and, therefore, the inequality (4.15) obviously follows from (4.17).

Next multiplying (4.1) by $\frac{5}{3}\big(v_x^2(t,x) + 2v(t,x)v_{xx}(t,x)\big)\rho(x)$ and integrating over $\mathbb{R}_-$ we derive an equality

$$\frac{5}{6}\frac{d}{dt}\int_{\mathbb{R}_-} v^2 v_{xx}\rho\,dx - 5\int_{\mathbb{R}_-} v_x v_{xx}^2\rho\,dx + 5\int_{\mathbb{R}_-} vv_x v_{xxx}\rho'\,dx \tag{4.19}$$
$$- \frac{5}{6}\int_{\mathbb{R}_-} v^2 v_{xx}\rho'''\,dx + \frac{5}{6}\int_{\mathbb{R}_-}(4vv_x\rho' + v^2\rho'')(av_x - F)\,dx$$
$$+ \frac{5}{3}a\int_{\mathbb{R}_-} v_x(v_x^2 + 2vv_{xx})\rho\,dx + \frac{5}{6}(2vv_{xx}^2\rho + v^2v_{xx}\rho'' + v^2v_t\rho')\big|_{x=0}$$
$$= \frac{5}{3}\int_{\mathbb{R}_-} F(v_x^2 + 2vv_{xx})\rho\,dx.$$

It is easy to show that any of the integrals in (4.19), except the first two on the left side, can be estimated by

$$\delta \int_{\mathbb{R}_-} v_{xxx}\rho' \, dx + c(\delta)\Big[\int_{\mathbb{R}_-} v_{xx}^2 \rho \, dx + \big(\int_{\mathbb{R}_-} v^2 \, dx\big)^5 + \Big(1 + \sup_{x\le 0} F^2\Big)\int_{\mathbb{R}_-} v^2 \, dx\Big],$$

where $\delta > 0$ can be chosen arbitrarily small. Applying the inequality (4.18) to estimate the integrated terms in (4.19) we derive (4.16). □

Now we pass to a problem in a bounded rectangle. In view of the further use we consider it in a rectangle of an arbitrary length $Q_{T,L} = (0,T)\times(0,L)$; namely, we study a problem

$$v_t + v_{xxx} + av_x = F(t,x), \tag{4.20}$$

$$v\big|_{t=0} = v_0(x), \quad v\big|_{x=0} = v_1(t), \quad v\big|_{x=L} = v_2(t), \quad v_x\big|_{x=L} = v_3(t). \tag{4.21}$$

It is known that a generalized solution of this problem is unique in $L_2(Q_{T,L})$ (see [14]).

Lemma 4.6. *Let $v_0 \in H^s(0,L)$, $v_1 \in H^{(s+1)/3}(0,T)$, $v_2 \in H^{(s+1)/3}(0,T)$, $v_3 \in H^{s/3}(0,T)$, $F \in Y_{s,a,b,\alpha}(Q_{T,L})$ for some $T > 0$, $L > 0$ and $s \geq 0$ such that $(s/3 - 1/6)$ and $(s/3 - 1/2)$ are noninteger numbers. Assume also that $v_1^{(m)}(0) = \widetilde{\Phi}_m(0)$, $v_2^{(m)}(0) = \widetilde{\Phi}_m(L)$ for any integer $m \in [0, s/3 - 1/6)$, $v_3^{(m)}(0) = \widetilde{\Phi}'_m(L)$ for any integer $m \in [0, s/3 - 1/2)$. Then in $Q_{T,L}$ there exists a unique solution $v(t,x)$ of the problem (4.20), (4.21) in the space $Z_s(Q_{T,L})$ and*

$$\|v\|_{Z_s(Q_{T,L})} \leq c(s,a,b,\alpha,T,L^{-1})\big(\|v_0\|_{H^s(0,L)} + \|v_1\|_{H^{(s+1)/3}(0,T)} + \|v_2\|_{H^{(s+1)/3}(0,T)} + \|v_3\|_{H^{s/3}(0,T)} + \|F\|_{Y_{s,a,b,\alpha}(Q_{T,L})}\big). \tag{4.22}$$

For any natural $m \leq s/3$ the function $D_t^m v$ is a solution in $Q_{T,L}$ of the problem of the (4.20), (4.21) type, where v_0, v_1, v_2, v_3, F are replaced by $\widetilde{\Phi}_m$, $v_1^{(m)}$, $v_2^{(m)}$, $v_3^{(m)}$, $D_t^m F$.

Proof. We construct the desired solution in the form

$$v(t,x) = w(t,x) + V(t,x), \tag{4.23}$$

where $w(t,x)$ is a solution of an initial–boundary-value problem in $\Pi_{T,L}^- = (0,T)\times(-\infty,L)$ for the equation (4.1) with initial and boundary conditions (4.2) (boundary data v_2 and v_3 are set, of course, for $x = L$). According to Lemma 4.2 such a solution $w \in Z_s(\Pi_{T,L}^-)$ exists and

$$\|w\|_{Z_s(\Pi_{T,L}^-)} \leq c(s,a,b,\alpha,T)\big(\|v_0\|_{H^s(0,L)} + \|v_2\|_{H^{(s+1)/3}(0,T)}$$

$$+ \|v_3\|_{H^{s/3}(0,T)} + \|F\|_{Y_{s,a,b,\alpha}(Q_{T,L})}\big). \quad (4.24)$$

In particular, due to compatibility conditions at the point $(0,0)$

$$V_1(t) \equiv v_1(t) - w(t,0) \in H_{0,+}^{(s+1)/3}|_{(0,T)}$$

and

$$\|V_1\|_{H^{(s+1)/3}(0,T)} \le c(s,a,b,\alpha,T)\big(\|v_0\|_{H^s(0,L)} + \|v_1\|_{H^{(s+1)/3}(0,T)} + \|v_2\|_{H^{(s+1)/3}(0,T)} + \|v_3\|_{H^{s/3}(0,T)} + \|F\|_{Y_{s,a,b,\alpha}(Q_{T,L})}\big). \quad (4.25)$$

Consider in $Q_{T,L}$ a problem for the function V:

$$V_t + V_{xxx} + aV_x = 0, \quad (4.26)$$

$$V\big|_{t=0} = 0, \qquad V\big|_{x=0} = V_1, \qquad V\big|_{x=L} = V_x\big|_{x=L} = 0. \quad (4.27)$$

In order to construct a solution of this problem we consider the boundary potential $J_a(t,x;\mu)$ for an arbitrary function $\mu \in H_{0,+}^{(s+1)/3}|_{(0,T)}$ (see (3.16)). According to (3.17) for any $\delta \in (0,T]$

$$\|J_a(\cdot,L;\mu)\|_{H^{(s+1)/3}(0,\delta)} + \|J_{a\,x}(\cdot,L;\mu)\|_{H^{s/3}(0,\delta)} \le c(s,a,L^{-1})\delta^{1/2}\|\mu\|_{L_2(0,\delta)}. \quad (4.28)$$

Moreover, $J_a(\cdot,L;\mu) \in H_{0,+}^{(s+1)/3}|_{(0,T)}$, $J_{a\,x}(\cdot,L;\mu) \in H_{0,+}^{s/3}|_{(0,T)}$.

Consider in the half-strip $\Pi_{\delta,L}^-$ the problem of the (4.1),(4.2) type, where $v_0 \equiv 0$, $F \equiv 0$, $v_2 \equiv -J_a(\cdot,L;\mu)$, $v_3 \equiv -J_{a\,x}(\cdot,L;\mu)$ (boundary data are set for $x = L$). Again from Lemma 4.2, it follows that a solution of this problem $W \in Z_s(\Pi_{\delta,L}^-)$ exists and, in particular,

$$\|W(\cdot,0)\|_{H^{(s+1)/3}(0,\delta)} \le c(s,a,T)\left(\|J_a(\cdot,L;\mu)\|_{H^{(s+1)/3}(0,\delta)} + \|J_{a\,x}(\cdot,L;\mu)\|_{H^{s/3}(0,\delta)}\right). \quad (4.29)$$

Moreover, it is obvious that $W(\cdot,0) \in H_{0,+}^{(s+1)/3}|_{(0,\delta)}$.

Consider a linear operator $\Gamma : \mu \mapsto W(\cdot,0)$ in the space $H_{0,+}^{(s+1)/3}|_{(0,\delta)}$. Then for small δ the estimates (4.28) and (4.29) provide that the operator $(E+\Gamma)$ is invertible (E is the identity operator) and setting $\mu \equiv (E+\Gamma)^{-1}V_1$ we obtain the desired solution of the problem (4.26),(4.27)

$$V(t,x) \equiv J_a(t,x;\mu) + W(t,x),$$

where

$$\|V\|_{Z_s(Q_{\delta,L})} \le c(s,a,T)\|V_1\|_{H^{(s+1)/3}(0,T)}. \quad (4.30)$$

Thus, the solution of the problem (4.20),(4.21) in the rectangle $Q_{\delta,L}$ is constructed and according to (4.23)–(4.25) and (4.30) is evaluated in the space $Z_s(Q_{\delta,L})$ by the right part of (4.22). Moving step by step we obtain the desired solution of the problem (4.20),(4.21) in the whole rectangle $Q_{T,L}$.

In order to finish the proof of the lemma, it is enough to note that the procedure of construction of the function V yields that $D_t^m V$ is a solution of the problem of the (4.26),(4.27) type, where V_1 is replaced by $V_1^{(m)}$. □

Next, along with (4.22) we also need an estimate on solutions without the use of the space Y_s.

Lemma 4.7. *Let $L = 1$, $v_0 \in L_2(0,1)$, $v_1 \in H^{1/3}(0,T)$, $v_2 \in H^{1/3}(0,T)$, $v_3 \in L_2(0,T)$, $F \in L_1(0,T;L_2(0,1))$ for some $T > 0$. Then there exists a unique solution $v(t,x)$ of the problem* (4.20), (4.21) *such that for any $t \in (0,T]$*

$$\|v\|_{C([0,t];L_2(0,1))} + \|v_x\|_{L_2(Q_t)} \le c(a,T)\big(\|v_0\|_{L_2(0,1)} + \|v_1\|_{H^{1/3}(0,T)} + \|v_2\|_{H^{1/3}(0,T)} + \|v_3\|_{L_2(0,T)} + \|F\|_{L_1(0,t;L_2(0,1))}\big), \tag{4.31}$$

and if $v_1 = v_2 \equiv 0$, then

$$\int_0^1 v^2(t,x)\,dx \le \int_0^1 v_0^2\,dx + \int_0^t v_3^2(\tau)\,d\tau + 2\int_0^t\int_0^1 Fv\,dx\,d\tau. \tag{4.32}$$

Moreover, if $F \in L_2(Q_T)$, then

$$\begin{aligned} &\|v\|_{C([0,t];L_2(0,1))} + \|v\|_{C([0,1];H^{1/3}(0,t))} + \|v_x\|_{C([0,1];L_2(0,t))} \\ &\le c(a,T)\Big(\|v_0\|_{L_2(0,1)} + \|v_1\|_{H^{1/3}(0,T)} + \|v_2\|_{H^{1/3}(0,T)} \\ &\qquad + \|v_3\|_{L_2(0,T)} + t^{1/3}\|F\|_{L_2(Q_t)}\Big). \end{aligned} \tag{4.33}$$

Proof. The estimate (4.31) was proved in [3] (Lemma 3.3, which is valid in the case $s = 0$) and [11]. The inequality (4.32) in the smooth case can be obtained via multiplication of (4.20) by $2v(t,x)$ and integration, and in the general case by closure on the base of the estimate (4.31). The estimate (4.33) is a consequence of the equality (4.23), the inequality (4.30) and the estimate (4.11), applied to the function $w(t,x)$ instead of (4.7). □

At the end of this section we consider initial–boundary-value problems in Π_T^- and Q_T for a linearized KdV equation with a variable coefficient

$$v_t + v_{xxx} + av_x + \big(h(t,x)v\big)_x = F(t,x). \tag{4.34}$$

Lemma 4.8. *If $h \in X_{0,a,b,\alpha}$ for some $b \in [7/16, 1/2)$, $\alpha \in (1/2, 2/3)$, then both solutions of the problem* (4.34), (4.2) *and the problem* (4.34), (4.21) *for $L = 1$ are unique in the spaces $X_{0,a,b,\alpha}((0,T) \times I)$ for any $T > 0$, where $I = \mathbb{R}_-$ or $I = (0,1)$, respectively.*

Proof. The proof of this lemma is based on the inequality (2.3) and word for word repeats the proof of an analogous lemma from [16] (Lemma 4.4) for the problem in Π_T^+. □

Lemma 4.9. *Let either $I = \mathbb{R}_-$ or $I = (0,1)$, $h \in Z_3((0,T) \times I)$ for some $T > 0$. Assume that $s \in [0,3]$, $s \neq 1/2$, $s \neq 3/2$, $v_0 \in H^s(I)$, $v_1 \in H^{(s+1)/3}(0,T)$ (in the case $I = (0,1)$), $v_2 \in H^{(s+1)/3}(0,T)$, $v_3 \in H^{s/3}(0,T)$, $F \in H^{s/3,s}((0,T) \times I)$ and, in addition, $v_0(0) = v_2(0)$ (in the case $I = \mathbb{R}_-$), $v_0(0) = v_1(0)$, $v_0(1) = v_2(0)$ (in the case $I = (0,1)$) if $s > 1/2$, $v_0'(0) = v_3(0)$ (in the case $I = \mathbb{R}_-$), $v_0'(1) = v_3(0)$ (in the case $I = (0,1)$) if $s > 3/2$. Then there exists a (unique) solution $v(t,x)$ of the problem* (4.34), (4.2) ((4.34), (4.21) *for $L = 1$, respectively) in the space $C([0,T]; H^s(I)) \cap X_{0,a,b,\alpha}((0,T) \times I)$ (for any $b \in [7/16, 1/2)$, $\alpha \in (1/2, \alpha_0(b))$, where $\alpha_0(b) \in (1/2, 2/3)$ is defined in* (2.3)) *and*

$$\|v\|_{C([0,T];H^s(I))} \leq c(s,a,b,\alpha,T,\|h\|_{Z_3((0,T)\times I)})\big(\|v_0\|_{H^s(I)} + \|v_1\|_{H^{(s+1)/3}(0,T)} + \|v_2\|_{H^{(s+1)/3}(0,T)} + \|v_3\|_{H^{s/3}(0,T)} + \|F\|_{H^{s/3,s}((0,T)\times I)}\big) \tag{4.35}$$

(*for the problem in Π_T^- the term containing v_1 must be, of course, omitted*).

Proof. The proof of this lemma is quite similar to the proof of Lemma 4.5 in [16]. With the use of the inequalities (4.11) or (4.33), respectively, the estimate (4.35) is at first established for $s = 0$, then for $s = 3$, and for intermediate values of s interpolation is used (in Lemma 4.5 from [16] one more value $s = 1$ was considered before interpolation, but it is not used here). □

5. Nonlinear case

First of all, we establish a result on uniqueness of solutions of the considered problems in the space X_0.

Lemma 5.1. *If $b \in [7/16, 1/2)$, $\alpha \in (1/2, 2/3)$, then for any $T > 0$ solutions of the problems* (1.1)–(1.3) *and* (1.1), (1.2), (1.4) *are unique in the spaces $X_{0,a,b,\alpha}(\Pi_T^-)$ and $X_{0,a,b,\alpha}(Q_T)$, respectively.*

Proof. This lemma easily follows from Lemma 4.8. □

Remark 5.1. Other uniqueness results were obtained earlier without any use of Bourgain-type spaces: in the paper [13] for the problem in Π_T^- in a class of functions $u \in L_\infty(0,T;L_{2,-})$ such that $u_x \exp(\varepsilon x) \in L_2(\Pi_T^-)$ for some $\varepsilon > 0$, and in the paper [14] for the problem in Q_T in a class of functions $u \in L_\infty(0,T;L_2(0,1))$ such, that $u_x \in L_2(Q_T)$.

Next, we establish local well-posedness of the considered problems.

Theorem 5.1. *Let either $I = \mathbb{R}_-$ or $I = (0,1)$. Assume that either the hypotheses of Theorem 2.1 or of Theorem 2.2 are, respectively, satisfied. Then there exists $t_0 \in (0,T]$, depending on a, T, s, $\|u_0\|_{H^s(I)}$, $u_1 \in H^{(s+1)/3}(0,T)$ (in the case $I = (0,1)$), $u_2 \in H^{(s+1)/3}(0,T)$, $u_3 \in H^{s/3}(0,T)$, $f \in H^{s/3,s}((0,T) \times I)$, such that in the domain $(0,t_0)\times I$ there exists a (unique) solution $u(t,x)$ of the problem (1.1)–(1.3) or (1.1), (1.2), (1.2), respectively, in the space $Z_s((0,t_0)\times I)$. The mapping $(u_0,u_1,u_2,u_3,f) \mapsto u$ is Lipschitz continuous on any ball in the same norms as in Theorems 2.1 or 2.2 with the natural substitution of $Z_s((0,T)\times I)$ by $Z_s((0,t_0)\times I)$.*

Proof. The proof will be reproduced for the case of a bounded rectangle (for another domain it is similar).

For some $\delta \in (0, T^{1/3}]$ define

$$v(t,x) \equiv \delta^2 u(\delta^3 t, \delta x), \qquad F(t,x) \equiv \delta^5 f(\delta^3 t, \delta x), \qquad v_0(x) \equiv \delta^2 u_0(\delta x),$$
$$v_1(t) \equiv \delta^2 u_1(\delta^3 t), \qquad v_2(t) \equiv \delta^2 u_2(\delta^3 t), \qquad v_3(t) \equiv \delta^3 u_3(\delta^3 t).$$

Then the function v is a solution in $Q_{1,1/\delta} = (0,1)\times(0,1/\delta)$ of a problem

$$v_t + v_{xxx} + a\delta^2 v_x + v v_x = F, \tag{5.1}$$

$$v\big|_{t=0} = v_0, \quad v\big|_{x=0} = v_1, \quad v\big|_{x=1/\delta} = v_2, \quad v_x\big|_{x=1/\delta} = v_3 \tag{5.2}$$

if and only if the function u is a solution in Q_{δ^3} of the original problem. In addition,

$$\|v_0\|_{H^s(0,1/\delta)} + \|v_1\|_{H^{(s+1)/3}(0,1)} + \|v_2\|_{H^{(s+1)/3}(0,1)} + \|v_3\|_{H^{s/3}(0,1)} + \|F\|_{H^{s/3,s}(Q_{1,1/\delta})} \le c\delta^{1/2} \tag{5.3}$$

for a certain constant c, depending on the same values as t_0 in the hypothesis of the theorem. Besides that, if one denotes by $\Phi_{\delta,m}$ corresponding functions from Definition 2.5 for the problem (5.1),(5.2), then $\Phi_{\delta,m}(x) \equiv \delta^{2+3m}\Phi_m(\delta x)$, and so necessary compatibility conditions are also satisfied for the problem (5.1),(5.2).

Let $b \in [7/16, 1/2)$, $\alpha \in (1/2, \alpha_0(b))$. Introduce a set of functions

$$\widetilde{X}_s(Q_{1,1/\delta}) = \big\{w \in X_{s,a,b,\alpha}(Q_{1,1/\delta}) : \\ D_t^m w\big|_{t=0} = \Phi_{\delta,m}(x) \text{ for } m \in [0, s/3 - 7/6), x \in [0, 1/\delta]\big\}.$$

For any function $w \in \widetilde{X}_s(Q_{1,1/\delta})$ consider a linear initial–boundary-value problem in $Q_{1,1/\delta}$ for an equation

$$v_t + v_{xxx} + a\delta^2 v_x = f - ww_x \tag{5.4}$$

with initial and boundary conditions (5.2). It is easy to see that if one constructs for this problem the corresponding functions $\widetilde{\Phi}_{\delta,m}$, then $\widetilde{\Phi}_{\delta,m}(x) = \Phi_{\delta,m}(x)$ for $m \in [0, s/3 - 1/6)$, $x \in [0, 1/\delta]$. According to (2.2) $ww_x \in Y_s(Q_{1,1/\delta})$, and by virtue of Lemma 4.6 there exists a unique solution of the problem (5.4),(5.2) $v \in Z_s(Q_{1,1/\delta}) \cap \widetilde{X}_s(Q_{1,1/\delta})$. Define $v \equiv \Lambda w$; then applying (5.3), (4.22) and (2.2) yields that (maybe for another constant c)

$$\|\Lambda w\|_{X_s(Q_{1,1/\delta})} \le c\delta^{1/2} + c\|w\|^2_{X_s(Q_{1,1/\delta})}.$$

By a standard argument the mapping Λ is a contraction on a certain ball in $\widetilde{X}_s(Q_{1,1/\delta})$ provided δ is small.

Continuous dependence can be estimated in a similar way. □

Now we establish certain global a priori estimates on solutions of the considered problems and start with the problem in Π_T^-.

Lemma 5.2. *Let the hypotheses of Theorem* 2.1 *be satisfied for* $s = 2$. *Assume that for some* $T' \in (0, T]$ *there exists a solution* $u(t,x)$ *of the problem* (1.1)–(1.3) *in the space* $Z_2(\Pi_{T'}^-)$. *Then for any* $\varepsilon > 0$

$$\|u\|_{C([0,T'];L_{2,-})} \le c\big(a, T, \varepsilon, \|u_2\|_{H^{1/3+\varepsilon}(0,T)}\big) \\ \times \Big(\|u_0\|_{L_{2,-}} + \|u_2\|_{H^{1/3}(0,T)} + \|u_3\|_{L_2(0,T)} + \|f\|_{L_2(\Pi_T^-)}\Big), \tag{5.5}$$

$$\|u\|_{C([0,T'];H^2_-)} \le c\big(a, T, \varepsilon, \|u_0\|_{L_{2,-}}, \|u_2\|_{H^{1/3+\varepsilon}(0,T)}, \|u_3\|_{H^{\varepsilon}(0,T)}, \|f\|_{L_2(\Pi_T^-)}\big) \\ \times \Big(\|u_0\|_{H^2_-} + \|u_2\|_{H^1(0,T)} + \|u_3\|_{H^{2/3}(0,T)} + \|f\|_{L_2(0,T;H^2_-)}\Big), \tag{5.6}$$

$$\|u\|_{L_2(\mathbb{R}^x_-;C[0,T'])} \le c\big(a, T, \|u_0\|_{H^2_-}, \|u_2\|_{H^1(0,T)}, \|u_3\|_{H^{2/3}(0,T)}, \|f\|_{L_2(0,T;H^2_-)}\big). \tag{5.7}$$

Proof. First of all we prove the estimate (5.5). Extend the function u_2 by zero to the whole line $\mathbb{R}$ and set

$$v(t,x) \equiv u(t,x) - \widetilde{J}(t,x;u_2), \tag{5.8}$$

where the function $\widetilde{J}$ is defined by the formula (3.34). Then the function v is a solution in $\Pi^-_{T'}$ of the problem (4.1),(4.2) for $v_0(x) \equiv u_0(x) - \widetilde{J}(0,x;u_2)$, $v_2(t) \equiv 0$, $v_3(t) \equiv u_3(t) - \widetilde{J}_x(t,0;u_2)$, $F(t,x) \equiv f(t,x) - u(t,x)u_x(t,x) - a\widetilde{J}_x(t,x;u_2)$. Write down the equality (4.14). Since

$$2uu_x v = \Big(\frac{2}{3}v^3 + \widetilde{J}v^2\Big)_x + \widetilde{J}_x\big(v^2 + 2\widetilde{J}v\big), \tag{5.9}$$

then

$$\Big|2\int_{\mathbb{R}_-} uu_x v\,dx\Big| = \Big|\int_{\mathbb{R}_-} \widetilde{J}_x\big(v^2+2\widetilde{J}v\big)\,dx\Big| \le c\sup_{x\le 0}|\widetilde{J}_x|\int_{\mathbb{R}_-}(v^2+\widetilde{J}^2)\,dx, \tag{5.10}$$

and using the estimates (3.19) for $s=0$, (3.22) for $s=1$ and (3.28) for $l=1$ (with the natural transformation for $\widetilde{J}$) we derive (5.5).

Next, we pass to the proof of (5.6). Extend the function u_3 to the whole line $\mathbb{R}$ in the class $H^{2/3}$ such that $u_3(t)=0$ for $|t|>2T$. Let

$$V(t,x) \equiv u(t,x) - \widetilde{J}^*(t,x;u_3) \tag{5.11}$$

where the function $\widetilde{J}^*$ is defined by the formula (3.35). Then the function V is a solution in $\Pi^-_{T'}$ of the problem (4.1),(4.2) for $v_0(x) \equiv u_0(x) - \widetilde{J}^*(0,x;u_3)$, $v_2(t) \equiv u_2(t) - \widetilde{J}^*(t,0;u_3)$,$v_3(t) \equiv 0$, $F(t,x) \equiv f(t,x) - u(t,x)u_x(t,x) - a\widetilde{J}(t,x;u_3)$. Write down for the function V the corresponding inequalities (4.15) and (4.16). Consider corresponding integrals, containing uu_x. Let us agree to denote by c_0 various constants, depending on the same values as the constant c in the right part of (5.6), and by $\gamma(\tau)$ arbitrary non-negative functions such that $\|\gamma\|_{L_1(0,T)} \le c_0$. We have

$$\begin{aligned} 2\int_{\mathbb{R}_-}(uu_x)_{xx}V_{xx}\rho\,dx &= 5\int_{\mathbb{R}_-} V_x V_{xx}^2\rho\,dx - \int_{\mathbb{R}_-} uV_{xx}^2\rho'\,dx + 5\int_{\mathbb{R}_-}\widetilde{J}V_{xx}^2\rho\,dx \\ &+ u_3\big(V_{xx}^2\rho\big)\big|_{x=0} + 6\int_{\mathbb{R}_-}\widetilde{J}_x V_x V_{xx}\rho\,dx + 2\int_{\mathbb{R}_-}\widetilde{J}_{xx}uV_{xx}\rho\,dx + 6\int_{\mathbb{R}_-}\widetilde{J}\widetilde{J}_x V_{xx}\rho\,dx. \end{aligned}$$

Taking into account the already proved estimate (5.5) we derive that for an arbitrary $\delta>0$

$$\begin{aligned} \Big|\int_0^t\int_{\mathbb{R}_-} uV_{xx}^2\rho'\,dx\,d\tau\Big| &\le \int_0^t \sup_{x\le 0}\Big|V_{xx}(\rho')^{1/2}\Big|\Big(\int_{\mathbb{R}_-} V_{xx}^2\rho'\,dx\Big)^{1/2}d\tau \\ \times \sup_{\tau\in[0,T']}\Big(\int_{\mathbb{R}_-} u^2\,dx\Big)^{1/2} &\le \delta\int_0^t\int_{\mathbb{R}_-} V_{xxx}^2\rho'\,dx\,d\tau + c(\delta)c_0\int_0^t\int_{\mathbb{R}_-} V_{xx}^2\rho\,dx\,d\tau; \end{aligned}$$

by virtue of (3.28) for $l=0$

$$\Big|\int_0^t\int_{\mathbb{R}_-}\widetilde{J}V_{xx}^2\rho\,dx\,d\tau\Big|\le\int_0^t\gamma(\tau)\int_{\mathbb{R}_-}V_{xx}^2\rho\,dx\,d\tau;$$

according to (4.18)

$$\int_0^t|u_3|\big(V_{xx}^2\rho\big)\big|_{x=0}\,d\tau\le\delta\int_0^t\int_{\mathbb{R}_-}V_{xxx}^2\rho'\,dxd\tau+c(\delta)\int_0^t\gamma(\tau)\int_{\mathbb{R}_-}V_{xx}^2\rho\,dxd\tau;$$

$$6\int_{\mathbb{R}_-}\widetilde{J}_xV_xV_{xx}\rho\,dx=-3\int_{-\infty}^{-1}(\widetilde{J}_x\rho)_xV_x^2\,dx-3\int_{-1}^0(\widetilde{J}_x\rho)_xV_x^2\,dx,$$

where by virtue of (3.17) and (3.30)

$$\begin{aligned}&\Big|3\int_0^t\int_{-\infty}^{-1}(\widetilde{J}_x\rho)_xV_x^2\,dxd\tau\Big|\\&\le c_0\Big(\int_0^t\int_{\mathbb{R}_-}V_{xx}^2\rho\,dx\,d\tau+\int_0^t\int_{\mathbb{R}_-}u^2\,dx\,d\tau+\|u_3\|_{L_2(0,T)}^2\Big)\end{aligned}$$

and by virtue of (3.19) for $s=1$ and (3.30)

$$\begin{aligned}\Big|3\int_0^t\int_{-1}^0(\widetilde{J}_x\rho)_xV_x^2\,dx\,d\tau\Big|&\le c\|u_3\|_{H^{2/3}(0,T)}\Big(\int_0^t\int_{-1}^0V_x^4\,dx\,d\tau\Big)^{1/2}\\&\le\delta\int_0^t\int_{\mathbb{R}_-}V_{xxx}^2\rho'\,dx\,d\tau+c(\delta)c_0\Big(\|u_3\|_{H^{2/3}(0,T)}^2+\int_0^t\int_{\mathbb{R}_-}u^2\,dx\,d\tau\Big);\end{aligned}$$

by analogy with the evaluation of the previous integral

$$2\int_{\mathbb{R}_-}\widetilde{J}_{xx}uV_{xx}\rho\,dx=2\int_{-\infty}^{-1}\widetilde{J}_{xx}uV_{xx}\rho\,dx+2\int_{-1}^0\widetilde{J}_{xx}uV_{xx}\rho\,dx,$$

where

$$\Big|2\int_0^t\int_{-\infty}^{-1}\widetilde{J}_{xx}uV_{xx}\rho\,dx\,d\tau\Big|\le c_0\Big(\int_0^t\int_{\mathbb{R}_-}V_{xx}^2\rho\,dxd\tau+\int_0^t\int_{\mathbb{R}_-}u^2\,dx\,d\tau\Big),$$

$$\begin{aligned}&\Big|2\int_0^t\int_{-1}^0\widetilde{J}_{xx}uV_{xx}\rho\,dx\,d\tau\Big|\\&\le c\|u_3\|_{H^{2/3}(0,T)}\sup_{\tau\in[0,T']}\Big(\int_{-1}^0u^2\,dx\Big)^{1/2}\Big(\int_0^t\sup_{x\in[-1,0]}V_{xx}^2\,d\tau\Big)^{1/2}\\&\le\delta\int_0^t\int_{\mathbb{R}_-}V_{xxx}^2\rho'\,dx\,d\tau+c(\delta)c_0\Big(\int_0^t\int_{\mathbb{R}_-}V_{xx}^2\rho\,dx\,d\tau+\|u_3\|_{H^{2/3}(0,T)}^2\Big);\end{aligned}$$

by virtue of (3.19) for $s = 1$ and (3.28) for $l = 0$

$$\left|6\int_0^t\int_{\mathbb{R}_-}\widetilde{J}\widetilde{J}_x V_{xx}\rho\,dx\,d\tau\right| \le \int_0^t\int_{\mathbb{R}_-}V_{xx}^2\rho\,dx\,d\tau + c_0\|u_3\|^2_{H^{2/3}(0,T)}.$$

Next, again taking into account the already-proved estimate (5.5), we have that

$$\begin{aligned}\int_0^t \sup_{x\le 0}(u^2u_x^2)\,d\tau &\le c\int_0^t\int_{\mathbb{R}_-}(u_{xx}^2+u^2)\,dxd\tau \times \sup_{\tau\in[0,T']}\int_{\mathbb{R}_-}u^2\,dx\\ &\le c_0\int_0^t\int_{\mathbb{R}_-}V_{xx}^2\rho\,dx\,d\tau + c_0\|u_3\|^2_{H^{2/3}(0,T)} + c_0\int_0^t\int_{\mathbb{R}_-}u^2\,dx\,d\tau.\end{aligned}$$

Finally, consider

$$\int_0^t\left[(uu_x)_x^2+(uu_x)^2\right]\big|_{x=0}\,d\tau \sim \int_0^t\left(u_2^2u_{xx}^2\big|_{x=0}+u_3^4+u_2^2u_3^2\right)d\tau.$$

Here, with the use of the inequality (4.18), we have that for an arbitrary $\delta > 0$

$$\begin{aligned}\int_0^t u_2^2u_{xx}^2\big|_{x=0}\,d\tau &\le \delta\int_0^t\int_{\mathbb{R}_-}u_{xxx}^2\rho'\,dx\,d\tau + c(\delta)\int_0^t(1+u_2^4)\int_{\mathbb{R}_-}u_{xx}^2\rho\,dx\,d\tau\\ \le \delta\int_0^t\int_{\mathbb{R}_-}V_{xxx}^2\rho'\,dx\,d\tau &+ c(\delta)c_0\|u_3\|^2_{H^{2/3}(0,T)} + c(\delta)\int_0^t\gamma(\tau)\int_{\mathbb{R}_-}V_{xx}^2\rho\,dx\,d\tau,\end{aligned}$$

because $H^{1/3}(0,T)$ is embedded in $L_4(0,T)$ (see, for example, [25], [26]);

$$\begin{aligned}\int_0^t(u_2^2+u_3^2)u_3^2\,d\tau &\le \sup_{t\in[0,T]}(u_2^2+u_3^2)\int_0^T u_3^2\,d\tau\\ &\le c_0\left(\|u_2\|^2_{H^1(0,T)}+\|u_3\|^2_{H^{2/3}(0,T)}\right),\end{aligned}$$

because $H^{2/3}(0,T) \subset L_\infty(0,T)$.

As the final result, summing the corresponding inequalities (4.15) and (4.16) and using the estimates (3.19), (3.22), (3.30), (3.32) and (3.33), we derive that

$$\begin{aligned}\int_{\mathbb{R}_-}V_{xx}^2(t,x)\rho\,dx + \frac{5}{6}\int_{\mathbb{R}_-}V^2(t,x)V_{xx}(t,x)\rho\,dx &\le \int_0^t\gamma(\tau)\int_{\mathbb{R}_-}V_{xx}^2\rho\,dx\,d\tau\\ + c_0\left(\|u_0\|^2_{H^2_-}+\|u_2\|^2_{H^1(0,T)}+\|u_3\|^2_{H^{2/3}(0,T)}+\|f\|^2_{L_2(0,T;H^2_-)}\right).\quad &(5.12)\end{aligned}$$

Since for an arbitrary $\delta > 0$ by virtue of (5.5) and (3.30)

$$\Big|\int_{\mathbb{R}_-} V^2 V_{xx}\rho\,dx\Big| \le c\Big(\int_{\mathbb{R}_-} V_{xx}^2\rho\,dx\Big)^{5/8}\Big(\int_{\mathbb{R}_-} V^2\rho\,dx\Big)^{7/8}$$

$$\le \delta\int_{\mathbb{R}_-} V_{xx}^2\rho\,dx + c(\delta)c_0\left(\|u_0\|^2_{L_{2,-}} + \|u_2\|^2_{H^{1/3}(0,T)} + \|u_3\|^2_{L_2(0,T)} + \|f\|^2_{L_2(\Pi_T^-)}\right)$$

the inequality (5.12) provides the estimate (5.6).

The estimate (5.7) obviously follows from (4.12) and (5.6). □

Remark 5.2. The estimates (5.5) and (5.6) are, of course, the analogs of the conservation laws for the KdV equation (1.1) ($f \equiv 0$):

$$\int_{\mathbb{R}} u^2\,dx = \text{const}, \qquad \int_{\mathbb{R}} \big(u_{xx}^2 + \frac{5}{6}u^2 u_{xx} + \frac{5}{36}u^4\big)\,dx = \text{const}.$$

Lemma 5.3. *Let the hypotheses of Theorem* 2.1 *be satisfied for* $s = 3k$, k *- natural. Assume, that for some* $T' \in (0,T]$ *there exists a solution* $u(t,x)$ *of the problem* (1.1)–(1.3) *in the space* $Z_{3k}(\Pi^-_{T'})$. *Then for any integer* $0 \le m \le k$

$$\|D_t^m u\|_{C([0,T'];H_-^{3(k-m)})} \le c\big(a,k,T,\|u_0\|_{H_-^{3k-1}},\|u_2\|_{H^k(0,T)}, \tag{5.13}$$
$$\|u_3\|_{H^{k-1/3}(0,T)},\|f\|_{H^{k-1/3,3k-1}(\Pi_T^-)},\|u\|_{Z_{3k-1}(\Pi^-_{T'})}\big)$$
$$\times\Big(\|u_0\|_{H_-^{3k}} + \|u_2\|_{H^{k+1/3}(0,T)} + \|u_3\|_{H^k(0,T)} + \|f\|_{H^{k,3k}(\Pi_T^-)} + \|u\|_{Z_{3k-1}(\Pi^-_{T'})}\Big).$$

Lemma 5.4. *Let the hypotheses of Theorem* 2.1 *be satisfied for* $s = 3k+2$, k *natural. Assume that for some* $T' \in (0,T]$ *there exists a solution* $u(t,x)$ *of the problem* (1.1)–(1.3) *in the space* $Z_{3k+2}(\Pi^-_{T'})$. *Then for any integer* $0 \le m \le k$

$$\|D_t^m u\|_{C([0,T'];H_-^{3(k-m)+2})} \le c\big(a,k,T,\|u_0\|_{H_-^{3k}},\|u_2\|_{H^{k+1/3}(0,T)},\|u_3\|_{H^k(0,T)},$$
$$\|f\|_{H^{k,3k}(\Pi_T^-)},\|u\|_{Z_{3k}(\Pi^-_{T'})}\big)\left(\|u_0\|_{H_-^{3k+2}} + \|u_2\|_{H^{k+1}(0,T)} + \|u_3\|_{H^{k+2/3}(0,T)}\right.$$
$$\left. + \|f\|_{H^{k+2/3,3k+2}(\Pi_T^-)} + \|u\|_{Z_{3k}(\Pi^-_{T'})}\right). \tag{5.14}$$

Proof of Lemmas 5.3 and 5.4. Proofs of these lemmas are carried out in a parallel way. First of all, by double induction: with respect to k and for fixed k with respect to $0 \le m \le k$, one can easily show that for $\sigma = 0$ or $\sigma = 2$

$$\|\Phi_m\|_{H_-^{3(k-m)+\sigma}} \le c\big(a,k,\|u_0\|_{H_-^{3(k-1)}},\|f\|_{H^{k-1,3(k-1)}(\Pi_T^-)}\big)$$

$$\times \Big(\|u_0\|_{H_-^{3k+\sigma}} + \|f\|_{H^{k+\sigma/3,3k+\sigma}(\Pi_T^-)} \Big). \quad (5.15)$$

Next, note that according to Lemma 4.2 and the inequality (2.2) a function $v(t,x) \equiv D_t^k u(t,x)$ is a solution in $\Pi_{T'}^-$ of a linear problem

$$v_t + v_{xxx} + av_x = D_t^k(f - uu_x), \quad (5.16)$$

$$v\big|_{t=0} = \Phi_k, \quad v\big|_{x=0} = u_2^{(k)}, \quad v_x\big|_{x=0} = u_3^{(k)}. \quad (5.17)$$

Let $\phi_k(t,x) \in Z_0(\Pi_T^-)$ be a solution of the problem (4.1),(4.2) for $v_0 \equiv \Phi_k$, $v_2 \equiv u_2^{(k)}$, $v_3 \equiv u_3^{(k)}$, $F \equiv D_t^k f$. Define

$$U_k(t,x) \equiv v(t,x) - \phi_k(t,x).$$

Then by virtue of (4.14) for any $t \in [0,T']$ (it is easy to show that $D_t^k(uu_x) \in L_2(\Pi_{T'}^-)$)

$$\int_{\mathbb{R}_-} U_k^2(t,x)\,dx = -2\int_0^t\!\!\int_{\mathbb{R}_-} D_\tau^k(uu_x)U_k\,dx\,d\tau. \quad (5.18)$$

Using (4.11) we find that

$$\Big|2\int_0^t\!\!\int_{\mathbb{R}_-}(uD_\tau^k u)_x U_k\,dx\,d\tau\Big| \le \Big|\int_0^t\!\!\int_{\mathbb{R}_-} u_x U_k^2\,dx\,d\tau\Big| + \Big|2\int_0^t\!\!\int_{\mathbb{R}_-}(u\phi_k)_x U_k\,dx\,d\tau\Big|$$

$$\le c\Big(\sup_{\overline{\Pi}_{T'}^-}|u_x| + 1\Big)\int_0^t\!\!\int_{\mathbb{R}_-}(U_k^2 + \phi_k^2)\,dx\,d\tau + c\|u\|^2_{L_2(\mathbb{R}_-^x;C[0,T'])}\sup_{x\le 0}\int_0^T \phi_{k\,x}^2\,dt$$

$$\le c\big(a,T,\|u\|_{Z_2(\Pi_{T'}^-)},\|u\|_{L_2(\mathbb{R}_-^x;C[0,T'])}\big)\Big(\int_0^t\!\!\int_{\mathbb{R}_-} U_k^2\,dx\,d\tau + \|\Phi_k\|^2_{L_{2,-}}$$

$$+ \|u_2^{(k)}\|^2_{H^{1/3}(0,T)} + \|u_3^{(k)}\|^2_{L_2(0,T)} + \|D_t^k f\|^2_{L_2(\Pi_T^-)}\Big) \quad (5.19)$$

and taking into account (5.7) and (5.15) for $\sigma = 0$ derive from (5.18) the estimate (5.13) for $m = k$.

If $m < k$, then expressing $D_t^m D_x^{3(k-m)}u$ from the equation (1.1) itself by derivatives with respect to x of lower orders and using induction with respect to $j = k - m$ we obtain (5.13) for the rest of the values of m. Lemma 5.3 is proved.

Next, if the hypothesis of Lemma 5.4 is satisfied, write down for the function U_k the corresponding inequality (4.15). Since

$$-2\int_{\mathbb{R}_-} uD_\tau^k u_{xxx}U_{k\,xx}\rho\,dx = -u_2(U_{k\,xx}^2\rho)(0) + \int_{\mathbb{R}_-} u_x U_{k\,xx}^2\rho\,dx$$

$$+\int_{\mathbb{R}_-} uU_{k\,xx}^2\rho'\,dx - 2\int_{\mathbb{R}_-} u\phi_{k\,xxx}U_{k\,xx}\rho\,dx,$$

taking into account the inequalities (4.18) and (4.13) we find that for an arbitrary $\delta > 0$

$$\Big|2\int_0^t\int_{\mathbb{R}_-}(uD_\tau^k u)_{xxx}U_{k\,xx}\rho\,dx\,d\tau\Big| \le \delta\int_0^t\int_{\mathbb{R}_-}U_{k\,xxx}^2\rho'\,dx\,d\tau$$
$$+ c(\delta)\Big(1 + \|u_2\|^2_{C[0,T]} + \|u\|^2_{C([0,T'];H^3_-)} + \|u\|^2_{L_2(\mathbb{R}^x_-;C[0,T'])}\Big)$$
$$\times\Big(\int_0^t\int_{\mathbb{R}_-}U_{k\,xx}^2\rho\,dx\,d\tau + \|\Phi_k\|^2_{H^2_-} + \|u_2^{(k)}\|^2_{H^1(0,T)} + \|u_3^{(k)}\|^2_{H^{2/3}(0,T)}$$
$$+ \|D_t^k f\|^2_{L_2(0,T;H^2_-)} + \|D_t^k u\|^2_{C([0,T'];L_{2,-})}\Big),$$

whence the estimate (5.14) for $m = k$ succeeds (also with the use of (5.15) for $\sigma = 2$). The end of the proof of Lemma 5.4 is carried out similarly to Lemma 5.3. □

Now we pass to global a priori estimates for the problem in the bounded rectangle. The method of proof follows main ideas from [14] and [3] with certain refinements, related to properties of a corresponding linear problem, established here in Sections 3 and 4.

Lemma 5.5. *Let the hypotheses of Theorem* 2.2 *be satisfied for* $s = 3k$, k *natural. Assume that for some* $T' \in (0, T]$ *there exists a solution* $u(t, x)$ *of the problem* (1.1), (1.2), (1.4) *in the space* $Z_{3k}(Q_{T'})$. *Then for any* $\varepsilon > 0$

$$\|u\|_{C([0,T'];L_2(0,1))} \le c\big(a, T, \varepsilon, \|u_0\|_{L_2(0,1)}, \|u_1\|_{H^{1/3+\varepsilon}(0,T)},$$
$$\|u_2\|_{H^{1/3+\varepsilon}(0,T)}, \|u_3\|_{L_2(0,T)}, \|f\|_{L_1(0,T;L_2(0,1))}\big), \quad (5.20)$$

and for any integer $0 \le m \le k$

$$\|D_t^m u\|_{C([0,T'];H^{3(k-m)}(0,1))} \le c\big(a, k, T, \|u_0\|_{H^{3(k-1)}(0,1)}, \|u_1\|_{H^{k-2/3}(0,T)},$$
$$\|u_2\|_{H^{k-2/3}(0,T)}, \|u_3\|_{H^{k-1}(0,T)}, \|f\|_{H^{k-1,3(k-1)}(Q_T)}, \|u\|_{Z_{3(k-1)}(Q_{T'})}\big)$$
$$\times\big(\|u_0\|_{H^{3k}(0,1)} + \|u_1\|_{H^{k+1/3}(0,T)} + \|u_2\|_{H^{k+1/3}(0,T)} + \|u_3\|_{H^k(0,T)}$$
$$+ \|f\|_{H^{k,3k}(Q_T)} + \|u\|_{Z_{3(k-1)}(Q_{T'})}\big). \quad (5.21)$$

Proof. First of all we prove the estimate (5.20). Let

$$\phi(t, x; u_1, u_2) \equiv J(t, x; u_1)\eta(3/2 - 2x) + \widetilde{J}(t, x-1; u_2)\eta(2x - 1/2),$$
$$v(t, x) \equiv u(t, x) - \phi(t, x; u_1, u_2).$$

Then the function v is a solution in $Q_{T'}$ of the problem (4.20),(4.21) for $L=1$, $v_0(x)\equiv u_0(x)-\phi(0,x)$, $v_1(t)=v_2(t)\equiv 0$, $v_3(t)\equiv u_3(t)-\widetilde{J}_x(t,0;u_2)$, $F(t,x)\equiv f(t,x)-u(t,x)u_x(t,x)-(\phi_t+\phi_{xxx}+a\phi_x)(t,x)$. Note that

$$\|\phi\|_{C([0,T];L_2(0,1))}+\|\phi_x\|_{C([0,1];L_2(0,T))}+\|\phi_t+\phi_{xxx}+a\phi_x\|_{L_2(Q_T)} \le c(a,T)\Big(\|u_1\|_{H^{1/3}(0,T)}+\|u_2\|_{H^{1/3}(0,T)}\Big).$$

Write down for the function v the inequality (4.32). Transforming the integral containing the nonlinear term similarly to (5.10) (with the substitution of $\widetilde{J}$ by ϕ) and using the properties of the potential J, we derive by analogy with (5.5) the inequality (5.20).

Next, note that similarly to (5.15)

$$\|\Phi_m\|_{H^{3(k-m)}(0,1)}\le c\big(a,k,\|u_0\|_{H^{3(k-1)}(0,1)},\|f\|_{H^{k-1,3(k-1)}(Q_T)}\big) \times\Big(\|u_0\|_{H^{3k}(0,1)}+\|f\|_{H^{k,3k}(Q_T)}\Big).$$

According to Lemma 4.6 and the inequality (2.2) a function

$$v(t,x)\equiv D_t^k u(t,x)$$

is a solution in $Q_{T'}$ of a linear problem

$$v_t+v_{xxx}+av_x=D_t^k(f-uu_x), \tag{5.22}$$

$$v\big|_{t=0}=\Phi_k,\quad v\big|_{x=0}=u_1^{(k)},\quad v\big|_{x=1}=u_2^{(k)},\quad v_x\big|_{x=1}=u_3^{(k)}. \tag{5.23}$$

Let for a certain function $g(t,x)$

$$\lambda(g;t)\equiv\|g\|_{C([0,t];L_2(0,1))}+\|g_x\|_{L_2(Q_t)}.$$

We use the inequality (for $t\in[0,T]$)

$$\int_0^t\|(g_1(\tau,\cdot)g_2(\tau,\cdot))_x\|_{L_2(0,1)}\,d\tau\le c(T)t^{1/4}\lambda(g_1;t)\lambda(g_2;t),$$

established in [3] (Lemma 3.1). Then

$$\|D_t^k(uu_x)\|_{L_1(0,t;L_2(0,1))}\le c(T,k)\|u\|_{Z_{3(k-1)}(Q_{T'})}t^{\frac14}(\lambda(D_t^k u;t)+\|u\|_{Z_{3(k-1)}(Q_{T'})}),$$

and applying the inequality (4.31) to the problem (5.22),(5.23), we derive the estimate (5.21) for $m=k$.

If $m<k$, then expressing $D_t^m D_x^{3(k-m)}u$ from the equation (1.1) itself by derivatives with respect to x of lower orders and using induction with respect to $j=k-m$ similarly to the end of the proof of Lemma 5.3, we derive (5.21) for the rest of the values of m. □

Now we can at last pass to the proof of the main results.

Proof of Theorems 2.1 and 2.2. For $s = 3k$ and $s = 3k + 2$, where $k \geq 0$ is an integer, Theorem 2.1 follows from the local well-posedness (Theorem 5.1) and the global a priori estimates, established in Lemmas 5.2–5.4. Similarly, Theorem 2.2 for $s = 3k$, $k \geq 0$ an integer, follows from Theorem 5.1 and Lemma 5.5.

Let $I = \mathbb{R}_-$ for the problem in Π_T^-, $I = (0,1)$ for the problem in Q_T. Fix $b \in [7/16, 1/2)$ and choose arbitrary $s \in (0, 3\alpha_0(b) - 3/2)$ and $\alpha \in (1/2, \alpha_0(b) - s/3)$ (see Lemma 2.1). Consider the local solution $u(t,x) \in Z_s((0,t_0) \times I)$ of the corresponding problem. Then this function is a solution of a linear problem in $(0,t_0) \times I$ for the equation (4.1), where $F \equiv f - \psi_{t_0}^2(t)(u^2/2)_x$. Therefore, according to Lemma 4.2 or Lemma 4.6 and Lemma 2.1,

$$\|u\|_{Z_s((0,t_0)\times I)} \leq c(1 + t_0^{\varepsilon}\|u\|_{X_{0,a,b,\alpha}}\|u\|_{Z_s((0,t_0)\times I)}).$$

This inequality together with the already-proved global well-posedness in the case $s = 0$ provide global well-posedness for small positive s under natural assumptions on boundary data.

For other values of $s \in (0,3)$ one can obtain the desired result via the interpolation theorem by Tartar (see [24]; the statement can also be found in [16], [3] and [5]).

If $s = 3k + \theta$, $\theta \in (0,3)$, $\theta \neq 1/2$, $\theta \neq 3/2$, with k natural ($\theta \neq 2$ for the problem in Π_T^-), then a function $v(t,x) \equiv D_t^k u(t,x)$ is a solution of a corresponding linear initial–boundary-value problem for an equation of the (4.34) type, where $h(t,x) \equiv u(t,x)$, $F \equiv D_t^k f - D_t^k(uu_x) + (uD_t^k u)_x$, and an initial data $v|_{t=0} = \Phi_k$. It is easy to show that

$$\|\Phi_k\|_{H^\theta(I)} \leq c(a, s, \|u_0\|_{H^s(I)}, \|f\|_{H^{s/3,s}((0,T)\times I)})$$

and for $1 \leq m \leq k$

$$\|D_t^m u D_t^{k-m} u_x\|_{H^{1,3}((0,T)\times I)} \leq c(T)\|u\|^2_{Z_{3k}((0,T)\times I)}.$$

According to Lemma 4.9

$$\|D_t^k u\|_{C([0,T];H^\theta(I))} \leq c(a, T, s, \|u_0\|_{H^s(I)}, \|u_1\|_{H^{(s+1)/3}(0,T)}, \|u_2\|_{H^{(s+1)/3}(0,T)}, \|u_3\|_{H^{s/3}(0,T)}, \|f\|_{H^{s/3,s}((0,T)\times I)}). \quad (5.24)$$

Finally, if $m < k$, then using the equation (1.1) itself by induction with respect to $j = k - m$ we evaluate $D_t^m u$ in $C([0,T]; H^{s-3m}(I))$ similarly to (5.24). □

Remark 5.3. If $s = 0$, the assumptions on boundary data in Theorems 2.1 and 2.2 can be weakened with the use of the Slobodetskii spaces: $u_1, u_2 \in (H^{1/3} \cap W_1^{1/3})(0,T)$, $u_3 \in L_2(0,T)$ (see Remark 3.1).

Remark 5.4. In the paper [16] global well-posedness of the problem (1.1), (1.2), (1.5) in Π_T^+ for $0 < s < 1$, $s \neq 1/2$ was established under ε-close-to-natural assumptions on boundary data via interpolation between the cases $s = 0$ and $s = 1$. Similarly to the present paper, global well-posedness of this problem can be proved in the case $0 < s < 1$, $s \neq 1/2$ under the natural assumption $u_1 \in H^{(s+1)/3}(0,T)$.

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