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RADÓ TYPE REMOVABILITY RESULT FOR FULLY NONLINEAR EQUATIONS

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Abstract. We consider the removability of a level set for solutions to fully nonlinear elliptic and parabolic equations. We prove that if a C^1 function u is a viscosity solution to the fully nonlinear equation $F(x,u,Du,D^2u)=0$ or $u_t+F(t,x,u,Du,D^2u)=0$ in a domain outside the zero-level set of u, then u is indeed a viscosity solution to the same equation in the whole domain, under some hypotheses on F. We also establish the removability result for singular fully nonlinear equations.

1. Introduction

In the early 20th century, Radó [20] proved the following theorem; *Let f be a continuous complex-valued function in a domain $\Omega \subset \mathbb{C}$. If f is analytic in $\Omega \setminus f^{-1}(0)$, then f is analytic in the whole domain Ω.* That is, a level set is always removable for continuous analytic functions. In this paper, we study the Radó type removability result for solutions to fully nonlinear equations. The equations which we are concerned with are the following degenerate elliptic, *fully nonlinear* equation

$$F(x,u,Du,D^2u)=0, \tag{1.1}$$

in $\Omega \subset \mathbb{R}^n$, or the parabolic one

$$u_t+F(t,x,u,Du,D^2u)=0, \tag{1.2}$$

in $\mathcal{O} \subset \mathbb{R}\times\mathbb{R}^n$. In both equations, D means the derivation with respect to the space variables, that is,

$$Du := \Big(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\Big)^T, \quad D^2u := \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}}. \tag{1.3}$$

Here A^T denotes the transpose of a matrix A.

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Such removability problems have been intensively studied. For the Laplace equation, it is known that if $u \in C^1(\Omega)$ is harmonic in $\Omega \setminus u^{-1}(0)$, then it is harmonic in the whole domain Ω; see [1, 8, 17]. The corresponding results for linear elliptic equations were proved by Šabat [21]. The case of the p-Laplace equation has been treated in [13, 16]. Recently, Juutinen and Lindqvist [14] proved the removability of a level set for *viscosity* solutions to general quasilinear elliptic and parabolic equations. However, to the best of our knowledge, there are no results concerning such problems for fully nonlinear PDEs. We shall obtain the removability results for (1.1) and (1.2). Our theorem covers most of the equations discussed in the literature.

Throughout this paper, we consider the class of *viscosity solutions*, which are solutions in a certain weak sense. In many nonlinear partial differential equations, the viscosity framework allows us to obtain existence and uniqueness results under mild hypotheses.

First we prove the removability of a level set for solutions to (1.1). We denote by $\mathbb{S}^{n\times n}$ the set of $n \times n$ real symmetric matrices and by I_n the $n \times n$ identity matrix.

Theorem 1.1. *Let Ω be a domain in $\mathbb{R}^n$. We suppose that $F = F(x, r, q, X)$ satisfies the following conditions.*

(A1) *F is a continuous function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n\times n}$.*

(A2) *F is degenerate elliptic; i.e.,*

$$F(x, r, q, X) \geq F(x, r, q, Y) \tag{1.4}$$

for every $x \in \Omega$, $r \in \mathbb{R}$, $q \in \mathbb{R}^n$, and $X, Y \in \mathbb{S}^{n\times n}$ with $X \leq Y$.

(A3) *$F(x, 0, 0, O) = 0$ for every $x \in \Omega$.*

(A4) *There exists a constant $\alpha > 2$ such that for every compact subset $K \Subset \Omega$ we can find positive constants ε, C and a continuous, non-decreasing function $\omega_K : [0, \infty) \to [0, \infty)$ which satisfy $\omega_K(0) = 0$ and the following:*

$$\begin{aligned} F(y, s, j|x-y|^{\alpha-2}(x-y), Y) - F(x, r, j|x-y|^{\alpha-2}(x-y), X) \\ \leq \omega_K(|r-s| + j|x-y|^{\alpha-1} + |x-y|) \end{aligned} \tag{1.5}$$

whenever $x, y \in K$, $r, s \in (-\varepsilon, \varepsilon)$, $j \geq C$, $X, Y \in \mathbb{S}^{n\times n}$ and

$$-\left(j + j(\alpha-1)|x-y|^{\alpha-2}\right) I_{2n} \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \tag{1.6}$$

$$\leq \left(j(\alpha-1)|x-y|^{\alpha-2} + 2j(\alpha-1)^2|x-y|^{2\alpha-4}\right) \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}$$

holds.

If $u \in C^1(\Omega)$ *is a viscosity solution to* (1.1) *in* $\Omega \setminus u^{-1}(0)$, *then* u *is a viscosity solution to* (1.1) *in the whole domain* Ω.

Remark 1.1. We remark about the regularity assumption on u. One can not weaken the differentiability assumption. More precisely, if we replace $u \in C^1(\Omega)$ by $u \in C^{0,1}(\Omega)$, the conclusion fails to hold. Define the function u by

$$u(x) = |x_1|, \quad x = (x_1, \dots, x_n) \in \Omega = B_1 = \{|x| < 1\}. \tag{1.7}$$

It is easily checked that u satisfies $-\Delta u = 0$ in $\Omega \setminus u^{-1}(0) = B_1 \setminus \{x_1 = 0\}$ in the classical sense as well as in the viscosity sense. But u does not satisfy $-\Delta u = 0$ in B_1 in the viscosity sense.

In Theorem 1.1, the conditions (A1) and (A2) are quite natural, and it is necessary to assume (A3) since the function $u \equiv 0$ must be a solution to (1.1). However, the condition (A4) seems to be complicated and artificial. For the particular case that F can be expressed as $F(x, r, q, X) = \tilde{F}(q, X)$ or $\tilde{F}(q, X) + f(r)$, the hypotheses can be simplified as follows.

Corollary 1.2. *Let* Ω *be a domain in* $\mathbb{R}^n$. *We suppose that* $\tilde{F} = \tilde{F}(q, X)$ *and* $f = f(r)$ *satisfy the following conditions.*

- (B1) $\tilde{F}$ *is a continuous function defined in* $\mathbb{R}^n \times \mathbb{S}^{n\times n}$ *and* f *is a continuous function defined in* $\mathbb{R}$.
- (B2) $\tilde{F}$ *is degenerate elliptic.*
- (B3) $\tilde{F}(0, O) + f(0) = 0$.

If $u \in C^1(\Omega)$ *is a viscosity solution to*

$$\tilde{F}(Du, D^2u) + f(u) = 0 \tag{1.8}$$

in $\Omega \setminus u^{-1}(0)$, *then* u *is a viscosity solution to* (1.8) *in the whole domain* Ω.

Example 1.1. Utilizing Theorem 1.1 or Corollary 1.2, one sees that our removability result can be applied to many well-known equations. Here are some examples.

- (i) Laplace equation $-\Delta u = 0$; cf. [1, 8, 17].
- (ii) Poisson equation $-\Delta u = f(u)$, where $f(0) = 0$ and f is continuous; for example, $f(u) = |u|^{p-1}u$ $(p > 0)$.
- (iii) Linear elliptic equations

$$-\sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^{n} b_i(x) D_i u(x) + c(x) u(x) = 0; \tag{1.9}$$

cf. Šabat [21].

(iv) Quasilinear elliptic equations

$$-\sum_{i,j=1}^{n} a_{ij}(x,u,Du)D_{ij}u(x)+b(x,u,Du)=0, \tag{1.10}$$

such as the minimal surface equation $-\operatorname{div}(Du/\sqrt{1+|Du|^2})=0$, p-Laplace equation $-\Delta_p u(x) := -\operatorname{div}(|Du|^{p-2}Du) = 0$ $(p \geq 2)$ and ∞-Laplace equation $\sum_{i,j=1}^{n} D_iuD_juD_{ij}u=0$; cf. Juutinen and Lindqvist [14]. We note that our result does not contain theirs, but that is because they utilize the quasilinear nature of the equation.

(v) Pucci's equation, which is an important example of fully nonlinear uniformly elliptic equations,

$$-\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u)=f(u), \qquad -\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u)=f(u), \tag{1.11}$$

where $\mathcal{M}^{+}_{\lambda,\Lambda}$ and $\mathcal{M}^{-}_{\lambda,\Lambda}$ are the so-called Pucci extremal operators with parameters $0<\lambda\leq\Lambda$ defined by

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X)=\Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i, \quad \mathcal{M}^{-}_{\lambda,\Lambda}(X)=\lambda\sum_{e_i>0}e_i+\Lambda\sum_{e_i<0}e_i, \tag{1.12}$$

for $X\in\mathbb{S}^{n\times n}$ (see [2, 19]). Here e_i $(i=1,\ldots,n)$ are the eigenvalues of X.

(vi) Monge-Ampère equation

$$\det(D^2u)=f(u). \tag{1.13}$$

When we are concerned with (1.13), we look for solutions in the class of convex functions. It is known that the equation (1.13) is not elliptic on all C^2 functions; it is degenerate elliptic for only C^2 *convex* functions. In this case, the condition (A2) is not satisfied. However, modifying our argument below appropriately, one can also apply Theorem 1.1 to (1.13) and obtain the removability result.

(vii) k-Hessian equation

$$F_k[u]=S_k(\lambda_1,\ldots,\lambda_n)=f(u), \tag{1.14}$$

where $\lambda=(\lambda_1,\ldots,\lambda_n)$ denotes the eigenvalues of D^2u and S_k $(k=1,\ldots,n)$ denotes the k-th elementary symmetric function, that is,

$$S_k(\lambda)=\sum\lambda_{i_1}\cdots\lambda_{i_k}, \tag{1.15}$$

where the sum is taken over increasing k-tuples, $1\leq i_1<\cdots<i_k\leq n$. Thus $F_1[u]=\Delta u$ and $F_n[u]=\det D^2u$, which we have seen

before. This equation has been intensively studied; see for example [3, 23, 24, 25].

(viii) Gauss curvature equation

$$\det(D^2u) = f(u)\big(1 + |Du|^{(n+2)/2}\big). \tag{1.16}$$

(ix) k-curvature equation

$$H_k[u] = S_k(\kappa_1, \ldots, \kappa_n) = f(u), \tag{1.17}$$

where $\kappa_1, \ldots, \kappa_n$ denote the principal curvatures of the graph of the function u, and S_k is the k-th elementary symmetric function. The mean, scalar and Gauss curvature equation correspond respectively to the special cases $k = 1, 2, n$ in (1.17). For the classical Dirichlet problem for k-curvature equations in the case that $2 \leq k \leq n-1$, see for instance [4, 11, 22].

In the last two sections, we also prove the removability of a level set for solutions to the fully nonlinear parabolic equations (1.2) such as the Gauss curvature flow equation, and the singular equations in the sense that F has a singularity at $Du = 0$ such as the p-Laplace equation where $1 < p < 2$. These results are also applicable to many PDEs. See Theorems 4.2, 5.2 and 5.4 and subsequent remarks.

This paper is divided as follows. In the following section we describe the definition and basic properties of viscosity solutions. Theorem 1.1 and Corollary 1.2 are proved in Section 3. We extend those removability results to the parabolic equations and the singular equations in Sections 4 and 5 respectively.

2. The notion of viscosity solutions

In this section we recall the notion of viscosity solutions to the fully nonlinear elliptic equation, (1.1). The parabolic equation's case is dealt with in the later section.

The theory of viscosity solutions to fully nonlinear equations was developed by Crandall, Evans, Ishii, Jensen, Lions and others. See, for example, [6, 7, 9, 12].

In the sequel, we denote by $\mathrm{USC}(\Omega)$ the set of all upper-semicontinuous functions $u : \Omega \to [-\infty, \infty)$, and by $\mathrm{LSC}(\Omega)$ the set of all lower-semicontinuous functions $u : \Omega \to (-\infty, \infty]$. We define a viscosity solution to (1.1).

Definition 2.1. Let Ω be a domain in $\mathbb{R}^n$. Assume that (A1) and (A2) in Theorem 1.1 are satisfied.

(i) A function $u \in \mathrm{USC}(\Omega)$ is said to be a *viscosity subsolution* to (1.1) in Ω if $u \not\equiv -\infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a maximum point of $u - \varphi$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0. \tag{2.1}$$

(ii) A function $u \in \mathrm{LSC}(\Omega)$ is said to be a *viscosity supersolution* to (1.1) in Ω if $u \not\equiv \infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a minimum point of $u - \varphi$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0. \tag{2.2}$$

(iii) A function $u \in C^0(\Omega)$ is said to be a *viscosity solution* to (1.1) in Ω if it is both a viscosity subsolution and supersolution to (1.1) in Ω.

Next we define the notion of the second-order "superjets" and that of "subjets." Let u be a real-valued function defined on Ω. For $\hat{x} \in \Omega$, $q \in \mathbb{R}^n$ and $X \in \mathbb{S}^{n\times n}$, we say $(q, X) \in J^{2,+}u(\hat{x})$ if it holds that

$$u(x) \leq u(\hat{x}) + \langle q, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(|x-\hat{x}|^2) \quad \text{as } x \to \hat{x}, \tag{2.3}$$

and $(q, X) \in J^{2,-}u(\hat{x})$ if it holds that

$$u(x) \geq u(\hat{x}) + \langle q, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(|x-\hat{x}|^2) \quad \text{as } x \to \hat{x}, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^n$. Next we define the closures of superjets and subjets as follows.

$$\begin{aligned} \overline{J}^{2,+}u(x) := \{&(q, X) \in \mathbb{R}^n \times \mathbb{S}^{n\times n} : \text{there exists a sequence } \{(x_n, q_n, X_n)\} \\ &\subset \Omega \times \mathbb{R}^n \times \mathbb{S}^{n\times n} \text{ such that } (q_n, X_n) \in J^{2,+}u(x_n) \\ &\text{and } x_n \to x, u(x_n) \to u(x), q_n \to q, X_n \to X\}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \overline{J}^{2,-}u(x) := \{&(q, X) \in \mathbb{R}^n \times \mathbb{S}^{n\times n} : \text{there exists a sequence } \{(x_n, q_n, X_n)\} \\ &\subset \Omega \times \mathbb{R}^n \times \mathbb{S}^{n\times n} \text{ such that } (q_n, X_n) \in J^{2,-}u(x_n) \\ &\text{and } x_n \to x, u(x_n) \to u(x), q_n \to q, X_n \to X\}. \end{aligned} \tag{2.6}$$

We omit the proof of the following proposition.

Proposition 2.2. *Let Ω be a domain in $\mathbb{R}^n$ and assume* (*A*1) *and* (*A*2) *in Theorem* 1.1 *are satisfied. If $u \in USC(\Omega)$ (respectively $u \in LSC(\Omega)$) is a viscosity subsolution* (*respectively viscosity supersolution*) *to* (1.1) *in Ω, then $F(\hat{x}, u(\hat{x}), q, X) \leq 0$* (*respectively $F(\hat{x}, u(\hat{x}), q, X) \geq 0$*) *for every $\hat{x} \in \Omega$ and every $(q, X) \in \overline{J}^{2,+}u(\hat{x})$ (respectively $(q, X) \in \overline{J}^{2,-}u(\hat{x})$).*

In the last part of this section, we introduce another notion of viscosity solutions to (1.1), which we call relaxed viscosity solutions. The difference between the definition of viscosity solutions and the following one is that nothing is required if the test function φ satisfies $D\varphi(x_0) = 0$.

Definition 2.3. Let Ω be a domain in $\mathbb{R}^n$. Assume that (A1) and (A2) in Theorem 1.1 are satisfied.

(i) A function $u \in \mathrm{USC}(\Omega)$ is said to be a *relaxed viscosity subsolution* to (1.1) in Ω if $u \not\equiv -\infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$, which is a maximum point of $u - \varphi$ and satisfies $D\varphi(x_0) \neq 0$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0. \tag{2.7}$$

(ii) A function $u \in \mathrm{LSC}(\Omega)$ is said to be a *relaxed viscosity supersolution* to (1.1) in Ω if $u \not\equiv \infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a minimum point of $u - \varphi$ and satisfies $D\varphi(x_0) \neq 0$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0. \tag{2.8}$$

(iii) A function $u \in C^0(\Omega)$ is said to be a *relaxed viscosity solution* to (1.1) in Ω if it is both a relaxed viscosity subsolution and supersolution to (1.1) in Ω.

It is trivial that if u is a viscosity solution, then it is a relaxed viscosity solution. We shall show in the following section that under some assumptions, the notion of viscosity solutions and that of relaxed viscosity solutions are equivalent, which is proved for the case of quasilinear equations in [14]. Namely, we require no testing at all at the points where the gradient of φ vanishes in the definition of viscosity solutions. See Proposition 3.1.

Furthermore, utilizing this definition, we can define the notion of viscosity solutions to singular equations in the sense that $F(x, r, q, X)$ in (1.1) is defined and degenerate elliptic only on $\{q \neq 0\}$, for example, the p-Laplace equation in the case $1 < p < 2$. In section 5, we shall obtain the Radó type removability result for singular equations.

3. Proof of Theorem 1.1 and Corollary 1.2

First we prove the removability of a level set for solutions to (1.1), Theorem 1.1. Our idea of the proof is adapted from that of Juutinen and Lindqvist [14].

We shall show that u is a viscosity subsolution to (1.1) in the whole domain Ω. To the contrary, we suppose that there exist a point $x_0 \in \Omega$ and a function $\varphi \in C^2(\Omega)$ such that

$$u(x_0) = \varphi(x_0), \tag{3.1}$$

$$u(x) < \varphi(x) \quad \text{for } x \in \Omega \setminus \{x_0\}, \tag{3.2}$$

and that

$$\mu := F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) > 0. \tag{3.3}$$

Here we note that $u(x_0)$ must be 0 since u is a viscosity subsolution to (1.1) in $\Omega \setminus u^{-1}(0)$.

Case 1. We assume that $D\varphi(x_0) \neq 0$. Then it holds from (3.1) and (3.2) that $Du(x_0) = D\varphi(x_0) \neq 0$. Here we used the assumption that u is a differentiable function.

Therefore, it follows from the implicit function theorem that $\{u = 0\}$ and $\{\varphi = 0\}$ are a C^1-hypersurface and a C^2-hypersurface in some neighborhood of x_0, respectively. This fact, together with (3.1) and (3.2), implies that there exist positive constants ρ and $\rho' \in (0, \rho/2)$ and a point $z \in \{\varphi < 0\}$ such that

$$B_{\rho'}(z) \subset \{\varphi < 0\} \cap B_\rho(x_0) \subset \{u < 0\} \cap B_\rho(x_0) \tag{3.4}$$

and $x_0 \in \partial B_{\rho'}(z)$ (see [14, Figure 3.1]). Without loss of generality, we may assume that $x_0 = 0$ and $z = (0, \ldots, 0, \rho')$.

For $\delta \in (0, \rho')$, we define ψ_δ by

$$\psi_\delta(x) = \varphi(x) - \left(\delta^2 x_n - \frac{\delta}{2}|x|^2\right). \tag{3.5}$$

Then $w_\delta := u - \psi_\delta$ satisfies the following:

(i) $D_n w_\delta(0) = D_n(u - \varphi)(0) + \delta^2 = \delta^2 > 0$.
(ii) $w_\delta(0) = u(0) - \varphi(0) = 0$.
(iii) if $\delta^2 x_n = \delta|x|^2/2$, i.e., $x \in \partial B_\delta(0, \ldots, 0, \delta)$, then

$$w_\delta(x) = u(x) - \varphi(x) \leq 0. \tag{3.6}$$

Thus, there exists a point $\tilde{x}_\delta \in B_\delta(0, \ldots, 0, \delta)$ such that

$$\sup\{w_\delta(x) : x \in \overline{B_\delta(0, \ldots, 0, \delta)}\} = w_\delta(\tilde{x}_\delta). \tag{3.7}$$

Since $\tilde{x}_\delta \in B_\delta(0, \ldots, 0, \delta) \subset B_{\rho'}(z) \subset \{u < 0\}$ and u is a viscosity subsolution to (1.1) in $\Omega \setminus u^{-1}(0)$, we have

$$F(\tilde{x}_\delta, u(\tilde{x}_\delta), D\psi_\delta(\tilde{x}_\delta), D^2\psi_\delta(\tilde{x}_\delta)) \leq 0. \tag{3.8}$$

We see that $\tilde{x}_\delta \to 0$ as $\delta \to +0$, and furthermore,

$$u(\tilde{x}_\delta) \to u(0) = 0, \tag{3.9}$$

$$D\psi_\delta(\tilde{x}_\delta) = D\varphi(\tilde{x}_\delta) - \delta^2(0, \ldots, 0, 1)^T + \delta\tilde{x}_\delta \to D\varphi(0), \tag{3.10}$$

$$D^2\psi_\delta(\tilde{x}_\delta) = D^2\varphi(\tilde{x}_\delta) + \delta I_n \to D^2\varphi(0) \tag{3.11}$$

as $\delta \to +0$. Taking $\delta \to +0$ in (3.8), we obtain by the condition (A1) that

$$F(0, 0, D\varphi(0), D^2\varphi(0)) = \mu \leq 0, \tag{3.12}$$

which is contradictory to (3.3).

Case 2. We assume that $D\varphi(x_0) = 0$. As is mentioned in the previous section, under some hypotheses we need no testing at all if $D\varphi = 0$ in the definition of viscosity solutions. Indeed, we have the following proposition.

Proposition 3.1. *Suppose that* (*A*1) *and* (*A*2) *in Theorem* 1.1 *and the conditions given below are satisfied.*

(*A*3)′ $F(x, r, 0, O) = 0$ *for every* $x \in \Omega$ *and every* $r \in \mathbb{R}$.

(*A*4)′ *There exists a constant* $\alpha > 2$ *such that for every compact subset* $K \Subset \Omega \times \mathbb{R}$ *we can find a constant* $C > 0$ *and a continuous, nondecreasing function* $\omega_K : [0, \infty) \to [0, \infty)$ *which satisfy* $\omega_K(0) = 0$ *and the following:*

$$\begin{aligned} F(y, s, j|x-y|^{\alpha-2}(x-y), Y) - F(x, r, j|x-y|^{\alpha-2}(x-y), X) \\ \leq \omega_K(|r-s| + j|x-y|^{\alpha-1} + |x-y|) \end{aligned} \tag{3.13}$$

whenever $(x, r), (y, s) \in K$, $j \geq C$, $X, Y \in \mathbb{S}^{n\times n}$ *and*

$$\begin{aligned} -\left(j + j(\alpha-1)|x-y|^{\alpha-2}\right) I_{2n} \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \\ \leq \left(j(\alpha-1)|x-y|^{\alpha-2} + 2j(\alpha-1)^2|x-y|^{2\alpha-4}\right) \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} \end{aligned} \tag{3.14}$$

holds.

Then $u \in C(\Omega)$ *is a relaxed viscosity subsolution* (*respectively supersolution, solution*) *to* (1.1) *if and only if it is a viscosity subsolution* (*respectively supersolution, solution*) *to* (1.1).

Proof. We prove the subsolution case only. Other cases can be proved similarly. The "if" part is trivial.

To prove the "only if" part, we argue by contradiction. We suppose that there exist a point $x_0 \in \Omega$ and a function $\varphi \in C^2(\Omega)$ such that

$$D\varphi(x_0) = 0, \tag{3.15}$$

$$u(x_0) = \varphi(x_0), \tag{3.16}$$

$$u(x) < \varphi(x) \quad \text{for } x \in \Omega \setminus \{x_0\}, \tag{3.17}$$

and that

$$\mu := F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) > 0. \tag{3.18}$$

Fix a constant $R > 0$ such that $B_R := B_R(x_0) \Subset \Omega$.

We use the technique that we double the number of variables and penalize the doubling, as discussed in [7]. For $j \in \mathbb{N}$, we define $\psi_j = \psi_j(x, y)$ by

$$\psi_j(x, y) = \frac{j}{\alpha}|x - y|^\alpha \tag{3.19}$$

and set

$$w_j(x, y) = u(x) - \varphi(y) - \psi_j(x, y). \tag{3.20}$$

Then there exists $(x_j, y_j) \in \overline{B_R} \times \overline{B_R}$ which satisfies

$$w_j(x_j, y_j) = \max_{(x,y)\in\overline{B_R}\times\overline{B_R}} w_j(x, y). \tag{3.21}$$

One can show the following:

$$\lim_{j\to\infty} \frac{j}{\alpha}|x_j - y_j|^\alpha = 0, \quad \lim_{j\to\infty} (x_j, y_j) = (x_0, x_0); \tag{3.22}$$

see [7, Proposition 3.7]. Thus $(x_j, y_j) \in B_R \times B_R$ for sufficiently large j. From now on we assume j is sufficiently large. Since $w_j(x_j, y) \le w_j(x_j, y_j)$ for every point $y \in B_R$, we have

$$\varphi(y) \ge \varphi(y_j) + \psi_j(x_j, y_j) - \psi_j(x_j, y) \tag{3.23}$$

for all $y \in B_R$. We denote the right-hand side of (3.23) by $\Psi_j(y)$.

It follows from (3.23) and the equality $\varphi(y_j) = \Psi_j(y_j)$ that

$$D\varphi(y_j) = D\Psi_j(y_j) = j|x_j - y_j|^{\alpha-2}(x_j - y_j), \tag{3.24}$$

$$\begin{aligned} D^2\varphi(y_j) \ge D^2\Psi_j(y_j) = &-j|x_j - y_j|^{\alpha-2} I_n \\ &- j(\alpha - 2)|x_j - y_j|^{\alpha-4}(x_j - y_j) \otimes (x_j - y_j). \end{aligned} \tag{3.25}$$

We first deal with the case that $x_j = y_j$ for infinitely many j's. Passing to a subsequence if necessary, we may assume that $x_j = y_j$ for all $j \in \mathbb{N}$. By (3.24) and (3.25), we obtain that $D\varphi(y_j) = 0$ and $D^2\varphi(y_j) \ge O$. Therefore the conditions (A2) and (A3)$'$ yield

$$F(y_j, \varphi(y_j), D\varphi(y_j), D^2\varphi(y_j)) \le F(y_j, \varphi(y_j), 0, O) = 0 \tag{3.26}$$

for all $j \in \mathbb{N}$. As $j \to \infty$, it follows from (3.22) and (A1) that

$$\mu = F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \le 0, \tag{3.27}$$

which contradicts (3.18).

Next we consider the case that there exists $j_0 \in \mathbb{N}$ such that $x_j \neq y_j$ for all $j \geq j_0$. By the maximum principle for semicontinuous functions (see [7]), we have that there exist $X_j, Y_j \in \mathbb{S}^{n\times n}$ such that

$$(D_x\psi_j(x_j,y_j), X_j) \in \overline{J}^{2,+}u(x_j), \tag{3.28}$$

$$(-D_y\psi_j(x_j,y_j), Y_j) \in \overline{J}^{2,-}\varphi(y_j), \tag{3.29}$$

$$-(j+\|A_j\|)I_{2n} \leq \begin{pmatrix} X_j & O \\ O & -Y_j \end{pmatrix} \leq A_j + \frac{1}{j}{A_j}^2, \tag{3.30}$$

where $A_j = D^2\psi_j(x_j,y_j) = \begin{pmatrix} D^2_{xx}\psi_j(x_j,y_j) & D^2_{xy}\psi_j(x_j,y_j) \\ D^2_{yx}\psi_j(x_j,y_j) & D^2_{yy}\psi_j(x_j,y_j) \end{pmatrix}$. In this case ψ_j is defined by (3.19), so that we can calculate the last inequality (3.30) as

$$\begin{aligned} -\left(j + j(\alpha-1)|x_j-y_j|^{\alpha-2}\right)I_{2n} &\leq \begin{pmatrix} X_j & O \\ O & -Y_j \end{pmatrix} \\ &\leq j\left(|x_j-y_j|^{\alpha-2} + 2|x_j-y_j|^{2\alpha-4}\right)\begin{pmatrix} I_n & -I_n \\ I_n & -I_n \end{pmatrix} \\ &\quad + j(\alpha-2)\left(|x_j-y_j|^{\alpha-4} + 2\alpha|x_j-y_j|^{2\alpha-6}\right) \\ &\quad \times \begin{pmatrix} (x_j-y_j)\otimes(x_j-y_j) & -(x_j-y_j)\otimes(x_j-y_j) \\ -(x_j-y_j)\otimes(x_j-y_j) & (x_j-y_j)\otimes(x_j-y_j) \end{pmatrix} \\ &\leq \left(j(\alpha-1)|x_j-y_j|^{\alpha-2} + 2j(\alpha-1)^2|x_j-y_j|^{2\alpha-4}\right)\begin{pmatrix} I_n & -I_n \\ I_n & -I_n \end{pmatrix}. \end{aligned} \tag{3.31}$$

Next, since $x_j \neq y_j$ for $j \geq j_0$, it holds that

$$D_x\psi_j(x_j,y_j) = -D_y\psi_j(x_j,y_j) = j|x_j-y_j|^{\alpha-2}(x_j-y_j) \neq 0, \tag{3.32}$$

for $j \geq j_0$. From (3.18), (3.28), (3.29) and the fact that u is a relaxed viscosity subsolution to (1.1), it follows that

$$F(x_j, u(x_j), j|x_j-y_j|^{\alpha-2}(x_j-y_j), X_j) \leq 0, \tag{3.33}$$

$$F(y_j, \varphi(y_j), j|x_j-y_j|^{\alpha-2}(x_j-y_j), Y_j) \geq \mu/2 \tag{3.34}$$

for sufficiently large j. Moreover, by (3.15), (3.22) and (3.24)

$$j|x_j-y_j|^{\alpha-2}(x_j-y_j) = D\varphi(y_j) \to D\varphi(x_0) = 0 \quad \text{as } j\to\infty, \tag{3.35}$$

and thus

$$j|x_j-y_j|^{\alpha-1} \to 0 \quad \text{as } j \to \infty. \tag{3.36}$$

Finally, by (3.16), (3.22), (3.33), (3.34), (3.36) and the condition (A4)′, we obtain

$$\mu/2 \leq F(y_j, \varphi(y_j), j|x_j-y_j|^{\alpha-2}(x_j-y_j), Y_j) \tag{3.37}$$

$$- F(x_j, u(x_j), j|x_j - y_j|^{\alpha-2}(x_j - y_j), X_j)$$
$$\leq \omega_K(|u(x_j) - \varphi(y_j)| + j|x_j - y_j|^{\alpha-1} + |x_j - y_j|) \to 0$$

as $j \to \infty$. We reach a contradiction. □

Let us mention again that if u is assumed to be a viscosity subsolution to (1.1) in $\{u \neq 0\}$, then $u(x_0)$ and $\varphi(x_0)$ must be 0. Therefore, in our setting the inequalities (3.27) and (3.37) hold if we only assume (A3) and (A4) instead of (A3)′ and (A4)′. Thus we conclude that u is a viscosity subsolution to (1.1) in the whole domain Ω, and it can be proved by analogous arguments that u is a supersolution to (1.1) in Ω. This completes the proof of Theorem 1.1.

Next we prove Corollary 1.2. It is enough to check that (A1), (A2), (A3) and (A4) are satisfied when we set $F(x, r, q, X) = \tilde{F}(q, X) + f(r)$. It is trivial that our conditions (B1), (B2) and (B3) imply (A1), (A2) and (A3) respectively. (A4) follows from the conditions (B1) and (B2), and the fact that (1.6) implies $X \leq Y$.

4. Removability results for parabolic equations

In this section we shall establish a Radó type removability result for solutions to the fully nonlinear parabolic equation, (1.2). First we recall a notion of viscosity solutions to parabolic equations. Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We suppose that $F = F(t, x, r, q, X)$ satisfies the following:

(C1) F is a continuous function defined in $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n\times n}$.
(C2) F is degenerate elliptic.

Definition 4.1. Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We assume (C1) and (C2) are satisfied.

(i) A function $u \in \mathrm{USC}(\mathcal{O})$ is said to be a *viscosity subsolution* to (1.2) in $\mathcal{O}$ if $u \not\equiv -\infty$ and for any function $\varphi \in C^2(\mathcal{O})$ and any point $(t_0, x_0) \in \mathcal{O}$ which is a maximum point of $u - \varphi$, we have

$$\varphi_t(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \leq 0. \tag{4.1}$$

(ii) A function $u \in \mathrm{LSC}(\mathcal{O})$ is said to be a *viscosity supersolution* to (1.2) in $\mathcal{O}$ if $u \not\equiv \infty$ and for any function $\varphi \in C^2(\mathcal{O})$ and any point $(t_0, x_0) \in \mathcal{O}$ which is a minimum point of $u - \varphi$, we have

$$\varphi_t(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \geq 0. \tag{4.2}$$

(iii) A function $u \in C^0(\mathcal{O})$ is said to be a *viscosity solution* to (1.2) in $\mathcal{O}$ if it is both a viscosity subsolution and supersolution to (1.2) in $\mathcal{O}$.

The parabolic variants of the superjets $J^{2,+}u$ and the subjets $J^{2,-}u$ are defined as follows.

$$(\tau, q, X) \in P^{2,+}u(\hat{t}, \hat{x}) \tag{4.3}$$

$$\overset{\text{def}}{\Longleftrightarrow} u(t,x) \le u(\hat{t}, \hat{x}) + \tau(t-\hat{t}) + \langle q, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(|t-\hat{t}| + |x-\hat{x}|^2) \quad \text{as } (t,x) \to (\hat{t}, \hat{x}),$$

$$(\tau, q, X) \in P^{2,-}u(\hat{t}, \hat{x}) \tag{4.4}$$

$$\overset{\text{def}}{\Longleftrightarrow} u(t,x) \ge u(\hat{t}, \hat{x}) + \tau(t-\hat{t}) + \langle q, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(|t-\hat{t}| + |x-\hat{x}|^2) \quad \text{as } (t,x) \to (\hat{t}, \hat{x}),$$

for $(\hat{t}, \hat{x}) \in \mathcal{O}$. The closures of parabolic semijets, $\overline{P}^{2,+}u(\hat{t}, \hat{x})$ and $\overline{P}^{2,-}u(\hat{t}, \hat{x})$, are defined similarly.

Now we state our removability result for parabolic equations.

Theorem 4.2. *Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We suppose that $(C1)$, $(C2)$ and the conditions given below are satisfied.*

(C3) *$F(t, x, 0, 0, O) = 0$ for every $(t, x) \in \mathcal{O}$.*

(C4) *There exists a constant $\alpha > 2$ such that for every compact subset $K \Subset \mathcal{O}$ we can find positive constants ε and C, and a continuous, non-decreasing function $\omega_K : [0, \infty) \to [0, \infty)$, which satisfy $\omega_K(0) = 0$ and the following:*

$$F(t', y, s, j|x-y|^{\alpha-2}(x-y), Y) - F(t, x, r, j|x-y|^{\alpha-2}(x-y), X) \le \omega_K(|t-t'| + |r-s| + j|x-y|^{\alpha-1} + |x-y|) \tag{4.5}$$

whenever $(t, x), (t', y) \in K$, $r, s \in (-\varepsilon, \varepsilon)$, $j \ge C$, $X, Y \in \mathbb{S}^{n\times n}$ and

$$-\left(j + j(\alpha-1)|x-y|^{\alpha-2}\right) I_{2n} \le \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \tag{4.6}$$

$$\le \left(j(\alpha-1)|x-y|^{\alpha-2} + 2j(\alpha-1)^2|x-y|^{2\alpha-4}\right) \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}$$

holds.

If $u \in C^1(\mathcal{O})$ is a viscosity solution to (1.2) *in $\mathcal{O} \setminus u^{-1}(0)$, then u is a viscosity solution to* (1.2) *in the whole domain $\mathcal{O}$.*

Remark 4.1. For F of the form $\tilde{F}(q, X) + f(r)$, the conditions (C1) and (C2) imply (C4) as in the elliptic case, so that a level set of a viscosity solution to (1.2) is always removable if we assume the continuity of $\tilde{F}$ and f, the degenerate ellipticity of $\tilde{F}$, and $\tilde{F}(0,O)+f(0)=0$ only. Therefore, Theorem 4.2 is applicable to a number of well-known parabolic equations, such as the heat equation $u_t-\Delta u=0$, the p-Laplace diffusion equation $u_t-\Delta_p u=0$ where $p>2$, the parabolic Monge-Ampère equation $u_t-(\det D^2u)^{1/n}=0$, and the Gauss curvature flow equation $u_t-\det D^2u/\left(1+|Du|^2\right)^{(n+1)/2}=0$.

Proof. The overall strategy of the proof of this theorem is the same as that of Theorem 1.1. We only show that u is a viscosity subsolution to (1.2) in the whole domain $\mathcal{O}$, since the supersolution case can be proved similarly. To the contrary, we suppose that there exist a point $(t_0,x_0)\in\mathcal{O}$ and a function $\varphi\in C^2(\mathcal{O})$ such that

$$u(t_0,x_0)=\varphi(t_0,x_0)=0, \tag{4.7}$$

$$u(t,x)<\varphi(t,x) \quad \text{for } (t,x)\in\mathcal{O}\setminus\{(t_0,x_0)\}, \tag{4.8}$$

and that

$$\mu:=\varphi_t(t_0,x_0)+F(t_0,x_0,u(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0))>0. \tag{4.9}$$

We first deal with the case $(\varphi_t(t_0,x_0),D\varphi(t_0,x_0))\neq(0,0)$. Without loss of generality, we may assume that $(t_0,x_0)=(0,0)$, $\varphi_t(t_0,x_0)=a$, $|D\varphi(t_0,x_0)|=D_n\varphi(t_0,x_0)=b$. By (4.7), (4.8) and the implicit function theorem, we can choose small numbers $\rho,\rho'>0$ such that

$$\begin{aligned} B_{\rho'}(z_{\rho'}):&=B_{\rho'}\Big(\frac{\rho'}{\sqrt{a^2+b^2}}(-a,0,\dots,0,-b)\Big) \\ &\subset\{\varphi<0\}\cap B_\rho(0,0)\subset\{u<0\}\cap B_\rho(0,0). \end{aligned} \tag{4.10}$$

For $\delta\in(0,\rho')$, we define ψ_δ by

$$\psi_\delta(t,x)=\varphi(t,x)+\delta^2(at+bx_n)+\frac{\delta\sqrt{a^2+b^2}}{2}(t^2+|x|^2). \tag{4.11}$$

Then $w_\delta=u-\psi_\delta$ satisfies the following:

(i) $-a(w_\delta)_t(0,0)-bD_nw_\delta(0,0)=(a^2+b^2)\delta^2>0$.
(ii) $w_\delta(0,0)=u(0,0)-\varphi(0,0)=0$.
(iii) if $\delta^2(at+bx_n)+\delta\sqrt{a^2+b^2}(t^2+|x|^2)/2=0$, i.e., $(t,x)\in\partial B_\delta(z_\delta)$, then

$$w_\delta(t,x)=u(t,x)-\varphi(t,x)\le 0. \tag{4.12}$$

Thus, there exists a point $(\tilde{t}_\delta,\tilde{x}_\delta)\in B_\delta(z_\delta)\subset B_{\rho'}(z_{\rho'})\subset\{u<0\}$ such that

$$\sup\{w_\delta(t,x):(t,x)\in\overline{B_\delta(z_\delta)}\}=w_\delta(\tilde{t}_\delta,\tilde{x}_\delta). \tag{4.13}$$

Since u is a viscosity subsolution in $\{u\neq 0\}$, it follows that

$$(\psi_\delta)_t(\tilde{t}_\delta,\tilde{x}_\delta)+F(\tilde{t}_\delta,\tilde{x}_\delta,u(\tilde{t}_\delta,\tilde{x}_\delta),D\psi_\delta(\tilde{t}_\delta,\tilde{x}_\delta),D^2\psi_\delta(\tilde{t}_\delta,\tilde{x}_\delta))\le 0. \tag{4.14}$$

Taking $\delta \to +0$, we obtain by the fact $(\tilde{t}_\delta, \tilde{x}_\delta) \to (0,0)$ and the condition (C1) that

$$\varphi_t(0,0) + F(0,0,u(0,0), D\varphi(0,0), D^2\varphi(0,0)) = \mu \leq 0. \tag{4.15}$$

We reach a contradiction.

Next we consider the case $(\varphi_t(t_0,x_0), D\varphi(t_0,x_0)) = (0,0)$. We use "doubling" argument again. Fix a constant $R > 0$ such that $B_R := B_R(t_0,x_0) \Subset \mathcal{O}$. For $j \in \mathbb{N}$, we consider the function $\psi_j : \overline{B_R} \times \overline{B_R} \to \mathbb{R}$ defined by

$$\psi_j(t,x,s,y) = \frac{j}{2}(t-s)^2 + \frac{j}{\alpha}|x-y|^\alpha \tag{4.16}$$

and set

$$w_j(t,x,s,y) = u(t,x) - \varphi(s,y) - \psi_j(t,x,s,y), \tag{4.17}$$

where α is a constant appearing in (C4). There exists $(t_j,x_j,s_j,y_j) \in \overline{B_R} \times \overline{B_R}$ such that

$$w_j(t_j,x_j,s_j,y_j) = \max_{(t,x,s,y)\in\overline{B_R}\times\overline{B_R}} w_j(t,x,s,y), \tag{4.18}$$

$$\lim_{j\to\infty}\left(\frac{j}{2}(t_j-s_j)^2 + \frac{j}{\alpha}|x_j-y_j|^\alpha\right) = 0, \tag{4.19}$$

$$\lim_{j\to\infty}(t_j,x_j,s_j,y_j) = (t_0,x_0,t_0,x_0), \tag{4.20}$$

$$(t_j,x_j,s_j,y_j) \in B_R \times B_R \quad \text{for sufficiently large } j. \tag{4.21}$$

Furthermore, we can see that

$$\varphi_t(s_j,y_j) = j(t_j-s_j), \tag{4.22}$$

$$D\varphi(s_j,y_j) = j|x_j-y_j|^{\alpha-2}(x_j-y_j), \tag{4.23}$$

$$\begin{aligned} D^2\varphi(s_j,y_j) \geq{}& -j|x_j-y_j|^{\alpha-2}I_n \\ &- j(\alpha-2)|x_j-y_j|^{\alpha-4}(x_j-y_j)\otimes(x_j-y_j). \end{aligned} \tag{4.24}$$

Then (4.20), (4.22) and (4.23) yield that

$$j(t_j-s_j) = \varphi_t(s_j,y_j) \to \varphi_t(t_0,x_0) = 0, \tag{4.25}$$

$$j|x_j-y_j|^{\alpha-1} = |D\varphi(s_j,y_j)| \to |D\varphi(t_0,x_0)| = 0 \tag{4.26}$$

as $j \to \infty$.

If $x_j = y_j$ for infinitely many j's, we can reach a contradiction as discussed in the elliptic case. If there exists $j_0 \in \mathbb{N}$ such that $x_j \neq y_j$ for all $j \geq j_0$, it follows from the maximum principle for semicontinuous functions and [18, Lemma 3.5] that there exist $X_j, Y_j \in \mathcal{S}^{n\times n}$ such that

$$(j(t_j - s_j), j|x_j - y_j|^{\alpha-2}(x_j - y_j), X_j) \in \overline{P}^{2,+}u(t_j, x_j), \tag{4.27}$$

$$(-j(t_j - s_j), -j|x_j - y_j|^{\alpha-2}(x_j - y_j), Y_j) \in \overline{P}^{2,-}\varphi(s_j, y_j) \tag{4.28}$$

and (3.30) hold. The rest of the proof is the same as before. □

5. Removability results for singular equations

In this section we focus on the fully nonlinear equations (1.1), (1.2) which are *singular* in the sense that F is not defined on $\{Du = 0\}$. Typical examples are the p-Laplace equation $-\Delta_p u = 0$ and the p-Laplace diffusion equation $u_t - \Delta_p u = 0$ where $1 < p < 2$, and the mean curvature flow equation

$$u_t - |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0, \tag{5.1}$$

which says that every level set $\Gamma_c := \{u(t, \cdot) = c\}$ moves by its mean curvature provided $|Du| \neq 0$ on Γ_c. It is important to study singular equations because such equations appear in physics and geometry.

Hereafter we deal with the particular case that F depends only on Du and D^2u variable. The equations we consider are

$$F(Du, D^2u) = 0, \tag{5.2}$$

$$u_t + F(Du, D^2u) = 0. \tag{5.3}$$

Let us remark that F is not necessarily geometric in the sense of [5]. The notion of viscosity solutions to singular equations, (5.2) and (5.3), is due to Ohnuma and Sato [18] (see also [10, 15]). Let us recall the definition. We introduce some notation and state the assumptions on F.

We define $\mathcal{F}(F)$ and Σ by

$$\mathcal{F}(F) = \Big\{f \in C^2([0,\infty)) : f(0) = f'(0) = f''(0) = 0, \tag{5.4}$$
$$f''(r) > 0 \text{ for all } r > 0, \text{ and } \lim_{x\to 0} F(Df(|x|), D^2f(|x|)) = 0\Big\},$$

$$\Sigma = \{\sigma \in C^1(\mathbb{R}) : \sigma(0) = \sigma'(0) = 0,\ \sigma(t) = \sigma(-t) > 0 \text{ for all } t > 0\}. \tag{5.5}$$

We suppose that $F = F(q, X)$ satisfies the following:

(D1) F is a continuous function defined in $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}^{n\times n}$.
(D2) F is degenerate elliptic.
(D3) $\mathcal{F}(F) \neq \emptyset$, and if $f \in \mathcal{F}(F)$ and $a > 0$ then $af \in \mathcal{F}(F)$.

A function u is said to be a viscosity solution to the singular elliptic equation (5.2) if u is a relaxed viscosity solution, which is defined in Definition 2.3, to (5.2). More precisely, we give a definition as follows.

Definition 5.1. Let Ω be a domain in $\mathbb{R}^n$. Assume that (D1), (D2) and (D3) are satisfied.

(i) A function $u \in \mathrm{USC}(\Omega)$ is said to be a *viscosity subsolution* to (5.2) in Ω if $u \not\equiv -\infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$, which is a maximum point of $u - \varphi$ and satisfies $D\varphi(x_0) \neq 0$, we have

$$F(D\varphi(x_0), D^2\varphi(x_0)) \leq 0. \tag{5.6}$$

(ii) A function $u \in \mathrm{LSC}(\Omega)$ is said to be a *viscosity supersolution* to (5.2) in Ω if $u \not\equiv \infty$ and for any function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a minimum point of $u - \varphi$ and satisfies $D\varphi(x_0) \neq 0$, we have

$$F(D\varphi(x_0), D^2\varphi(x_0)) \geq 0. \tag{5.7}$$

(iii) A function $u \in C^0(\Omega)$ is said to be a *viscosity solution* to (5.2) in Ω if it is both a viscosity subsolution and supersolution to (5.2) in Ω.

Here is our Radó type removability result for (5.2).

Theorem 5.2. *Let Ω be a domain in $\mathbb{R}^n$. We suppose that* (D1), (D2) *and* (D3) *are satisfied. If $u \in C^1(\Omega)$ is a viscosity solution to* (5.2) *in $\Omega \setminus u^{-1}(0)$, then u is a viscosity solution to* (5.2) *in the whole domain Ω.*

Since the proof of this theorem is the same as Case 1 in the proof of Theorem 1.1, we omit the proof. Theorem 5.2 can be applied, for example, to the p-Laplace equation where $1 < p < 2$. We note that for $p \geq 2$, the p-Laplace equation has no singularity at $Du = 0$ and has been already covered by Theorem 1.1.

Next we give the notion of viscosity solutions to the singular parabolic equation (5.3). Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We say that a function $\varphi \in C^2(\mathcal{O})$ is *admissible* if for any $(\hat{t}, \hat{x}) \in \mathcal{O}$ with $D\varphi(\hat{t}, \hat{x}) = 0$, there exist $f \in \mathcal{F}(F)$, $\sigma \in \Sigma$ and a constant $\rho > 0$ such that $B_\rho(\hat{t}, \hat{x}) \subset \mathcal{O}$ and

$$|\varphi(t,x) - \varphi(\hat{t}, \hat{x}) - \varphi_t(\hat{t}, \hat{x})(t - \hat{t})| \leq f(|x - \hat{x}|) + \sigma(t - \hat{t}) \tag{5.8}$$

for all $(t,x) \in B_\rho(\hat{t}, \hat{x})$.

Definition 5.3. Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We assume (D1), (D2) and (D3) are satisfied.

(i) A function $u \in \mathrm{USC}(\mathcal{O})$ is said to be a *viscosity subsolution* to (5.3) in $\mathcal{O}$ if $u \not\equiv -\infty$ and for any admissible function $\varphi \in C^2(\mathcal{O})$ and any point $(t_0, x_0) \in \mathcal{O}$ which is a maximum point of $u - \varphi$, we have

$$\begin{cases} \varphi_t(t_0, x_0) + F(D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \leq 0 & \text{if } D\varphi(t_0, x_0) \neq 0, \\ \varphi_t(t_0, x_0) \leq 0 & \text{if } D\varphi(t_0, x_0) = 0. \end{cases} \tag{5.9}$$

(ii) A function $u \in \mathrm{LSC}(\mathcal{O})$ is said to be a *viscosity supersolution* to (5.3) in $\mathcal{O}$ if $u \not\equiv \infty$ and for any admissible function $\varphi \in C^2(\mathcal{O})$ and any point $(t_0, x_0) \in \mathcal{O}$ which is a minimum point of $u - \varphi$, we have

$$\begin{cases} \varphi_t(t_0, x_0) + F(D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \geq 0 & \text{if } D\varphi(t_0, x_0) \neq 0, \\ \varphi_t(t_0, x_0) \geq 0 & \text{if } D\varphi(t_0, x_0) = 0. \end{cases} \tag{5.10}$$

(iii) A function $u \in C^0(\mathcal{O})$ is said to be a *viscosity solution* to (5.3) in $\mathcal{O}$ if it is both a viscosity subsolution and supersolution to (5.3) in $\mathcal{O}$.

We shall show the removability of a level set for (5.3).

Theorem 5.4. *Let $\mathcal{O}$ be a domain in $\mathbb{R} \times \mathbb{R}^n$. We suppose that* (D1), (D2) *and* (D3) *are satisfied. If $u \in C^1(\mathcal{O})$ is a viscosity solution to* (5.3) *in $\mathcal{O} \setminus u^{-1}(0)$, then u is a viscosity solution to* (5.3) *in the whole domain $\mathcal{O}$.*

Remark 5.1. This theorem is applicable to various equations such as the p-Laplace diffusion equation where $1 < p < 2$ and the mean curvature flow equation (5.1).

Proof. We prove the subsolution case only. We fix a point $(t_0, x_0) \in \mathcal{O}$ and an admissible function $\varphi \in C^2(\mathcal{O})$ which satisfy

$$u(t_0, x_0) = \varphi(t_0, x_0) = 0, \tag{5.11}$$

$$u(t, x) < \varphi(t, x) \quad \text{for } (t, x) \in \mathcal{O} \setminus \{(t_0, x_0)\}. \tag{5.12}$$

It is sufficient to show that (5.9) holds. If $D\varphi(t_0, x_0) \neq 0$, the argument is the same as in the proof of Theorem 4.2. We consider the case $D\varphi(t_0, x_0) = 0$ and suppose that $\varphi_t(t_0, x_0) > 0$. It follows from (5.11) and the admissibility of φ that one can take $f \in \mathcal{F}(F)$, $\sigma \in \Sigma$ and $\rho > 0$ such that

$$|\varphi(t, x) - \varphi_t(t_0, x_0)(t - t_0)| \leq f(|x - x_0|) + \sigma(t - t_0) \tag{5.13}$$

for all $(t, x) \in B_\rho(t_0, x_0)$. Without loss of generality we may assume $(t_0, x_0) = (0, 0)$. By (5.11), (5.12) and the implicit function theorem, we can take a sufficiently small number $\rho' \in (0, \rho)$ such that

$$\{\varphi < 0\} \cap B_{\rho'}(0, 0) \subset \{u < 0\} \cap B_{\rho'}(0, 0). \tag{5.14}$$

For $\delta > 0$, we define ψ_δ by

$$\psi_\delta(t,x) = \varphi_t(0,0)t + f(|x|) + \sigma(t) + \Big(\delta^2 t + \frac{\delta}{2}t^2 + f(|x|)\Big). \tag{5.15}$$

We set $w_\delta = u - \psi_\delta$ and $\mathcal{C}_\delta = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : \delta^2 t + \delta t^2/2 + f(|x|) < 0\}$. Then the following properties hold.

(i) $(w_\delta)_t(0,0) = u_t(0,0) - (\varphi_t(0,0) + \delta^2) = -\delta^2 < 0$.

(ii) $w_\delta(0,0) = u(0,0) = 0$.

(iii) if $\delta^2 t + \delta t^2/2 + f(|x|) = 0$, i.e., $(t,x) \in \partial\mathcal{C}_\delta$, then it follows from (5.11), (5.12), (5.13) and (5.15) that

$$\begin{aligned} w_\delta(t,x) &= u(t,x) - \varphi_t(0,0)t - f(|x|) - \sigma(t) \\ &\le \varphi(t,x) - \varphi_t(0,0)t - f(|x|) - \sigma(t) \le 0. \end{aligned} \tag{5.16}$$

(iv) The set $\overline{\mathcal{C}_\delta}$ is compact in $\mathbb{R} \times \mathbb{R}^n$.

(v) $(-\varepsilon, 0) \in \mathcal{C}_\delta$ for sufficiently small $\varepsilon > 0$.

These properties imply that there exists a point $(\tilde{t}_\delta, \tilde{x}_\delta) \in \mathcal{C}_\delta$ such that

$$\sup\{w_\delta(t,x) : (t,x) \in \overline{\mathcal{C}_\delta}\} = w_\delta(\tilde{t}_\delta, \tilde{x}_\delta). \tag{5.17}$$

Let $\{\sigma_k\}_{k\in\mathbb{N}}$ be a sequence in $C^2(\mathbb{R})$ such that

$$\sigma_k(0) = \sigma_k'(0) = 0 \quad \text{for all } k \in \mathbb{N}, \tag{5.18}$$

$$\sigma_k(t) = \sigma_k(-t) > 0 \quad \text{for all } k \in \mathbb{N} \text{ and all } t > 0, \tag{5.19}$$

and that

$$\lim_{k\to\infty} \|\sigma_k - \sigma\|_{C^1([a,b])} = 0, \quad \text{for any bounded closed interval } [a,b]. \tag{5.20}$$

Now we set

$$\psi_{\delta,k}(t,x) = \varphi_t(0,0)t + f(|x|) + \sigma_k(t) + \Big(\delta^2 t + \frac{\delta}{2}t^2 + f(|x|)\Big). \tag{5.21}$$

By (5.20), we can find a sequence $\{(\tilde{t}_{\delta,k}, \tilde{x}_{\delta,k})\}_{k\in\mathbb{N}} \subset \mathcal{C}_\delta$ such that $w_{\delta,k} = u - \psi_{\delta,k}$ attains a local maximum at $(\tilde{t}_{\delta,k}, \tilde{x}_{\delta,k})$ (at least for sufficiently large k) and that

$$\lim_{k\to\infty} (\tilde{t}_{\delta,k}, \tilde{x}_{\delta,k}) = (\tilde{t}_\delta, \tilde{x}_\delta). \tag{5.22}$$

Here we prove the following lemma.

Lemma 5.5. *(i) $\psi_{\delta,k} \in C^2(\mathcal{O})$ is admissible.*

(ii) There exists a positive constant δ_0 such that for every $\delta \in (0, \delta_0)$, $u(\tilde{t}_{\delta,k}, \tilde{x}_{\delta,k}) < 0$.

Proof. First we prove (i). One sees that $D\psi_{\delta,k}(t,x)=0$ if and only if $x=0$. We fix a point $(\hat{t},0)\in\mathcal{O}$. Then we can take $\hat{\rho}>0$ such that $B_{\hat{\rho}}(\hat{t},0)\subset\mathcal{O}$. For every $(t,x)\in B_{\hat{\rho}}(\hat{t},0)$, it holds that

$$\begin{aligned}
&|\psi_{\delta,k}(t,x)-\psi_{\delta,k}(\hat{t},0)-(\psi_{\delta,k})_t(\hat{t},0)(t-\hat{t})| \\
&\quad=\left|2f(|x|)+\sigma_k(t)-\sigma_k(\hat{t})-\sigma_k'(\hat{t})(t-\hat{t})+\frac{\delta}{2}(t-\hat{t})^2\right| \\
&\quad\le 2f(|x|)+\frac{M_{k,\hat{\rho}}+\delta}{2}(t-\hat{t})^2,
\end{aligned}\tag{5.23}$$

where

$$M_{k,\hat{\rho}}=\max_{0\le s\le\hat{\rho}}|\sigma_k''(s)|.\tag{5.24}$$

It follows from (D3) that $2f\in\mathcal{F}(F)$, and one can easily see that $\tilde{\sigma}\in\Sigma$ where $\tilde{\sigma}(t):=(M_{k,\hat{\rho}}+\delta)t^2/2$. Therefore the assertion (i) follows.

Next we show (ii). It is sufficient to prove that $\mathcal{C}_\delta\subset\{\varphi<0\}\cap B_{\rho'}(0,0)$. By (5.14) and the fact that $(\tilde{t}_{\delta,k},\tilde{x}_{\delta,k})\in\mathcal{C}_\delta$, the assertion (ii) follows.

It holds that

$$(t,x)\in\mathcal{C}_\delta \implies -2\delta<t<0,\ f(|x|)<\frac{\delta^3}{2}.\tag{5.25}$$

Hence, we can take a positive constant δ_1 such that for all $\delta\in(0,\delta_1)$, $\mathcal{C}_\delta\subset B_{\rho'}(0,0)$. Let (t,x) be a point in $\mathcal{C}_\delta$. Since $\sigma\in\Sigma$, there exists $\delta_2>0$ which satisfies

$$|\sigma(s)|\le\frac{\varphi_t(0,0)}{2}|s|\quad\text{for all } s\in(-\delta_2,\delta_2).\tag{5.26}$$

Set $\delta_0:=\min\{\delta_1,\delta_2/2,\sqrt{\varphi_t(0,0)/2}\}$. If $\delta\in(0,\delta_0)$, then it follows from (5.13), (5.25) and (5.26) that

$$\begin{aligned}
\varphi(t,x)&\le\varphi_t(0,0)t+f(|x|)+\sigma(t) \\
&<\varphi_t(0,0)t-\left(\delta^2t+\frac{\delta}{2}t^2\right)-\frac{\varphi_t(0,0)}{2}t<\left(\frac{\varphi_t(0,0)}{2}-\delta_0^2\right)t\le 0.
\end{aligned}\tag{5.27}$$

(Note that t is a negative number.) The proof of (ii) is complete. □

We proceed to prove Theorem 5.4. We divide the proof into two cases. First we consider the case that $\tilde{x}_{\delta_m}\neq 0$ for some sequence $\{\delta_m\}_{m\in\mathbb{N}}$ such that $\delta_m\searrow 0$ as $m\to\infty$. It holds from (5.22) that $\tilde{x}_{\delta_m,k}\neq 0$ for sufficiently large k. Then $D\psi_{\delta_m,k}(\tilde{t}_{\delta_m,k},\tilde{x}_{\delta_m,k})\neq 0$. Since u is a viscosity subsolution to (5.3) in $\{u\neq 0\}$, we have that

$$\varphi_t(0,0)+\sigma_k'(\tilde{t}_{\delta_m,k})+\delta_m^2+\delta_m\tilde{t}_{\delta_m,k}+F\left(2Df(|\tilde{x}_{\delta_m,k}|),2D^2f(|\tilde{x}_{\delta_m,k}|)\right)\le 0.\tag{5.28}$$

As $k \to \infty$, we obtain

$$\varphi_t(0,0) + \sigma'(\tilde{t}_{\delta_m}) + \delta_m^2 + \delta_m \tilde{t}_{\delta_m} + F\left(2Df(|\tilde{x}_{\delta_m}|), 2D^2 f(|\tilde{x}_{\delta_m}|)\right) \leq 0. \tag{5.29}$$

Since $(\tilde{t}_{\delta_m}, \tilde{x}_{\delta_m}) \in \mathcal{C}_{\delta_m}$, it follows from (5.25) that $(\tilde{t}_{\delta_m}, \tilde{x}_{\delta_m}) \to (0,0)$ as $m \to \infty$. Taking $m \to \infty$ in (5.29), we get

$$\varphi_t(0,0) \leq 0, \tag{5.30}$$

since $\sigma \in \Sigma$ and $2f \in \mathcal{F}(F)$. This is a contradiction.

Next we deal with the case that there exists $\delta' > 0$ such that $\tilde{x}_\delta = 0$ for all $\delta \in (0, \delta')$. Fix $\delta \in (0, \delta')$. If $\tilde{x}_{\delta,k} = 0$ for infinitely many k's, then we may assume $\tilde{x}_{\delta,k} = 0$ for all $k \in \mathbb{N}$ so that $D\psi_{\delta,k}(\tilde{t}_{\delta,k}, \tilde{x}_{\delta,k}) = 0$. This yields

$$\varphi_t(0,0) + \sigma_k'(\tilde{t}_{\delta,k}) + \delta^2 + \delta \tilde{t}_{\delta,k} \leq 0. \tag{5.31}$$

As $k \to \infty$, we get

$$\varphi_t(0,0) + \sigma_k'(\tilde{t}_\delta) + \delta^2 + \delta \tilde{t}_\delta \leq 0. \tag{5.32}$$

If there exists $k_0 \in \mathbb{N}$ such that $\tilde{x}_{\delta,k} \neq 0$ for all $k \geq k_0$, then we have

$$\varphi_t(0,0) + \sigma_k'(\tilde{t}_{\delta,k}) + \delta^2 + \delta \tilde{t}_{\delta,k} + F\left(2Df(|\tilde{x}_{\delta,k}|), 2D^2 f(|\tilde{x}_{\delta,k}|)\right) \leq 0. \tag{5.33}$$

Since $(\tilde{t}_{\delta,k}, \tilde{x}_{\delta,k}) \to (\tilde{t}_\delta, 0)$ as $k \to \infty$, we obtain (5.32) again. Finally, taking $\delta \to +0$ in (5.32), we get (5.30) and reach a contradiction. □

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