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THE DIRICHLET PROBLEM IN CONVEX BOUNDED DOMAINS FOR OPERATORS IN NON-DIVERGENCE FORM WITH L^∞-COEFFICIENTS

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Abstract. Consider the Dirichlet problem for elliptic and parabolic equations in non-divergence form with variable coefficients in convex bounded domains of $\mathbb{R}^n$. We prove solvability of the elliptic problem and maximal L^q-L^p-estimates for the solution of the parabolic problem provided the coefficients $a_{ij} \in L^\infty$ satisfy a Cordes condition and $p \in (1, 2]$ is close to 2. This implies that in two dimensions, i.e., $n = 2$, the elliptic Dirichlet problem is always solvable if the associated operator is uniformly strongly elliptic, and $p \in (1, 2]$ is close to 2, for maximal L^q-L^p-regularity in the parabolic case an additional assumption on the growth of the coefficients is needed.

1. Introduction

In this paper we consider elliptic and parabolic equations in convex bounded domains Ω of $\mathbb{R}^n$ for second-order differential operators in non-divergence form with coefficients $a_{ij} \in L^\infty(\Omega)$. More precisely, we are interested in the L^p-theory of the elliptic problem $\mathcal{A}u = f$ with homogeneous Dirichlet data and in the parabolic problem $u_t - \mathcal{A}u = f$, also with homogeneous Dirichlet data, where

$$\mathcal{A} := \sum_{i,j=1}^{n} a_{ij}(x,t) D_{ij}$$

is a second-order differential operator with coefficients $a_{ij} \in L^\infty(\Omega \times \mathbb{R}_+, \mathbb{C})$.

Whereas the autonomous situation for operators in divergence form, i.e., $\mathcal{A}u = \mathrm{div}\,(a_{ij}(x)\nabla u)$, is fairly well understood (see [4] for the latest results),

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the situation is less clear for operators in non-divergence form. If Ω is a bounded domain with smooth boundary, i.e., of class $C^{2+\alpha}$ and the coefficients a_{ij} are continuous on $\overline{\Omega}$, then by classical results of Agmon, Douglis and Nirenberg [3] and Ladyzhenskaya, Solonnikov and Ural'tesva [12] a very satisfactory L^p-regularity theory in the elliptic and the parabolic case is known for all p satisfying $1 < p < \infty$.

On the other hand, there exist examples of strongly elliptic operators $\mathcal{A}$ in non-divergence form, i.e., $\mathcal{A} := \sum_{i,j=1}^n a_{ij}(x)D_{ij}$ with $a_{ij} \in L^\infty(\Omega)$ such that for $f \in L^2(\Omega)$ the elliptic Dirichlet problem does not admit a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$. In fact, let $n > 2$, Ω be the unit ball in $\mathbb{R}^n$ and set

$$\mathcal{A} = \sum_{i,j=1}^n \Big(\frac{x_i x_j}{|x|^2}(1-cn) + \delta_{ij}c\Big)D_{ij},$$

for $0 < c < \frac{n-2}{n(n-1)}$. Then $\mathcal{A} : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is not an isomorphism. We refer to Talenti [17] and Maugeri, Palagachev and Softova [13] and the references therein.

Starting from this situation elliptic operators $\mathcal{A}$ with $a_{ij} \in L^\infty(\Omega)$ satisfying a so-called Cordes condition [6] were investigated in detail on smooth domains by Campanato [5] and Talenti [17]. They proved existence and uniqueness results as well as regularity properties in the L^p-context for p close to 2. For a systematic treatment and a survey of these results, we refer to Maugeri, Palagachev and Softova [13].

Our main aim here is to show that results of the above type hold true also for domains with non-smooth boundary, i.e., more precisely for bounded and convex domains Ω and for $p \in (p_0, 2]$ where p_0 is sufficiently close to 2. Our analysis relies on former results on the Laplacian in convex domains. In fact, the Dirichlet problem for the Laplacian in $L^p(\Omega)$ for convex bounded domains Ω was studied first by Adolfsson [1] and [2] and Fromm [9]. Adolfsson proved that if $f \in L^p(\Omega)$, then the solution to

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has second-order derivatives in $L^p(\Omega)$ provided $1 < p \le 2$. Fromm's results say that if $f \in W^{s,p}(\Omega)$ for $-1 \le s \le 0$ then the solution u lies in $W^{s+2,p}(\Omega)$ for $1 < p < \frac{2}{s+1}$ and for $p = 2$ when $s = 0$. Both results were obtained by estimating the Green function. Fromm also proved that the restriction to the range $1 < p \le 2$ is necessary to control the second derivatives. In [11], Jerison and Kenig show that for bounded Lipschitz domains, the solution gains two degrees of smoothness when the right-hand side f of (1.1) is in a negative Sobolev space and p satisfies certain assumptions.

Resolvent and gradient estimates in $L^p(\Omega)$ in bounded Lipschitz domains for real constant coefficient elliptic systems were obtained by Shen [15]. He was therefore able to show that these operators with maximal domain generate bounded analytic semigroups on $L^p(\Omega)$ provided $1 \leq p \leq \infty$ for $n = 3$ and $2n/(n+3) - \delta < p < 2n/(n-3) + \delta$ for $n \geq 4$ for some $\delta = \delta(\Omega) > 0$. Of course, if $p = \infty$, the semigroups are not strongly continuous.

In [18] the second author proved that the Laplacian Δ in convex bounded domains Ω with domain $D(\Delta) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is a closed operator provided $1 < p \leq 2$. Note that this precise description of the domain of the Laplacian is not true for arbitrary Lipschitz domains. He then even obtained estimates of the form

$$\left\|u'\right\|_{L^q(J,L^p(\Omega))} + \|u\|_{L^q(J,W^{2,p}(\Omega))} \leq C \|f\|_{L^q(J,L^p(\Omega))}, \tag{1.2}$$

for the solution u of the parabolic problem $u_t - \Delta u = f$ in Ω subject to Dirichlet boundary conditions. The proof relies on Fromm's results [9] for the Laplacian and recent results on L^q-L^p estimates for parabolic problems associated to positive contraction semigroups [7]. It would be interesting to see whether this result could also be obtained by pure potential theoretical arguments.

In this paper we generalize the above estimates for the Laplacian. Not only are we able to treat variable coefficient operators depending on x and t, but in addition to Shen's results we are able to give a precise description of the domain of the operators and show maximal regularity estimates in convex bounded domains.

Before giving the precise assumptions on the coefficients and on the operators $\mathcal{A}$, a remark on the particular situation of two space dimensions, i.e., $n = 2$, seems to be in order. It follows from our approach that strongly elliptic operators in two dimensions generate analytic semi-groups on $L^p(\Omega)$ provided $p \in (p_0, 2]$ and p_0 is sufficiently close to 2 if an additional growth condition is satisfied by the coefficients. This is remarkable, since results of this type cannot be obtained by the usual localization technique.

2. Elliptic Equations

Let Ω be a bounded, convex domain in $\mathbb{R}^n$. Note that any such domain is a Lipschitz domain (cf. [10, Corollary 1.2.2.3]). Consider the operator

$$\mathcal{A}u(x) := \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u(x).$$

We assume that the coefficients $a_{ij} \in L^\infty(\Omega)$ are complex-valued, that

$$\sum_{i=1}^{n} \mathrm{Re}\ a_{ii}(x) \neq 0 \text{ a.e. } x \in \Omega. \tag{2.1}$$

and that they satisfy a Cordes condition; i.e., there exists $\varepsilon > 0$ such that

$$\frac{\sum_{i,j=1}^{n} |a_{ij}(x)|^2}{(\sum_{i=1}^{n} \mathrm{Re}\ a_{ii}(x))^2} \leq \frac{1}{n-1+\varepsilon} \text{ a.e. } x \in \Omega. \tag{2.2}$$

Moreover, assume that

$$\alpha(x) := \frac{\sum_{i=1}^{n} \mathrm{Re}\ a_{ii}(x)}{\sum_{i,j=1}^{n} |a_{ij}(x)|^2} \in L^\infty(\Omega). \tag{2.3}$$

Note that (2.1) and (2.3) are satisfied whenever $\mathcal{A}$ has real-valued coefficients and satisfies the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2, \text{ a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n. \tag{2.4}$$

Example 2.1. Let $b \in L^\infty(\Omega; \mathbb{C})$. Then the operator $\mathcal{A} = b(x)\Delta$ satisfies the Cordes condition (2.2) if b takes values in the double sector given by

$$\Big|\frac{\mathrm{Im}\ b(x)}{\mathrm{Re}\ b(x)}\Big| \leq \sqrt{\frac{1-\varepsilon}{n-1+\varepsilon}} \text{ a.e. } x \in \Omega.$$

The conditions (2.1) and (2.3) are satisfied as long as b is bounded away from zero.

Our aim is to solve the following problem. Let $1 < p \leq 2$ and let A be the operator defined by

$$Au = \mathcal{A}u \tag{2.5}$$

$$D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{2.6}$$

Given $f \in L^p(\Omega)$, find $u \in D(A)$ such that $Au = f$.

To this end, we introduce the following Banach space V_p.

Proposition 2.2. *Let $1 < p \leq 2$ and $V_p := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ equipped with the norm*

$$\|u\|_{V_p} = \Big(\int_\Omega \Big(\sum_{i,j=1}^{n} |D_{ij}u|^2\Big)^{p/2}\Big)^{1/p}.$$

Then, $(V_p, \|\cdot\|_{V_p})$ is a Banach space.

Proof. This follows from the fact that for functions in V_p, $\|\cdot\|_{V_p}$ is equivalent to the usual Sobolev norm in $W^{2,p}(\Omega)$, which follows easily from Fromm's result [9]. □

We first consider the case $p = 2$.

Proposition 2.3. *The norm in V_2 is bounded by the L^2-norm of the Laplacian, i.e.,*

$$\int_\Omega \sum_{i,j=1}^n |D_{ij}u|^2 \le \int_\Omega |\Delta u|^2.$$

Proof. This is due to the convexity of the domain Ω. See [9] or [10]. □

Our first result is the following.

Proposition 2.4. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. Let A be defined as in* (2.5) *and* (2.6) *and assume that* (2.1), (2.2) *and* (2.3) *are satisfied. Given $f \in L^2(\Omega)$, there is a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying $Au = f$ in Ω.*

Proof. Set $V_2 = H^2(\Omega) \cap H_0^1(\Omega)$ and denote by $\Phi : V_2 \to V_2$ the map defined by $u = \Phi w$, where u is the unique solution of

$$\Delta u = \alpha f + \Delta w - \alpha A w, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.7}$$

where α is the function defined in (2.3). As the right-hand side above is in $L^2(\Omega)$, by [9] there exists a unique function $u \in V_2$ satisfying (2.7). Let $u = \Phi w$ and $\overline{u} = \Phi \overline{w}$ for $w, \overline{w} \in V_2$. Using Proposition 2.3, we obtain

$$\begin{aligned} \|\Phi w - \Phi \overline{w}\|_{V_2}^2 &\le \|\Delta(u - \overline{u})\|_{L^2}^2 = \|\Delta(w - \overline{w}) - \alpha A(w - \overline{w})\|_{L^2}^2 \\ &= \int_\Omega \Big| \sum_{i,j=1}^n (\delta_{ij} - \alpha a_{ij}) D_{ij}(w - \overline{w}) \Big|^2 \\ &\le \int_\Omega \Big(\sum_{i,j=1}^n |\delta_{ij} - \alpha a_{ij}|^2 \Big) \Big(\sum_{i,j=1}^n |D_{ij}(w - \overline{w})|^2 \Big). \end{aligned}$$

Since

$$\begin{aligned} &\sum_{i,j=1}^n |\delta_{ij} - \alpha a_{ij}|^2 = \sum_{i,j=1}^n \big(\delta_{ij}^2 - 2\alpha\delta_{ij} \mathrm{Re}\ a_{ij} + \alpha^2 |a_{ij}|^2\big) \\ &= n - 2\alpha \sum_{i=1}^n \mathrm{Re}\ a_{ii} + \alpha^2 \sum_{i,j=1}^n |a_{ij}|^2 = n - \frac{(\sum_{i=1}^n \mathrm{Re}\ a_{ii})^2}{\sum_{i,j=1}^n |a_{ij}|^2} \le 1 - \varepsilon, \end{aligned}$$

we obtain $\|\Phi w - \Phi \overline{w}\|_{V_2}^2 \le (1-\varepsilon)\|w-\overline{w}\|_{V_2}^2$. By the Banach fixed-point theorem, this gives us a unique solution $u \in V_2$ such that $\Phi u = u$, which is the desired solution to $Au = f$. $\square$

We next consider the case $p \neq 2$. We first need to transfer Proposition 2.3 to the L^p context.

Proposition 2.5. *Let $1 < p \le 2$. Given $f \in L^p(\Omega)$, there exists a constant C_p and a unique $u \in V_p$ satisfying $\Delta u = f$ with $\|u\|_{V_p} \le C_p \|f\|_p$. Furthermore, for any $\varepsilon \in (0,1)$ there exists $\delta > 0$ such that for $p \in (2-\delta, 2]$, $C_p < (1-\varepsilon)^{-1/2}$.*

Proof. By the results by Fromm [9], we may define a bounded operator T on $L^p(\Omega)$ which maps f to $(\sum |D_{ij}u|^2)^{1/2}$, where u is the unique solution of $\Delta u = f$ in V_p. Thus, $\|Tf\|_p \le C_p \|f\|_p$ for some C_p. By Proposition 2.3, $C_2 = 1$. Hence, by the Riesz-Thorin interpolation theorem, C_p is close to 1 for p near 2. $\square$

Remark 2.6. There exist a bounded convex domain Ω and an $f \in C^\infty(\overline{\Omega})$ such that for any $p > 2$ the solution to

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is not in $W^{2,p}(\Omega)$ (cf. [9]). Therefore, an extension of the results to $p > 2$ is impossible.

Our main result for the elliptic situation is the following.

Theorem 2.7. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ and let A be an operator defined as in (2.5) and (2.6) satisfying (2.1), (2.2) and (2.3). Then there exists $\delta > 0$ such that for $p \in (2-\delta, 2]$ and $f \in L^p(\Omega)$, there is a unique $u \in V_p$ satisfying $Au = f$ in Ω.*

Proof. Define Φ as in the proof of Theorem 2.4. A similar calculation shows that Φ is again a contraction on V_p. The Banach fixed-point theorem then proves the theorem. $\square$

The case of two dimensions is of particular interest.

Corollary 2.8. *Let Ω be a bounded convex domain in $\mathbb{R}^2$. Assume that the coefficients of A defined as in (2.5) and (2.6) satisfy $a_{ij} \in L^\infty(\Omega, \mathbb{R})$ and that $a_{12} = a_{21}$. Assume further that A is a strongly elliptic operator; i.e., (2.4) is satisfied. Then there exists $\delta > 0$ such that for $p \in (2-\delta, 2]$ and $f \in L^p(\Omega)$, there is a unique $u \in V_p := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying $Au = f$ in Ω.*

Proof. Obviously, (2.1) and (2.3) are satisfied. It remains to show that (2.2) holds. For this, we need that $(1+\varepsilon)(a_{11}^2+2a_{12}^2+a_{22}^2) \le a_{11}^2+2a_{11}a_{22}+a_{22}^2$ for some $\varepsilon > 0$. Due to the boundedness of the coefficients, it is enough to show that $a_{12}^2 < a_{11}a_{22}$. This follows from the ellipticity condition, as is easily seen by setting $\xi_1 = \sqrt{a_{22}}$ and $\xi_2 = \pm\sqrt{a_{11}}$. □

Remark 2.9. The condition that $a_{12} = a_{21}$ is necessary here for the Cordes condition to be satisfied. Furthermore, we cannot replace strong ellipticity by parameter ellipticity (cf. [7] for a definition of parameter ellipticity). This can be seen from simple examples.

3. Non-autonomous parabolic equations

In this section, we show that similar results to those for elliptic equations also hold in the parabolic case. Let Ω be a bounded convex domain in $\mathbb{R}^n$, $J = [0,T]$, or $J = [0,\infty)$ and $Q = \Omega \times J$. Consider the operator

$$\mathcal{A}(t) := \sum_{i,j=1}^{n} a_{ij}(x,t)D_{ij},$$

with $a_{ij} \in L^\infty(Q,\mathbb{C})$. We assume that $\mathcal{A}$ satisfies a *modified parabolic Cordes condition*. By this we mean that there exists $\lambda > 0$ such that

$$\sum_{i=1}^{n} \mathrm{Re}\ a_{ii}(x,t) + \lambda^{-1} \neq 0 \text{ a.e. } (x,t) \in Q, \tag{3.1}$$

and there exists $\varepsilon > 0$ such that

$$\frac{\sum_{i,j=1}^{n} |a_{ij}(x,t)|^2 + \lambda^{-2}}{(\sum_{i=1}^{n} \mathrm{Re}\ a_{ii}(x,t) + \lambda^{-1})^2} \le \frac{1}{n+\varepsilon} \text{ a.e. } (x,t) \in Q. \tag{3.2}$$

For a similar version of the parabolic Cordes condition we refer to [13].

Example 3.1. Let $b \in \mathbb{C}$, $\mathrm{Re}\ b > 0$. Then if b lies inside the sector Σ given by

$$\left|\frac{\mathrm{Im}\ b}{\mathrm{Re}\ b}\right| < \sqrt{\frac{1}{n-1}},$$

we can find $\lambda > 0$ and $\varepsilon > 0$ such that the operator $\mathcal{A} = b\Delta$ satisfies (3.2).

Lemma 3.2. *If $\mathcal{A}$ satisfies $\sum_{i=1}^{n} \mathrm{Re}\ a_{ii}(x,t) > 0$ almost everywhere and the modified parabolic Cordes condition for some λ_0, $\varepsilon > 0$, then*

$$\frac{\sum_{i,j=1}^{n} |a_{ij}(x,t)|^2}{(\sum_{i=1}^{n} \mathrm{Re}\ a_{ii}(x,t))^2} \le \frac{1}{n-1+\varepsilon} \quad \text{a.e. } x \in \Omega, t \in J;$$

i.e., $\mathcal{A}$ satisfies the elliptic Cordes condition (2.2) *for the same ε.*

Proof. Consider the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $f(\lambda) = \frac{(b+\lambda^{-1})^2}{a+\lambda^{-2}}$, where $a, b > 0$. Then f takes its sole maximum at $\lambda = \frac{b}{a}$. Fix $(x,t) \in Q$ and let $\lambda_{(x,t)} = \frac{\sum \mathrm{Re}\ a_{ii}(x,t)}{\sum |a_{ij}(x,t)|^2}$. Then for almost every $(x,t) \in Q$,

$$\begin{aligned} n + \varepsilon &\leq \frac{(\sum_{i=1}^n \mathrm{Re}\ a_{ii}(x,t) + \lambda_0^{-1})^2}{\sum_{i,j=1}^n |a_{ij}(x,t)|^2 + \lambda_0^{-2}} \\ &\leq \frac{(\sum_{i=1}^n \mathrm{Re}\ a_{ii}(x,t) + \lambda_{(x,t)}^{-1})^2}{\sum_{i,j=1}^n |a_{ij}(x,t)|^2 + \lambda_{(x,t)}^{-2}} = \frac{(\sum_{i=1}^n \mathrm{Re}\ a_{ii}(x,t))^2}{\sum_{i,j=1}^n |a_{ij}(x,t)|^2} + 1. \qquad \square \end{aligned}$$

Fix $\lambda > 0$ so that $\mathcal{A}$ satisfies (3.1) and (3.2) with λ. Let

$$\begin{aligned} Y_p &= L^p(J, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap W^{1,p}(J, L^p(\Omega)) \\ Z_p &= \{u \in Y_p : u(x,0) = 0 \text{ on } \Omega\}, \end{aligned}$$

and let $\|u\|_\lambda = \left(\int_Q (\sum_{i,j=1}^n |D_{ij}u|^2 + \lambda^2 |u_t|^2)^{p/2} dxdt\right)^{1/p}$ for $u \in Y_p$.

The following regularity result on the Dirichlet Laplacian in bounded convex domains is due to the second author.

Proposition 3.3. *Wood* [18] *Let $\Omega \subset \mathbb{R}^n$ be bounded and convex and $1 < p \leq 2$. Then, given $f \in L^p(Q)$, the problem*

$$\begin{cases} u_t - \Delta u = f & \text{in } Q, \\ u(t,\cdot) = 0 & \text{on } \partial\Omega, \\ u(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

has a unique solution in Z_p which satisfies the estimate

$$\|u_t\|_{L^p(Q)} + \|u\|_{L^p(J,W^{2,p}(\Omega))} \leq C \|f\|_{L^p(Q)}.$$

Remark 3.4. (a) This shows that the Laplacian with domain $D(\Delta) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ has maximal L^q-regularity for $1 < p \leq 2$. Therefore, given $f \in L^p(Q)$, there exists a unique $u \in Z_p$ with $u_t - \Delta u = f$ in this range of p. By rescaling the time, we see that for every $\lambda > 0$ and $f \in L^p(Q)$ there exists a unique solution to $\lambda u_t - \Delta u = f$ in Z_p.

(b) $(Z_p, \|\cdot\|_\lambda)$ is a Banach space, as the norm $\|\cdot\|_\lambda$ is equivalent to the usual norm given by

$$\|u\| = \Big(\int_Q \Big(\sum_{|\alpha| \leq 2} |D^\alpha u|^2 + |u_t|^2 \Big)^{p/2} dxdt \Big)^{1/p}.$$

This can be seen using Proposition 3.3 and the previous remark.

For $t \in J$ denote by $A(t)$ the operator defined by

$$A(t)u = \mathcal{A}(t)u \tag{3.3}$$

$$D(A(t)) = Z_p. \tag{3.4}$$

We consider the following non-autonomous parabolic problem. Given $f \in L^p(Q)$, find $u \in Z_p$ such that

$$u'(t) - A(t)u = f \text{ in } Q. \tag{3.5}$$

Our result for $p = 2$ is the following.

Proposition 3.5. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. Let $A(t)$ be the operator defined as in* (3.3) *and* (3.4) *and assume that* (3.1) *and* (3.2) *are satisfied. Then, given $f \in L^2(Q)$, there is a unique $u \in Z_2$ solving* (3.5).

Before proving the proposition, we show two useful lemmas.

Lemma 3.6.

$$\mathrm{Re} \int_Q \Delta u \; \overline{u}_t \; dxdt \;\le\; 0 \textit{ for } u \in Z_2.$$

Proof. By [14, Theorem 1.1.6/2], it is enough to consider $u \in C^\infty(\overline{Q})$ with $u(x,0) = 0$ and $u|_{\partial\Omega} = 0$. Then we have

$$\mathrm{Re} \int_Q \Delta u \; \overline{u}_t \; dxdt \;=\; -\frac{1}{2}\int_Q \frac{d}{dt}|\nabla u|^2 dtdx \;=\; -\frac{1}{2}\int_\Omega |\nabla u(x,T)|^2 dx \;\le\; 0.$$

□

Lemma 3.7. *Let $f \in L^2(Q)$, $\lambda > 0$ and suppose that u satisfies $\lambda u_t - \Delta u = f$. Then $\|u\|_\lambda \le \|f\|_{L^2}$.*

Proof. Using Proposition 2.3 and Lemma 3.6, we obtain

$$\begin{aligned}
\|u\|_\lambda^2 &= \int_Q \Big(\sum_{i,j=1}^n |D_{ij}u|^2 + \lambda^2 |u_t|^2 \Big) dxdt \le \int_Q \left(|\Delta u|^2 + \lambda^2 |u_t|^2 \right) dxdt \\
&\le \int_Q \left(|\Delta u|^2 + \lambda^2 |u_t|^2 - 2\lambda \mathrm{Re}\; \Delta u \; \overline{u}_t \right) dxdt \\
&= \int_Q |\Delta u - \lambda u_t|^2 \, dxdt = \|f\|_{L^2}^2 .
\end{aligned}$$

□

Proof of Proposition 3.5. We rewrite problem (3.5) as in the elliptic case. It is equivalent to finding $u \in Z_2$ such that

$$\lambda u_t - \Delta u = \alpha f + \sum_{i,j=1}^n (\alpha a_{ij} - \delta_{ij}) D_{ij} u + (\lambda - \alpha) u_t,$$

where we choose

$$\alpha(x) = \frac{\sum_{i=1}^n \mathrm{Re}\ a_{ii}(x) + \lambda^{-1}}{\sum_{i,j=1}^n |a_{ij}(x)|^2 + \lambda^{-2}} \in L^\infty(Q).$$

Now, we define a map $\Phi : Z_2 \to Z_2$ by $\Phi w = u$, where u is the unique solution in Z_2 to

$$\lambda u_t - \Delta u = \alpha f + \sum_{i,j=1}^n (\alpha a_{ij} - \delta_{ij}) D_{ij} w + (\lambda - \alpha) w_t.$$

Let $u = \Phi w$ and $\overline{u} = \Phi \overline{w}$ for $w, \overline{w} \in Z_2$. Then, by Lemma 3.7,

$$\begin{aligned}
&\|\Phi w - \Phi \overline{w}\|_\lambda^2 = \|u - \overline{u}\|_\lambda^2 \\
&\le \Big\| \sum_{i,j=1}^n (\alpha a_{ij} - \delta_{ij}) D_{ij}(w - \overline{w}) + (\lambda - \alpha)(w_t - \overline{w}_t) \Big\|_{L^2}^2 \\
&\le \int_Q \Big(\sum_{i,j=1}^n |\alpha a_{ij} - \delta_{ij}|^2 + \frac{(\lambda - \alpha)^2}{\lambda^2} \Big) \Big(\sum_{i,j=1}^n |D_{ij}(w - \overline{w})|^2 + \lambda^2 |w_t - \overline{w}_t|^2 \Big) \\
&\le (1 - \varepsilon) \|w - \overline{w}\|_\lambda^2 ,
\end{aligned}$$

as

$$\begin{aligned}
&\sum_{i,j=1}^n |\alpha a_{ij} - \delta_{ij}|^2 + \frac{(\lambda - \alpha)^2}{\lambda^2} \\
&= \alpha^2 \sum_{i,j=1}^n |a_{ij}|^2 - 2\alpha \sum_{i=1}^n \mathrm{Re}\ a_{ii} + n + 1 - \frac{2\alpha}{\lambda} + \frac{\alpha^2}{\lambda^2} \\
&= -\frac{(\sum_{i=1}^n \mathrm{Re}\ a_{ii} + \lambda^{-1})^2}{\sum_{i,j=1}^n |a_{ij}|^2 + \lambda^{-2}} + n + 1 \ \le\ 1 - \varepsilon.
\end{aligned}$$

Applying the Banach fixed-point theorem to Φ ends the proof. □

We now examine the parabolic case for $p \neq 2$.

Proposition 3.8. *Let $1 < p \le 2$ and $\lambda > 0$. Then, given $f \in L^p(Q)$, there exists a constant C_p and a unique $u \in Z_2$ satisfying $(\lambda \partial_t - \Delta) u = f$ with $\|u\|_\lambda \le C_p \|f\|_{L^p}$. Furthermore, for any $\varepsilon \in (0, 1)$ there exists $\delta > 0$ such that for $p \in (2 - \delta, 2]$, $C_p < (1 - \varepsilon)^{-1/2}$.*

Proof. By Proposition 3.3, we can define an operator T on $L^p(Q)$ mapping f to $(\sum |D_{ij} u|^2 + \lambda^2 |u_t|^2)^{1/2}$, where u is the unique solution of $(\lambda \partial_t - \Delta) u = f$ in Z_p. Then $\|Tf\|_p \le C_p \|f\|_p$. Moreover, by Lemma 3.7, $C_2 = 1$. Thus, by the Riesz-Thorin interpolation theorem, C_p is close to 1 for p near 2. □

A similar calculation as in the proof of Proposition 3.5 then proves the following.

Theorem 3.9. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. For $t \in J$ let $A(t)$ be an operator as in* (3.3) *and* (3.4) *satisfying* (3.1) *and* (3.2). *Then there exists $\delta > 0$ such that for $p \in (2-\delta, 2]$ and $f \in L^p(Q)$, there is a unique $u \in Z_p$ satisfying $(\partial_t - A(t))u = f$ in Q.*

Corollary 3.10. *Under the assumptions of the theorem, A has maximal L^p-regularity on $L^p(\Omega)$ and the solution to* (3.5) *satisfies*

$$\|u\|_{L^p(Q)} + \|u'\|_{L^p(Q)} + \|Au\|_{L^p(Q)} \leq C\,\|f\|_{L^p(Q)}\,.$$

In the autonomous case, i.e., when the coefficients of $\mathcal{A}$ are independent of t, we also obtain that A is the generator of a semigroup.

Corollary 3.11. *Assume that the coefficients a_{ij} of $\mathcal{A}$ do not depend on t. Then, under the assumptions of the theorem, A generates an analytic semigroup T on $L^p(\Omega)$ for $p \in (2-\delta, 2]$. Moreover, if $J = [0, \infty)$, the growth bound $w(T)$ of the semigroup T is negative.*

Proof. This is a well-known consequence from maximal L^p-regularity estimates; see [7]. □

For the non-autonomous case, we have the following abstract result.

Proposition 3.12. *Let X be a Banach space and $\{B(t),\ t \in J\}$ a family of operators on X with domain $D(B(t)) = D$ for all $t \in J$. Assume that for every $f \in L^q(J, X)$,*

$$u'(t) - B(t)u(t) = f(t), \tag{3.6}$$

has a unique solution $u \in Y := L^q(J, D) \cap W^{1,q}(J, X)$. Furthermore, assume the map $\phi : J \to \mathcal{L}(Y, L^q(J, X))$, $\phi(t) = B(t)$ is continuous in $t_0 \in J$. Then for every $f \in L^q(J, X)$, there is a unique solution in Y to

$$u'(t) - B(t_0)u(t) = f(t),$$

Proof. Let $M : L^q(J, X) \to Y$ be the solution operator to (3.6). Then by the closed graph theorem

$$\|Mf\|_Y \leq C\,\|f\|_{L^q(J,X)}\,.$$

By continuity of ϕ, there exists $\delta > 0$ such that for $|t - t_0| < \delta$,

$$\|B(t_0) - B(t)\|_{\mathcal{L}(Y, L^q(J,X))} \leq \frac{1}{2C}.$$

Then

$$\|(B(t_0) - B(t))M\|_{\mathcal{L}(L^q(J,X))} \leq \frac{1}{2}.$$

This implies the invertibility in $\mathcal{L}(Y, L^q(J,X))$ of

$$\partial_t - B(t_0) = (1 - (B(t_0) - B(t))M)(\partial_t - B(t)). \qquad \square$$

In our situation, this result implies maximal regularity for $A(t_0)$ whenever we have maximal regularity for the non-autonomous problem and continuity in t of the coefficients a_{ij}. However, as for almost all t_0, $A(t_0)$ obviously satisfies the Cordes condition whenever $A(t)$ does, the continuity condition is not needed here and we get the following result.

Corollary 3.13. *Under the assumptions of the theorem, $A(t_0)$ generates an analytic semigroup T_{t_0} on $L^p(\Omega)$ for $p \in (2-\delta, 2]$ and almost every $t_0 \in J$. Furthermore, $A(t_0)$ has maximal L^p-regularity and the solution to*

$$\begin{cases} u' - \mathcal{A}(t_0)u = f & \text{in } Q, \\ u(t,\cdot) = 0 & \text{on } \partial\Omega, \\ u(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

satisfies

$$\|u\|_{L^p(Q)} + \|u'\|_{L^p(Q)} + \|\mathcal{A}(t_0)u\|_{L^p(Q)} \le C\|f\|_{L^p(Q)}.$$

In the autonomous case, we also get L^q-L^p-estimates.

Corollary 3.14. *Assume that the coefficients a_{ij} of $\mathcal{A}$ do not depend on t. If A and p satisfy the assumptions of the theorem, then for every $f \in L^q(J, L^p(\Omega))$ the equation*

$$\begin{cases} u' - \mathcal{A}u = f & \text{in } Q, \\ u(t,\cdot) = 0 & \text{on } \partial\Omega, \\ u(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

has a unique solution $u \in L^q(J, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap W^{1,q}(J, L^p(\Omega))$ and

$$\|u\|_{L^q(J,L^p(\Omega))} + \|u'\|_{L^q(J,L^p(\Omega))} + \|\mathcal{A}u\|_{L^q(J,L^p(\Omega))} \le C\|f\|_{L^q(J,L^p(\Omega))}.$$

Proof. This is due to Sobolevskii [16] and Dore [8]; see also [7]. $\square$

The following corollary finally deals with the particular situation of two dimensions. Unfortunately, unlike in the case of elliptic equations, symmetry of the operator together with strong ellipticity is not sufficient to verify the modified parabolic Cordes condition.

Corollary 3.15. *Let Ω be a bounded convex domain in $\mathbb{R}^2$. Let A be defined as in* (3.3) *and* (3.4) *and assume that $a_{ij} \in L^\infty(Q,\mathbb{R})$ such that $a_{12} = a_{21}$.*

Assume further that A is strongly elliptic; i.e., there exists $\mu > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x,t)\xi_i\xi_j \geq \mu|\xi|^2, \text{ a.e.}(x,t) \in Q,$$

and that

$$\begin{aligned} &\operatorname*{ess\,sup}_{(x,t)\in Q} \left(a_{11}(x,t) + a_{22}(x,t) - 2\sqrt{a_{11}(x,t)a_{22}(x,t) - a_{12}(x,t)^2}\right) \qquad (3.7)\\ &< \operatorname*{ess\,inf}_{(x,t)\in Q} \left(a_{11}(x,t) + a_{22}(x,t) + 2\sqrt{a_{11}(x,t)a_{22}(x,t) - a_{12}(x,t)^2}\right). \end{aligned}$$

Then, there exists $\delta > 0$ such that for $p \in (2-\delta, 2]$ and $f \in L^p(Q)$, there is a unique $u \in Z_p$ satisfying $(\partial_t - A)u = f$ in Q.

Proof. Obviously, (3.1) is satisfied. It remains to show that (3.2) holds. For this, we need to show that there exists $\lambda > 0$ such that

$$(2+\varepsilon)(a_{11}^2 + 2a_{12}^2 + a_{22}^2 + \lambda^2) \leq a_{11}^2 + 2a_{11}a_{22} + a_{22}^2 + 2\lambda(a_{11} + a_{22}) + \lambda^2.$$

Due to the boundedness of the coefficients, this is equivalent to finding $\lambda > 0$ such that

$$\lambda^2 - 2\lambda(a_{11} + a_{22}) + a_{11}^2 + 4a_{12}^2 + a_{22}^2 - 2a_{11}a_{22} < 0.$$

We consider the equation

$$\lambda^2 - 2\lambda(a_{11} + a_{22}) + a_{11}^2 + 4a_{12}^2 + a_{22}^2 - 2a_{11}a_{22} = 0.$$

Its roots are given by

$$\lambda_\pm = a_{11}(x,t) + a_{22}(x,t) \pm 2\sqrt{a_{11}(x,t)a_{22}(x,t) - a_{12}(x,t)^2},$$

and due to the ellipticity condition, $\lambda_\pm \in \mathbb{R}$. Also, λ_+ is positive, as $\lambda_+ > 2\mu$. However, we still have the problem that $\lambda_\pm$ depends on $(x,t) \in Q$. The condition (3.7) now guarantees that we can choose $\lambda > 0$ independent of $(x,t) \in Q$ such that $\lambda_-(x,t) < \lambda < \lambda_+(x,t)$. Having done this, we can choose $\varepsilon > 0$ so that (3.2) is satisfied. The statement then follows from Theorem 3.9. □

Remark 3.16. (a) The condition (3.7) reflects the fact that in the modified Cordes condition (3.2), λ needs to be chosen according to the size of the coefficients $a_{ij}(x,t)$. If the coefficients vary too much over time or space, it may not be possible to find one fixed λ to "suit all sizes." This however is needed to be able to solve the equation $\lambda u' - \Delta u = f$ in Z_p.

(b) Obviously for constant coefficients a_{ij}, (3.7) is satisfied for any strongly elliptic operator.

(c) If A is strongly elliptic with ellipticity constant μ and

$$M := \max\{\|a_{11}\|_\infty, \|a_{22}\|_\infty\} < 2\mu,$$

then (3.7) is satisfied.

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