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REMARKS ON THE OSTROVSKY EQUATION

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Abstract. The main result of this paper concerns the limit of the solution of the Ostrovsky equation as the rotation parameter γ goes to zero. We are interested also in the ill-posedness of the Cauchy problem associated with this equation. First, using a compactness method, we show that the initial-value problem of Ostrovsky equation is locally well-posed in $H^s(\mathbb{R})$ for $s > 3/4$. The compactness method is essentially used to prove that the solution of the Ostrovsky equation converges to that of the Korteweg-de Vries equation, as γ tends to zero, locally in time, in $H^s(\mathbb{R})$ for $s > 3/4$. Thanks to some conservation laws and estimates, we will prove a persistence property of the solutions. Therefore, we show the convergence of the solutions in $L^\infty_{loc}(\mathbb{R}, H^s(\mathbb{R}))$ for $s \geq 3/4$. In the case of positive dispersion, we gain a strong convergence in $C(\mathbb{R}, H^1(\mathbb{R}))$. The last section is devoted to studying the ill-posedness of the Cauchy problem associated with the Ostrovsky equation.

1. Introduction

In this paper, we study the Cauchy problem for the Ostrovsky equation, which governs the propagation of weakly nonlinear long surface and internal waves of small amplitude in a rotating fluid. The liquid is assumed to be incompressible and inviscid. After a suitable scaling, the Ostrovsky equation can be written as [7]

$$(u_t - \beta u_{xxx} + uu_x)_x = \gamma u, \quad x \in \mathbb{R}, \tag{1.1}$$

where $\gamma, \beta = cte$. The parameter $\gamma > 0$ measures the effect of rotation, and β determines the type of dispersion. Namely, for $\beta < 0$ (negative dispersion), the equation models surface and internal waves in the ocean and surface waves in a shallow channel with uneven bottom [1], while for $\beta > 0$, it models capillary waves on the surface of a liquid and magneto-acoustic waves in a plasma [6]. Here the dissipation is ignored.

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We are interested, in this paper, in the local and global well posedness of the Ostrovsky equation, and in their behavior as the rotation parameter γ vanishes. If we set $\gamma = 0$ in (1.1) and integrate, we obtain the Korteweg-de Vries (KDV) equation

$$u_t - \beta u_{xxx} + u u_x = 0. \tag{1.2}$$

Thus it is natural to wonder whether the solution of the Ostrovsky equation converges to the one of the KDV equation as γ approaches zero.

There are many results concerning these subjects. Liu and Varlamov [18] showed that the initial-value problem for (1.1) is locally well posed in the space

$$X_s = \Big\{ f \in H^s(\mathbb{R}) / \mathcal{F}^{-1}\Big(\frac{\hat{f}(\xi)}{\xi}\Big) \in H^s(\mathbb{R}) \Big\}$$

with norm $\|f\|_{X^s} = \|f\|_{H^s} + \|\mathcal{F}^{-1}\big(\frac{\hat{f}(\xi)}{\xi}\big)\|_{H^s}$ for $s > 3/2$, where $\mathcal{F}$ or $\hat{\cdot}$ denotes the Fourier transform. They proceed by using a classical compactness method. This method is quite general and does not use the dispersive nature of the equation.

Linares and Milanes, see [5], proved that equation (1.1) is locally well-posed in X_s for $s \geq 3/4$. They use some regularizing effects on the linear part of the equation. A global existence is deduced thanks to the conservations laws satisfied by (1.1). In the first part of my thesis, to be defended in December 2007, the global well-posedness of the Ostrovsky equation, with negative dispersion, for initial data in a suitable Sobolev space $\tilde{H}^s(\mathbb{R})$, for $s \geq 0$ is shown. The Sobolev space $\tilde{H}^s(\mathbb{R})$ is defined by

$$\tilde{H}^s(\mathbb{R}) = \Big\{ f \in H^s(\mathbb{R}) / \mathcal{F}^{-1}\Big(\frac{\hat{f}(\xi)}{\xi}\Big) \in L^2(\mathbb{R}) \Big\}.$$

The method used to prove the local well-posedness is an iterative method based on the dispersive properties of (1.1). The global existence in $\tilde{H}^s(\mathbb{R})$, for $s \geq 0$, is deduced, thanks to some conservation laws and estimates.

Huo and Jia, in [8], prove that the Cauchy problem of (1.1) is well posed in X_s for $s \geq \frac{-1}{8}$. They use an iterative method introduced by Bourgain; see [3, 4]. Due to the conservation of moment, the global existence in $L^2(\mathbb{R})$ is deduced. Isaza and Mejia, in [9], showed that the Ostrovsky equation is locally well-posed in $H^s(\mathbb{R})$, for $s > -1/2$ if $\beta > 0$ (positive dispersion), and for $s > -3/4$ if $\beta < 0$ (negative dispersion). They used an iterative method and applied the technique of elementary calculus inequalities introduced by Kenig, Ponce and Vega [14] for the KDV equation.

Levandosky and Liu [15] prove that, in the case of positive dispersion ($\beta > 0$), the solitary waves of (1.1) converge strongly in the space $H^1(\mathbb{R})$ to those of the KDV equation as γ goes to zero. They used some variational methods.

We organize this paper as follows. In the first section, we prove the local well-posedness, for initial data u_0 in the Sobolev space $H^s(\mathbb{R})$, with $s > 3/4$, such that $\partial_x^{-1}u_0 \in L^2(\mathbb{R})$. We proceed by a compactness method based on the dispersive property of (1.1). This method is inspired by the work of Kenig, Ponce and Vega on the KDV equation; see [12].

Equation (1.1) has two conserved quantities, the conservation of momentum

$$m(u(t)) = \|u(t)\|_{L^2}^2 = const,$$

and the conservation of energy

$$E(u(t)) = \int_{\mathbb{R}} \Big[\beta u_x^2 + \frac{\gamma}{2}(D_x^{-1}u)^2 + \frac{1}{3}u^3\Big](t,x)dx = const.$$

Moreover, we have the estimate

$$\|D_x^{-1}u(t)\|_{L^2} \leq ct^{2/3} + \|D_x^{-1}\varphi\|_{L^2}.$$

Due to the conservation of moment and the last estimate, we show the persistence property of solutions of the Ostrovsky equation. We extend the local solution proved in [8, 9] to the global one in $H^s(\mathbb{R})$, for $s \geq 0$.

The compactness method used in the first section is the key in the proof of the convergence solution of (1.1) to the one of the KDV equation as γ tends to zero. The third section is devoted to this aim. First, we show that we have the local convergence when the initial data $u_0^\epsilon \in H^\infty$, with $\partial_x^{-1}u_0^\epsilon \in L^2(\mathbb{R})$. Consequently, we deduce the local convergence for any data $u_0 \in H^s(\mathbb{R})$ such that $\partial_x^{-1}u_0 \in L^2(\mathbb{R})$, for $s > 3/4$.

Our second goal, in this part, is the convergence in $L^\infty_{loc}(\mathbb{R}, H^s(\mathbb{R}))$, for $s \geq 3/4$. We obtain this result by using the conservation laws and some estimates satisfied by (1.1). We show also strong convergence in $C(\mathbb{R}, H^1(\mathbb{R}))$ in the case of positive dispersion.

In the last section of this paper, we study the ill-posedness result of the Ostrovsky equation. The iterative method, used in [8, 9] to prove the well-posedness of the Cauchy problem associated with (1.1), is essentially based on a bilinear estimate in suitable Bourgain spaces. Here, we show that this estimate fails for $s < -3/4$. This result seems to be sharp in the case of positive dispersion. We establish also that the Cauchy problem for

(1.1) cannot be solved by an iteration method in $H^s(\mathbb{R})$, for $s < -1$. As a consequence, the flow map $\phi \longrightarrow u(t, \cdot)$ cannot be C^2 in $H^s(\mathbb{R})$.

Notation. We denote by $\mathcal{F}$ or $\widehat{\cdot}$ the Fourier transform. We define the operators ∂_x^{-1} or D_x^{-1} and $|D_x|^\alpha$, $\alpha \in \mathbb{R}$, by $\widehat{D_x^{-1} f} = \widehat{\partial_x^{-1} f}(\xi) = \dfrac{1}{i\xi}\hat{f}(\xi)$ and $\widehat{|D_x|^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$. The Hilbert transform operator H is defined as $\widehat{Hf} = -\dfrac{i\xi}{|\xi|}\hat{f}$; we have that $H\partial_x = HD_x = |D_x|$. We denote by $\| \cdot \|_{m,p}$ (respectively $\| \cdot \|_{H^s}$, $\| \cdot \|_2$ or $\| \cdot \|_{L^2}$ and $\| \cdot \|_\infty$ or $\| \cdot \|_{L^\infty}$) the norm in the Sobolev space $W^{m,p}$ (respectively in H^s, L^2 and L^∞).

2. Local well posedness of the Ostrovsky equation via compactness arguments

In this section, we prove the local existence solution for the Ostrovsky equation in $H^s(\mathbb{R})$, for $s > 3/4$, based on a compactness method. This method relies heavily on the dispersive character of the Ostrovsky equation. We solve first a regularized version of (1.1) involving a small parameter. After, we prove the convergence of these regularized solutions in $H^s(\mathbb{R})$, for $s > 3/4$. Rewrite the initial-value problem of (1.1) in the following way:

$$\begin{cases} u_t - \beta u_{xxx} + uu_x = \gamma \partial_x^{-1} u, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \tag{2.1}$$

We will give now the main theorem of this section.

Theorem 2.1. *Let $s > 3/4$. For any $u_0 \in H^s(\mathbb{R})$ with $D_x^{-1}u_0 \in L^2$, there exist a positive $T = T(\|u_0\|_{H^s} + \gamma\|\partial_x^{-1}u_0\|_{L^2})$ $(\lim_{\rho\to 0} T(\rho) = \infty)$ and a unique solution $u(t)$ of the initial-value problem* (1.1) *satisfying*

$$u \in C([-T,T], H^s(\mathbb{R})) \qquad \text{and} \qquad \partial_x u \in L^4([-T,T], L^\infty).$$

Moreover, if $u_0 \in H^{s'}(\mathbb{R})$ with $s' > s$, we have a unique solution of (1.1) *with the same existence time T.*

To prove Theorem 2.1, we proceed as in the case of the KDV equation; see [12]. First, we consider the initial-value problem

$$v_t - \beta v_{xxx} = 0, \qquad v(0,x) = v_0(x), \tag{2.2}$$

whose solution is given by the unitary group $S(t)$ in $H^s(\mathbb{R})$, where $S(t) = e^{i\beta t\partial_x^3}$ (i.e., $v(\cdot,t) = S(t)v_0$) is defined by its Fourier transform $\widehat{S(t)v_0}(\xi) = e^{-i\beta t\xi^3}$. We shall state some preliminary estimates, due to the dispersive character of (2.2). After, we prove the local existence of (2.1) via compactness method by considering $\partial_x^{-1}u$ in the second member.

2.1. **Linear estimates and local smoothing.** In this subsection, we provide the linear estimates and local smoothing properties for (2.2). We refer to [12] for the proof of some lemmas. We begin by recalling the sharp version of the local smoothing effect, proved in [12].

Lemma 2.1. *Let $v_0 \in L^2(\mathbb{R})$. Then*

$$\int_{-\infty}^{+\infty} ||D_x|S(t)v_0(x)|^2 dt = c\|v_0\|_{L^2},$$

for any $x \in \mathbb{R}$.

Next, we state a version of the dispersive inequality corresponding to the group $S(t)$.

Lemma 2.2. *For any $(a,b) \in [0,1] \times [0,1/2]$, we have*

$$\||D_x|^{ab} S(t)v_0\|_{2/(1-a)} \leq C|t|^{-a(b+1)/3}\|v_0\|_{2/(1+a)}.$$

For the proof, we refer to [11].

Theorem 2.2. *For any $(a,b) \in [0,1] \times [0,1/2]$*

$$\left(\int_{\infty}^{\infty} \||D_x|^{ab/2} S(t)v_0\|_p^q\right)^{1/q} \leq C\|v_0\|_2, \tag{2.3}$$

and

$$\left(\int_{\infty}^{\infty} \|\int |D_x|^{ab} S(t-\tau)f(\tau,\cdot)d\tau\|_p^q dt\right)^{1/q} \leq C\left(\int_{-\infty}^{\infty} \|f(t,\cdot)\|_{p'}^{q'} dt\right)^{1/q'}, \tag{2.4}$$

where $(q,p) = (6/a(b+1), 2/(1-a))$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

For a detailed proof see [12]. Next, we recall the maximal function estimate proved in [12].

Lemma 2.3. *For any $s > 3/4$ and $\rho > 3/4$, we have*

$$\left(\int_{-\infty}^{\infty} \sup_{[-T,T]} |S(t)v_0|^2 dx\right)^{1/2} \leq (1+T)^{\rho}\|v_0\|_{s,2}. \tag{2.5}$$

To analyze the products that arise from the nonlinear term of the Ostrovsky equation, we require the following Leibniz rules for fractional derivatives. For detailed proofs of these facts, see [13].

Lemma 2.4. *Let* $\alpha = \alpha_1 + \alpha_2 \in (0,1)$ *with* $\alpha_i \in (0,\alpha)$, $p \in [1,\infty)$ *and* $p_1, p_2 \in (1,\infty)$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then*

$$\||D_x|^\alpha (fg) - f|D_x|^\alpha g - g|D_x|^\alpha f\|_{L^p} \le c\||D_x|^{\alpha_1} f\|_{L^{p_1}} |D_x|^{\alpha_2} g\|_{L^{p_2}}.$$

Now, we turn to the energy estimate satisfied by the solution of (1.1) .

Lemma 2.5. *Let* $u \in C([-T,T], H^\infty)$ *be a solution of the initial-value problem* (2.1). *Then for any* $s \ge 0$, *we have*

$$\sup_{[-T,T]} \|u(t)\|_{s,2} \le c_s \|u_0\|_{s,2} \exp\left(\int_{-T}^{T} \|\partial_x u(\tau)\|_{L^\infty} d\tau\right). \tag{2.6}$$

Proof. Applying D_x^s to (1.1) leads to the identity

$$\partial_t D^s u - \beta \partial_x^3 D^s u + u \partial_x D^s u + (D^s u \partial_x u - u \partial_x D^s u) = \gamma D_x^{-1} D_x^s u.$$

Multiplying by $D_x^s u$, integrating by parts and applying Lemma (2.4) yields

$$\frac{d}{dt}\|D_x^s u(t)\|_{L^2}^2 \le c\|\partial_x u\|_{L^\infty} \|D_x^s u\|_{L^2}^2 + \|[D^s; u]\partial_x u\|_{L^2} \le \|\partial_x u\|_{L^\infty} \|D_x^s u\|_{L^2}^2.$$

Thus, Gronwall's inequality yields (2.6).

We state now the conservation laws and some estimates satisfied by equation (1.1).

Lemma 2.6. *Let* $u \in C([0,T]; H^s(\mathbb{R}))$ *be a solution of* (1.1) *associated with the initial data* $u_0 \in H^s(\mathbb{R})$, *with* $D_x^{-1} u_0 \in L^2$. *For all* $t \in [0,T]$, *one has*

$$m(u(t)) = \|u(t)\|_{L^2}^2 = const, \tag{2.7}$$

$$\|D_x^{-1} u(t)\|_{L^2} \le ct^{2/3} + \|D_x^{-1} u_0\|_{L^2}, \tag{2.8}$$

and

$$E(u(t)) = \int_{\mathbb{R}} \left[\beta u_x^2 + \frac{\gamma}{2}(D_x^{-1} u)^2 + \frac{1}{3} u^3\right](t,x) dx = const. \tag{2.9}$$

(2.7) *and* (2.8) *hold for* $s \ge 0$; *the conservation of energy* (2.9) *holds for* $s \ge 1$.

Proof. We first prove these estimates for $u \in C([0,T]; H^\infty(\mathbb{R}))$ such that $D_x^{-2} u(t) \in L^2(\mathbb{R})$. The general case can be deduced by using a regularized sequence. We refer to [17] for more details of the proof of this type of estimate.

We begin by showing the conservation of the norm L^2. Multiplying (1.1) by u and integrating over x, noticing that $u_t \in L^\infty([-T,T];L^2(\mathbb{R}))$ to obtain

$$\frac{d}{dt}\int |u(x,t)|^2 dx = \beta \int u_{xxx} u\, dx - \int (u^2)_x u\, dx - \int \gamma D_x^{-1} uu\, dx = 0,$$

we then have (2.7). To prove the second estimate (2.8), we multiply (1.1) by $D_x^{-2}u$ and integrating over x, we obtain

$$-\frac{d}{dt}\int |D_x^{-1}u(x,t)|^2 dx$$
$$= \int \beta u_{xxx} D_x^{-2} u\, dx - \int (u^2)_x D_x^{-2} u + \int \gamma D_x^{-1} u D_x^{-2} u\, dx = \int u^2 D_x^{-1} u\, dx.$$

By using Hölder's inequality and the Gagliardo-Nirenberg inequality, we obtain

$$\frac{d}{dt}\|D_x^{-1}u\|_{L^2}^2 \le \|u\|_{L^2}^2 \|D_x^{-1}u\|_{L^\infty} \le c\|u\|_{L^2}^{5/2}\|D_x^{-1}u\|_{L^2}^{1/2} = c\|\varphi\|_{L^2}^{5/2}\|D_x^{-1}u\|_{L^2}^{1/2}.$$

Therefore,

$$\|D_x^{-1}u\|_{L^2}^{3/2} \le ct + \|D_x^{-1}\varphi\|_{L^2}^{3/2}.$$

To establish the conservation of energy, we multiply (1.1) by $-\beta u_{xx} + u^2 - \gamma D_x^{-2}u$ and integrate over $\mathbb{R}$ to get

$$\frac{d}{dt}\int \Big[\beta u_x^2 + \frac{\gamma}{2}(D_x^{-1}u)^2 + \frac{1}{3}u^3\Big](t,x)dx = 0;$$

thus, we obtain (2.9). □

2.2. **Proof of Theorem 2.1.** We consider the initial-value problem

$$\begin{cases} u_t - \beta u_{xxx} + uu_x = \gamma D_x^{-1}u, & x,t \in \mathbb{R}, \\ u(0,x) = \varphi_\epsilon * u_0 = u_0^\epsilon, \end{cases} \tag{2.10}$$

where $\varphi \in S(\mathbb{R})$ with $\int \varphi(x)dx = 1$, $\int x^k \varphi(x)dx = 0$, $k = 1,2,\ldots,$ and $\varphi_\epsilon = \epsilon^{-1}\varphi(x/\epsilon)$ for $\epsilon > 0$.

Our first goal will be the proof of Theorem 2.1 with initial data u_0^ϵ, on an interval of time $[-T,T]$, where T depends only on $(\|u_0\|_s + \gamma\|D_x^{-1}u_0\|_{L^2})$ and does not depend on ϵ, for any $\epsilon > 0$, whenever $u_0 \in H^s(\mathbb{R})$ with $s > 3/4$. Then we have the next lemma.

Lemma 2.7. *For any $u_0 \in H^s(\mathbb{R})$, with $s > 3/4$, there exists $T = T(\|u_0\|_s + \gamma\|D_x^{-1}u_0\|_{L^2})$ and $u = u^\epsilon \in C([-T,T], H^s(\mathbb{R}))$ a solution of the initial-value problem* (2.10).

Proof. We perform the change of variables $v(t,x) = \lambda^2 u(\lambda^3 t, \lambda x)$, with initial data $v_0(x) = \lambda^2 u_0(\lambda x)$. Then v satisfies the equation

$$v_t - \beta v_{xxx} + v v_x = \gamma \lambda^4 D_x^{-1} v, \qquad v(0,x) = \lambda^2 u_0(\lambda x). \tag{2.11}$$

Since $\|v_0\|_s = \lambda^{3/2}(1+\lambda^s)\|u_0\|_s$ and $\|D_x^{-1} v_0\|_{L^2} = \lambda^{1/2}\|D_x^{-1} u_0\|$, we can always reduce our problem to the case of small initial data in (2.11). Moreover, observe that if there exists a solution for (2.11) $v \in C([-T,T], H^s(\mathbb{R}))$ whenever $(\|v_0\|_s + \gamma\|D_x^{-1}v_0\|_{L^2}) \leq \delta$, then for arbitrary data, there exists a solution of (2.1) for time $T \lesssim \|u_0\|_s + \gamma\|D_x^{-1}u_0\|_{L^2})^{-2}$. In the following, we always assume that $T \leq 1$, and we frequently apply the Cauchy-Schwarz inequality in the integrations with respect to t without further comment.

In view of the energy estimate (2.6), the key is to establish a priori control of $\|\partial_x u\|_{L_t^4 L_x^\infty}$. Then using the Duhamel formula, we can write the solution of (2.11) as

$$v(t) = S(t)v_0 - \int_0^t S(t-t')(v v_x(t'))dt' - \gamma\lambda^4 \int_0^t S(t-t') D_x^{-1} v(t')dt'. \tag{2.12}$$

Then

$$\begin{aligned} \partial_x v(t) = &- |D_x|^{1/4} S(t) |D_x|^{3/4} H v_0 \\ &+ \int_0^t |D_x|^{1/4} S(t-t') |D_x|^{3/4} H(v v_x(t'))dt' - \gamma\lambda^4 \int_0^t S(t-t') v(t')dt', \end{aligned} \tag{2.13}$$

where H is the Hilbert transform. Hence, using Theorem 4.1 with $(a,b) = (1,1/2)$ in (2.2), it follows that

$$\begin{aligned} \|\partial_x v\|_{L_T^4 L_x^\infty} \leq &\ c\|v_0\|_{3/4,2} + c\int_0^T \||D_x|^{3/4}(v v_x(t'))dt'\|_2 \\ &+ c\lambda^4\gamma \Big\| \int_0^T S(t-t')v(t')dt' \Big\|_{L_T^6 L_x^\infty}. \end{aligned}$$

Applying Theorem 2.2 again to the last term in the right-hand side member with $(a,b) = (1,0)$ and $(p,q) = (6,\infty)$, and the conservation of the norm L^2, we get

$$\begin{aligned} \|\partial_x v\|_{L_T^4 L_x^\infty} &\leq c\|v_0\|_{3/4,2} + c\int_0^T \||D_x|^{3/4}(v v_x(t'))\|_2 dt' + c\lambda^4\gamma \int_0^T \|v(t')\|_2 dt' \\ &\leq c\|v_0\|_{3/4,2} + c\int_0^T \||D_x|^{3/4}(v v_x(t'))\|_2 dt'. \end{aligned}$$

By Lemma 2.4, for $\eta \geq 0$, we have

$$\begin{aligned}
&\||D_x|^{3/4+\eta}(v\partial_x v)(t)\|_2 \leq \|(|D_x|^{3/4+\eta}(v\partial_x v)(t) - v|D_x|^{3/4+\eta}\partial_x v \\
&- |D_x|^{3/4+\eta} v\partial_x v)(t)\|_2 + \||D_x|^{3/4+\eta} v\partial_x v(t)\|_2 + \|v|D_x|^{3/4+\eta}\partial_x v(t)\|_2 \\
&\leq c\|\partial_x v(t)\|_{L^\infty}\||D_x|^{3/4+\eta} v(t)\|_2 + \|v|D_x|^{3/4+\eta}\partial_x v(t)\|_2.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^T \||D_x|^{3/4+\eta}(v\partial_x v)(t')\|_2 \, dt' \\
&\leq c\int_0^T \|\partial_x v(t')\|_{L^\infty}\||D_x|^{3/4+\eta} v(t')\|_2 \, dt' + \int_0^T \|v|D_x|^{3/4+\eta}\partial_x v(t')\|_2 \, dt' \\
&\leq c\sup_{[0,T]} \|v(t)\|_{3/4+\eta,2} T^{3/4}\Big(\int_0^T \|\partial_x v(t')dt'\|_\infty^4 \, dt'\Big)^{1/4} \\
&\quad + \int_0^T \|v|D_x|^{3/4+\eta}\partial_x v(t')\|_2 \, dt'.
\end{aligned}$$

Using Lemma 2.5, we obtain

$$\begin{aligned}
&\int_0^T \||D_x|^{3/4+\eta}(v\partial_x v)(t')\|_2 \, dt' \\
&\leq c\|v_0\|_{3/4+\eta,2} \exp\Big(c\|\partial_x v\|_{L^4_T L^\infty_x}\Big) + \int_0^T \|v|D_x|^{3/4+\eta}\partial_x v(t')\|_2 \, dt'.
\end{aligned}$$

By the Cauchy-Schwarz and Hölder inequalities, we get

$$\begin{aligned}
&\int_0^T \||D_x|^{3/4+\eta}(v\partial_x v)(t')dt'\|_2 \leq T^{1/2}\Big(\int_0^T \|v|D_x|^{3/4+\eta}\partial_x v(t')\|_2 \, dt'\Big)^{1/2} \\
&= T^{1/2}\Big(\int_0^T \int |v|D_x|^{3/4+\eta}\partial_x v(t',x)|^2 dx dt'\Big)^{1/2} \qquad (2.14) \\
&\leq T^{1/2}\Big(\int \int_0^T |v|D_x|^{3/4+\eta}\partial_x v(t',x)|^2 dt' dx\Big)^{1/2} \\
&\leq T^{1/2}\Big(\int \sup_{[0,T]} |v(x,t)|^2 dx\Big)^{1/2}\Big(\sup_x \int_0^T ||D_x|^{3/4+\eta}\partial_x v(t',x)|^2 \, dt'\Big)^{1/2}.
\end{aligned}$$

Now, we treat the second member of (2.14). Using Lemma 2.3 in (2.12), we obtain

$$\int \sup_{[0,T]} |v(x,t)|^2 dx$$

$$\leq c\left\{\|v_0\|_{3/4+\eta,2}+\int_0^T\left(\|v\partial_x v(t)\|_2+\||D_x|^{3/4+\eta}v\partial_x v(t)\|\right)dt\right.$$
$$\left.+\gamma\lambda^4\int_0^T\left(\|D_x^{-1}v(t)\|_2+\||D_x|^{3/4+\eta-1}v(t)\|_2\right)dt\right\},$$

for $\eta>0$ such that $0<\eta\leq 1/4$; hence, from the estimate (2.8) and the inequality

$$\||D_x|^{3/4+\eta-1}v(t)\|_2\leq\|D_x^{-1}v(t)\|_2+\|v(t)\|_2,$$

we have

$$\begin{aligned}\int\sup_{[0,T]}|v(x,t)|^2dx &\leq c\left\{\left(\|v_0\|_{3/4+\eta,2}+\lambda^4\gamma\|D_x^{-1}v_0\|\right)\right.\\ &+\left.\int_0^T\left(\|v\partial_x v(t)\|_2+\||D_x|^{3/4+\eta}v\partial_x v(t)\|_2\right)dt\right\}.\end{aligned}$$

For the second term of (2.14), applying Lemma 2.1 to the integral equation (2.12), we obtain

$$\begin{aligned}&\sup_x\left(\int_0^T||D_x|^{3/4+\eta}\partial_x v(t',x)|^2dt'\right)^{1/2}\leq c\|v_0\|_{3/4+\eta,2}\\ &+\int_0^T\||D_x|^{3/4+\eta}(v\partial_x v(t'))\|_2dt'+c\lambda^4\gamma\int_0^T\left(\|D_x^{-1}v(t')\|_2+\|v(t')\|_2\right)dt'\\ &\leq c\left(\|v_0\|_{3/4+\eta,2}+\gamma\|D_x^{-1}v_0\|\right)+\int_0^T\||D_x|^{3/4+\eta}(v\partial_x v(t'))\|_2dt'.\end{aligned}$$

Then, we choose η such that $3/4<3/4+\eta\leq s$ and set

$$\begin{aligned}F(T)=\|\partial_x v\|_{L_T^4L_x^\infty}&+\left(\int\sup_{[0,T]}|v(t,x)|^2dx\right)^{1/2}\\ &+\sup_x\left(\int_0^T||D_x|^{3/4+\eta}\partial_x v(t',x)|^2dt'\right)^{1/2}\end{aligned}$$

and $\nu=\|v_0\|_{3/4+\eta,2}+\gamma\lambda^4\|D_x^{-1}v_0\|_2$. $F(T)$ is a continuous, nondecreasing function of T. Combining all of the previous estimates, we know that $F(T)\leq c\nu+c\nu\exp cF(T)$, provided that $\nu\leq\delta\leq 1$. Note that $F(0)=\|v_0\|_2\leq\nu$. By using a standard argument, see [10], we show that there exists $\delta>0$ and a constant $M>0$ such that if $\nu\leq\delta$, then $F(1)\leq M$. Hence we get the control on $\|\partial_x v\|_{L_T^4L_x^\infty}$. The energy estimate (2.6) completes the proof of this lemma. □

Remark 2.1. Note that, in the proof of Lemma 2.7, we use only the norm in $H^{3/4+\eta}$ with $3/4 < 3/4+\eta \le s$ for any $\eta > 0$. Then, for any $s' > s$, we have the same time $T = T(\|u_0\|_{3/4+\eta,2} + \gamma\|D_x^{-1}v\|_2)$.

Lemma 2.8. *There exists a constant $K > 0$ independent of ϵ such that the solution of the initial-value problem* (2.10) *satisfies*

$$\Big(\int_0^T \Big\||D_x|^{s+ab/2}u^\epsilon(t)\Big\|_p^q\,dt\Big)^{1/q} \le K,$$

for any $(a,b) \in [0,1]\times[0,1/2]$ with $(q,p) = (6/a(b+1), 2/1-a)$.

Proof. Applying $|D_x|^{s+ab/2}$to the integral equation, we get

$$\begin{aligned}|D_x|^{s+ab/2}u^\epsilon(t) = {}&|D_x|^{ab/2}S(t)|D_x|^s u_0^\epsilon\\ &-\int_0^t |D_x|^{ab/2}S(t-t')|D_x|^s(u^\epsilon\partial_x u^\epsilon(t'))dt'\\ &-\gamma\int_0^t S(t-t')|D_x|^{s+ab/2}\partial_x^{-1}u^\epsilon(t')dt'.\end{aligned}$$

Hence, Theorem 2.2 and Lemma 2.6 yield

$$\begin{aligned}&\Big(\int_0^T \Big\||D_x|^{s+ab/2}u^\epsilon(t)\Big\|_p^q\,dt\Big)^{1/q}\\ &\le c\Big\{\|u_0\|_{s,2} + \int_0^T \||D_x|^s(u^\epsilon\partial_x u^\epsilon(t'))\|_2 dt' + \gamma\left(\|u_0\|_{s,2} + \gamma\|D_x^{-1}u_0\|_2\right)\Big\}\\ &\le c\left(\|u_0\|_{s,2} + \gamma\|D_x^{-1}u_0\|_2\right) + \int_0^T \||D_x|^s u^\epsilon\partial_x u^\epsilon(t'))\|_2\,dt'.\end{aligned}$$

The proof of Lemma 2.7 leads to

$$\int_0^T \big\||D_x|^s(u^\epsilon\partial_x u^\epsilon(t'))\big\|_2\,dt' \le C.$$

Combining all these estimates, Lemma 2.8 follows. □

Proof of Theorem 2.1. From Lemma (2.7), it follows that for any $\epsilon \in (0,1)$, the corresponding solution $u^\epsilon(\cdot)$ of the initial-value problem (2.10) satisfies

$$u^\epsilon \in C([0,T];H^\infty(\mathbb{R})), \qquad \sup_{[0,T]}\|u^\epsilon(t)\|_{s,2} \le K, \tag{2.15}$$

and

$$\int_0^T \|\partial_x u^\epsilon(t)\|_{L^\infty}\,dt \le K, \tag{2.16}$$

with T and K depending on $\|u_0\|_{s,2}$ and $\|D_x^{-1}u_0\|_2$. Defining $\omega(t) = \omega^{\epsilon,\epsilon'}(t) = (u^\epsilon - u^{\epsilon'})(t)$ for $\epsilon > \epsilon' > 0$, we have that $\omega(t)$ satisfies the equation

$$\partial_t\omega - \beta\partial_x^3\omega + u^{\epsilon'}\partial_x\omega + \omega\partial_x u^\epsilon = \gamma D_x^{-1}\omega. \tag{2.17}$$

Thus, a standard argument shows that

$$\frac{d}{dt}\|\omega(t)\|_2 \leq c(\|\partial_x u^\epsilon\|_\infty\| + \partial_x u^{\epsilon'}\|_\infty)(t)\|\omega(t)\|_2.$$

Hence,

$$\sup_{[0,T]}\|(u^\epsilon - u^{\epsilon'})(t)\|_2 \leq c\|u_0^\epsilon - u_0^{\epsilon'}\|_2 = 0(\epsilon^s), \tag{2.18}$$

as ϵ tends to zero.

The above inequality proves the existence and uniqueness of a strong solution $u(\cdot)$ of the initial-value problem (2.1). To obtain the persistency property, we use the argument given by Bona and Smith in [2] to show that the sequence $(u^\epsilon)_\epsilon$ converges in $L^\infty([0,T]; H^s(\mathbb{R}))$ as ϵ tends to zero.

We shall need the following estimates:

$$\sup_{[0,T]}\|u^\epsilon(t)\|_{s+l} \leq c\epsilon^{-l}, \tag{2.19}$$

and

$$\left(\int_0^T \left\||D_x|^{s+ab/2+l}u^\epsilon(t)\right\|_p^q dt\right)^{1/q} \leq c\epsilon^{-l}, \tag{2.20}$$

for any $l > 0$, with $(a,b) \in [0,1] \times [0,1/2]$, $(q,p) = (6/a(b+1), 2/(1-a))$, and where the constant c depends only on $\|u_0\|_{s,2}$ and l.

The estimate (2.19) follows from Lemma 2.5 and (2.16). The proof of (2.20) can be obtained by inserting (2.19) into the proof of Lemma 2.8, where all the estimates depend linearly on the highest derivatives.

Now using Lemma 2.4 in the equation(2.17), we find that

$$\begin{aligned}\frac{d}{dt}\|\omega(t)\|_{s,2} &\leq c\left(\|\partial_x u^{\epsilon'}\|_\infty + |\partial_x u^\epsilon\|_\infty\right)(t)\|\omega(t)\|_{s,2}\\ &+ \|u^{\epsilon'}(t)\|_{s,2}\|\partial_x\omega(t)\|_\infty + \|\partial_x u^\epsilon\|_{s,r_1}\|\omega(t)\|r_2,\end{aligned}$$

with $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{2}$. Now we state a lemma which will be crucial for the sequel.

Lemma 2.9. *we have*

$$\int_0^T \|u^{\epsilon'}(t)\|_{s,2}\|\partial_x\omega(t)\|_\infty dt = 0(1), \tag{2.21}$$

and

$$\int_0^T \|\partial_x u^\epsilon(t)\|_{s,r_1}\|\omega(t)\|r_2 = 0(1), \tag{2.22}$$

as ϵ tends to zero, where r_1 is sufficiently large.

We shall prove this lemma later. Then by Lemma 2.9 and Gronwall inequality, we get

$$\sup_{[0,T]} \|(u^\epsilon - u^{\epsilon'})(t)\|_{s,2} = 0(1).$$

This completes the proof of Theorem 2.1. □

To prove Lemma 2.9, we need the following Gagliardo-Nirenberg inequality.

Lemma 2.10. *Let $f \in S(\mathbb{R}^n)$; we have*

$$\|f\|_{m,p} \le c\|f\|^{\theta}_{m_0,p_0}\|f\|^{1-\theta}_{m_1,p_1},$$

for p_0, $p_1 \in (1,\infty)$ and m_0, $m_1 \in [0,\infty]$ such that $m = \theta m_0 + (1-\theta)m_1$ and $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$.

Proof of Lemma 2.9. Let us prove equality (2.21). Using the Sobolev embedding theorem, we get

$$\|\partial_x \omega(t)\|_\infty \le c\|\omega\|_{1+\delta,1}, \qquad 0 < \delta \ll 1.$$

Applying Lemma 2.10 with $p = 1$ and $m = 1+\delta$, we have

$$\|\omega(t)\|_\infty \le c\|\omega(t)\|^{\theta}_{s_0,p_0}\|\omega(t)\|^{1-\theta}_2, \tag{2.23}$$

with

$$1+\delta = \theta s_0, \qquad 1 = \frac{\theta}{p_0} + \frac{1-\theta}{2}. \tag{2.24}$$

Thus, using the above estimates, along with the Sobolev embedding theorem and Hölder's inequality, we get

$$\begin{aligned} \int_0^T \|\partial_x \omega(t)\|_\infty dt &\le \sup_{[0,T]} \|\omega(t)\|^{1-\theta}_2 \int_0^T \|\omega(t)\|^{\theta}_{s_0,p_0} dt \\ &\le c\epsilon^{s(1-\theta)} + c\epsilon^{s(1-\theta)}\Big(\int_0^T \||D_x|^{s_0}\omega(t)\|^q_{p_0}\, dt\Big)^{\theta/q}, \end{aligned}$$

where (q,p_0) is such that we can apply Lemma 2.8. Let $(a,b) \in [0,1] \times [0,1/2]$, the couple in Lemma 2.8 such that $(q,p_0) = (6/a(b+1, 2/(1-a)))$ and $s_0 = s + a/4 + l$. Thus, we obtain

$$\int_0^T \|\partial_x \omega(t)\|_\infty dt \le c\epsilon^{s(1-\theta)} + c\epsilon^{s(1-\theta)} \Big(\int_0^T \Big\| |D_x|^{s+a/4+l} \omega(t) \Big\|_{p_0}^q dt \Big)^{\theta/q}$$
$$\le c\epsilon^{s(1-\theta)} \epsilon^{-l\theta}.$$

Using (2.24), we get $s(1-\theta) - l\theta = s + a/4 - s_0\theta = s + (a\theta/4 - 1 - \delta)$. Then, for $s > 3/4$, we can choose a, δ and $\theta \sim 1$ such that $s > 1 + \delta - a\theta/4$. Thus we obtain $s(1-\theta) - l\theta > 0$. We conclude that

$$\int_0^T \|\partial_x \omega(t)\|_\infty = 0(1).$$

By the same method, we can prove (2.22), for r_1 sufficiently large. $\square$

3. Global solution

In this section, starting from the global existence result for the Ostrovsky equation in $\tilde{L}^2(\mathbb{R})$ due to Huo and Jia (see [8]), we will prove a persistence property in $\tilde{H}^s(\mathbb{R})$, for $s \ge 0$. The spaces $\tilde{L}^2(\mathbb{R})$ and $\tilde{H}^s(\mathbb{R})$ are defined such that

$$\|\varphi\|_{\tilde{L}^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})} + \||D_x|^{-1}\varphi\|_{L^2(\mathbb{R})},$$

and

$$\|\varphi\|_{\tilde{H}^s(\mathbb{R})} = \|\varphi\|_{H^s(\mathbb{R})} + \||D_x|^{-1}\varphi\|_{H^s(\mathbb{R})}.$$

We will extend the local solution obtained in [8] to the time interval $[0,T]$, for an arbitrary $T > 0$. In the case of positive dispersion, we show the global existence in $C(\mathbb{R}, H^1(\mathbb{R}))$.

Huo and Jia proved that the Cauchy problem for the Ostrovsky equation is locally well-posed in $\tilde{H}^s(\mathbb{R})$, for $s \ge \frac{-1}{8}$. To this aim, they used an iterative method introduced by Bourgain, see [3, 4], in a suitable Bourgain space $\tilde{X}_{s,b}$, which is defined as

$$\|u\|^2_{\tilde{X}_{s,b}} = \|U(-t)u\|^2_{H^s_x H^b_t} + \|U(-t)|D_x|^{-1}u\|^2_{H^s_x H^b_t},$$

where $U(t)$ is the unitary group associated with the Ostrovsky equation, defined by its Fourier transform $\widehat{U(t)\varphi}(\xi) = e^{-it(\beta\xi^3 + \frac{\gamma}{\xi})}\hat{\varphi}(\xi)$. They extend the local solution to a global one in $C(\mathbb{R}, L^2(\mathbb{R}))$ thanks to the conservation of moment (2.7). Here, we show that, for $s \ge 0$, the local solution can be extended to a global one in $L^\infty_{loc}(\mathbb{R}, \tilde{H}^s(\mathbb{R})) \cap \tilde{X}^{s,b}$. In the case of positive dispersion, we obtain the global existence in $C(\mathbb{R}, \tilde{H}^1(\mathbb{R}))$. Then we have the next theorem.

Theorem 3.1. (*a*) *For $s \geq 0$, $\frac{1}{2} < b < 1$ and $u_0 \in \tilde{H}^s(\mathbb{R})$, there exists a unique global solution for the Ostrovsky equation $u \in L^\infty_{loc}(\mathbb{R}, \tilde{H}^s(\mathbb{R}) \cap \tilde{X}_{s,b}$ such that $\sup_{[0,T]} \|u(t)\|_{\tilde{H}^s} \leq C$, where C depends on T, $\|u_0\|_{\tilde{H}^s}$.*

(*b*) *For $s = 1$, in the case of positive dispersion (i.e., $\beta > 0$), we have a unique global solution $u \in C(\mathbb{R}, H^1(\mathbb{R})) \cap \tilde{X}_{1,b}$ for initial data $u_0 \in \tilde{H}^1(\mathbb{R})$*

To prove this theorem, we shall need the following lemma.

Lemma 3.1. *Let $s \geq 1$, $u_0 \in H^s$ be such that $D_x^{-1} u_0 \in L^2(\mathbb{R})$. If u is the solution of (1.1) in the case of positive dispersion with initial data u_0, then there exists $C = C\left(\|u_0\|_{H^1} + \|D_x^{-1} u_0\|_2\right)$ such that*

$$\|u(t)\|_{H^1} + \|D_x^{-1} u(t)\|_2 \leq C. \tag{3.1}$$

Proof. Using the conservation law of energy, we get

$$E(u(t)) = \int_{\mathbb{R}} \left[\beta u_x^2 + \frac{\gamma}{2}(D_x^{-1}u)^2 + \frac{1}{3}u^3\right](t,x)dx = E(u_0).$$

The cubic term with can be estimated in the following way:

$$\left|\int u^3(t)dx\right| = \left|\int (D_x^{-1} u \partial_x(u^2))(t)dx\right| \leq 2\|D_x^{-1}u(t)\|_\infty \int (|u||\partial_x u|)(t)dx.$$

Using the Gagliardo-Nirenberg and Hölder inequalities and the conservation of momentum, we get

$$\begin{aligned}
&\left|\int u^3(t)dx\right| \leq c\|D_x^{-1}u(t)\|_2^{1/2}\|u(t)\|_2^{1/2}\|u(t)\|_2\|\partial_x u(t)\|_2 \\
&= c\|D_x^{-1}u(t)\|_2^{1/2}\|u(t)\|_2^{3/2}\|\partial_x u(t)\|_2 \leq \frac{3}{4}|\beta|\|\partial_x u(t)\|_2^2 + C\|D_x^{-1}u(t)\|_2 \\
&\leq \frac{3}{4}|\beta|\|\partial_x u(t)\|_2^2 + 3\frac{\gamma}{4}\|D_x^{-1}u(t)\|_2^2 + C,
\end{aligned}$$

where C depends only on $\|u_0\|_2$.Therefore, it follows from the conservation of energy that

$$\begin{aligned}
&\frac{1}{2}\Big(\beta\int |u_x|^2(t)dx + \gamma\int |D_x^{-1}u|^2(t)dx\Big) \\
&\leq C + \frac{\beta}{4}\int u_x^2(t)dx + \frac{\gamma}{4}\int |D_x^{-1}u|^2(t)dx.
\end{aligned}$$

Then, it turns out that for β, $\gamma > 0$

$$\|\partial_x u(t)\|_2 + \|D_x^{-1}u(t)\|_2 \leq C = C\left(\|u_0\|_{H^1} + \|D_x^{-1}u_0\|_2\right).$$

Proof of Theorem 3.1. (a) Let $s > 0$, $u_0 \in \tilde{H}^s(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R})$, with $\psi = 1$ on $[\frac{-1}{2}, \frac{1}{2}]$ and *supp* $\psi \subset [-1, 1]$. Define $\psi_t = \psi(T^{-1}(\cdot))$, for $T > 0$. We define the operator

$$\Phi_{u_0}(u) = \psi_1(t)U(t)u_0 + \psi_1(t)\int_0^t U(t-t')\psi_\delta(t')(uu_x)(t')dt'.$$

Assume that u_0 satisfies $\|u_0\|_{\tilde{H}^s} \leq M_s$. Using the conservation of momentum (2.7) and (2.8), we get $\|u(t,\cdot)\|_{\tilde{L}^2} \leq M_0$, $\forall t \in [0,1]$, where M_0 depends on $\|u_0\|_{\tilde{L}^2}$. Next we extend the local existence result to the interval $I = [0,1]$ by applying the contraction method in the space with norm

$$|\cdot|_\alpha = \|\cdot; \tilde{X}_{0,b}\| + \alpha\|\cdot; \tilde{X}_{s,b}\|,$$

where $\alpha = \alpha(M_s)$ to be chosen later. Let $B_r = \{u \in \tilde{X}_{0,b} \cap \tilde{X}_{s,b} / |u|_\alpha < r\}$, with $r = 2(M_0 + \alpha M_s)$. In order to prove that Φ_{u_0} is a contraction mapping on B_r, we first prove $\Phi_{u_0}(B_r) \subset B_r$. By using the linear and bilinear estimates proved by Huo and Jia, see [8] Theorem 4.1 and Lemmas 3.1–3.3, we obtain

$$\begin{aligned}|\Phi_{u_0}(u)(t)|_\alpha &\leq M_0 + \alpha M_s + c\delta^{b'-b}\Big[\|u; \tilde{X}_{0,b}\|^2 + \alpha\|u; \tilde{X}s, b\|^2\Big]\\ &\leq M_0 + \alpha M_s + c\delta^{b'-b}|u|_\alpha^2,\end{aligned}$$

for $\frac{1}{2} < b < b'$. Choose δ such that $c\delta^{b'-b}r^2 < r/4$. Then $|\Phi_{u_0}(u)(t)|_\alpha < r$, and we have $\Phi_{u_0}(B_r) \subset B_r$. For $u, v \in B_r$, reasoning as above, it follows that

$$|\Phi_{u_0}(u) - \Phi_{u_0}(v)|_\alpha \leq c\delta^{b'-b}(|u|_\alpha + |v|_\alpha)|u-v|_\alpha \leq \frac{1}{2}|u-v|_\alpha.$$

Therefore, Φ_{u_0} is a contraction mapping on B_r, and there exists a unique solution on the interval $[0, T_1]$ with $T_1 \leq \delta$, satisfying $|u|_\alpha \leq r$. By the Sobolev embedding theorem, we have that $u \in C([0,T_1], H^s(\mathbb{R}))$. Hence, we have a born on $\|u(T_1,\cdot)\|_{\tilde{H}^s}$. Let M'_s be this born, and choose α' such that $\alpha' M'_s = \alpha M_s$.

By using the same argument, we prove that $\Phi_{u(T_1,\cdot)}$ is a contraction on $B'_r = \{u \in \tilde{X}_{0,b} \cap \tilde{X}_{0,b} / |u|_{\alpha'} < r' = 2(M_0 + \alpha' M'_s) = 2(M_0 + \alpha M_s) = r\}$. Noticing that the time of existence is also T_1, we have a solution on the interval $[T_1, 2T_1]$ satisfying $|u|_\alpha \leq r$. Therefore, we construct a successive local solution on consecutive (overlapping) intervals $[iT_1, (i+1)T_1]$ to get a local solution on $I_0 = [0,1]$.

In view of (2.7) and (2.8), we have a born on the norm $\tilde{L^2}$ on $[k, k+1]$ depending only on $\|u(k,.)\|_{\tilde{L}^2}$ and $T = k+1$. Then we can apply the same method on consecutive intervals $I_k = [k, k+1]$ to get a local solution on I_k.

One gets thus a unique global solution u such that on each finite interval the norm $\|\cdot\|_{\tilde{H}^s}$ is finite .

(b) This result is a direct consequence of (a) and Lemma 3.1. □

4. Limit to the KDV equation

In this section, we show that the solution of the initial-value problem (1.1), of parameter γ_n, converges to that of the (KDV) equation as γ_n tends to zero. Let γ_n such that $0 < \gamma_n < 1$ and γ_n tends to zero as $n \to \infty$. consider the initial-value problem associated with γ_n,

$$\partial_t u^n - \beta \partial_x^3 u^n + u^n \partial_x u^n = \gamma_n D_x^{-1} u^n, \qquad u^n(0,x) = u_0(x), \tag{4.1}$$

and the initial-value problem associated with the (KDV) equation:

$$\partial_t v - \beta \partial_x^3 v + v \partial_x v = 0, \qquad v(0,x) = u_0(x). \tag{4.2}$$

Then, we have

Theorem 4.1. *For $s > 3/4$, $u_0 \in H^s(\mathbb{R})$ with $D_x^{-1} u \in L^2$, there exists $T > 0$ such that the solution of the initial-value problem* (4.1) *converges to the corresponding one of the* (KDV) *equation in $C([0,T], H^s(\mathbb{R}))$ as γ_n tends to zero.*

Theorem 4.2. (*a*) *For $s > 3/4$, u_n converges to v in $L^\infty_{loc}(\mathbb{R}, H^s(\mathbb{R}))$.* (*b*) *In the case of positive dispersion, the convergence is strong in $C(\mathbb{R}, H^1(\mathbb{R}))$.*

Proof of Theorem 4.1. We prove this theorem in two steps.

Step 1. We show in this part the convergence of $u^{\epsilon n}$ to v^ϵ in $C([0,T], H^s(\mathbb{R}))$ as γ_n tends to zero, where $u^{\epsilon n}$ and v^ϵ are respectively the solutions of (4.1) and (4.2) with initial data $u^{\epsilon n}(0,x) = v^\epsilon(0,x) = (\varphi_\epsilon * u_0)(x)$.

Let T be the common time of existence to $u^{\epsilon n}$ and v^ϵ (we can take T independent of n), and set $\omega(t) = (u^{\epsilon n} - v^\epsilon)(t)$. Then ω satisfies the equation

$$\partial_t \omega - \beta \partial_x^3 \omega + u^{\epsilon n} \partial_x \omega + \omega \partial_x v^\epsilon = \gamma_n D_x^{-1} u^{\epsilon n}. \tag{4.3}$$

Multiplying (4.3) by $\omega(t)$ and integrating by parts, we get

$$\frac{d}{dt}\|\omega(t)\|_{L^2}^2 \leq \left(\|\partial_x u^{\epsilon n}\|_\infty + \|\partial_x v^\epsilon\|_\infty\right)\|\omega(t)\|^2 + c\gamma_n \|D_x^{-1} u^{\epsilon n}(t)\|_{L^2}\|\omega(t)\|_{L^2}.$$

Using the Gronwall inequality and the fact that $\|\partial_x u^{\epsilon n}(t)\|_\infty$, $\|\partial_x u^{\epsilon n}(t)\|_\infty \leq C$, where C depends on $\|u_0\|_{s,2} + \|D_x^{-1} u\|_2$ and T, we obtain

$$\sup_{[0,T]} \|\omega(t)\|_{L^2} \leq C\gamma_n \sup_{[0,T]} \|D_x^{-1} u^{\epsilon n}(t)\|_2. \tag{4.4}$$

Applying now Lemma 2.10 with $p = p_0 = p_1 = 2$, we get

$$\|\omega(t)\|_{H^s} \le \|\omega(t)\|^{\theta}_{H^{s'}} \|\omega(t)\|^{1-\theta}_{L^2},$$

where $s = \theta s'$, $\theta < 1$. The energy estimate yields

$$\|\omega(t)\|_{H^{s'}} \le \|u^{\epsilon n}\|_{H^{s'}} + \|v^{\epsilon}\|_{H^{s'}} \le c\epsilon^{s-s'}, \qquad t \in [0,T].$$

Then using the last estimates, we conclude

$$\sup_{[0,T]} \|\omega(t)\|_{H^s} \le C\Big(\gamma_n \sup_{[0,T]} \|D_x^{-1} u^{\epsilon n}\|_{L^2}\Big)^{1-\theta} \epsilon^{(s-s')\theta}.$$

Using now (2.8), we obtain

$$\sup_{[0,T]} \|\omega(t)\|_{H^s} \le C\gamma_n^{1-\theta} \epsilon^{(s-s')\theta},$$

where c depends on T, $\|u_0\|_{H^s}$ and $\|D_x^{-1}u_0\|_2$. Then for fixed ϵ, we have $\|\omega(t)\|_{H^s}$ tends to zero as γ_n tends to zero.

Step 2. In the proof of Theorem 2.1, we showed that $u^{\epsilon n}$ converges to u^n, where u^n is the solution of the initial-value problem (4.1). We remark that, if γ_n is bounded, the convergence is independent of γ_n; then, we deduce that the convergence is uniform on n. For the (KDV) equation, we have that v^{ϵ} converges to v in $C([0,T], H^s(\mathbb{R}))$ (take $\gamma = 0$), [12]. Then for $\eta > 0$, there exist $\epsilon_0 > 0$ such that

$$\sup_{[0,T]} \|(u^{\epsilon_0 n} - u^n)(t)\|_{H^s} \le c\frac{\eta}{3}. \qquad \forall n \in \mathbb{N}.$$

and

$$\sup_{[0,T]} \|(v^{\epsilon_0} - v)(t)\|_{H^s} \le c\frac{\eta}{3}.$$

It follows, from the first steps, that there exists $n_0 > 0$ such that for $n \ge n_0$

$$\sup_{[0,T]} \|(u^{\epsilon_0 n} - v^{\epsilon_0})(t)\| \le c(\gamma_n)^{1-\theta} < \frac{\eta}{3}.$$

Combining all these estimates, we obtain that for $n \ge n_0$

$$\sup_{[0,T]} \|(u^n - v)(t)\|_{H^s} \le \sup_{[0,T]} [\|(u^n - u^{\epsilon_0 n})(t)\|_{H^s} + \|(u^{\epsilon_0 n} - v^{\epsilon_0})(t)\|_{H^s} + \|(v^{\epsilon_0} - v)(t)\|_{H^s}] \le c\left(\frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3}\right) = c\eta. \qquad \square$$

Proof of Theorem 4.2. (a) By Theorem 3.1, we have

$$\sup_{[0,T]} \|u(t)\|_{H^s} \le C,$$

and

$$\sup_{[0,T]} \|v(t)\|_{H^s} \leq C_0, \qquad \forall T > 0,$$

where C depends on T, $\|u_0\|_{H^s}$ and $\|D_x^{-1}u_0\|_{L^2}$, C_0 depends on T and $\|u_0\|_{H^s}$. Thus, we have $\|\partial_x v^\epsilon(t)\|_\infty$, $\|\partial_x u^{\epsilon n}(t)\|_\infty \leq C$, with C depends on T and $\|u_0\|_{s,2} + \|D_x^{-1}u\|_2$. Then by the first estimates, in the first step of the proof of the Theorem 4.1, we can write

$$\sup_{[0,T_0]} \|\omega(t)\|_{H^s} \leq c\Big(\gamma_n \sup_{[0,T_0]} \|D_x^{-1}u^{\epsilon n}\|_{L^2}\Big)^{1-\theta} \epsilon^{(s-s')\theta} \leq C\gamma_n^{1-\theta}\epsilon^{(s-s')\theta},$$

where C depends on $T > T_0$, $\|u_0\|_{H^1}$ and $\|D_x^{-1}u_0\|_2$, and T_0 is the common time of existence. We conclude that all the other estimates in the first and second steps of Theorem 4.1 depend on T, $\|u_0\|_{H^1}$ and $\|D_x^{-1}u_0\|_2$ and do not depend on T_0. Then by the bootstrapping method, we obtain the convergence on $[0,T]$, and we have

$$\sup_{[0,T]} \|(u^n - v)(t)\|_{H^s} \leq C\eta, \qquad \forall n \geq n_0,$$

where C depends on T, $\|u_0\|_{H^1}$ and $\|D_x^{-1}u_0\|_2$.

(b) In the case of positive dispersion, by Lemma 3.1, we have

$$\|u(t)\|_{H^1} + \|D_x^{-1}u(t)\|_2 \leq C,$$

where C depends only on $\big(\|u_0\|_{H^1} + \|D_x^{-1}u_0\|_2\big)$. Then in the first steps of the proof of Theorem 4.1, we have

$$\sup_{[0,T]} \|\omega(t)\|_{H^s} \leq c\Big(\gamma_n \sup_{[0,T]} \|D_x^{-1}u^{\epsilon n}\|_{L^2}\Big)^{1-\theta} \epsilon^{(s-s')\theta} \leq C\gamma_n^{1-\theta}\epsilon^{(s-s')\theta},$$

where C depends only on $\big(\|u_0\|_{H^1} + \|D_x^{-1}u_0\|_2\big)$. We conclude that all the other estimates, in the first and the second steps of Theorem 4.1, are independent of T. Then by the bootstrapping method, we get convergence on $\mathbb{R}$, and we have

$$\sup_{\mathbb{R}} \|(u^n - v)(t)\|_{H^s} \leq C\eta, \qquad \forall n \geq n_0,$$

where C depends only on $\|u_0\|_{H^s}$ and $\|D_x^{-1}u_0\|_2$.

5. Ill-posedness result

In this section, we prove that the crucial bilinear estimate, which is the key of the iterative method used to prove the local well-posedness result (see [8, 9]), fails for $s < -3/4$. We show also that the Cauchy problem of the Ostrovsky equation cannot be solved by an iteration method for $u_0 \in H^s(\mathbb{R})$, $s < -1$. We deduce that the flow map is not C^2 from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.

Note that we shall use the same notation as in the other section. Consider the initial-value problem for the Ostrovsky equation,

$$\begin{cases} u_t - \beta u_{xxx} + uu_x = \gamma D_x^{-1} u, & x \in \mathbb{R}, \\ u(0,x) = \phi(x). \end{cases} \tag{5.1}$$

We have the following result.

Theorem 5.1. *The estimate*

$$\|\partial_x(uv); \tilde{X}_{s,b-1}\| \le c\|u; \tilde{X}_{s,b}\|\|v; \tilde{X}_{s,b}\| \tag{5.2}$$

fails for $s < -3/4$.

Theorem 5.2. *Let* $s < -1$; *then there does not exist* $T > 0$ *and a space* X_t *continuously embedded in* $C([-T,T], H^s(\mathbb{R}))$ *such that there exists* $C > 0$ *with*

$$\|U(t)\phi\|_{X_T} \le C\|\phi\|_{H^s(\mathbb{R})}, \qquad \phi \in H^s(\mathbb{R}) \tag{5.3}$$

and

$$\left\| \int_0^T U(t-t')u(t')u_x(t')dt' \right\|_{X_T} \le \|u\|_{X_T}^2, \qquad u \in X_T, \tag{5.4}$$

where $U(t)$ *is the semigroup associated with* (5.1) *defined by*

$$\widehat{U(t)\phi}(\xi) = e^{-it(\beta\xi^3 + \frac{\gamma}{\xi})}\hat{\phi}(\xi).$$

Note that (5.3) and (5.4) would be needed to implement a Picard iterative scheme on (5.1) in the space X_T. As a consequence of Theorem 5.2, we can obtain the following result.

Theorem 5.3. *Let* $s < -1$. *Then there does not exist a* $T > 0$ *such that* (5.1) *admits a unique local solution defined on the interval* $[-T,T]$ *and such that the flow-map data solution* $\phi \longrightarrow u(t)$, $t \in [-T,T]$, *is* C^2 *differentiable at zero from* $H^s(\mathbb{R})$ *to* $H^s(\mathbb{R})$.

Proof of Theorem 5.1. We construct examples where the bilinear estimate, which is crucial for local well-posedness, fails by considerations similar to ([14], p. 591).

Let $u \in \tilde{X}_{s,b}$; we define f such that

$$f(\tau,\xi) = \langle \tau + \beta\xi^3 + \frac{\gamma}{\xi} \rangle^b \langle \xi \rangle^s (1+|\xi|^{-1}) \hat{u}(\tau,\xi).$$

Therefore, we have $\|f\|_{L^2_\tau L^2_\xi} = \|u; \tilde{X}_{s,b}\|$. Using the definition of the spaces $\tilde{X}_{s,b}$ and $\tilde{X}_{s,b-1}$, one sees that (5.2) implies that

$$\Big\| \xi \langle \tau + \beta\xi^3 + \frac{\gamma}{\xi} \rangle^{b-1} \langle \xi \rangle^s (1+|\xi|^{-1}) \int\int \frac{f(\tau_1,\xi_1) f(\tau-\tau_1,\xi-\xi_1)}{\langle \tau_1 + \beta\xi_1^3 + \frac{\gamma}{\xi_1} \rangle^b \langle \xi_1 \rangle^s (1+|\xi_1|^{-1})}$$
$$\frac{d\tau_1 d\xi_1}{\langle \tau - \tau_1 + \beta(\xi-\xi_1)^3 + \frac{\gamma}{\xi-\xi_1} \rangle^b \langle \xi-\xi_1 \rangle^s (1+|\xi-\xi_1|^{-1})} \Big\|_{L^2_\tau L^2_\xi} \leq c\|f\|^2_{L^2_\tau L^2_\xi}. \tag{5.5}$$

Let $A, B \subset \mathbb{R}^2$, defined by $A = \{(\tau,\xi) \in \mathbb{R}^2 / N \leq \xi \leq N + \frac{1}{\sqrt{N}},\ |\tau + \beta\xi^3 + \frac{\gamma}{\xi}| \leq 1\}$, $B = -A = \left\{(\tau,\xi) \in \mathbb{R}^2 / (-\tau,-\xi) \in A\right\}$. Let $f(\tau,\xi) = \chi_A(\tau,\xi) + \chi_B(\tau,\xi)$. Clearly, we have $\|f\|_{L^2_\tau L^2_\xi} \leq cN^{-\frac{1}{4}}$. On the other hand, A contains a rectangle with (N^3, N) as a vertex, with dimensions $C_0 N^{3/2} \times N^{-2}$ and longest side pointing in the $(3N^2, 1)$ direction. Let D be the rectangle centered at the origin, of dimensions $C_0 N^{3/2} \times N^{-2}$ and longest side pointing in the $(3N^2, 1)$ direction. Therefore, we have

$$|(f * f)(\tau,\xi)| \geq \frac{C}{\sqrt{N}} \chi_D(\tau,\xi).$$

Consider the set $S = D \cap \left\{(\tau,\xi) / |\xi| \geq C_0 N^{-1}\right\}$. For $(\tau,\xi) \in S$, we have $|\xi| \geq C_0 N^{-1}$, ξ belongs to an interval of radius $\sim N^{-\frac{1}{2}}$ and τ belongs to an interval of radius $\sim N^{\frac{3}{2}}$. Hence, (5.5) implies that

$$cN^{-1/2} \geq cN^{-2s} N^{-1/2} \Big(\int_S |\xi \langle \tau + \beta\xi^3 + \frac{\gamma}{\xi} \rangle^{b-1} \langle \xi \rangle^s (1+|\xi|^{-1})|^2 d\tau d\xi \Big)^{1/2}$$
$$\geq cN^{-2s} N^{-1/2} \Big(\int_c^{cN^{3/2}} \int_{cN^{-1}}^{cN^{-1/2}} |\xi \langle \tau + \beta\xi^3 + \frac{\gamma}{\xi} \rangle^{b-1} \langle \xi \rangle^s (1+|\xi|^{-1})|^2 d\xi\, d\tau \Big)^{1/2}$$
$$\geq cN^{-2s} N^{-1/2} N^{3/4} N^{-1/2} N^{-1} N^{3/2(b-1)},$$

which implies the conditions $\frac{3}{2} b \leq 2s + \frac{3}{4} + \frac{3}{2}$. Then for $s < -3/4$, we have

$$b < 1/2. \tag{5.6}$$

Now, we show the necessity of $b > 1/2$. By polarization and duality, (5.5) is equivalent to

$$\Big\| \frac{1}{\langle \tau_1 + \beta\xi_1^3 + \frac{\gamma}{\xi_1}\rangle^b \langle \xi_1\rangle^s (1+|\xi_1|^{-1})} \times \tag{5.7}$$
$$\int\int \frac{|\xi|\langle\xi\rangle^s(1+|\xi|^{-1}) g(\tau,\xi) h(\tau-\tau_1,\xi-\xi_1) d\tau\, d\xi}{\langle \tau+\beta\xi^3+\frac{\gamma}{\xi}\rangle^{1-b}\langle \tau-\tau_1+\beta(\xi-\xi_1)^3+\frac{\gamma}{\xi-\xi_1}\rangle^b \langle\xi-\xi_1\rangle^s(1+|\xi-\xi_1|^{-1})} \Big\|_{L^2_\tau L^2_\xi}$$
$$\le c\|g\|_{L^2_\tau L^2_\xi}\|h\|_{L^2_\tau L^2_\xi}.$$

Let $D' \subset \mathbb{R}^2$ such that $D' = \left\{(\tau+2N^3, \xi+2N)/(\tau,\xi)\in D\right\}$. We take

$$g(\tau,\xi) = \chi_A(\tau,\xi), \qquad f(\tau,\xi) = \chi_B(\tau,\xi). \tag{5.8}$$

We have

$$\|g\|_{L^2_\tau L^2_\xi}\|h\|_{L^2_\tau L^2_\xi} \le cN^{-1/2},$$

and

$$(\chi_A * \chi_A)(\tau_1,\xi_1) \ge \frac{C}{\sqrt{N}}\chi_{D'}(\tau_1,\xi_1).$$

Then

$$\int_{\mathbb{R}^2} g(\tau,\xi)h(\tau-\tau_1)d\tau\, d\xi = \chi_A * \chi_A(\tau_1,\xi_1) \ge \frac{C}{\sqrt{N}}\chi_{D'}(\tau_1,\xi_1).$$

Now substituting (5.8) into (5.7) and using the above estimate, we get

$$cN^{-1/2} \ge cNN^{-1/2}\times$$
$$\Big(\int_{2N^3-cN^{3/2}}^{2N^3+cN^{3/2}}\int_{N-cN^{-1/2}}^{N+cN^{-1/2}} \frac{1}{\langle \tau_1+\beta\xi_1^3+\frac{\gamma}{\xi_1}\rangle^b\langle\xi_1\rangle^s(1+|\xi_1|^{-1})} d\xi_1\, d\tau_1\Big)^{1/2}$$
$$\ge cN^{1/2}N^{-s}N^{-3b}N^{3/4}N^{-1},$$

which implies the conditions $3b \ge -s + \frac{3}{4}$. Then if $s < -3/4$, we have $b \ge \frac{1}{2}$, which implies a contradiction in view of (5.6). This completes the proof of Theorem 5.1. □

Proof of Theorem 5.2. Suppose that there exists a space X_T such that (5.3) and (5.4) hold. Take $u = U(t)\phi$ in (5.4). Then

$$\Big\|\int_0^t U(t-t')[(U(t')\phi)(U(t')\phi_x)]dt'\Big\|_{X_T} \le C\|U(t)\phi\|^2_{X_T}.$$

Now using (5.3), along with the fact that X_T is continuously embedded in $C([-T,T], H^s(\mathbb{R}))$, we obtain for any $t \in [-T,T]$ that

$$\left\| \int_0^t U(t-t')[(U(t')\phi)(U(t')\phi_x)]dt' \right\|_{H^s(\mathbb{R})} \leq C\|U(t)\phi\|^2_{H^s(\mathbb{R})}. \tag{5.9}$$

We show that (5.9) fails by choosing an appropriate ϕ. Take ϕ defined by its Fourier transform as

$$\hat{\phi}(\xi) = \alpha^{-1/2} N^{-s} \left\{ 1_{I_1}(\xi) + 1_{I_2}(\xi) \right\}, \qquad N \gg 1, \quad 0 < \alpha \ll 1,$$

where I_1 and I_2 are the intervals $I_1 = [-N, -N+\alpha]$ and $I_2 = [N+\alpha, N+2\alpha]$. Clearly, we have $\|\phi\|_{H^s} \sim 1$. We will use the next lemma.

Lemma 5.1. *The following identity holds:*

$$\int_0^t U(t-t')[(U(t')\phi)(U(t')\phi_x)]dt'$$
$$= c \int_{\mathbb{R}^2} e^{i(x\xi)-itp(\xi)} \xi \hat{\phi}(\xi_1)\hat{\phi}(\xi-\xi_1) \frac{e^{-it(p(\xi_1)+p(\xi-\xi_1)-p(\xi))}-1}{p(\xi_1)+p(\xi-\xi_1)-p(\xi))} d\xi_1\, d\xi,$$

where $p(\xi) = \beta\xi^3 + \frac{\gamma}{\xi}$.

Proof. The proof is very similar to that of Lemma 1 in [16]. We will give the details. Taking the inverse Fourier transform with respect to x, it is easy to see that

$$\int_0^t U(t-t')[(U(t')\phi)(U(t')\phi_x)]dt'$$
$$= c \int_0^t \int_{\mathbb{R}} e^{ix\xi-itp(\xi)} e^{it'p(\xi)} \xi \left[e^{-it'p(\cdot)}\hat{\phi}(\cdot) * e^{-it'p(\cdot)}\hat{\phi}(\cdot) \right](\xi) d\xi\, dt'$$
$$= c \int_0^t \int_{\mathbb{R}} e^{ix\xi-itp(\xi)} e^{it'p(\xi)} \xi \int_{\mathbb{R}} e^{-it'p(\xi_1)}\hat{\phi}(\xi_1) e^{-it'p(\xi-\xi_1)}\hat{\phi}(\xi-\xi_1) d\xi_1\, d\xi\, dt'$$
$$= c \int_0^t \int_{\mathbb{R}^2} e^{ix\xi-itp(\xi)} \xi \hat{\phi}(\xi_1)\hat{\phi}(\xi-\xi_1) \int_0^t e^{-it'(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} dt'\, d\xi_1\, d\xi$$
$$= c \int_0^t \int_{\mathbb{R}^2} e^{ix\xi-itp(\xi)} \xi \hat{\phi}(\xi_1)\hat{\phi}(\xi-\xi_1) \frac{e^{-it(p(\xi_1)+p(\xi-\xi_1)-p(\xi)))}-1}{p(\xi_1)+p(\xi-\xi_1)-p(\xi)} d\xi_1\, d\xi. \qquad \square$$

Set $\chi(\xi,\xi_1) = p(\xi_1) + p(\xi-\xi_1) - p(\xi) = -3\beta\xi\xi_1(\xi-\xi_1) + \gamma\frac{\xi^2+\xi_1^2-\xi\xi_1}{\xi\xi_1(\xi-\xi_1)}$. According to the above lemma, we can write

$$\int_0^t U(t-t')[(U(t')\phi)(U(t')\phi_x)]dt' = c\Big(f_1(t,x) + f_2(t,x) + f_3(t,x) \Big),$$

where, from the definition of ϕ, we have the following representations for f_1, f_2, and f_3:

$$f_1(t,x,y) = \frac{c}{\alpha N^{2s}} \int_{\substack{\xi_1 \in I_1 \\ \xi-\xi_1 \in I_1}} e^{ix\xi - itp(\xi)} \frac{e^{-it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)} d\xi_1\, d\xi,$$

$$f_2(t,x,y) = \frac{c}{\alpha N^{2s}} \int_{\substack{\xi_) \in I_2 \\ \xi-\xi_1 \in I_2}} e^{ix\xi - itp(\xi)} \frac{e^{-it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)} d\xi_1\, d\xi,$$

$$\begin{aligned} f_3(t,x,y) = {} & \frac{c}{\alpha N^{2s}} \int_{\substack{\xi_1 \in D_1 \\ \xi-\xi_1 \in D_2}} e^{ix\xi - itp(\xi)} \frac{e^{-it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)} d\xi_1\, d\xi \\ & + \frac{c}{\alpha N^{2s}} \int_{\substack{\xi_1 \in I_2 \\ \xi-\xi_1 \in I_1}} e^{ix\xi - itp(\xi)} \frac{e^{-it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)} d\xi_1\, d\xi. \end{aligned}$$

Then clearly,

$$\mathcal{F}_{x\to\xi}(f_1)(t,\xi) = \frac{c\xi e^{-itp(\xi)}}{\alpha N^{2s}} \int_{\substack{\xi_1 \in I_1 \\ \xi-\xi_1 \in I_1}} \frac{e^{-it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)} d\xi_1\, d\xi,$$

$$\mathcal{F}_{x\to\xi}(f_))(t,\xi) = \frac{c\xi e^{-itp(\xi)}}{\alpha N^{2s}} \int_{\substack{\xi_1 \in I_2 \\ \xi-\xi_1 \in I_2}} \frac{e^{-it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)} d\xi_1\, d\xi,$$

$$\begin{aligned} \mathcal{F}_{x\to\xi}(f_3)(t,\xi) = {} & \frac{c\xi e^{-itp(\xi)}}{\alpha N^{2s}} \int_{\substack{\xi_1 \in I_1 \\ \xi-\xi_1 \in I_2}} \frac{e^{-it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)} d\xi_1\, d\xi \\ & + \frac{c\xi e^{-itp(\xi)}}{\alpha N^{2s}} \int_{\substack{\xi_1 \in I_2 \\ \xi-\xi_1 \in I_1}} \frac{e^{-it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)} d\xi_1\, d\xi. \end{aligned}$$

Note that, for a fixed t, the supports of $\mathcal{F}(f_j)(t,\xi)$, $j = 1,2,3$, are disjoint. Therefore,

$$\Big\| \int_0^T U(t-t')u(t')u_x(t')dt' \Big\|_{H^s(\mathbb{R})} \geq \|f_3\|_{H^s(\mathbb{R})}.$$

We now give a lower bound for $\|f_3\|_{H^s(\mathbb{R})}$. Then, for $(\xi_1, \xi-\xi_1) \in I_1 \times I_2$ or $(\xi_1, \xi-\xi_1) \in I_2 \times I_1$, one has $\xi(\xi,\xi_1) \sim \alpha N^2 + \frac{1}{\alpha}$. Take a particular

$t_\alpha = \alpha N^{-2}$. Therefore, $t_\alpha \chi(\xi,\xi_1) \sim 0(1)$ and we have

$$\Big|\frac{e^{-it_\alpha \chi(\xi,\xi_1)} - 1}{\chi(\xi,\xi_1)}\Big| \geq c\alpha N^{-2},$$

for $\xi_1 \in I_1$, $\xi - \xi_1 \in I_2$ or $\xi_1 \in I_2$, $\xi - \xi_1 \in I_1$. Hence,

$$\|f_3(t_\alpha,\cdot)\|_{H^s(\mathbb{R})} \gtrsim \frac{\alpha^{1/2}\alpha\alpha N^{-2}\alpha}{\alpha N^{2s}} = \alpha^{5/2}N^{-2-2s}.$$

For $s < -1$, we have $-2-2s > 0$. If we take α such that $\alpha \sim N^{(2+2s+2\epsilon)\frac{2}{5}}$, with $0 < \epsilon \ll 1$, we obtain $\alpha^{5/2}N^{-2-2s} \sim N^{2\epsilon}$. Now, using (5.9) we get

$$1 \sim \|\phi\|^2_{H^s(\mathbb{R})} \gtrsim \|f_3(t_\alpha,\cdot)\|_{H^s(\mathbb{R})} \gtrsim N^{2\epsilon},$$

which is a contradiction for $N \gg 1$ and $\epsilon \ll 1$. This completes the proof of Theorem 5.2. □

Proof of Theorem 5.3. Consider the Cauchy problem

$$\begin{cases} u_t - \beta u_{xxx} + uu_x = \gamma D_x^{-1}u, & x \in \mathbb{R}, \\ u(0,x) = \delta\phi(x), \qquad \delta \ll 1, & \phi \in H^s(\mathbb{R}). \end{cases} \tag{5.10}$$

Suppose that $u(\delta,t,x)$ is a local solution of (5.10) and that the flow map is C^2 at the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$. We have

$$u(\delta,t,x) = \delta U(t)\phi(x) + \int_0^t U(t-t')(u(\delta,t'x)u_x(\delta,t',x))dt'.$$

Then

$$\frac{\partial u}{\partial \delta}(0,t,x) = U(t)\phi(x),$$

$$\frac{\partial^2 u}{\partial \delta^2}(0,t,x) = 2\int_0^t U(t-t')[U(t')\phi U(t')\phi_x]dt'.$$

The assumption of C^2 regularity yields

$$\Big\|\int_0^t U(t-t')[(U(t')\phi)(U(t')\phi_x)]dt'\Big\|_{H^s(\mathbb{R})} \leq \|\phi\|^2_{H^s(\mathbb{R})}.$$

But the above estimate is (5.9), which has been shown to fail in the proof of Theorem 5.2. □

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