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MORAWETZ-STRICHARTZ ESTIMATES FOR SPHERICALLY SYMMETRIC SOLUTIONS TO WAVE EQUATIONS AND APPLICATIONS TO SEMILINEAR CAUCHY PROBLEMS

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Abstract. We prove some space-time estimates with homogeneous weight functions for the wave equation with radially symmetric data in order to explain a connection between the Morawetz-type space-time estimate and the Strichartz-type space-time estimate. As an application of such linear estimates, we give a new proof of global existence of small radial solutions to semilinear wave equations. Local existence of radial solutions is also studied for low-regularity data.

1. Introduction

In this paper we are concerned with the local smoothness of solutions to the free wave equation

$$\Box u = 0, \quad t \in \mathbb{R}, \;\; x \in \mathbb{R}^n, \tag{1.1}$$

subject to the initial data

$$u(0,x) = f(x), \;\; \partial_t u(0,x) = g(x). \tag{1.2}$$

There has been a lot of activity in studying the smoothness of solutions to (1.1)–(1.2). Among other things, the Morawetz estimate [20]

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^n} |x|^{-3} |u(t,x)|^2 dx \le C\big(\|\nabla f\|^2_{L^2(\mathbb{R}^n)} + \|g\|^2_{L^2(\mathbb{R}^n)}\big), \quad n \ge 4 \tag{1.3}$$

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($C = C(n)$), is fundamental in the L^2-theory. Actually, because of the translation invariance one has more generally

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^n} |x-x_0|^{-3}|u(t,x)|^2 dx \le C\big(\|\nabla f\|^2_{L^2(\mathbb{R}^n)} + \|g\|^2_{L^2(\mathbb{R}^n)}\big), \quad n \ge 4 \quad (1.4)$$

($C = C(n)$) for any $x_0 \in \mathbb{R}^n$. The feature of (1.4) lies in that, for any fixed $x_0 \in \mathbb{R}^n$ ($n \ge 4$), the solution u of (1.1)–(1.2) remarkably satisfies

$$\||\cdot - x_0|^{-3/2}u(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \infty, \quad \text{a.a. } t \in \mathbb{R}, \quad (1.5)$$

though the Hardy inequality along with the energy conservation only shows

$$\begin{aligned} \||\cdot - x_0|^{-1}u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\le C\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \quad (1.6)\\ &\le C\big(\|\nabla f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}\big), \quad t \in \mathbb{R} \end{aligned}$$

for $n \ge 3$. The Morawetz estimate shows that the integration in time yields a gain of regularity. Defining the Fourier transform $\mathcal{F}$

$$(\mathcal{F}\varphi)(\xi) \equiv \hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

and the operators

$$|D_x|^\alpha \varphi := \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}\varphi, \quad (W\varphi)(t,x) \equiv e^{i|D_x|t}\varphi := \mathcal{F}^{-1}e^{i|\xi|t}\mathcal{F}\varphi \quad (\varphi \in \mathcal{S}(\mathbb{R}^n)),$$

we also have the extended estimate

$$\begin{aligned} &\int_{\mathbb{R}} dt \int_{\mathbb{R}^n} |x-x_0|^{-2\alpha}|(W\varphi)(t,x)|^2 dx \quad (1.7)\\ &\le C\||D_x|^{\alpha-(1/2)}\varphi\|^2_{L^2(\mathbb{R}^n)}, \quad n \ge 2, \quad \frac{1}{2} < \alpha < \frac{n}{2}, \end{aligned}$$

($C = C(n,\alpha)$) via the trace inequality and a duality argument. See Theorem 2 of Ben-Artzi [1] (see also Section 3 below). We also refer to the estimate of Hoshiro, which explains a gain of regularity in the angular variables [9].

Turning our attention to the L^q-theory, we know the Strichartz estimate [26]: for $C = C(n,q)$

$$\begin{aligned} &\|W\varphi\|_{L^q(\mathbb{R}\times\mathbb{R}^n)} \le C\||D_x|^s\varphi\|_{L^2(\mathbb{R}^n)} \quad (1.8)\\ &n \ge 2, \quad \frac{1+n}{q} = \frac{n}{2} - s, \quad \frac{2(n+1)}{n-1} \le q < \infty \quad \big(\Leftrightarrow \tfrac{1}{2} \le s < \tfrac{n}{2}\big). \end{aligned}$$

Observe that the Sobolev inequality together with the energy conservation only leads to

$$\|(W\varphi)(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C\||D_x|^s(W\varphi)(t,\cdot)\|_{L^2(\mathbb{R}^n)} = C\||D_x|^s\varphi\|_{L^2(\mathbb{R}^n)}, \ t \in \mathbb{R}, \tag{1.9}$$

$n \ge 1$, $\frac{n}{q} = \frac{n}{2} - s$, $2 \le q < \infty$. Since the Hölder inequality shows $L^{q_1}_{\mathrm{loc}} \subset L^{q_0}_{\mathrm{loc}}$ for $q_1 > q_0$, the Strichartz estimate also proves that the integration in time yields a gain of regularity in the spatial variables. We refer to [22], [4], [18], and [12] for the generalization of the original estimate of Strichartz.

As we have seen, there is a remarkable similarity between the Morawetz estimate and the Strichartz estimate in that both the estimates manifest the gain of regularity in the spatial variables. While Ben-Artzi raises the problem on page 1 of [1], whether or not space-time L^2 Morawetz estimates are related in any way to the L^q Strichartz estimates, little is known so far about relations between these two different types of estimates. Ben-Artzi also stresses the importance of understanding the connection between the two different types of estimates. (See [1] on page 2.) Echoing the view of Ben-Artzi [1], we aim at explaining a connection between the L^2-type estimate and the L^q-type estimate by showing some space-time L^q $(q \ge 2)$ estimates with homogeneous weight function $|x|^{-\alpha}$ for the equation (1.1) with spherically symmetric data (see (2.1) below). Our proof, which is inspired by Sogge [24], shows that one can in fact obtain the L^2-estimate (1.7) and the extended L^q-estimate of (1.8) in a unified way for spherically symmetric solutions. Under the assumption of spherical symmetry the set of our estimates (2.1) includes the original Morawetz estimates (1.3) and Strichartz estimates (1.8), and hence the estimates (2.1) can be regarded as the generalization of them. This is the reason why we name the estimates (2.1) Morawetz-Strichartz estimates.

We are also concerned with some local-in-time estimates. It is known that the estimate (1.8) is false in general if $q < 2(n+1)/(n-1)$ (equivalently, if $s < 1/2$), for a counterexample is shown by the well-known method of Knapp. Keeping in mind that some non-radial solutions yield this counterexample, we note the fact that the Strichartz estimate (1.8) remains true even for $2n/(n-1) < q < 2(n+1)/(n-1)$ (equivalently, for $1/(2n) < s < 1/2$) if the initial data are spherically symmetric. We know from a recent observation of [6] that the estimate (1.8) breaks down for $2 \le q \le 2n/(n-1)$ (equivalently, for $0 \le s \le 1/(2n)$) even if one assumes spherical symmetry of data. In this regard we make a remark. Indeed the global-in-time estimate (1.8) is false even for spherically symmetric data if $q \le 2n/(n-1)$, but the local-in-time estimate

$$\|W\varphi\|_{L^q((0,T)\times\mathbb{R}^n)} \le CT^\mu \||D_x|^{(1/2)-(1/q)+\delta}\varphi\|_{L^2(\mathbb{R}^n)} \tag{1.10}$$
$$\frac{1+n}{q} = \mu + \frac{n}{2} - \left(\frac{1}{2} - \frac{1}{q} + \delta\right), \quad 2 < q \le \frac{2n}{n-1},$$

holds for spherically symmetric data by the interpolation between the Strichartz estimate (1.8) with $q = \big(2n/(n-1)\big) + \varepsilon$ and the energy estimate

$$\|W\varphi\|_{L^2((0,T)\times\mathbb{R}^n)} \le T^{1/2}\|W\varphi\|_{L^\infty((0,T);L^2(\mathbb{R}^n))} = T^{1/2}\|\varphi\|_{L^2(\mathbb{R}^n)}. \tag{1.11}$$

Here, by δ and ε we mean sufficiently small positive numbers. Interestingly, Sogge proved that, for $n = 3$ and $2 < q < 3$, the estimate (1.10) is actually true even for $\delta = 0$ when the data is spherically symmetric. The proof of this fact is based on his very clever idea and relies upon the explicit form of the Fourier transform of the surface measure on $S^2 \subset \mathbb{R}^3$. By careful analysis near the origin we intend to generalize the sharp estimate of Sogge and show a local-in-time analogue of our estimate (2.1) in three space dimensions (see (2.8) below).

The estimate (2.8) below is interesting from the point of view of loss of derivatives. In particular, choosing $q = 2$, we see that the estimate yields

$$\||x|^{-\alpha}W\varphi\|_{L^2((0,T)\times\mathbb{R}^3)} \le CT^{(1/2)-\alpha}\|\varphi\|_{L^2(\mathbb{R}^3)}, \quad 0 \le \alpha < \tfrac{1}{2} \tag{1.12}$$

($C = C(\alpha)$) for spherically symmetric data. It is worth while to pay attention to this estimate. Following Kurokawa's observation [14], we first explain that the global-in-time generalized Morawetz estimate (1.7) breaks down for $\alpha = 1/2$. If the estimate (1.7) were true for $\alpha = 1/2$, then we could easily obtain the estimate with no loss of derivatives

$$\||x|^{-1/2}\partial_t u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} + \||x|^{-1/2}\nabla u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \le C\big(\|\nabla f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}\big) \tag{1.13}$$

for the solution u of (1.1)–(1.2). On the other hand, if $\operatorname{supp} f \cup \operatorname{supp} g \subset \{\, x \in \mathbb{R}^n : |x| < R \,\}$ for some $R > 0$, then we see by the finite speed of propagation and the energy conservation

$$\begin{aligned}
&\||x|^{-1/2}\partial_t u\|^2_{L^2((0,T)\times\mathbb{R}^n)} + \||x|^{-1/2}\nabla u\|^2_{L^2((0,T)\times\mathbb{R}^n)} \\
&= \int_0^T dt \int_{|x|<t+R} |x|^{-1}\big(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2\big)dx \\
&\ge \int_0^T \frac{dt}{t+R}\int_{|x|<t+R}\big(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2\big)dx
\end{aligned}$$

$$= \left(\log \frac{T+R}{R}\right)\big(\|\nabla f\|^2_{L^2(\mathbb{R}^n)} + \|g\|^2_{L^2(\mathbb{R}^n)}\big),$$

which contradicts the estimate (1.13) and, hence, shows that the estimate (1.7) is false for $\alpha = 1/2$.

Though the global-in-time generalized Morawetz estimate (1.7) breaks down for $\alpha = 1/2$, yet we know from the estimate (1.12) that if localized in time, space-time L^2-estimates with weight $|x|^{-\alpha}$ $(0 \le \alpha < 1/2)$ remain true for $L^2(\mathbb{R}^3)$-radial data. In fact, this is the case without the assumption of spherical symmetry. Making use of integrability estimate of the local energy due to Kenig, Ponce and Vega [13] and Smith and Sogge [23], we will show that the estimate (1.12) is actually true for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ in any space dimension $n \ge 1$ (see (3.3) below).

We apply the weighted estimates (2.1), (2.8) and (3.3) to semilinear Cauchy problems. Among other things, combined with the method of Li-Zhou [15], our estimate (2.1) is useful in giving a new proof of the classic of John [10] in the setting of spherical symmetry. In fact, this is a different proof of Theorem 6.1 of Chapter 3 of Sogge [24], and we naturally intend to generalize his (3+1)-dimensional result into the (2+1)- or (4+1)-dimensional one. Using (3.3), we also propose a method to solve the problem of unique existence of local solutions for $L^2(\mathbb{R}^n) \times \dot{H}^{-1}(\mathbb{R}^n)$-data. This leads to an extension of the previous result of Lindblad and Sogge [18].

This paper is organized as follows. Section 2 is devoted to the proof of the global-in-time Morawetz-Strichartz estimate (2.1). We also prove a local-in-time analogue of (2.1) in three space dimensions. Section 3 is concerned with local-in-time space-time L^2 estimates for $L^2(\mathbb{R}^n)$-data. In Sections 4 and 5 we apply these estimates to unique solvability of semilinear Cauchy problems.

2. Morawetz-Strichartz estimate

We start with global-in-time estimates. Following the proof of Proposition 6.3 of Chapter 3 of Sogge [24], we show

Theorem 2.1. *Suppose $n \ge 2$. The estimate*

$$\||x|^{-\alpha} W\varphi\|_{L^q(\mathbb{R}\times\mathbb{R}^n)} \le C\||D_x|^s \varphi\|_{L^2(\mathbb{R}^n)} \tag{2.1}$$

$(C = C(n,q,\alpha))$ *holds for spherically symmetric data $\varphi \in \mathcal{S}(\mathbb{R}^n)$ if*

$$\frac{n+1}{q} - \alpha = \frac{n}{2} - s,\ \alpha \ge 0,\ q \ge 2,\ \frac{n}{\alpha + \frac{n-1}{2}} < q < \frac{n}{\alpha}. \tag{2.2}$$

Here, the last condition $q < n/\alpha$ is interpreted as $q < \infty$ for $\alpha = 0$.

Proof. Let $d\sigma$ denote Lebesgue measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and let $\widehat{d\sigma}$ denote the Fourier transform of $d\sigma$, that is,

$$\widehat{d\sigma}(|\xi|) = \int_{S_\omega^{n-1}} e^{-i\omega\cdot\xi} d\sigma \quad (d\sigma = d\sigma(\omega)). \tag{2.3}$$

Writing $\hat{\varphi}(\xi)$ as $\psi(\rho)$ $(\rho = |\xi|)$, we have

$$\begin{aligned}(W\varphi)(t,x) &= (2\pi)^{-n} \int_0^\infty e^{it\rho}\psi(\rho)\rho^{n-1}\widehat{d\sigma}(\rho|x|)d\rho \qquad (2.4)\\ &= (2\pi)^{-n+1}\left((2\pi)^{-1}\int_{\mathbb{R}} e^{it\rho}H(\rho)\psi(\rho)\rho^{n-1}\widehat{d\sigma}(\rho|x|)d\rho\right).\end{aligned}$$

Here $H(\rho)$ denotes the Heaviside function. Viewing the right-hand side of (2.4) as an inverse Fourier transform in t, we have by the Sobolev embedding and the Plancherel theorems

$$\begin{aligned}&\|(W\varphi)(\cdot,x)\|_{L^q(\mathbb{R})} \qquad (2.5)\\ &\le C_{q,n}\left\| |D_t|^{1/2-1/q}\left((2\pi)^{-1}\int_{\mathbb{R}} e^{it\rho}H(\rho)\psi(\rho)\rho^{n-1}\widehat{d\sigma}(\rho|x|)d\rho\right)\right\|_{L^2(\mathbb{R})}\\ &= C\|\psi(\rho)\rho^{(1/2-1/q)+n-1}\widehat{d\sigma}(\rho|x|)\|_{L^2(\mathbb{R}^+_\rho)},\end{aligned}$$

for fixed $|x|$.

To proceed, we recall that $\widehat{d\sigma}$ is a spherically symmetric smooth function in $\mathbb{R}^n$ satisfying $\widehat{d\sigma}(|\xi|) = O(|\xi|^{-(n-1)/2})$ as $|\xi| \to \infty$ (see, e.g., Stein [25] on page 348). Using the Minkowski inequality and the change of variables $y = \rho x$, we obtain

$$\begin{aligned}&\||x|^{-\alpha}W\varphi\|_{L^q(\mathbb{R}\times\mathbb{R}^n)} \qquad (2.6)\\ &\le C\big\|\psi(\rho)\rho^{(1/2-1/q)+n-1}\||x|^{-\alpha}\widehat{d\sigma}(\rho|x|)\|_{L^q(\mathbb{R}^n_x)}\big\|_{L^2(\mathbb{R}^+_\rho)}\\ &= C\big\|\psi(\rho)\rho^{(1/2-1/q)+\alpha-(n/q)+n-1}\||y|^{-\alpha}\widehat{d\sigma}(|y|)\|_{L^q(\mathbb{R}^n_y)}\big\|_{L^2(\mathbb{R}^+_\rho)}\\ &\le C\|\psi(\rho)\rho^{(1/2-1/q)+\alpha-(n/q)+n-1}\|_{L^2(\mathbb{R}^+_\rho)}\\ &= C\||\xi|^{(1/2-1/q)+\alpha-(n/q)+(n-1)/2}\hat{\varphi}\|_{L^2(\mathbb{R}^n)}\\ &= C\||D_x|^{(n/2)-((n+1)/q)+\alpha}\varphi\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Here we have handled the L^q integral of $|y|^{-\alpha}\widehat{d\sigma}(|y|)$ as

$$\||y|^{-\alpha}\widehat{d\sigma}(|y|)\|_{L^q(\{y\in\mathbb{R}^n:|y|<1\})} + \||y|^{-\alpha}\widehat{d\sigma}(|y|)\|_{L^q(\{y\in\mathbb{R}^n:|y|>1\})} \le C, \tag{2.7}$$

using the decay property of $\widehat{d\sigma}$ and the last condition in (2.2). We have finished the proof. □

For $\alpha = 0$ the estimate (2.1) is false if the last condition of (2.2) is replaced by $q = 2n/(n-1)$ (see [6]). In quite the same way as in [6] it is possible to show that, for $\alpha > 0$ as well, the estimate (2.1) is false if the last condition of (2.2) is replaced by $q = n/(\alpha + (n-1)/2)$. Using the fact that the estimate (2.1) is scale-invariant (see the first condition of (2.2)), we can also show that, for $q = n/(\alpha + (n-1)/2)$, the estimate (2.1) is false even if we replace the global-in-time L^q norm on the left-hand side by the local-in-time L^q norm such as $L^q((0,1))$. For details, see the discussion in Section 4 of [6].

Similarly, we can no longer expect the global-in-time estimate (2.1) if the last condition of (2.2) is replaced by $q < n/(\alpha + (n-1)/2)$. It is, however, possible to recover sharp space-time L^q estimates with weight $|x|^{-\alpha}$ under the condition $q < n/(\alpha + (n-1)/2)$ in space dimensions $n = 3$, if we localize time. Indeed, we prove

Theorem 2.2. *Suppose that $n = 3$. The estimate*

$$\||x|^{-\alpha} W\varphi\|_{L^q((0,T)\times\mathbb{R}^3)} \le CT^{\mu} \||D_x|^{(1/2)-(1/q)}\varphi\|_{L^2(\mathbb{R}^3)} \tag{2.8}$$

($C = C(q,\alpha)$) holds for spherically symmetric data $\varphi \in \mathcal{S}(\mathbb{R}^3)$ if

$$\frac{3+1}{q} - \alpha = \mu + \frac{3}{2} - \left(\frac{1}{2} - \frac{1}{q}\right), \ \ \alpha \ge 0, \ \ 2 \le q < \frac{3}{\alpha+1}. \tag{2.9}$$

Proof. The estimate (2.8) is scale-invariant. It hence suffices to show (2.8) only for $T = 1$. We follow the proof of (6.12) of Chapter 3 of Sogge [24].

Let us first recall

$$\widehat{d\sigma}(|\xi|) = \int_{S^2} e^{-i\omega\cdot\xi} d\sigma = 4\pi \frac{\sin|\xi|}{|\xi|} \quad (d\sigma = d\sigma(\omega), \ \ \omega \in S^2). \tag{2.10}$$

Since

$$\left\| |x|^{-\alpha} \frac{\sin(\rho|x|)}{\rho|x|} \right\|_{L^q(B)} \le \rho^{-1} \||x|^{-\alpha-1}\|_{L^q(B)} \le C\rho^{-1} \tag{2.11}$$

($B = \{ x \in \mathbb{R}^3 : |x| < 1 \}$) due to the assumption $q < 3/(\alpha+1)$, we get as in (2.5)–(2.6)

$$\begin{aligned} &\||x|^{-\alpha} W\varphi\|_{L^q(\mathbb{R}\times B)} \le C \big\| \psi(\rho)\rho^{(1/2-1/q)+2} \||x|^{-\alpha}\widehat{d\sigma}(\rho|x|)\|_{L^q(B)} \big\|_{L^2(\mathbb{R}^+_\rho)} \\ &\le C\|\psi(\rho)\rho^{(1/2-1/q)+1}\|_{L^2(\mathbb{R}^+_\rho)} = C\||D_x|^{(1/2)-(1/q)}\varphi\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{2.12}$$

On the other hand, defining $\Omega = \{\, x \in \mathbb{R}^3 : |x| > 1 \,\}$, we have by the 1-dimensional Sobolev embedding and the Plancherel theorems, for fixed $t \in (0,1)$,

$$
\begin{aligned}
&\| |\cdot|^{-\alpha}(W\varphi)(t,\cdot)\|_{L^q(\Omega)} \qquad (2.13)\\
&= C\left\| r^{-\alpha}\int_{\mathbb{R}} e^{it\rho}H(\rho)\psi(\rho)\rho^2\left(\frac{e^{i\rho r}-e^{-i\rho r}}{\rho r}\right)d\rho\right\|_{L^q((1,\infty);r^2dr)}\\
&= C\left\| \int_{\mathbb{R}} \big(e^{i\rho r}-e^{-i\rho r}\big)e^{it\rho}H(\rho)\psi(\rho)\rho\, d\rho\right\|_{L^q((1,\infty);r^{2-q(1+\alpha)}dr)}\\
&\le C\left\| \int_{\mathbb{R}} e^{-i\rho r}\big(e^{it\rho}H(\rho)\psi(\rho)\rho\big)d\rho\right\|_{L^q(\mathbb{R};dr)}\\
&\le C\left\| |D_r|^{1/2-1/q}\int_{\mathbb{R}} e^{-i\rho r}\big(e^{it\rho}H(\rho)\psi(\rho)\rho\big)d\rho\right\|_{L^2(\mathbb{R};dr)}\\
&= C\|\rho^{1/2-1/q}\psi(\rho)\|_{L^2(\mathbb{R}^+;\rho^2d\rho)} = C\||D_x|^{1/2-1/q}\varphi\|_{L^2(\mathbb{R}^3)},
\end{aligned}
$$

which together with (2.12) yields the desired result. □

Remark. As is mentioned, the proof of (2.8) is essentially in line with that of (6.12) of Chapter 3 of Sogge [24]:

$$
\begin{aligned}
&\|W\varphi\|_{L^q((0,T)\times\mathbb{R}^3)} \le CT^{\mu}\||D_x|^{1/2-1/q}\varphi\|_{L^2(\mathbb{R}^3)}, \qquad (2.14)\\
&\frac{3+1}{q} = \mu + \frac{3}{2} - \left(\frac{1}{2}-\frac{1}{q}\right), \quad 2 \le q < 3.
\end{aligned}
$$

A somewhat interesting feature of the estimate (2.8) lies in that the presence of $|x|^{-\alpha}$ on the left-hand side causes no influence over the loss of derivatives on the right-hand side.

3. Space-time L^2 estimates

Let us first recall the space-time L^2 estimate

$$
\||x|^{-\alpha}W\varphi\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \le C\||D_x|^{\alpha-(1/2)}\varphi\|_{L^2(\mathbb{R}^n)} \qquad (3.1)
$$

($C = C(n,\alpha)$), which is sometimes called generalized Morawetz estimate and is true for $1/2 < \alpha < n/2$ ($n \ge 2$). The estimate (3.1) follows from the trace inequality

$$
\sup_{\lambda>0} \lambda^{(n/2)-s}\int_{S^{n-1}} |\widehat{w}(\lambda\omega)|^2 d\sigma \le C\||D_\xi|^s\widehat{w}\|_{L^2(\mathbb{R}^n)}\big(= C'\||x|^s w\|_{L^2(\mathbb{R}^n)}\big), \qquad (3.2)
$$

which holds for $1/2 < s < n/2$. See Ben-Artzi [1], Ben-Artzi and Klainerman [2], and Hoshiro [9] for the proof (3.1) via the trace inequality such as (3.2) and a duality argument. For the proof of (3.2) see, e.g., [5]. As we have observed in Section 1, the estimate (3.1) breaks down for $\alpha = 1/2$. If we localize time, it is still possible to show

Theorem 3.1. *Suppose $n \geq 1$ and let $0 < \mu \leq 1/2$. The estimate*

$$\||x|^{-(1/2)+\mu} W\varphi\|_{L^2((0,T)\times\mathbb{R}^n)} \leq CT^\mu \|\varphi\|_{L^2(\mathbb{R}^n)} \tag{3.3}$$

$(C = C(n,\mu))$ holds for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

For the proof we need

Lemma 3.2. *Let $n \geq 1$ and fix $\beta \in C_0^\infty(\mathbb{R}^n)$. The estimate*

$$\|\beta W\varphi\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)} \tag{3.4}$$

$(C = C(n,\beta))$ holds for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

We refer to (4.2)–(4.3) of Kenig, Ponce and Vega [13] and Lemma 2.2 of Smith and Sogge [23] for the proof of Lemma 3.2. See also Appendix B of Hidano and Yokoyama [8] and Metcalfe [19] for further related results.

Proof of Theorem 3.1. Observe that the estimate (3.3) is scale-invariant. It hence suffices to prove the estimate only for $T = 1$. It follows from (3.4) that for $D = \{\, x \in \mathbb{R}^n : 1/2 < |x| < 1 \,\}$

$$\|W\varphi\|_{L^2(\mathbb{R}\times D)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)}, \tag{3.5}$$

which yields by scaling

$$\||x|^{-1/2} W\varphi\|_{L^2(\mathbb{R}\times D_j)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)} \tag{3.6}$$

for a constant C independent of j. Here we have used the notation $D_j = \{x \in \mathbb{R}^n : 2^{j-1} < |x| < 2^j\}$. Therefore we obtain for $B = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\begin{aligned} \||x|^{-(1/2)+\mu} W\varphi\|^2_{L^2(\mathbb{R}\times B)} &= \sum_{j=-\infty}^{0} \||x|^{-(1/2)+\mu} W\varphi\|^2_{L^2(\mathbb{R}\times D_j)} \\ &\leq C\|\varphi\|^2_{L^2(\mathbb{R}^n)} \sum_{j=-\infty}^{0} (2^j)^{2\mu} = C\|\varphi\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{3.7}$$

On the other hand, we easily get for $\Omega = \{\, x \in \mathbb{R}^n : |x| > 1 \,\}$

$$\||x|^{-(1/2)+\mu} W\varphi\|_{L^2((0,1)\times\Omega)} \leq \sup_{0<t<1} \|(W\varphi)(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\varphi\|_{L^2(\mathbb{R}^n)} \tag{3.8}$$

by the energy conservation. The proof has been finished. □

4. Semilinear Cauchy problem 1

We turn our attention to the long-time existence of small solutions to the Cauchy problem for semilinear wave equations

$$\Box u = F_p(u), \quad t > 0, \quad x \in \mathbb{R}^n, \tag{4.1}$$

subject to the real-valued initial data

$$u(0) = f, \quad \partial_t u(0) = g. \tag{4.2}$$

Here, $F_p(u) = \lambda |u|^{p-1}u$ or $F_p(u) = \lambda |u|^p$ ($\lambda \in \mathbb{R} \setminus \{0\}$, $p > 1$). Using the extended estimate (2.1) for radial data, we aim at an alternative approach, in the setting of radial solutions, to the classic result of John [10] on the existence of global small solutions to (4.1) in three space dimensions under the assumption $p > 1 + \sqrt{2}$. Our argument is actually valid for similar results in space dimensions $n = 2$ or $n = 4$. We will work with the scale-invariant homogeneous Sobolev space $\dot{H}^{s_0}(\mathbb{R}^n) := |D_x|^{-s_0} L^2(\mathbb{R}^n)$ ($s_0 = n/2 - 2/(p-1)$), just in line with [18], [24] and [21] (see [3] for the definition and fundamental properties of homogeneous Sobolev spaces). Letting $p_0(n)$ be the larger root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$, we will prove

Theorem 4.1. *Suppose that $n = 2, 3, 4$ and $p_0(n) < p < 1 + 4/(n-1)$. Let $s_0 = (n/2) - (2/(p-1))$. There exists a constant $\delta = \delta(n, p, \lambda) > 0$ so that if the spherically symmetric data (f, g) satisfy $\|f; \dot{H}^{s_0}(\mathbb{R}^n)\| + \|g; \dot{H}^{-1+s_0}(\mathbb{R}^n)\| \leq \delta$, then the equation (4.1)–(4.2) admits a unique global spherically symmetric solution satisfying $(u, \partial_t u) \in C\big([0,\infty); \dot{H}^{s_0}(\mathbb{R}^n) \times \dot{H}^{-1+s_0}(\mathbb{R}^n)\big)$ and*

$$\||x|^{-\alpha_0} u\|_{L^p((0,\infty)\times\mathbb{R}^n)} \leq C\delta \tag{4.3}$$

for $\alpha_0 = ((n+1)/p) - (2/(p-1))$.

Remark. Sogge has already proved a result similar to Theorem 4.1 in three space dimensions (see Theorem 6.1 in Chapter 3 of [24]). Since $s_0 \geq 0$ for $p \geq 1 + (4/n)$ and $p_0(n) \geq 1 + (4/n)$ for $n = 2, 3, 4$, it is a natural attempt to extend the theorem of Sogge into the two or four space dimensional result. Though we have succeeded in such an attempt by employing the approach based on (2.1), it seems interesting to exploit further the method of Sogge using the $L^q_t L^p_x$ Strichartz estimate for $\Box u = 0$ with radial data together with the extended estimates for inhomogeneous equation $\Box w = G$ with radial (in x) forcing term $G(t, x)$ (see (6.4) and (6.6) of Chapter 3 of [24]).

Proof of Theorem 4.1. First of all, let us remark that the estimate (2.1) yields

$$\| |x|^{-\alpha} u \|_{L^q(\mathbb{R}\times\mathbb{R}^n)} \le C\big(\|f; \dot{H}^s(\mathbb{R}^n)\| + \|g; \dot{H}^{-1+s}(\mathbb{R}^n)\|\big) \tag{4.4}$$

for $\square u = 0$ with radial data (f, g) at $t = 0$ under the assumption (2.2). By the standard argument we also have

$$\| |x|^{-\alpha} v \|_{L^q((0,T)\times\mathbb{R}^n)} \le C \int_0^T \|G(\tau); \dot{H}^{-1+s}(\mathbb{R}^n)\| d\tau \tag{4.5}$$

(C is independent of $T > 0$) for $\square v = G$ with zero data at $t = 0$ and a spherically symmetric (in x) term G. Using (4.4)–(4.5), we find the solutions via the contraction mapping principle. For this purpose let us choose $q = p$, $s = s_0 = (n/2) - (2/(p-1))$ and $\alpha = \alpha_0 = ((n+1)/p) - (2/(p-1))$. It is easy to see that the condition (2.2) is certainly satisfied. In particular, we note that the condition $n/(\alpha + (n-1)/2) < q$ is equivalent to $p > p_0(n)$. We introduce

$$X = \{\, u = u(t,x) : |x|^{-\alpha_0} u \in L^p((0,\infty)\times\mathbb{R}^n), \quad u(t,x) \text{ is spherically symmetric in } x \,\} \tag{4.6}$$

and define $\|u\|_X = \| |x|^{-\alpha_0} u \|_{L^p((0,\infty)\times\mathbb{R}^n)}$ for $p_0(n) < p < 1 + 4/(n-1)$. We also need

$$X_M = \{\, u \in X \,:\, \|u\|_X \le M \,\}, \tag{4.7}$$

($M > 0$). For any $M > 0$ the set (X_M, d) is a complete metric space, where the metric is defined as $d(u,v) := \|u - v\|_X$. Defining the operator

$$\Phi(u) := u_0(t) + \int_0^t \frac{\sin |D_x|(t-\tau)}{|D_x|} F_p(u(\tau)) d\tau, \quad u \in X \tag{4.8}$$

($u_0(t) = (\cos |D_x| t) f + (|D_x|^{-1} \sin |D_x| t) g)$, we will show that, for the constants C_1 and δ to be chosen later, Φ has a unique fixed point in $X_{2C_1\delta}$ which is the solution we seek, provided that $\|f; \dot{H}^{s_0}(\mathbb{R}^n)\| + \|g; \dot{H}^{-1+s_0}(\mathbb{R}^n)\| \le \delta$.

We first see that the estimate (4.4) yields

$$\| |x|^{-\alpha_0} u_0 \|_{L^p((0,\infty)\times\mathbb{R}^n)} \le C_1\big(\|f; \dot{H}^{s_0}(\mathbb{R}^n)\| + \|g; \dot{H}^{-1+s_0}(\mathbb{R}^n)\|\big) \tag{4.9}$$

for a suitable constant C_1. We secondly observe that the estimate (4.5) together with the dual version of the trace inequality (3.2) yields for $u \in X$

$$\left\| \, |x|^{-\alpha_0} \int_0^t \frac{\sin |D_x|(t-\tau)}{|D_x|} F_p(u(\tau)) d\tau \, \right\|_{L^p((0,\infty)\times\mathbb{R}^n)} \tag{4.10}$$

$$\leq C\int_0^\infty \|F_p(u);\dot H^{-1+s_0}(\mathbb{R}^n)\|dt$$
$$\leq C_2\int_0^\infty dt\int_{\mathbb{R}^n}|x|^{-(n/2)+1-s_0}|u(t,x)|^p dx = C_2\||x|^{-\alpha_0}u\|^p_{L^p((0,\infty)\times\mathbb{R}^n)}.$$

Here, inspired by Li and Zhou [15], we have used the dual version of (3.2)

$$\|w;\dot H^{-s}(\mathbb{R}^n)\| \leq C\big\|r^{-(n/2)+s}\|w(r\cdot)\|_{L^2(S^{n-1})}\big\|_{L^1((0,\infty);r^{n-1}dr)},\quad \frac{1}{2}<s<\frac{n}{2}, \tag{4.11}$$

for $s = 1-s_0$ by taking account of the fact that the condition $1/2 < s$ is equivalent to $p-1 < 4/(n-1)$.

Repeating essentially the same computation as in (4.10), we obtain

$$\|\Phi(u)-\Phi(v)\|_X \leq C_3(\|u\|_X+\|v\|_X)^{p-1}\|u-v\|_X,\quad u,\, v\in X, \tag{4.12}$$

for a suitable constant C_3.

Choose a constant $\delta>0$ so that $C_2(2C_1\delta)^p \leq C_1\delta$, $C_3(4C_1\delta)^{p-1}<1$. We see by (4.9), (4.10) and (4.12) that if $\|f;\dot H^{s_0}(\mathbb{R}^n)\| + \|g;\dot H^{-1+s_0}(\mathbb{R}^n)\| \leq \delta$, then Φ maps $X_{2C_1\delta}$ into itself and it is a contraction mapping on $X_{2C_1\delta}$. There exists a unique fixed point in $X_{2C_1\delta}$ which satisfies the equivalent integral equation

$$u = u_0(t) + \int_0^t \frac{\sin|D_x|(t-\tau)}{|D_x|}F_p(u(\tau))d\tau. \tag{4.13}$$

The continuity of the map $t\mapsto (u(t),\partial_t u(t))$ from $[0,\infty)$ to $\dot H^{s_0}(\mathbb{R}^n)\times \dot H^{-1+s_0}(\mathbb{R}^n)$ is an immediate consequence of the fact

$$F_p(u)\in L^1(\mathbb{R}^+;\dot H^{-1+s_0}(\mathbb{R}^n)).$$

We have completed the proof. □

We are next concerned with the application of Theorem 2.2 to the semilinear problem (4.1)–(4.2). The estimate (2.8) is useful for the study of life-span of the solutions to (4.1)–(4.2) when $2\leq p<1+\sqrt{2}$. We will prove

Theorem 4.2. *Suppose that $n=3$ and $2\leq p<1+\sqrt{2}$. Let $s_1 = (1/2)-(1/p)$, $\alpha_1 = (1/p)-(1/p^2)$ and $\mu_1 = (1+2p-p^2)/p^2$. Suppose that $(f,g)\in \dot H^{s_1}(\mathbb{R}^3)\times \dot H^{-1+s_1}(\mathbb{R}^3)$ and they are spherically symmetric. Set $\Lambda = \|f;\dot H^{s_1}(\mathbb{R}^3)\| + \|g;\dot H^{-1+s_1}(\mathbb{R}^3)\|$. The equation (4.1)–(4.2) admits a unique spherically symmetric solution satisfying $(u,\partial_t u)\in C([0,T_\Lambda];\dot H^{s_1}(\mathbb{R}^3)\times \dot H^{-1+s_1}(\mathbb{R}^3))$ and*

$$T_\Lambda^{-\mu_1}\||x|^{-\alpha_1}u\|_{L^p((0,T_\Lambda)\times\mathbb{R}^3)} \leq C\Lambda \tag{4.14}$$

for a suitable constant $C > 0$ independent of Λ. Here

$$T_\Lambda = C\Lambda^{-(p(p-1))/(1+2p-p^2)}, \tag{4.15}$$

for a suitable constant $C > 0$ independent of Λ.

Before the proof of Theorem 4.2, we make three remarks. First, this result shows the lower bound of life-span as $\Lambda \to +0$ or $\Lambda \to +\infty$. Using a different way, Sogge also obtained a result similar to (4.15) only in the setting of small data (see Theorem 6.1 in Chapter 3 of [24]). We take this opportunity to mention that the assumption of smallness of data is not essential in his proof. Indeed, modifying the definition of B_m on page 124 of [24] into

$$B_m := T^{-(1+2\kappa-\kappa^2)/\kappa^2}\|u_m - u_{m-1}\|_{L_t^{\kappa^2}L_x^{\kappa}([0,T]\times\mathbb{R}^3)},$$

in his notation and repeating essentially the same computation as on page 124 of [24], we can handle the problem without the smallness of data. Thus it is actually possible to show (4.15) even for large data by the same method as in [24]. While Sogge's proof uses the $L_t^{p^2}L_x^p$ Strichartz estimates (see (6.5), (6.7) of Chapter 3 of [24]), our proof relies on the Morawetz-Strichartz estimate (2.8).

Secondly, we note that Theorem 4.2 shows the local existence and life-span for $\Box u = u^2$ with $L^2(\mathbb{R}^n) \times \dot{H}^{-1}(\mathbb{R}^n)$ radial data. Therefore the use of (2.8) enables us to give a different proof of a result due to Lindblad, who obtained local well-posedness in $L^2(\mathbb{R}^3)\times\dot{H}^{-1}(\mathbb{R}^3)$ for $\Box u = u^2$ in the setting of spherical symmetry by using the sharp estimate

$$\|w\|_{L^\infty((0,T);L^2(\mathbb{R}^3))} \le C\|G\|_{L^2((0,T);L^1(\mathbb{R}^3))}$$

for $\Box w = G$ with zero data and spherically symmetric (in x) G (see Theorems 0.2 and 2.1 of [17]).

Thirdly, it seems interesting to mention that the lower bound of life-span has been investigated thoroughly in the setting of classical solutions, and the same order as in (4.15) is known for small C^2-solutions (see, e.g., [10] and [16]).

Proof of Theorem 4.2. Note that the estimate (2.8) yields

$$\||x|^{-\alpha}u\|_{L^q((0,T)\times\mathbb{R}^3)} \le CT^\mu\big(\|f;\dot{H}^{(1/2)-(1/q)}(\mathbb{R}^3)\| + \|g;\dot{H}^{-(1/2)-(1/q)}(\mathbb{R}^3)\|\big) \tag{4.16}$$

for $\Box u = 0$ with radial data (f,g) at $t=0$ under the assumption (2.9). We also have by the standard argument

$$\||x|^{-\alpha}v\|_{L^q((0,T)\times\mathbb{R}^3)} \le CT^\mu\int_0^T \|G(\tau);\dot{H}^{-(1/2)-(1/q)}(\mathbb{R}^3)\|d\tau \tag{4.17}$$

for $\Box v = G$ with zero data at $t=0$ and a spherically symmetric (in x) G.

We again use the contraction mapping method. For the purpose let us introduce for $T>0$

$$Y_T = \{\, u = u(t,x) : |x|^{-\alpha_1} u \in L^p((0,T)\times\mathbb{R}^3), \quad u(t,x) \text{ is spherically symmetric in } x \,\} \tag{4.18}$$

and define $\|u\|_{Y_T} = T^{-\mu_1}\||x|^{-\alpha_1}u\|_{L^p((0,T)\times\mathbb{R}^3)}$ for $2 \le p < 1+\sqrt{2}$ and α_1, μ_1 given in Theorem 4.2. We also need for $M>0$

$$Y_{T,M} = \{\, u \in Y_T : \|u\|_{Y_T} \le M \,\}. \tag{4.19}$$

The set $Y_{T,M}$ is complete with the metric $d(u,v) = \|u-v\|_{Y_T}$ for every $T, M > 0$. Defining the operator

$$\Psi(u) := u_0(t) + \int_0^t \frac{\sin|D_x|(t-\tau)}{|D_x|} F_p(u(\tau))d\tau, \quad u \in Y_T \tag{4.20}$$

($u_0(t) = (\cos|D_x|t)f + (|D_x|^{-1}\sin|D_x|t)g)$, we will show that Ψ has a unique fixed point in $Y_{T_\Lambda, C\Lambda}$, where T_Λ is given in (4.15) and C is a suitable constant independent of Λ.

Observing that the assumption $p < 3/(\alpha_1+1)$ is equivalent to $p < 1+\sqrt{2}$ as well as that the equality $(4/p) - \alpha_1 = \mu_1 + (3/2) - ((1/2)-(1/p))$ holds, we get by (4.16)

$$T^{-\mu_1}\||x|^{-\alpha_1}u_0\|_{L^p((0,T)\times\mathbb{R}^3)} \le C_4(\|f;\dot{H}^{s_1}(\mathbb{R}^3)\| + \|g;\dot{H}^{-1+s_1}(\mathbb{R}^3)\|) \tag{4.21}$$

for a suitable constant C_4. Moreover, it follows from (4.17) and the dual version of (3.2) (see (4.11)) that

$$\begin{aligned} &T^{-\mu_1}\left\| |x|^{-\alpha_1}\int_0^t \frac{\sin|D_x|(t-\tau)}{|D_x|}F_p(u(\tau))d\tau \right\|_{L^p((0,T)\times\mathbb{R}^3)} \\ &\le C\int_0^T \|F_p(u);\dot{H}^{-1+s_1}(\mathbb{R}^3)\|dt \le C_5\int_0^T dt\int_{\mathbb{R}^3}|x|^{-1+(1/p)}|u(t,x)|^p dx \\ &= C_5 T^{p\mu_1}\big(T^{-\mu_1}\||x|^{-\alpha_1}u\|_{L^p((0,T)\times\mathbb{R}^3)}\big)^p \end{aligned} \tag{4.22}$$

for $u \in Y_T$. Repeating the same computation as in (4.22), we also get

$$\|\Psi(u)-\Psi(v)\|_{Y_T} \le C_6 T^{p\mu_1}(\|u\|_{Y_T}+\|v\|_{Y_T})^{p-1}\|u-v\|_{Y_T}, \quad u,\, v \in Y_T, \tag{4.23}$$

for a suitable constant C_6.

We define $T_\Lambda > 0$ according to

$$T_\Lambda^{p\mu_1} = \min\left\{\frac{1}{2C_5(2C_4)^{p-1}},\ \frac{1}{C_6(4C_4)^{p-1}}\right\}\Lambda^{-(p-1)} \tag{4.24}$$

so that

$$C_5 T_\Lambda^{p\mu_1}(2C_4\Lambda)^p \le C_4\Lambda \quad \text{and} \quad C_6 T_\Lambda^{p\mu_1}(4C_4\Lambda)^{p-1} \le \frac{1}{2}. \tag{4.25}$$

It then follows from (4.21)–(4.23) that Ψ maps $Y_{T_\Lambda,2C_4\Lambda}$ into itself and it is a contraction mapping on $Y_{T_\Lambda,2C_4\Lambda}$. The map Ψ has a unique fixed point there, and it is the solution we seek. The continuity of the map $t \mapsto (u(t), \partial_t u(t))$ from $[0, T_\Lambda]$ to $\dot{H}^{s_1}(\mathbb{R}^3) \times \dot{H}^{-1+s_1}(\mathbb{R}^3)$ can be shown in the same way as in the proof of Theorem 4.1. Finally, we note that (4.15) is an immediate consequence of (4.24). We have finished the proof. □

5. Semilinear Cauchy problem 2

The problem we address in this section is local-in-time existence of unique solutions to the Cauchy problem (4.1)–(4.2) for initial data (f, g) in $L^2(\mathbb{R}^n) \times \dot{H}^{-1}(\mathbb{R}^n)$. In [18] Lindblad and Sogge solved this problem for $1 < p \le 1 + (3/n)$ $(n \ge 4)$, showing an improvement over the previous result of Kapitanski [11] for $p < 1 + (2/(n-1))$ $(n \ge 4)$. Lindblad and Sogge raised the question whether the bound $1+(3/n)$ is optimal or not for local existence of unique solutions for $L^2(\mathbb{R}^n) \times \dot{H}^{-1}(\mathbb{R}^n)$-data (see page 360 of [18]). In this section we revisit this problem and prove that local existence of unique solutions actually holds for $L^2(\mathbb{R}^n) \times \dot{H}^{-1}(\mathbb{R}^n)$-data for $p < 1 + (3/(n-1))$ $(n \ge 4)$ as far as spherically symmetric data (hence spherically symmetric solutions) are concerned. The main theorem of this section is stated as follows.

Theorem 5.1. *Suppose that $n \ge 4$, $2/n < p - 1 < 3/(n-1)$. Let μ_0 $(0 < \mu_0 \le 1/2)$ be a small constant satisfying $p < 1 + \big((3 - 2\mu_0)/(n - 1 + 2\mu_0)\big)$. Suppose also that $f \in L^2(\mathbb{R}^n)$, $g \in \dot{H}^{-1}(\mathbb{R}^n)$, and they are spherically symmetric. Set $\Lambda := \|f; L^2(\mathbb{R}^n)\| + \|g; \dot{H}^{-1}(\mathbb{R}^n)\|$. Then the Cauchy problem (4.1)–(4.2) admits a unique spherically symmetric solution $(u, \partial_t u) \in C([0, T_\Lambda]; L^2(\mathbb{R}^n) \times \dot{H}^{-1}(\mathbb{R}^n))$ satisfying*

$$\sup_{0<t<T_\Lambda} \|u(t)\|_{L^2(\mathbb{R}^n)} + T_\Lambda^{-\mu_0} \||x|^{-(1/2)+\mu_0} u\|_{L^2((0,T_\Lambda)\times\mathbb{R}^n)} \le C\Lambda, \tag{5.1}$$

for some $T_\Lambda > 0$ with $T_\Lambda \to +\infty$ as $\Lambda \to +0$, $T_\Lambda \to +0$ as $\Lambda \to +\infty$.

In the study of unique solvability via the contraction mapping principle the crucial first step usually lies in the estimate of free solutions in appropriate space-time norms. Recall that Strichartz-type estimates for free wave equations with $L^2(\mathbb{R}^n) \times \dot{H}^{-1}(\mathbb{R}^n)$-data are not available. Therefore, in the proof of the local solvability for $L^2(\mathbb{R}^n) \times \dot{H}^{-1}(\mathbb{R}^n)$-data we can no longer use machinery built upon space-time estimates of Strichartz type for free solutions. Lindblad and Sogge got over this difficulty by exploiting the method based on L^2 (in time) Strichartz-type estimates for inhomogeneous equations $\Box w = G$ with zero data, such as

$$\|w\|_{L^\infty((0,T);L^2(\mathbb{R}^n))} \le C\|G\|_{L^2((0,T);L^{2n/(n+3)}(\mathbb{R}^n))} \quad (n \ge 4) \tag{5.2}$$

(see, e.g., (3.18) of [18]). In the present section we propose another approach to this problem, using local-in-time space-time L^2 estimates (3.3).

We note that $3/(n-1) = 4/n$ for $n = 4$. Since the exponent $p = 1 + (4/n)$ corresponds to the L^2-critical case from the point of view of the standard scaling argument, our condition $p - 1 < 3/(n-1)$ covers whole of the L^2-subcritical case for $n = 4$. The issue naturally arises whether the Cauchy problem (4.1)–(4.2) admits unique solutions for radial data in $L^2(\mathbb{R}^n) \times \dot{H}^{-1}(\mathbb{R}^n)$ when the exponent p satisfies $3/(n-1) \le p - 1 < 4/n$ for $n \ge 5$. This issue has been recently resolved affirmatively in a joint work with Yuki Kurokawa. The proof will appear in [7].

Proof of Theorem 5.1. Let us first observe that the inequality (3.3) yields

$$\begin{aligned} &\||x|^{-(1/2)+\mu} w\|_{L^2((0,T)\times\mathbb{R}^n)} \\ &\le CT^\mu \left(\|f; L^2(\mathbb{R}^n)\| + \|g; \dot{H}^{-1}(\mathbb{R}^n)\| + \int_0^T \|G(\tau); \dot{H}^{-1}(\mathbb{R}^n)\| d\tau \right) \end{aligned} \tag{5.3}$$

$(n \ge 1,\ 0 < \mu \le 1/2)$ for the equation $\Box w = G$ with data $(w(0), \partial_t w(0)) = (f, g)$ at $t = 0$.

For any p with $1 < p < 1 + (3/(n-1))$, we choose $0 < \mu_0 \le 1/2$ small so that $p < 1 + ((3 - 2\mu_0)/(n - 1 + 2\mu_0))$. The proof of Theorem 5.1 also uses the contraction mapping principle. For $M > 0$ and $T > 0$ we define

$$\begin{aligned} Z_{T,M} = \{\, u = u(t,x) : {} & u \in C([0,T]; L^2(\mathbb{R}^n)), \\ & u(t) \text{ is spherically symmetric in } x \text{ for any } t \in [0,T], \\ & \|u\|_{Z_T} := \sup_{0<t<T} \|u(t)\|_{L^2(\mathbb{R}^n)} \\ & \qquad + T^{-\mu_0} \||x|^{-(1/2)+\mu_0} u\|_{L^2((0,T)\times\mathbb{R}^n)} \le M \,\}. \end{aligned} \tag{5.4}$$

The set $(Z_{T,M}, d)$ is a complete metric space, where the metric is defined as $d(u, v) = \|u - v\|_{Z_T}$. For $u \in Z_{T,M}$ we define the operator Γ

$$\Gamma[u](t) = u_0(t) + \int_0^t \frac{\sin |D_x|(t-\tau)}{|D_x|} F_p(u(\tau)) d\tau, \tag{5.5}$$

where $u_0(t) = (\cos |D_x| t) f + (|D_x|^{-1} \sin |D_x| t) g$.

It is easy to show

Proposition 5.2. *The inequality*

$$\|u_0\|_{Z_T} \le C_7\big(\|f; L^2(\mathbb{R}^n)\| + \|g; \dot{H}^{-1}(\mathbb{R}^n)\|\big) \tag{5.6}$$

holds for a constant $C_7 > 0$ independent of T.

Writing the second term on the right-hand side of (5.5) as $I[u](t)$, we may focus our attention on the proof of the following:

Proposition 5.3. *The inequality*

$$\|I[u]\|_{Z_T} \le C_8\big(T^{(2-p)/2} + T\big)\|u\|_{Z_T}^p \tag{5.7}$$

holds for a constant $C_8 > 0$ independent of T.

Proof of Proposition 5.3. We use (5.3) and the dual version of the trace inequality (3.2) (see (4.11)) to obtain

$$\|I[u]\|_{Z_T} \le C\int_0^T \|F_p(u(\tau)); \dot{H}^{-1}(\mathbb{R}^n)\| d\tau \tag{5.8}$$

$$\le C\int_0^T \||x|^{-(n/2)+1}|u(\tau)|^p\|_{L^1(\mathbb{R}^n)} d\tau = C\int_0^T \||x|^{-((n/2)-1)/p}u(\tau)\|_{L^p(\mathbb{R}^n)}^p d\tau.$$

We divide the last integral into two pieces. Over $B = \{\, x \in \mathbb{R}^n : |x| < 1\,\}$ we see

$$\begin{aligned}&\||x|^{-((n/2)-1)/p}u(\tau)\|_{L^p(B)} \\ &\le \||x|^{-(((n/2)-1)/p)+(1/2)-\mu_0}\|_{L^{2p/(2-p)}(B)}\||x|^{-(1/2)+\mu_0}u\|_{L^2(B)}.\end{aligned} \tag{5.9}$$

Our assumption $p < 1 + ((3-2\mu_0)/(n-1+2\mu_0))$ is equivalent to

$$\left(\frac{\frac{n}{2}-1}{p} - \frac{1}{2} + \mu_0\right)\frac{2p}{2-p} < n, \tag{5.10}$$

by which we find that the first norm on the right-hand side of (5.9) is finite.

On the other hand, we see over $\Omega = \{\, x \in \mathbb{R}^n : |x| > 1\,\}$

$$\||x|^{-((n/2)-1)/p}u\|_{L^p(\Omega)} \le \||x|^{-((n/2)-1)/p}\|_{L^{2p/(2-p)}(\Omega)}\|u\|_{L^2(\Omega)}. \tag{5.11}$$

Our assumption $p > 1 + (2/n)$ is equivalent to

$$\frac{\frac{n}{2}-1}{p} \times \frac{2p}{2-p} > n, \tag{5.12}$$

which implies that the first norm on the right-hand side of (5.11) is finite.

Combining (5.8) with (5.9) and (5.11), we can continue the estimate of (5.8) as

$$\begin{aligned}\|I[u]\|_{Z_T} &\le C\int_0^T \||x|^{-(1/2)+\mu_0}u(\tau)\|^p_{L^2(\mathbb{R}^n)}d\tau + C\int_0^T \|u(\tau)\|^p_{L^2(\mathbb{R}^n)}d\tau\\ &\le CT^{(2-p)/2}\||x|^{-(1/2)+\mu_0}u\|^p_{L^2((0,T)\times\mathbb{R}^n)} + CT\Big(\sup_{0<t<T}\|u(t)\|_{L^2(\mathbb{R}^n)}\Big)^p\\ &\le C\big(T^{(2-p)/2}+T\big)\|u\|^p_{Z_T}, \end{aligned}\tag{5.13}$$

which leads to (5.7). We have finished the proof. □

In the same way as above we can also obtain

$$\begin{aligned}&\|\Gamma[u_1]-\Gamma[u_2]\|_{Z_T} \\ &\le C\int_0^T \||x|^{-(n/2)+1}(|u_1|+|u_2|)^{p-1}(u_1-u_2)\|_{L^1(\mathbb{R}^n)}d\tau\\ &\le C\int_0^T \||x|^{-(n/2)+1-p(-(1/2)+\mu_0)}\|_{L^{2/(2-p)}(B)}\\ &\qquad\times\Big(\sum_{i=1}^2 \||x|^{-(1/2)+\mu_0}u_i\|_{L^2(B)}\Big)^{p-1}\||x|^{-(1/2)+\mu_0}(u_1-u_2)\|_{L^2(B)}d\tau\\ &+C\int_0^T \||x|^{-(n/2)+1}\|_{L^{2/(2-p)}(\Omega)}\Big(\sum_{i=1}^2\|u_i\|_{L^2(\Omega)}\Big)^{p-1}\|u_1-u_2\|_{L^2(\Omega)}d\tau\\ &\le C_9\big(T^{(2-p)/2}+T\big)\big(\|u_1\|_{Z_T}+\|u_2\|_{Z_T}\big)^{p-1}\|u_1-u_2\|_{Z_T}\end{aligned}\tag{5.14}$$

for a constant $C_9>0$ independent of $T>0$.

For any $\Lambda>0$ we choose $T_\Lambda>0$ so that

$$C_8\big(T_\Lambda^{(2-p)/2}+T_\Lambda\big)\big(2C_7\Lambda\big)^{p-1}\le\frac12 \quad\text{and}\quad C_9\big(T_\Lambda^{(2-p)/2}+T_\Lambda\big)\big(4C_7\Lambda\big)^{p-1}\le\frac12.$$

We see that Γ maps $Z_{T_\Lambda,2C_7\Lambda}$ into itself and it is a contraction mapping on $Z_{T_\Lambda,2C_7\Lambda}$. The unique fixed point is the solution we seek. □

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