LARGE-TIME BEHAVIOR OF SOLUTIONS TO A DISSIPATIVE BOUSSINESQ SYSTEM

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(Submitted by: Roger Temam)

Abstract. In this article we consider the Boussinesq system supplemented with some dissipation terms. These equations model the propagation of a water wave in shallow water. We prove the existence of a global smooth attractor for the corresponding dynamical system.

1. Introduction

This article is concerned with the long-time behavior of the solutions to a damped-forced Boussinesq system that reads

$$\begin{cases} \eta_t + u_x + (\eta u)_x - \eta_{xx} &= 0, \\ u_t - u_{txx} - u_{xx} + \eta_x + u_x u &= f. \end{cases} \tag{1.1}$$

Here, we have an incompressible fluid on a channel. $u(t,x)$ is the horizontal velocity at the top of the fluid, η is the fluctuation of the height of the fluid with respect to the rest position, which is $z = \eta(t,x) = 0$, assuming that the bottom of the channel is at $z = -1$. Observe that in our model we have to ensure that $\eta(t,x) > -1\ \forall t, x$.

Here, $f(x)$ is an external force that does not depend on time, and the damping terms are respectively $-u_{xx}, -\eta_{xx}$. In the conservative case, that reads

$$\begin{cases} \eta_t + u_x + (\eta u)_x &= 0, \\ u_t - u_{txx} + \eta_x + u_x u &= 0; \end{cases} \tag{1.2}$$

this system was introduced by Boussinesq in 1877 to model the fluctuation of a water wave in shallow water. Other well-known asymptotic models are

Accepted for publication: March 2007.
AMS Subject Classifications: 37L30, 37L50, 35Q35, 35Q53.

Korteweg-de Vries equations and the Benjamin-Bona-Mahony equation, also known as the regularized long-wave equation. For these asymptotic models we would like to refer to [5, 19] and to the references therein.

In this article we are interested in the dissipative case. In the case where $f = 0$, the solutions converge to the equilibrium, and the issue is to find out the rate of convergence. Following the pioneering work of Amick, Bona and Schonbek [4], this issue has been addressed in the case where $x \in \mathbb{R}^D$, $D \geq 2$ using the famous Schonbek splitting method [16]. Here we plan to study the dynamical system provided by (1.2) in the framework of an infinite-dimensional dynamical system [17, 15, 11]. Our main result is stated as follows:

Theorem 1.1. *The dynamical system provided by* (1.1) *features a compact global attractor into a suitable energy space. Moreover, this compact global attractor has finite fractal and Hausdorff dimension.*

This result compares with previous results obtained for dissipative KdV equations [7, 8, 9, 10] or dissipative BBM equations [18, 2]. This article is organized as follows: In the next section we introduce the mathematical framework that we have chosen to study this dynamical system. In a third section we address the initial-value problem for the evolution equation. In a fourth section we prove the existence of a smooth finite-dimensional attractor.

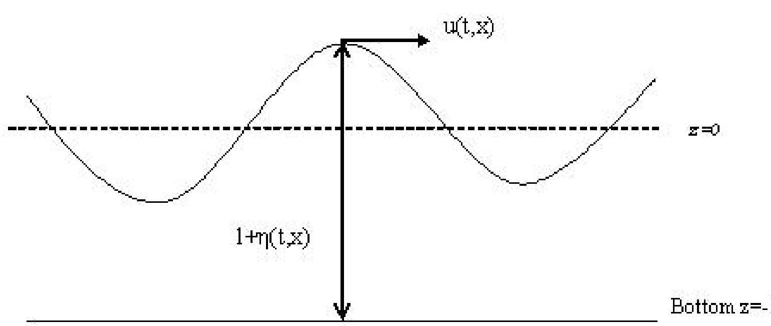

Figure 1. Fluid at rest $(u, \eta) = (0, 0)$.

2. MATHEMATICAL FRAMEWORK

2.1. **Initial data.** For the sake of convenience, we are interested in considering periodic boundary conditions. We now consider functions for $x \in [0,1]$ that are 1-periodic. We also assume that $\int_0^1 f(x)dx = 0$ and $f \in L^2(0,1)$.

Introducing $w(t,x) = 1 + \eta(t,x)$, we instead use the following system:

$$\begin{cases} u_t - u_{txx} - u_{xx} + w_x + u_x u & = & f \quad \text{for } (x,t) \in (0,1) \times \mathbb{R}_+\,, \\ w_t + (wu)_x - w_{xx} & = & 0 \quad \text{for } (x,t) \in (0,1) \times \mathbb{R}_+\,. \end{cases} \tag{2.1}$$

The natural space for the velocity $u(t,x)$ is then

$$\dot{H}^1_{per} = \Big\{ v \in H^1_{per} / \int_0^1 v(x)dx = 0 \Big\}. \tag{2.2}$$

We then assume that $u_0 \in \dot{H}^1_{per}$. We now proceed to the assumptions on $w_0(x)$. The first physical assumption is that

$$\inf w_0(x) > 0. \tag{2.3}$$

This assumption ensures that the top of the fluid does not hit the bottom of the channel. The second assumption is that

$$\int_0^1 w_0 = 1, \tag{2.4}$$

which describes that w_0 fluctuates around 1, the height at rest of the fluid, and that the fluids have constant volume.

The third assumption is related to the very definition of the entropy for convection equation; see [16].

Introducing $Q(y) = y \ln y - y + 1$, which is convex and non-negative, we assume that

$$\int_0^1 Q(w_0(x))dx < +\infty. \tag{2.5}$$

Remark 2.1. Assume that $f = 0$ here and that we are given a regular enough solution (u,w) to (2) such that $w > 0 \ \forall x,t$. Multiplying (2) by $(u, 1 + \ln w)$ and summing the two resulting equations, we thus obtain

$$\frac{d}{dt}\Big[\frac{1}{2}|u|^2_{H^1} + \int_0^1 w \ln w\Big] + |u_x|^2_{L^2} + \int_0^1 \frac{w_x^2}{w} = 0; \tag{2.6}$$

then, using $\int_0^1 w = 1$,

$$\lim_{t\to\infty} \Big[\frac{1}{2}\int_0^1 (u^2 + u_x^2) + \int_0^1 Q(w)\Big] = 0,$$

and the fluid converges to the equilibrium $(u, w) = (0, 1)$.

2.2. **Functional analysis.** We set

$$\mathring{K} = \{w(x) > 0 : \int_0^1 w(x)dx = 1 \text{ and } \int_0^1 Q(w(x))dx < \infty\}; \tag{2.7}$$

one may wonder which kind of topology we shall use in $\mathring{K}$. First of all, we observe that $\mathring{K}$ is a convex set (this is obvious since the map $y \to Q(y)$ is a convex function). Furthermore, $\mathring{K}$ is related to the following Orlicz space (see [1]). Introducing

$$H(y) = (1 + y)\ln(1 + y) - y, \tag{2.8}$$

which is a convex function, we observe that H and Q are related by the following inequalities.

Lemma 2.2. $\exists C_0 > 0$ *such that* $\forall y \geq 0$

$$Q(y) - 1 \leq H(y) \leq C_0(Q(y) + 1). \tag{2.9}$$

Proof. On the one hand, if $y \in [0, 1]$, then $Q(y) \leq 1 - y \leq 1 + H(y)$. If $y \geq 1$, then

$$Q(y) = y \ln y + 1 - y \leq (y + 1)\ln(y + 1) - y + 1.$$

To establish the reverse inequality, we observe that $y \ln y \sim_\infty (1+y)\ln(y+1)$; then, for large y's , say $y \geq R$, $H(y) \leq 2(y \ln y - y + 1)$.

For $y \in [0, R]$, $H(y)$ is bounded by $H(R)$. Then the proof of the lemma is completed. □

Therefore, $w > 0$ belongs to $\mathring{K}$ if and only if $\int_0^1 w = 1$ and $w \in L_H$, the Orlicz space, whose norm is defined by

$$\|w\|_{L_H} = \inf\Big\{\lambda > 0,\ \int_0^1 H(\frac{w}{\lambda})(x)dx \leq 1\Big\}. \tag{2.10}$$

We now endow $\mathring{K}$ with the topology of L_H, which is given by the distance

$$d(w_1, w_2) = \|w_1 - w_2\|_{L_H}. \tag{2.11}$$

Remark 2.3. $\mathring{K}$ is not a closed subset of L_H, but $K = \{w \geq 0;\ \int_0^1 w = 1$ *and* $\int_0^1 Q(w) < \infty\}$ is. Consider w_n in K such that $\|w_n - w\|_{L_H} \to 0$. There exists $\lambda_n \to 0$ such that

$$\int_0^1 H(\frac{|w_n - w|}{\lambda_n})(x)dx \leq 2. \tag{2.12}$$

We shall use

$$H(y) \geq \frac{1}{2}(\sqrt{y+1}-1)^2. \tag{2.13}$$

Then, setting $v_n(x) = \lambda_n^{-1}|w_n(x) - w(x)|$,

$$\int_0^1 \frac{|w_n - w|}{\lambda_n} = \int_0^1 v_n = \int_0^1 (\sqrt{v_n+1}-1)^2 + 2\int_0^1 (\sqrt{v_n+1}-1)$$
$$\leq \quad 4\int_0^1 H(v_n) + 4(\int_0^1 (\sqrt{v_n+1}-1)^2)^{\frac{1}{2}} \leq 4(1+\sqrt{2}).$$

Then $w_n \to w$ in L^1, and, up to a subsequence extraction $w_n \to w$ almost everywhere.

3. The initial-value problem

3.1. **Main theorem.**

Theorem 3.1. *Consider the initial data* (u_0, w_0) *in* $H^1_{per} \times \mathring{K}$. *Then there exists a unique solution for* (2.1),

$$(u(t), w(t)) \in C(\mathbb{R}_+; H^1_{per}) \times C(\mathbb{R}_+; \mathring{K}),$$

which satisfies moreover $\sqrt{w} \in L^2_{loc}(\mathbb{R}_+; H^1_{per})$.

Remark 3.2. In the theorem above, we would like to point out two important facts:

$*$ If $\inf w_0(x) > 0$, then, for any $t, x > 0$, $w(t,x) > 0$, which is physically relevant.

$*$ The dissipative Boussinesq system provides a smoothing effect in the w variable. In fact, $\sqrt{w}$ belongs $C(\mathbb{R}_+ - \{0\}; H^1_x)$.

Proof of Theorem 3.1. Existence: We first regularize the initial data $(u_0^\epsilon, w_0^\epsilon)$ to construct smooth solutions $(u^\epsilon(t), w^\epsilon(t))$in $C([0,T], H^1_{per} \times H^1_{per})$ for instance. We then prove some a priori estimates, and finally pass to limit. Since these methods are classical, we just indicate below how to derive the a priori estimates, referring the reader to [3] for details. For the sake of simplicity, we drop the subscript ε throughout the proof of the theorem.

First step: We first prove $w(t,x)$ is positive. Consider $\alpha = \inf w_0 > 0$. Introduce

$$J(t,x) = \max(0, \alpha - w(t,x)). \tag{3.1}$$

Using Kato's inequality (see [12, 13]), which reads (in the distributions sense)

$$w_{xx} sgn(w) \leq (|w|)_{xx}, \tag{3.2}$$

we thus obtain

$$J_t + (uJ)_x - J_{xx} \leq 0. \tag{3.3}$$

Then, integrating in x, we have that

$$\int_0^1 J(t,x)dx \leq \int_0^1 J(0,x)dx = 0, \tag{3.4}$$

and $w(t,x) \geq \alpha$ almost everywhere. This result is related to parabolic Harnack's inequalities; see [6].

Second step: A priori estimate in $H^1_{per} \times \mathring{K}$. We begin with a technical lemma

Lemma 3.3. *Consider $w > 0$, a smooth periodic function such that*

$$\int_0^1 w(x)dx = 1;$$

then

$$\int_0^1 Q(w)dx \leq \Big(\int_0^1 \frac{w_x^2}{w} \Big)^{\frac{1}{2}}. \tag{3.5}$$

Proof. Since $-\ln$ is convex, $Q(w) \leq w(w-1) - w + 1$. Then, using once more $\int_0^1 w = 1$,

$$\int_0^1 Q(w) \leq \int_0^1 w^2 - 1. \tag{3.6}$$

On the other hand, for any φ-smooth periodic function,

$$\varphi^2(x) \leq 2\Big(\int_0^1 \varphi^2 \Big)^{\frac{1}{2}} \Big(\int_0^1 \dot{\varphi}^2 \Big)^{\frac{1}{2}} + \int_0^1 \varphi^2. \tag{3.7}$$

We apply this to $\varphi = \sqrt{w}$. Then

$$\int_0^1 w^2 \leq \|w\|_{L^\infty} \Big(\int_0^1 w \Big) \leq \Big(\int_0^1 \frac{w_x^2}{w} \Big)^{\frac{1}{2}} + 1; \tag{3.8}$$

this concludes the proof of the lemma □

We now proceed to the a priori estimates. We multiply (2.1) by $(u, 1 + \ln w)$. We integrate in x over $[0,1]$ the resulting equations and then sum to obtain

$$\frac{d}{dt}\Big[\int_0^1 \Big(w \ln w - w + 1 \Big) + \frac{1}{2}\|u\|^2_{H^1} \Big] + \int (uw)_x \ln w \\ + \int_0^1 \frac{w_x^2}{w} + \int_0^1 w_x u + ||u_x||^2_{L^2} = \int_0^1 fu. \tag{3.9}$$

Integrating by parts, we observe that

$$\int_0^1 (uw)_x \ln w + \int_0^1 w_x u = 0. \tag{3.10}$$

On the other hand, using the Young and Poincaré-Wirtinger inequalities, we obtain

$$\begin{aligned} -\int_0^1 fu + \|u_x\|_{L^2}^2 \quad & \geq \|u_x\|_{L^2}^2 - (\pi - \frac{1}{2})\|u\|_{L^2}^2 - \frac{1}{4\pi - 2}\|f\|_{L^2}^2 \\ & \geq \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{4\pi - 2}\|f\|_{L^2}^2. \end{aligned} \tag{3.11}$$

We thus obtain

$$\frac{d}{dt}\Big[\int_0^1 Q(w) + \frac{1}{2}\|u\|_{H^1}^2\Big] + \frac{1}{2}\|u\|_{H^1}^2 + \int_0^1 \frac{w_x^2}{w} \leq \frac{1}{4\pi - 2}\|f\|_{L^2}^2. \tag{3.12}$$

We now infer from (3.6) and (3.8) that

$$\int_0^1 Q(w) \leq \Big(\int_0^1 \frac{w_x^2}{w}\Big) + \frac{1}{4}. \tag{3.13}$$

We combine this inequality together with (3.9), and we integrate with respect to t (thanks to the Gronwall lemma), and thus obtain

$$\int_0^1 Q(w(t)) + \frac{1}{2}\|u(t)\|_{H^1}^2 \leq \frac{1}{4} + \frac{1}{4\pi - 2}\|f\|_{L^2}^2 + e^{-t}(\int_0^1 Q(w_0) + \frac{1}{2}\|u_0\|_{H^1}^2). \tag{3.14}$$

Since H is a convex function, for $\lambda \geq 1$ (observe $H(0) = 0$), and due to Lemma 2.2,

$$\int_0^1 H(\frac{w}{\lambda})dx \leq \frac{1}{\lambda}\int_0^1 H(w)dx \leq \frac{C_0}{\lambda}\Big[1 + \sup_{t\geq 0}\int_0^1 Q(w(t))\Big] \leq 1, \tag{3.15}$$

for λ large enough. Then $w(t)$ remains bounded in the Orlicz space L_H. We then have established an a priori estimate for (u, w) in $L^\infty(R_+; H^1_{per}) \times L^\infty(R_+; L_H)$.

Remark 3.4. Actually (3.12) implies that $\sqrt{w}$ is almost everywhere in t in H^1_x. Since this Sobolev space is an algebra and since we can solve the evolution equation under consideration with initial data w in H^1_x, this implies that for all $t > 0$ w is in H^1_x (smoothing effect). We make this fact precise below.

Third step: Smoothing effect; $\sqrt{w}$ belongs to H^1 for $t > 0$. We set $v = \sqrt{w}$, which solves

$$v_t - v_{xx} - \frac{v_x^2}{v} + \frac{1}{2}u_x v + u v_x = 0. \tag{3.16}$$

Multiply (3.16) by $-v_{xx}$ and integrate. We then get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_x\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2 + \frac{1}{3}\int \frac{v_x^4}{v^2} &= \frac{1}{2}\int u_x v v_{xx} + \int u v_x v_{xx} \\ &\le \frac{1}{2}\|v_{xx}\|_{L^2}^2 + c\Big[\|v\|_\infty^2\|u_x\|_{L^2}^2 + \|u\|_\infty^2\|v_x\|_{L^2}^2\Big]. \end{aligned} \tag{3.17}$$

Then, since $\|v\|_{L^2} = 1$,

$$\frac{d}{dt}\|v_x\|_{L^2}^2 \le c_1\|u\|_{H^1}^2\|v\|_{H^1}^2 \le c_1\|u\|_{H^1}^2(1 + \|v_x\|_{L^2}^2). \tag{3.18}$$

Due to Gronwall's lemma and since u is bounded in H^1, we then get the H^1 bound on $v = \sqrt{w}$.

Remark 3.5. Actually, for $T < +\infty$ $\sqrt{w}$ is in $L^2(0, T; H_x^1)$, and then w is in $L^1(0, T; H_x^1)$.

Fourth step: Uniqueness. Consider two trajectories (u_2, w_2) and (u_1, w_1) that start from the same initial data. Due to the previous estimates both (u_2, w_2) and (u_1, w_1) remain bounded in $L^1(0, T; L_x^\infty)$. We set $u = u_2 - u_1$ and $w = w_2 - w_1$, solutions to

$$\begin{cases} u_t - u_{txx} - u_{xx} + w_x + \frac{1}{2}(u_2^2 - u_1^2)_x = 0, \\ w_t - w_{xx} + (u_2 w_2 - u_1 w_1)_x = 0. \end{cases} \tag{3.19}$$

Then multiply these equations by (u, w) and integrate the resulting equation to obtain

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{H^1} + \|w\|_{L^2}) \\ &\le C(1 + \max(\|u_2\|_{L^\infty}, \|w_2\|_{L^\infty}, \|u_1\|_{L^\infty}, \|w_1\|_{L^\infty})(\|u\|_{H^1} + \|w\|_{L^2}).\end{aligned}$$

The results follow promptly. □

4. The global attractor

4.1. **Existence of the global attractor.** To begin with we state and prove

Proposition 4.1. *The semigroup $S(t)$ defined on $\dot{H}^1_{per} \times \mathring{K}$ possesses an absorbing set that is bounded in $\dot{H}^1_{per} \times H^1$*

Proof. The existence of a bounded absorbing set in $\dot{H}^1_{per} \times \mathring{K}$ comes from the estimate (3.14) of the previous section. Let t_0 be the entrance time into this absorbing ball. Going back to (3.18) and applying the uniform Gronwall lemma (see Lemma III.1.1 in [17]), we thus obtain that for $t > 0$, for some numerical constant c,

$$t\|(\sqrt{w})_x(t+t_0)\|^2_{L^2} \leq c(1+t)(1+\|f\|^2_{L^2}) \exp(c + c\|f\|^2_{L^2}). \tag{4.1}$$

Therefore, $\sqrt{w}$ is bounded for large times into H^1. Since H^1 is an algebra, then w is also bounded for large times in H^1. □

Theorem 4.2. *The semigroup $S(t)$ possesses a global attractor $\mathcal{A}$ in $\dot{H}^1_{per} \times L_H$, which is a compact subset of $H^2 \times H^2$.*

Proof. We introduce the splitting $(u, w) = (u^1, w) + (u^2, 0)$, where u^1 satisfies

$$u^1_t - u^1_{txx} - u^1_{xx} + w_x + u_x u = f, \qquad u^1(0) = 0, \tag{4.2}$$

and u^2 is solution to

$$u^2_t - u^2_{txx} - u^2_{xx} = 0, \qquad u^2(0) = u_0. \tag{4.3}$$

We now define the families $\{S_1(t)\}_{t\geq 0}$ and $\{S_2(t)\}_{t\geq 0}$ of maps in $H^1 \times L_H$, where $S_1(t)(u_0, w_0) = (u^1, w)$ and $S_1(t)(u_0, w_0) = (u^2, 0)$.

First step: We prove that u^1 is bounded in H^2. For this we multiply (4.2) by $-u^1_{xx}$ and integrate between 0 and 1 to obtain

$$\frac{1}{2}\frac{d}{dt}\|u^1_x\|^2_{H^1} + \|u^1_{xx}\|^2_{L^2} = -\int f u^1_{xx} + \int w_x u^1_{xx} + \int u u_x u^1_{xx},$$

due to the Young and Cauchy-Schwarz inequalities, then

$$\frac{d}{dt}\|u^1_x\|^2_{H^1} + \|u^1_{xx}\|^2_{L^2} \leq c\Big(\|f\|^2_{L^2} + \|w_x\|^2_{L^2} + \|u_x\|^2_{L^2}\|u\|^2_{L^\infty}\Big), \tag{4.4}$$

due to Proposition 4.1, (3.14), the Gronwall and Poincaré inequalities, we obtain that u^1 remains in a bounded set of H^2 for large times ($t > t_0$ the entrance time into the absorbing ball).

On the other hand, it is an exercise to prove that

$$u^2(t) \to 0 \text{ strongly in } H^1 \text{ when } t \to \infty. \tag{4.5}$$

Then $S_1(t)(u_0, w_0)$ is bounded in $H^2 \times H^1$ then compact in $H^1 \times L_H$, and $S_2(t)(u_0, w_0) \to 0$ in $H^1 \times L_H$ uniformly on bounded sets.

Then from Theorem I.1.1 in [T] we have the existence of a global attractor $\mathcal{A}$ in $H^1 \times L_H$ that is moreover a bounded set in $H^2 \times H^1$.

We now prove that for a trajectory (u,w) in the global attractor, w remains bounded in H^2. For that purpose, multiply the second equation in (2.1) by w_{4x} and integrate by parts to obtain

$$\frac{d}{dt}\|w_{xx}\|_{L^2}^2 + \|w_{xxx}\|_{L^2}^2 = \int_0^1 u_{xx} w w_{3x} + 2\int_0^1 u_x w_x w_{3x} - \frac{1}{2}\int_0^1 u_x w_{2x}^2$$
$$\leq c\|u\|_{H^2}^2 \|w\|_{H^1}^2 + \frac{1}{2}\|w_{xxx}\|_{L^2}^2. \tag{4.6}$$

Then the results follow promptly. It remains to prove that the global attractor, which is bounded in $H^2 \times H^2$, is in fact a compact subset of this space. This can be performed by the energy equation method of [14], which is a suitable adaptation of the famous J. Ball argument. This is standard and will not be reproduced here; we refer the reader to [3] for details. □

4.2. **Dimension of the attractor.** In this section we are going to prove that the global attractor $\mathcal{A}$ has a finite dimension in $\mathcal{E} = \dot{H}^1 \times \{w \in L^2; \int_0^1 w = 1\}$. $\mathcal{E}$ is an affine space whose associated vector space is $E = \dot{H}^1 \times \dot{L}^2$. To begin with, we need a result on the differentiability of the semigroup $S(t)$ on the global attractor. Consider the non-autonomous linearized system

$$\begin{cases} v_t - v_{txx} - v_{xx} + h_x + (uv)_x = 0 \\ h_t + (uh + vw)_x - h_{xx} = 0, \end{cases} \tag{4.7}$$

where $(u(t), w(t)) = S(t)(u_0, w_0)$, $(u_0, w_0) \in \mathcal{E}$, is a trajectory solution of (2.1) and $(v_0, h_0) \in E$. Actually, the linear mapping $DS(t)(u_0, w_0)(v_0, h_0) = (v(t), h(t))$ is the uniform differential of $S(t)$, as stated below.

Theorem 4.3. *The non-autonomous PDE* (4.7) *provides a well-posed initial value problem in E. Moreover, for $T > 0$, $(v_0, h_0) \in E$, $(u_0, w_0) \in \mathcal{A}$, $t \leq T$ there exists a constant $C = C(T)$ such that*

$$\|S(t)(u_0 + v_0, w_0 + h_0) - S(t)(u_0, w_0) - DS(t)(u_0, w_0)(v_0, h_0)\|_E$$
$$\leq C(T)\|(v_0, h_0)\|_E^\delta, \tag{4.8}$$

where $1 < \delta < 2$.

Proof. To prove that the initial-value problem is well-posed is standard and thus omitted. Consider the solutions $(u_1(t), w_1(t)) = S(t)(u_0, w_0)$, $(u_2(t), w_2(t)) = S(t)(u_0 + v_0, w_0 + h_0)$, $(v(t), h(t)) = \big(DS(t)(u_0, w_0)\big)(v_0, h_0)$.

Then $(p,q)=(u_2,w_2)-(u_1,w_1)-(v,h)$ satisfies the system

$$\left\{\begin{array}{lcl} p_t-p_{txx}-p_{xx}+q_x+(\frac{1}{2}v^2+vp+u_1p+\frac{1}{2}p^2)_x & = & 0 \\ q_t-q_{xx}+(pq+ph+vq+hv+qu_1+w_1p)_x & = & 0. \end{array}\right. \tag{4.9}$$

We shall use in the sequel that $\int_0^1 p=\int_0^1 q=0$; then $\|p\|_{H^1}$ and $\|p_x\|_{L^2}$ define equivalent norms. Multiply (4.9) by (p,q) and integrate to obtain (due to straightforward computations)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\|q\|_{L^2}^2+\|p\|_{H^1}^2\Big]+\|q_x\|_{L^2}^2+\|p_x\|_{L^2}^2 \\ &=-\int q_xp+\frac{1}{2}\int (p+v)^2p_x+\int u_1pp_x \\ &\quad +\int (pq+ph+vq+vh+w_1p+qu_1)q_x \\ &\leq (\|v\|_{H^1}+\|h\|_{L^2})\|v\|_{H^1}\Big[\|q_x\|_{L^2}+\|p_x\|_{L^2}\Big] \\ &\quad +(1+\|u_1\|_{H^1}+\|v\|_{H^1}+\|w_1\|_{L^2}+\|h\|_{L^2}) \\ &\quad \times\Big[\|q_x\|_{L^2}^2+\|p_x\|_{L^2}^2\Big]+\Big[\|p\|_{L^2}\|q_x\|_{L^2}^2\Big]. \end{aligned}$$

We thus obtain, using the bounds on the attractor and the local-in-time bounds on (v,h),

$$\frac{d}{dt}\Big[\|(p,q)\|_E^2\Big]\leq K_1(T)\|(p,q)\|_E^2+K_2(T)\|(v_0,h_0)\|_E^4+K_3(T)\|(p,q)\|_E^4. \tag{4.10}$$

Consider a given interval of time $[0,T]$. Set $\varepsilon^2=K_2(T)\|(v_0,h_0)\|_E^4$ small. Then $\phi(t)=\exp(-tK_1(T))\|(p,q)\|_E^2(t)$ satisfies the ODE

$$\dot{\phi}\leq K\phi^2+\varepsilon^2, \tag{4.11}$$

supplemented with $\phi(0)=0$. Then $E(t)\leq 2\varepsilon$ if ε is small enough. □

We now give the main result of this section:

Theorem 4.4. *The fractal and Hausdorff dimensions in $\mathcal{E}$ of the attractor $\mathcal{A}$ are finite.*

Proof. Set $\xi=(u,w)$ and $\beta=(v,h)$. Now we study the operators $DS(t)\xi_0$ that contract the m-dimensional volumes in $\mathcal{E}$. Let $\beta_0^1,\dots,\beta_0^m$ in E. We study the following quantities:

$$G_m=\|\beta^1(t)\wedge\cdots\wedge\beta^m\|_E^2=\det_{1\leq i,j\leq m}\Big(\beta^i(t),\beta^j(t)\Big)_E, \tag{4.12}$$

where $\beta^i(t) = (DS(t)\xi_0)\beta_0^i$. The Gram determinant G_m represents the volume of an m-dimensional polyhedron defined by the vectors $\beta^1(t), \dots, \beta^m(t)$. We will show that for sufficiently large m this determinant decays exponentially as $t \to \infty$.

We consider $\beta(t) = (DS(t)\xi_0)\beta_0$ a solution of (4.7); we multiply by $\beta = (v, h)$ and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\|\beta\|_E^2 + \|v_x\|_{L^2}^2 + \|h_x\|_{L^2}^2 = \int_0^1 uhh_x + \int_0^1 vwh_x + \int_0^1 uvv_x + \int_0^1 hv_x. \tag{4.13}$$

Recall that (u, w) is a trajectory that belongs to the global attractor. Introduce $M = c(1 + \|f\|_{L^2}^2)$, that is, the H^1 bound for u in the attractor (see (3.14)). We do not want to use estimates that involve H^2 norms of u such as (4.1).

We bound the fourth term in the right-hand side of (4.13) by $\frac{1}{4}\|h_x\|_{L^2}^2 + \|v\|_{L^2}^2$. The third term can be bounded as follows:

$$\Big|\int_0^1 uvv_x\Big| \le \frac{1}{2}\|u\|_{H^1}\|v\|_{L^2}\|v\|_{L^\infty}$$
$$\le c\|u\|_{H^1}\|v\|_{L^2}^{3/2}\|v_x\|_{L^2}^{1/2} \le \frac{1}{4}\|v_x\|_{L^2}^2 + cM^{4/3}\|v\|_{L^2}^2.$$

We now proceed to the first term as follows:

$$\Big|\int_0^1 uhh_x\Big| \le \frac{1}{2}\|u\|_{H^1}\|h\|_{L^4}^2$$
$$\le c\|u\|_{H^1}\|h\|_{H^{-1}}^{3/4}\|h_x\|_{L^2}^{5/4} \le \frac{1}{4}\|h_x\|_{L^2}^2 + cM^{8/3}\|h\|_{H^{-1}}^2.$$

For the second term, we have

$$|\int_0^1 vwh_x| \le \frac{1}{4}\|h_x\|_{L^2}^2 + \|w\|_{L^2}^2\|v\|_{L^\infty}^2$$
$$\le \frac{1}{4}\|h_x\|_{L^2}^2 + \frac{1}{8}\|v_x\|_{L^2}^2 + c\|w\|_{L^2}^4\|v\|_{L^2}^2.$$

To go further, we need a new estimate on w that reads

Lemma 4.5. *For any (u, w) in $\mathcal{A}$, $\|w(t)\|_{L^2} \le c(1 + M^2)$.*

Proof. for a given trajectory in the attractor multiply the second equation by w and integrate to obtain

$$\frac{d}{dt}\|w\|_{L^2}^2 + 2\|w_x\|_{L^2}^2 = 2\int_0^1 uww_x \le \|w_x\|_{L^2}^2 + \|u\|_{L^\infty}^2\|w\|_{L^\infty}; \tag{4.14}$$

here we have used that $\int_0^1 w = 1$. We then infer from (4.14) that, using the Poincaré-Wirtinger inequality,

$$\int_0^1 (w-1)^2 = \|w\|_{L^2}^2 - 1 \leq \|w_x\|_{L^2}^2, \tag{4.15}$$

that

$$\frac{d}{dt}\|w\|_{L^2}^2 + \frac{1}{4}\|w\|_{L^2}^2 \leq c(1+M^4). \tag{4.16}$$

Then the classical Gronwall lemma leads to the result. □

We then have

$$\frac{1}{2}\frac{d}{dt}\|\beta\|_E^2 + \|\beta\|_E^2 = c\left((1+M^8)\|\beta\|_{L^2\times\dot{H}^{-1}}^2\right). \tag{4.17}$$

We introduce the Gram determinant

$$G_m(t) = \det_{1\leq i,j\leq m}\left(\Lambda(\beta^i(t),\beta^j(t))\right)_E,$$

where $\Lambda(a,b) = \frac{\|a+b\|_E^2 - \|a-b\|_E^2}{4}$, and that represents the m-dimensional volume. Then we can proceed as in [17, 7] to establish that

$$\frac{dG_m}{dt} + mG_m \leq c(1+M^8)\left(\sum_{l=1}^{m} \max_{A\subset\mathbb{R}^m, dim A = l} \min_{v\in A, v\neq 0} \frac{\|\beta\|_{L^2\times\dot{H}^{-1}}^2}{\|\beta\|_E^2}\right) G_m. \tag{4.18}$$

Since the eigenvalues of the Laplace periodic operator are $4\pi^2k^2$, each of multiplicity 2,

$$\sum_{l=1}^{m} \max_{A\subset\mathbb{R}^m, dim A = l} \min_{v\in A, v\neq 0} \frac{\|\beta\|_{L^2\times\dot{H}^{-1}}^2}{\|\beta\|_E^2} \sim 2\pi^2 \sum_{k=1}^{m/2} (2\pi k)^{-2} \leq \frac{1}{12}. \tag{4.19}$$

Therefore, for $m \geq c(1+M^8)$ the m-dimensional volume G_m decays and the attractor has finite dimension. □

Acknowledgment. This work was performed under the research program "Dissipation et Asymptotique" supported by CNRS (France) and CNRST (Morocco). The third author would like to thank L. Dupaigne for pointing out to him the reference [6]. Thanks also to E. Zahrouni for helpful remarks.

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