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ENERGY ESTIMATES FOR STRICTLY HYPERBOLIC EQUATIONS WITH LOW REGULARITY IN COEFFICIENTS

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Abstract. An energy inequality is derived for a strictly hyperbolic operator under some assumptions related to the low regularity of the coefficients and its oscillating character.

1. Introduction

Well-posedness results for strictly hyperbolic Cauchy problems with non-regular time-dependent coefficients has a long tradition. For the Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \tag{1.1}$$

we know from the results of [3] that the critical condition for C^∞ well-posedness is the *global condition* $a_{kl} \in \mathrm{LogLip}[0,T]$. Global condition means that the condition is prescribed on the whole interval $[0,T]$ under consideration. Less regularity leads to well-posedness in subspaces of infinitely differentiable functions. The same holds of course if we consider the strictly

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hyperbolic Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x) u_{x_k x_l} = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \tag{1.2}$$

with coefficients depending on spatial variables, too (see e.g. [12]). To get at least Sobolev solutions for (1.2) we recall the results from [11]. If the coefficients belong to $C([0,T],\mathcal{B}^k) \cap C^1([0,T],\mathcal{B}^0)$ and the data belong to H^{k+1}, H^k, respectively, then there exists a uniquely determined solution belonging to $C([0,T],H^{k+1}) \cap C^1([0,T],H^k)$ for an appropriate k depending on the dimension n. By $\mathcal{B}^k$ we denote the space of functions on $\mathbb{R}^n$ consisting of functions with global bounded derivatives up to the order k. In the paper [1] it was shown that one can allow the critical global regularity $\mathrm{LogLip}[0,T]$ in t and $\mathcal{B}^\infty$ in x to prove the well-posedness in Sobolev classes in x but with a so-called *loss of regularity*.

A second condition to express a non-standard (standard means $a_{kl} \in \mathrm{LogLip}[0,T]$) behavior in t goes back to [5]. This condition consists of two parts, a very weak global condition $a_{kl} \in C[0,T]$ and a *local condition* $a_{kl} \in C^1(0,T]$ with $|a'_{kl}| \leq \frac{C}{t}$ for $t \in (0,T]$ with a constant C independent of t. Local condition means that the derivative $|a'_{kl}|$ is bounded only on each compact subinterval of $(0,T]$. If we suppose more regularity in t, let us say $a_{kl} \in C^2(0,T]$, then one can prescribe local conditions for the first two derivatives with a weaker bound than $\frac{C}{t}$ and with a special description of the loss of derivatives (see e.g. [8] and [6] for time-dependent coefficients and [9] for coefficients depending on time and smoothly on spatial variables and references therein).

In the present paper we are interested in the Cauchy problem with very low regularity of coefficients in both variables of the form

$$Pu := \partial_t^2 u - \partial_x(a(t,x)\partial_x u) = f(t,x),\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \tag{1.3}$$

where $a(t,x) \geq \lambda_0 > 0$. In the paper [4], under the assumption $a \in \mathrm{LogLip}\,(\mathbb{R}^2)$, at least an energy inequality was derived for functions $u \in C^2([0,T^*],H^\infty(\mathbb{R}))$. This inequality reads as follows:

$$\sup_{0\leq s\leq t} \|u(s)\|_{H^{1-\theta-\beta s}(\mathbb{R})} + \sup_{0\leq s\leq t} \|u_t(s)\|_{H^{-\theta-\beta s}(\mathbb{R})} \tag{1.4}$$

$$\leq C\Big(\|u(0)\|_{H^{1-\theta}(\mathbb{R})} + \|u_t(0)\|_{H^{-\theta}(\mathbb{R})} + \int_0^t \|Pu(s)\|_{H^{-\theta-\beta s}(\mathbb{R})} ds \Big)$$

for all $t \in [0,T^*]$, where $\theta \in (0,1/4]$, $T^* = 1/\beta$ and β depends on the LogLip norm of $a(t,x)$.

The estimate (1.4) hints at some observations:

- although the coefficients are globally defined the energy inequality is valid only in a small time interval $[0, T^*]$;
- the low regularity (LogLip) of $a = a(t, x)$ with respect to x allows only the derivation of energy estimates with a loss of time-dependent regularity βt, $t \in [0, T^*]$, T^* small, in the data space $H^{1-\theta} \times H^{-\theta}$, $\theta > 0$ small; if we assume more regularity (e.g. $B^{1+\varepsilon}$ (see [4])), then we can choose $\theta = 0$;
- a loss of derivatives proportional to the time t appears; the maximal loss is equal to 1 due to the choice of T^*;
- the low regularity of the coefficient allows only to prove the existence of solutions in distributional sense.

The coupling of global conditions (i.e., conditions of regularity in the variable t of the coefficient $a(t, x)$) and local ones (i.e., bounds on the oscillations of $a(t, x)$ near $t = 0$) was in the time-dependent case the aim of [7] and in the general case (1.2) with smooth coefficients in x the aim of [2] and [10]. Let us explain the strategy of [10]. In that paper, under a B^∞ regularity assumption in x, the question of well-posedness and precise loss of derivatives for solutions of (1.3) was studied with hypotheses of the following type:

$$\text{global condition :} \qquad \sup_{s>0, t, t+s \in [0,T]} \frac{\|a(t+s, \cdot) - a(t, \cdot)\|_{B^p(\mathbb{R})}}{s\,\mu(s)/\eta(s)} \le C_p, \tag{1.5}$$

$$\text{local condition :} \qquad \|\partial_t a(t, \cdot)\|_{B^p(\mathbb{R})} \le -C_p \frac{d}{dt}\Big(\mu(\eta^{-1}(t))\Big), \tag{1.6}$$

where μ and η are real functions with particular properties (see [10, Par. 2]) and the function $s \mapsto s\mu(s)/\eta(s)$ has the meaning of a modulus of continuity. In [10] the case $\eta(t) \equiv 1$ was excluded by the appearance of $\eta^{-1}(t)$ in (1.6). In any case, in this situation the estimate (1.4) and the well-posedness for (1.3) in the case of B^∞ regularity in the x variable was proved in [4] under the assumption (1.5) with $\mu(s) = \log(1/s)$.

The aim of the present work is here summarized:

- for the problem (1.3) we propose a set of assumptions on $a(t, x)$ which allow us to derive an energy inequality similar to (1.4); these assumptions are stated coupling global and local conditions with respect to t and keeping the LogLip regularity with respect to x in the coefficient $a(t, x)$.

The paper is organized as follows: in Section 2, we state the main results and we present some significant examples; Section 3 contains the proof of Theorems 2.1, 2.2, and 2.3; some concluding remarks are contained in Section 4.

2. Main results and examples

Let $\mu : (0,1] \to \mathbb{R}$ be a C^1 function such that $\mu(t) > 0$ and $\mu'(t) \leq 0$ for all $t \in (0,1]$. Let $\eta : (0,1] \to \mathbb{R}$ be a C^2 function such that $\lim_{t\to 0^+} \eta(t) = 0$, $\eta(t) > 0$, $\eta'(t) > 0$, $\eta''(t) < 0$ and $-\eta''(t) < \frac{\eta'(t)}{t}$ for all $t \in (0,1]$. We suppose that the function $t \mapsto t\mu(t)/\eta(t)$ is increasing for small positive t and that $\lim_{t\to 0^+} t\mu(t)/\eta(t) = 0$. We remark that under this last assumption and the assumptions on the function μ it will be not restrictive to suppose that $\eta(t) \geq t$ for all $t \in (0,1]$.

Our first result shows that if we suppose for the coefficient $a(t,x)$ the LogLip regularity with respect to the x variable and we assume conditions similar to (1.5) and (1.6) in which only the B^0 norm of $a(t,x)$ is concerned, then we can derive an energy inequality as (1.4) with an additional small loss of derivatives δ.

Theorem 2.1. *Let μ and η be as above. Suppose that*

$$\lim_{\varepsilon\to 0^+} \varepsilon\Big(-\mu'(\varepsilon) + \mu(\varepsilon)\frac{\eta'(\varepsilon)}{\eta(\varepsilon)}\Big) = 0. \tag{2.1}$$

Let $0 < T \leq 1$. Consider the operator $Pu := \partial_t^2 u - \partial_x(a(t,x)\partial_x u)$ under the following assumptions:

$$a \in C([0,T], \mathrm{LogLip}\ (\mathbb{R})) \cap C^1((0,T], \mathrm{LogLip}\ (\mathbb{R})), \quad a(t,x) \geq \lambda_0 > 0; \tag{2.2}$$

$$\textit{global condition:} \quad \sup_{s>0,t,t+s\in[0,T]} \frac{\|a(t+s,\cdot) - a(t,\cdot)\|_{B^0(\mathbb{R})}}{s\ \mu(s)/\eta(s)} \leq C; \tag{2.3}$$

$$\textit{local condition:} \quad \|\partial_t a(t,\cdot)\|_{B^0(\mathbb{R})} \leq -C\frac{d}{dt}\Big(\mu(\eta^{-1}(t))\Big), \quad \textit{for all } t \in (0,T]. \tag{2.4}$$

Then there exist positive constants θ, δ_0, β and T^* such that the following energy inequality holds for all functions $u \in C^2([0,T^*], H^\infty(\mathbb{R}))$:

$$\sup_{0\leq s\leq t} \|u(s)\|_{H^{1-\theta-\delta-\beta s}(\mathbb{R})} + \sup_{0\leq s\leq t} \|u_t(s)\|_{H^{-\theta-\delta-\beta s}(\mathbb{R})}$$

$$\leq C_\delta\Big(\|u(0)\|_{H^{1-\theta}(\mathbb{R})} + \|u_t(0)\|_{H^{-\theta}(\mathbb{R})} + \int_0^t \|Pu(s)\|_{H^{-\theta-\beta s}(\mathbb{R})}ds\Big)$$

for all $\delta \in (0,\delta_0]$ *and for all* $t \in [0,T^*]$.

Remark 2.1. The goal of Theorem 2.1 is to fix conditions like (2.1) to (2.4) which allow us to derive an energy inequality with an additional arbitrary small loss of derivatives δ in comparison with the energy inequality from [4]. If we weaken (2.1) to

$$\lim_{\varepsilon\to 0^+} \varepsilon\Big(-\mu'(\varepsilon) + \mu(\varepsilon)\frac{\eta'(\varepsilon)}{\eta(\varepsilon)}\Big) \leq \tilde{\delta} \tag{2.5}$$

with a small positive $\tilde{\delta}$, then the same energy inequality holds with a fixed $\delta = \delta(\tilde{\delta})$.

In our second result we simplify the hypotheses of Theorem 2.1 expressing the global and the local condition only in terms of a function ψ which essentially will measure how far the regularity of the coefficient $a(t,x)$ is from the LogLip-property.

Let $\psi : (0,\infty) \to \mathbb{R}$ be a C^2 function such that

$$\lim_{t\to\infty} \psi(t) = 0, \quad \lim_{t\to\infty} \frac{t}{\psi(t)\exp t} = 0$$

and $0 < \psi''(t) < -\psi'(t) \leq Ct^{-1}\psi(t)$ for sufficiently large $t > 0$.

Theorem 2.2. *Let* ψ *be as above. Let* $0 < T \leq 1$. *Consider the operator* $Pu := \partial_t^2 u - \partial_x(a(t,x)\partial_x u)$ *under the following assumptions:*

$$a \in C([0,T], \mathrm{LogLip}\ (\mathbb{R})) \cap C^1((0,T], \mathrm{LogLip}\ (\mathbb{R})), \quad a(t,x) \geq \lambda_0 > 0; \tag{2.6}$$

global condition:
$$\sup_{s>0,t,t+s\in[0,T]} \frac{\psi(\log 1/s)\|a(t+s,\cdot) - a(t,\cdot)\|_{B^0(\mathbb{R})}}{s\ \log 1/s} \leq C; \tag{2.7}$$

for all $\tilde{\delta} > 0$ *there exists* $0 < r_{\tilde{\delta}} \leq T$ *such that*

local condition:
$$\|\partial_t a(t,\cdot)\|_{B^0(\mathbb{R})} \leq -\frac{C}{\psi'\big(\psi^{-1}\big(\frac{t}{\tilde{\delta}}\big)\big)}, \quad \textit{for all } t \in (0, r_{\tilde{\delta}}]. \tag{2.8}$$

Then for every sufficiently small $\tilde{\delta} > 0$ *there exists a small positive* $\delta = \delta(\tilde{\delta})$ *such that the energy inequality of Theorem* 2.1 *holds with* δ.

We can also obtain an energy inequality if we assume only a local condition. This is contained in the following theorem.

Theorem 2.3. *Let* $0 < T \leq 1$. *Consider the operator* $Pu := \partial_t^2 u - \partial_x(a(t,x)\partial_x u)$ *under the following assumptions:*

$$a \in C([0,T], \text{LogLip}\,(\mathbb{R})) \cap C^1((0,T], \text{LogLip}\,(\mathbb{R})), \quad a(t,x) \geq \lambda_0 > 0; \tag{2.9}$$

$$\textit{local condition:} \qquad \Big\| \frac{\partial_t a(t,\cdot)}{a(t,\cdot)} \Big\|_{B^0(\mathbb{R})} \leq \frac{1}{2t} \qquad \textit{for all } t > 0. \tag{2.10}$$

Then there exist positive constants θ, δ *and* T^* *such that the following energy inequality holds for all functions* $u \in C^2([0,T^*], H^\infty(\mathbb{R}))$*:*

$$\sup_{0\leq s\leq t} \|u(s)\|_{H^{1-\theta-\delta}(\mathbb{R})} + \sup_{0\leq s\leq t} \|u_t(s)\|_{H^{-\theta-\delta}(\mathbb{R})}$$
$$\leq C\Big(\|u(0)\|_{H^{1-\theta}(\mathbb{R})} + \|u_t(0)\|_{H^{-\theta}(\mathbb{R})} + \int_0^t \|Pu(s)\|_{H^{-\theta}(\mathbb{R})} ds\Big)$$

for all $t \in [0,T^*]$.

Now we present some examples for the choice of functions μ and η from Theorem 2.1.

Example 2.1. We choose, for s near 0,

$$\mu(s) = \log_2 \frac{1}{s}\Big(\log\log_2 \frac{1}{s}\Big)^{-1} \text{ and } \eta(s) = \Big(\log_2 \frac{1}{s}\Big)^{-m},\ m > 0.$$

The condition (2.1) is satisfied and the local condition (2.4) is approximately the estimate

$$\|\partial_t a(t,\cdot)\|_{B^0(\mathbb{R})} \leq C\Big(\frac{1}{t}\Big)^{\frac{m+1}{m}} \text{ for all } t \in (0,T].$$

Example 2.2. We choose, for s near 0,

$$\mu(s) = \log_2 \frac{1}{s}\Big(\log\log\log_2 \frac{1}{s}\Big)^{-1} \text{ and } \eta(s) = \Big(\log_2 \log \frac{1}{s}\Big)^{-1}.$$

The condition (2.1) is satisfied and the local condition (2.4) is approximately

$$\|\partial_t a(t,\cdot)\|_{B^0(\mathbb{R})} \leq C\frac{1}{t^2}e^{\frac{1}{t}} \text{ for all } t \in (0,T].$$

We have then the following corollary.

Corollary 2.1. *Let us choose in the assumptions* (2.3) *and* (2.4) *one of the Examples* 2.1 *or* 2.2 *for the functions* μ *and* η. *Then the energy inequality of Theorem* 2.1 *holds.*

Now we present some examples for the choice of the function ψ from Theorem 2.2.

Example 2.3. We choose, for s near 0, $\psi(s) = s^{-m}$, $m > 0$. The local condition (2.8) is the estimate

$$\|\partial_t a(t,\cdot)\|_{B^0(\mathbb{R})} \le C\Big(\frac{\tilde{\delta}}{t}\Big)^{\frac{m+1}{m}} \text{ for all } t \in (0, r_{\tilde{\delta}}].$$

Example 2.4. We choose, for s near 0,

$$\psi(s) = \Big(\log^{[m]}\frac{1}{s}\Big)^{-1},\ m \in \mathbb{N}.$$

The local condition (2.8) is the estimate

$$\|\partial_t a(t,\cdot)\|_{B^0(\mathbb{R})} \le -C\frac{d}{dt}\Big(\exp^{[m]}\frac{\tilde{\delta}}{t}\Big) \text{ for all } t \in (0, r_{\tilde{\delta}}].$$

We have then the following corollary.

Corollary 2.2. *Let us choose in the assumptions* (2.7) *and* (2.8) *one of the Examples* 2.3 *or* 2.4 *for the function* ψ. *Then the energy inequality of Theorem* 2.2 *holds.*

3. Proofs of Theorems 2.1, 2.2 and 2.3

The following lemma is important for the proofs.

Lemma 3.1. *If the assumptions for the functions* μ *and* η *are satisfied, then condition* (2.1) *implies*

$$\lim_{\varepsilon\to 0+} \varepsilon\Big(\sup_{t\in[2\eta(\varepsilon),T]} -\frac{d}{dt}\Big(\mu(\eta^{-1}(t-\varepsilon))\Big)\Big) = 0. \tag{3.1}$$

Proof. The function $\rho = \rho(t) := t\eta'(t)$ is strictly increasing because of

$$\rho'(t) = \eta'(t) + t\eta''(t) > \eta'(t) + t\Big(\frac{-\eta'(t)}{t}\Big) = 0.$$

Here we used the assumption $-\eta''(t) < \frac{\eta'(t)}{t}$ for small t. In particular, putting $t = \eta^{-1}(s)$ gives the strictly increasing behavior of $\rho(\eta^{-1}(s))$ by taking into account the strictly increasing behavior of $\eta^{-1}(t)$.

Moreover, we have $\lim_{t\to 0+}\eta'(t) = \infty$. Otherwise, by the monotonicity of $\eta'(t)$ and $\mu(t)$ we conclude $\lim_{t\to 0+}\eta'(t) = \eta_0$ and $\lim_{t\to 0+}\mu(t) \ge \mu_0\,(> 0)$.

Then, since $\eta \in C^1(0,t)$ and $\lim_{t\to 0+}\eta(t) = 0$ the application of the mean value theorem gives

$$\lim_{t\to 0+}\frac{t\mu(t)}{\eta(t)} = \lim_{t\to 0+}\frac{t\mu(t)}{t\eta'(\theta(t)t)} = \lim_{t\to 0+}\frac{\mu(t)}{\eta'(\theta(t)t)} \geq \frac{\mu_0}{\eta_0},$$

which contradicts the assumption $\lim_{t\to 0+}\frac{t\mu(t)}{\eta(t)} = 0$. From (2.1) we get $\lim_{\varepsilon\to 0+}\varepsilon(-\mu'(\varepsilon)) = 0$. Thus $-\varepsilon\mu'(\varepsilon) \leq C$ for all $\varepsilon \in (0,\varepsilon_0]$. Setting $\varepsilon = \eta^{-1}(t-\varepsilon)$ for $t \in [2\eta(\varepsilon),T]$, T small, it implies $-\eta^{-1}(t-\varepsilon)\mu'(\eta^{-1}(t-\varepsilon)) \leq C$ for $t \in [2\eta(\varepsilon),T]$ and $\varepsilon \in (0,\varepsilon_0]$. To prove the statement of the lemma we conclude as follows:

$$\begin{aligned}
-\frac{d}{dt}\Big(\mu(\eta^{-1}(t-\varepsilon))\Big) &= -\frac{\mu'(\eta^{-1}(t-\varepsilon))}{\eta'(\eta^{-1}(t-\varepsilon))} = -\frac{\eta^{-1}(t-\varepsilon)\mu'(\eta^{-1}(t-\varepsilon))}{\eta^{-1}(t-\varepsilon)\eta'(\eta^{-1}(t-\varepsilon))}\\
&\leq \frac{C}{\rho(\eta^{-1}(t-\varepsilon))} \leq \frac{C}{\rho(\eta^{-1}(2\eta(\varepsilon)-\varepsilon))} \leq \frac{C}{\rho(\eta^{-1}(2\eta(\varepsilon)-\eta(\varepsilon))} = \frac{C}{\rho(\varepsilon)}.
\end{aligned}$$

Here, we used the monotonicity of ρ and the relation $\eta(\varepsilon) \geq \varepsilon$. Taking the sup over $t \in [2\eta(\varepsilon),T]$ on the left-hand side and multiplying by ε brings

$$\lim_{\varepsilon\to 0+}\varepsilon\Big(\sup_{t\in[2\eta(\varepsilon),T]} -\frac{d}{dt}\Big(\mu(\eta^{-1}(t-\varepsilon))\Big)\Big) \leq C\lim_{\varepsilon\to 0+}\frac{\varepsilon}{\rho(\varepsilon)} = C\lim_{\varepsilon\to 0+}\frac{1}{\eta'(\varepsilon)} = 0.$$

This is the statement we wanted to prove. □

Proof of Theorem 2.1. The proof is divided into several steps.

Step 1: Regularization. Let $\phi(r)$ be an even nonnegative function with support on $[-\rho,\rho]$ and which satisfies $\int_{\mathbb{R}}\phi(r)dr = 1$. We set $a(t,x) = a(0,x)$ for $t < 0$ and $a(t,x) = a(T,x) + (t-T)\partial_t a(T,x)$ for $t > T$. For $\varepsilon > 0$ we define

$$a_\varepsilon(t,x) := \int_{\mathbb{R}} a(t+\tau\varepsilon,x)\phi(\tau)\,d\tau \quad \text{for all } (t,x)\in[0,T]\times\mathbb{R}.$$

We introduce the *pseudodifferential zone* $Z_{pd} = \{(\varepsilon,t)\in\mathbb{R}^+\times[0,T] : t \leq 4\eta(\varepsilon)\}$ and the *hyperbolic zone* $Z_{hyp} = \{(\varepsilon,t)\in\mathbb{R}^+\times[0,T] : t \geq 2\eta(\varepsilon)\}$.

The following lemma states some useful properties of a_ε.

Lemma 3.2. *The regularization a_ε satisfies*

$$a_\varepsilon(t,x) \geq \lambda_0\ ;$$

$$\|a_\varepsilon(t,\cdot) - a(t,\cdot)\|_{B^0(\mathbb{R})} \leq C\varepsilon\frac{\mu(\varepsilon)}{\eta(\varepsilon)} \quad \text{in } Z_{pd},$$

$$\|\partial_t a_\varepsilon(t,\cdot)\|_{B^0(\mathbb{R})} \leq C\frac{\mu(\varepsilon)}{\eta(\varepsilon)} \quad \text{in } Z_{pd};$$

$$\|a_\varepsilon(t,\cdot)-a(t,\cdot)\|_{B^0(\mathbb{R})} \le -C\varepsilon \frac{d}{dt}\left(\mu(\eta^{-1}(t-\varepsilon))\right) \ \text{ in } \ Z_{hyp},$$

$$\|\partial_t a_\varepsilon(t,\cdot)\|_{B^0(\mathbb{R})} \le -C\frac{d}{dt}\left(\mu(\eta^{-1}(t-\varepsilon))\right) \ \text{ in } \ Z_{hyp}.$$

Here the constant C is independent of ε.

Proof. See [10, Lemma 4.1]. □

Step 2: A-priori estimates for approximate microlocalized energies. We introduce a Littlewood–Paley spectral decomposition (with respect to x) and we define the approximate microlocalized energy

$$E_{\nu,\varepsilon}(u)(t) = (\dot u_\nu, \dot u_\nu) + (a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x u_\nu), \ \ u_\nu = \varphi_\nu u.$$

Here $\dot u_\nu$ is the partial derivative of u_ν with respect to t. By $\{\varphi_\nu, \nu \ge 0\}$ we denote the usual family of functions generating the spectral decomposition, in particular $\varphi_0 \in C^\infty(\mathbb{R})$, φ_0 even, $0 \le \varphi_0 \le 1$, $\varphi_0(\xi) = 1$ for all $|\xi| \le 1/2$ and $\operatorname{supp}\varphi_0 \subseteq [-1,1]$, $\varphi(\xi) = \varphi_0(\xi/2) - \varphi_0(\xi)$ and finally $\varphi_\nu(\xi) = \varphi(\xi/2^\nu)$ for all $\nu \ge 1$ (see [4, Par. 3]). After differentiation of $E_{\nu,\varepsilon}(u)(t)$ with respect to t we obtain

$$\begin{aligned}
&\frac{d}{dt}E_{\nu,\varepsilon}(u)(t) = 2Re(\ddot u_\nu, \dot u_\nu) + (\dot a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x u_\nu) + 2Re(a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x \dot u_\nu)\\
&= 2Re(\varphi_\nu(\partial_x(a(t,\cdot)\partial_x u) + Pu), \dot u_\nu) + (\dot a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x u_\nu)\\
&\quad + 2Re(a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x \dot u_\nu)\\
&= 2Re(\partial_x(a(t,\cdot)\partial_x u_\nu), \dot u_\nu) + 2Re(\partial_x[\varphi_\nu, a(t,\cdot)]\partial_x u, \dot u_\nu)\\
&\quad + 2Re((Pu)_\nu, \dot u_\nu) + (\dot a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x u_\nu) + 2Re(a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x \dot u_\nu)\\
&= (\dot a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x u_\nu) + 2Re((a_\varepsilon(t,\cdot) - a(t,\cdot))\partial_x u_\nu, \partial_x \dot u_\nu)\\
&\quad - 2Re([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot u_\nu) + 2Re((Pu)_\nu, \dot u_\nu)\ .
\end{aligned}$$

Let us fix $\varepsilon_\nu = 2^{-\nu}$. Exploiting the results of Lemma 3.2 in the pseudodifferential zone we obtain

$$\begin{aligned}
&\frac{d}{dt}E_{\nu,\varepsilon_\nu}(u)(t) \le C\frac{\mu(\varepsilon_\nu)}{\eta(\varepsilon_\nu)}E_{\nu,\varepsilon_\nu}(u)(t) + 2\|(Pu)_\nu\|_{L^2}\|\dot u_\nu\|_{L^2}\\
&\quad + 2Re((a_{\varepsilon_\nu}(t,\cdot) - a(t,\cdot))\partial_x u_\nu, \partial_x \dot u_\nu) - 2Re([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot u_\nu).
\end{aligned}$$

The term $2Re((a_{\varepsilon_\nu}(t,\cdot) - a(t,\cdot))\partial_x u_\nu, \partial_x \dot u_\nu)$ can be estimated in the following way:

$$\begin{aligned}
&|2Re((a_{\varepsilon_\nu}(t,\cdot) - a(t,\cdot))\partial_x u_\nu, \partial_x \dot u_\nu)| \le 2C\varepsilon_\nu \frac{\mu(\varepsilon_\nu)}{\eta(\varepsilon_\nu)}\|\partial_x u_\nu\|_{L^2}\|\partial_x \dot u_\nu\|_{L^2}\\
&\le 2C\varepsilon_\nu \frac{\mu(\varepsilon_\nu)}{\eta(\varepsilon_\nu)}\|\partial_x u_\nu\|_{L_2}\varepsilon_\nu^{-1}\|\dot u_\nu\|_{L^2} \le C\ \frac{\mu(\varepsilon_\nu)}{\eta(\varepsilon_\nu)}E_{\nu,\varepsilon_\nu}(u)(t),
\end{aligned}$$

so that in Z_{pd} we have

$$\frac{d}{dt}E_{\nu,\varepsilon_\nu}(u)(t) \le C_0\frac{\mu(\varepsilon_\nu)}{\eta(\varepsilon_\nu)}E_{\nu,\varepsilon_\nu}(u)(t) + 2\|(Pu)_\nu\|_{L^2}\|\dot{u}_\nu\|_{L^2} - 2Re([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu). \qquad (3.2)$$

Similarly, we get in the hyperbolic zone

$$\frac{d}{dt}E_{\nu,\varepsilon_\nu}(u)(t) \le -C_0\frac{d}{dt}\Big(\mu(\eta^{-1}(t-\varepsilon_\nu))\Big)\, E_{\nu,\varepsilon_\nu}(u)(t) + 2\|(Pu)_\nu\|_{L^2}\|\dot{u}_\nu\|_{L^2} - 2Re([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu). \qquad (3.3)$$

From (3.2) and (3.3) we want to obtain an estimate for the approximate microlocalized energy $E_{\nu,\varepsilon_\nu}(u)(t)$ for all $(\varepsilon_\nu, t) \in (0,1] \times [0,T]$. To this end we introduce a C^∞ function $\chi(r)$ with $\chi(r) = 1$ for $r \le 1$ and $\chi(r) = 0$ for $r \ge 2$, χ decreasing and $0 \le \chi(r) \le 1$. Then multiplying (3.2) by $\chi\big(\frac{t}{2\eta(\varepsilon_\nu)}\big)$, (3.3) by $1 - \chi(\frac{t}{2\eta(\varepsilon_\nu)})$ and adding both inequalities we have

$$\frac{d}{dt}E_{\nu,\varepsilon_\nu}(u)(t) \le C_0\omega(\varepsilon_\nu, t)E_{\nu,\varepsilon_\nu}(u)(t) + 2\|(Pu)_\nu\|_{L^2}\|\dot{u}_\nu\|_{L^2} - 2Re([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu), \qquad (3.4)$$

where

$$\omega(\varepsilon_\nu, t) := \frac{\mu(\varepsilon_\nu)}{\eta(\varepsilon_\nu)}\chi\Big(\frac{t}{2\eta(\varepsilon_\nu)}\Big) - \frac{d}{dt}\Big(\mu(\eta^{-1}(t-\varepsilon_\nu))\Big)\Big(1 - \chi\Big(\frac{t}{2\eta(\varepsilon_\nu)}\Big)\Big).$$

Step 3: A-priori estimate for the total energy. For the functions $u \in C^2([0,T], H^\infty(\mathbb{R}))$ we introduce the energy

$$E(u)(t) := \sum_{\nu=0}^{\infty} e^{-C_0\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t}2^{-2\nu\theta}E_{\nu,\varepsilon_\nu}(u)(t),$$

where β and θ are positive constants to be fixed later on. We differentiate $E(u)(t)$ with respect to t. From (3.4) we obtain

$$\begin{aligned} E'(u)(t) \le & -2\beta\sum_{\nu=0}^{\infty} e^{-C_0\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t}2^{-2\nu\theta}\nu E_{\nu,\varepsilon_\nu}(u)(t) \\ & + \sum_{\nu=0}^{\infty} e^{-C_0\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t}2^{-2\nu\theta}2\|(Pu)_\nu\|_{L^2}\|\dot{u}_\nu\|_{L^2} \qquad (3.5) \\ & + \sum_{\nu=0}^{\infty} e^{-C_0\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t}2^{-2\nu\theta}2|([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu)|. \end{aligned}$$

We will use the structure of the first series to compensate for the influence of series appearing in the estimate of the commutator (see Step 5 below). The second series is related to the right-hand side f from (1.3).

Step 4: Properties of the weight. We point out some properties of the function $\int_0^t \omega(\varepsilon,\tau)d\tau$. The first one is established by the following lemma.

Lemma 3.3. *Let*

$$F(\varepsilon,t)=\int_0^t \omega(\varepsilon,\tau)d\tau.$$

Then there exists $\tilde{C}>0$ *such that* $F(\varepsilon,t)\le \tilde{C}\,\mu(\varepsilon)$ *for all* $(\varepsilon,t)\in(0,r_0]\times[0,T]$.

Proof. If $t\le 4\eta(\varepsilon)$ the statement is trivial in view of the definition of ω. Let then $t>4\eta(\varepsilon)$. We have

$$\begin{aligned}\int_0^t \omega(\varepsilon,\tau)d\tau &\le \int_0^{4\eta(\varepsilon)} \frac{\mu(\varepsilon)}{\eta(\varepsilon)}ds+\int_{2\eta(\varepsilon)}^t -C\frac{d}{ds}\Big(\mu(\eta^{-1}(s-\varepsilon))\Big)ds\\ &\le 4\mu(\varepsilon)+C\,\mu(\eta^{-1}(2\eta(\varepsilon)-\varepsilon))-C\mu(\eta^{-1}(t-\varepsilon)).\end{aligned}$$

Since $2\eta(\varepsilon)-\varepsilon\ge\eta(\varepsilon)$ we deduce that

$$\int_0^t \omega(\varepsilon,\tau)d\tau\le 4\mu(\varepsilon)+C\,\mu(\varepsilon)$$

and the conclusion follows. □

We recall now that from (2.1) we have $\lim_{s\to 0^+} -s\mu'(s)=0$. Consequently, for all $\delta>0$ there exists $r_\delta\in(0,1]$ such that

$$0\le-\mu'(t)\le\frac{\delta}{\tilde{C}t}$$

for all $t\in(0,r_\delta]$. Hence,

$$\mu(t)\le\mu(r_\delta)+\frac{\delta}{\tilde{C}}\log\Big(\frac{1}{t}\Big).$$

In view of the result of Lemma 3.3 it follows that for all $\delta>0$ there exists $\nu_0(\delta)$ such that

$$\frac{C_0}{2}\int_0^t \omega(\varepsilon_\nu,\tau)\,d\tau\le\delta\nu \tag{3.6}$$

for all $\nu\ge\nu_0(\delta)$ and for all $t\in[0,T]$. Remark that another consequence will be that for all $\delta>0$ the term $\exp(-C_0\int_0^t\omega(\varepsilon_\nu,\tau)d\tau)$ in the energy weight can be controlled by a loss of δ derivatives.

The property stated in the following lemma will be crucial in obtaining the estimate for the commutator.

Lemma 3.4. *For all $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that $\varepsilon|\partial_\varepsilon F(\varepsilon, t)| \le \delta$ for all $(\varepsilon, t) \in (0, \varepsilon_\delta] \times [0, T]$.*

Proof. From the definition of $\omega(\varepsilon, t)$ we deduce that

$$\begin{aligned} F(\varepsilon,t) = \ & \int_0^t \frac{\mu(\varepsilon)}{\eta(\varepsilon)}\chi\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\,d\tau - \int_0^t \mu(\eta^{-1}(\tau-\varepsilon))\chi'\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\frac{1}{2\eta(\varepsilon)}\,d\tau \\ & -\mu(\eta^{-1}(t-\varepsilon))\Big(1-\chi\Big(\frac{t}{2\eta(\varepsilon)}\Big)\Big). \end{aligned}$$

A straightforward computation gives

$$\begin{aligned} \partial_\varepsilon F(\varepsilon,t) = \ & \int_0^t \frac{\mu'(\varepsilon)\eta(\varepsilon)-\mu(\varepsilon)\eta'(\varepsilon)}{(\eta(\varepsilon))^2}\chi\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\,d\tau \\ & -\int_0^t \frac{\mu(\varepsilon)}{\eta(\varepsilon)}\chi'\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\frac{\eta'(\varepsilon)}{2(\eta(\varepsilon))^2}\tau\,d\tau \\ & -\int_0^t (\mu(\eta^{-1}))'(\tau-\varepsilon)\ \chi'\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\Big(\frac{\eta'(\varepsilon)}{2(\eta(\varepsilon))^2}\tau - \frac{1}{2\eta(\varepsilon)}\Big)\,d\tau \\ & +(\mu(\eta^{-1}))'(t-\varepsilon)\Big(1-\chi\Big(\frac{t}{2\eta(\varepsilon)}\Big)\Big). \end{aligned}$$

We have

$$\Big|\int_0^t \frac{\mu'(\varepsilon)\eta(\varepsilon)-\mu(\varepsilon)\eta'(\varepsilon)}{(\eta(\varepsilon))^2}\chi\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\,d\tau\Big| \le 2\Big(-\mu'(\varepsilon)+\mu(\varepsilon)\frac{\eta'(\varepsilon)}{\eta(\varepsilon)}\Big)$$

and

$$\begin{aligned} & \Big|\int_0^t \frac{\mu(\varepsilon)}{\eta(\varepsilon)}\chi'\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\frac{\eta'(\varepsilon)}{2(\eta(\varepsilon))^2}\tau\,d\tau\Big| \\ & \le -4\mu(\varepsilon)\frac{\eta'(\varepsilon)}{\eta(\varepsilon)}\int_{2\eta(\varepsilon)}^{4\eta(\varepsilon)} \chi'\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\frac{1}{2\eta(\varepsilon)}\,d\tau \le 4\mu(\varepsilon)\frac{\eta'(\varepsilon)}{\eta(\varepsilon)}. \end{aligned}$$

Since $2\eta(\varepsilon)-\varepsilon \ge \eta(\varepsilon)$ and $\mu(\eta^{-1})$ is a decreasing function we obtain

$$\begin{aligned} & \Big|\int_0^t (\mu(\eta^{-1}))'(\tau-\varepsilon)\ \chi'\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\ \frac{\eta'(\varepsilon)}{2(\eta(\varepsilon))^2}\tau\,d\tau\Big| \\ & \le 2(\max_{\mathbb{R}}|\chi'|)\frac{\eta'(\varepsilon)}{\eta(\varepsilon)}\int_{2\eta(\varepsilon)}^{4\eta(\varepsilon)} -(\mu(\eta^{-1}))'(\tau-\varepsilon)\,d\tau \end{aligned}$$

$$\leq 2(\max_{\mathbb{R}}|\chi'|)\frac{\eta'(\varepsilon)}{\eta(\varepsilon)}\mu(\eta^{-1}(2\eta(\varepsilon)-\varepsilon)) \leq 2(\max_{\mathbb{R}}|\chi'|)\mu(\varepsilon)\frac{\eta'(\varepsilon)}{\eta(\varepsilon)}.$$

Moreover,

$$\Big|\int_0^t (\mu(\eta^{-1}))'(\tau-\varepsilon)\,\chi'\Big(\frac{\tau}{2\eta(\varepsilon)}\Big)\,\frac{1}{2\eta(\varepsilon)}\,d\tau\Big| \leq \sup_{t\in[2\eta(\varepsilon),T]} -(\mu(\eta^{-1}))'(t-\varepsilon),$$

and finally

$$\Big| -(\mu(\eta^{-1}))'(t-\varepsilon)\Big(1-\chi\Big(\frac{t}{2\eta(\varepsilon)}\Big)\Big)\Big| \leq \sup_{t\in[2\eta(\varepsilon),T]} -(\mu(\eta^{-1}))'(t-\varepsilon).$$

The conclusion of the proof follows from the assumption (2.1) and from Lemma 3.1. □

Let us consider $G(\nu,t)=-\frac{C_0}{2}F(2^{-\nu},t)$. We have

$$\partial_\nu G(\nu,t)=\frac{C_0}{2}(\log 2)2^{-\nu}\partial_\varepsilon F(2^{-\nu},t).$$

The statement from Lemma 3.4 gives that for all $\delta>0$ there exists $\nu_1(\delta)$ such that $|\partial_\nu G(\nu,t)|\leq\delta$ for all $\nu\geq\nu_1(\delta)$ and for all $t\in[0,T]$. As a consequence we will have that for all $\delta>0$ there exists $\nu_1(\delta)$ such that

$$\frac{C_0}{2}\Big|\int_0^t(\omega(\varepsilon_\nu,\tau)-\omega(\varepsilon_\mu,\tau))\,d\tau\Big|\leq\delta|\nu-\mu| \tag{3.7}$$

for all ν, $\mu\geq\nu_1(\delta)$ and for all $t\in[0,T]$. We will denote by $\nu(\delta)$ the maximum value between $\nu_0(\delta)$ and $\nu_1(\delta)$.

Step 5: Estimates for the commutator term. We have to derive a suitable estimate for the series

$$\sum_{\nu=0}^{\infty}e^{-C_0\int_0^t\omega(\varepsilon_\nu,\tau)d\tau-2\beta\nu t}2^{-2\nu\theta}2Re([\varphi_\nu,a(t,\cdot)]\partial_x u,\partial_x\dot{u}_\nu),$$

where β and θ are positive constants. This is the subject of the following lemma.

Lemma 3.5. *Let $\theta=1/8$ and $\delta\leq 1/16$. There exist positive constants C_δ and C'_δ such that*

$$\begin{aligned}\sum_{\nu=0}^{\infty}&e^{-C_0\int_0^t\omega(\varepsilon_\nu,\tau)d\tau-2\beta\nu t}2^{-2\nu\theta}2|([\varphi_\nu,a(t,\cdot)]\partial_x u,\partial_x\dot{u}_\nu)|\\&\leq C_\delta\sum_{\nu=0}^{\infty}e^{-C_0\int_0^t\omega(\varepsilon_\nu,\tau)d\tau-2\beta\nu t}2^{-2\nu\theta}\nu E_{\nu,\varepsilon_\nu}(u)(t),\end{aligned} \tag{3.8}$$

and

$$e^{\frac{C_0}{2}\int_0^t \omega(\varepsilon_\nu,\tau)d\tau} \le C'_\delta c^{\nu\delta} \tag{3.9}$$

for all $\nu \ge 0$, for all $\beta \in (0,+\infty)$ and for all $t \in [0,T^*_\beta]$, where $T^*_\beta = 1/(8\beta)$.

Proof. The proof is inspired by the similar one in [4, Lemma 4.4]. Setting $\psi_\mu := \varphi_{\mu-1} + \varphi_\mu + \varphi_{\mu+1}$ $(\varphi_{-1} \equiv 0)$, the left-hand side of (3.8) can be estimated by

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} e^{-C_0\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t}2^{-2\nu\theta}2|([\varphi_\nu, a(t,\cdot)]\psi_\mu \partial_x u_\mu, \partial_x \dot{u}_\nu)| \\
&\le \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} e^{-C_0\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t}2^{-2\nu\theta}2\|[\varphi_\nu, a(t,\cdot)]\psi_\mu\|_{L^2\to L^2}\|\partial_x u_\mu\|_{L^2}2^\nu\|\dot{u}_\nu\|_{L^2} \\
&\le \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} 2\|[\varphi_\nu, a(t,\cdot)]\psi_\mu\|_{L^2\to L^2}2^\nu e^{-\frac{C_0}{2}\int_0^t(\omega(\varepsilon_\nu,\tau)-\omega(\varepsilon_\mu,\tau))d\tau - \beta(\nu-\mu)t} \\
&\quad \times 2^{-(\nu-\mu)\theta}(\nu+1)^{-\frac{1}{2}}(\mu+1)^{-\frac{1}{2}}e^{-\frac{C_0}{2}\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - \beta\nu t}2^{-\nu\theta}(\nu+1)^{\frac{1}{2}}\|\dot{u}_\nu\|_{L^2} \\
&\quad \times e^{-\frac{C_0}{2}\int_0^t \omega(\varepsilon_\mu,\tau)d\tau - \beta\mu t}2^{-\mu\theta}(\mu+1)^{\frac{1}{2}}\|\partial_x u_\mu\|_{L^2}.
\end{aligned}$$

We consider the following kernel on $l^2(\mathbb{N})$:

$$\begin{aligned}
k_{\nu\mu}(t) &:= \|[\varphi_\nu, a(t,\cdot)]\psi_\mu\|_{L^2\to L^2}2^\nu e^{-\frac{C_0}{2}\int_0^t(\omega(\varepsilon_\nu,\tau)-\omega(\varepsilon_\mu,\tau))d\tau - \beta(\nu-\mu)t}2^{-(\nu-\mu)\theta} \\
&\quad \times (\nu+1)^{-\frac{1}{2}}(\mu+1)^{-\frac{1}{2}}.
\end{aligned}$$

Our first goal is to prove that for all $\delta \le 1/16$ there exists C_δ positive such that

$$\sup_\mu \sum_\nu |k_{\nu\mu}(t)| + \sup_\nu \sum_\mu |k_{\nu\mu}(t)| \le C_\delta \tag{3.10}$$

for all $\beta \in (0,+\infty)$ and for all $t \in [0,T^*_\beta]$. First of all we recall that

$$\|[\varphi_\nu, a(t,\cdot)]\psi_\mu\|_{L^2\to L^2} \le \tilde{C}\|a\|_{C([0,T],\mathrm{LogLip}(\mathbb{R}))}\nu 2^{-\nu} \tag{3.11}$$

and moreover, if $|\nu-\mu| \ge 2$,

$$\|[\varphi_\nu, a(t,\cdot)]\psi_\mu\|_{L^2\to L^2} \le \tilde{C}\|a\|_{C([0,T],\mathrm{LogLip}(\mathbb{R}))}\mu 2^{-\mu} \tag{3.12}$$

(see [4, Proposition 3.6] and [4, Lemma 4.4]).

For $i = 1,\dots,6$ we denote by $k^{(i)}_{\nu\mu}(t)$ the function $k_{\nu\mu}(t)\chi_{A_i}(\nu,\mu)$, where χ_{A_i} is the characteristic function of the set

$$A_1 = \{(\nu,\mu) \in \mathbb{N}\times\mathbb{N} \;:\; \nu,\mu \le \nu(\delta)+2\},$$

$$A_2 = \{(\nu,\mu) \in \mathbb{N} \times \mathbb{N} \ : \ \nu \le \nu(\delta),\ \mu \ge \nu(\delta)+2\},$$
$$A_3 = \{(\nu,\mu) \in \mathbb{N} \times \mathbb{N} \ : \ \nu \ge \nu(\delta)+2,\ \mu \le \nu(\delta)\},$$
$$A_4 = \{(\nu,\mu) \in \mathbb{N} \times \mathbb{N} \ : \ \nu \ge \nu(\delta),\ \mu \ge \nu+2\},$$
$$A_5 = \{(\nu,\mu) \in \mathbb{N} \times \mathbb{N} \ : \ \nu \ge \nu(\delta),\ \mu \ge \nu(\delta),\ \nu-1 \le \mu \le \nu+1\},$$
$$A_6 = \{(\nu,\mu) \in \mathbb{N} \times \mathbb{N} \ : \ \nu \ge \nu(\delta)+2,\ \nu(\delta) \le \mu \le \nu-2\},$$

respectively; here $\nu(\delta)$ is the value from (3.6) and (3.7).

We will show that

$$\sup_\mu \sum_\nu |k^{(i)}_{\nu\mu}(t)| + \sup_\nu \sum_\mu |k^{(i)}_{\nu\mu}(t)| \le C^{(i)}_\delta \tag{3.13}$$

for $i = 1, \dots, 6$, and (3.10) will follow immediately. For $i = 1$ the result is clear since the series are actually finite sums. Let us consider the case $i = 2$. From (3.12) and (3.6) we have

$$\begin{aligned}
|k^{(2)}_{\nu\mu}(t)| &\le \tilde{C}\|a\|\mu 2^{-\mu}2^{\nu}e^{-\frac{C_0}{2}\int_0^t(\omega(\varepsilon_\nu,\tau)-\omega(\varepsilon_\mu,\tau))d\tau-\beta(\nu-\mu)t}2^{-(\nu-\mu)\theta} \\
&\quad \times (\nu+1)^{-\frac{1}{2}}(\mu+1)^{-\frac{1}{2}} \\
&\le \tilde{C}\|a\|\mu^{\frac{1}{2}}2^{-\mu(1-\theta)}e^{\beta\mu t}e^{\frac{C_0}{2}\int_0^t\omega(\varepsilon_\mu,\tau)d\tau}(\nu+1)^{-\frac{1}{2}}2^{\nu(1-\theta)}e^{-\beta\nu t}e^{-\frac{C_0}{2}\int_0^t\omega(\varepsilon_\nu,\tau)d\tau} \\
&\le \tilde{C}\|a\|\mu^{\frac{1}{2}}2^{-\frac{7}{8}\mu}e^{\frac{1}{8}\mu}e^{\frac{1}{16}\mu}(\nu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\nu}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_\nu |k^{(2)}_{\nu\mu}(t)| &= \sum_{\nu=0}^{\nu(\delta)} |k^{(2)}_{\nu\mu}(t)| \\
&\le \tilde{C}\|a\|_{C([0,T],\mathrm{LogLip}(\mathbb{R}))}\,\mu^{\frac{1}{2}}2^{-\frac{7}{8}\mu}e^{\frac{1}{4}\mu}\sum_{\nu=0}^{\nu(\delta)}(\nu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\nu}
\end{aligned}$$

and

$$\begin{aligned}
\sum_\mu |k^{(2)}_{\nu\mu}(t)| &= \sum_{\mu=\nu(\delta)+2}^{\infty} |k^{(2)}_{\nu\mu}(t)| \\
&\le \tilde{C}\|a\|_{C([0,T],\mathrm{LogLip}(\mathbb{R}))}(\nu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\nu}\sum_{\mu=\nu(\delta)+2}^{\infty}\mu^{\frac{1}{2}}2^{-\frac{7}{8}\mu}e^{\frac{1}{4}\mu}.
\end{aligned}$$

Hence, (3.13) holds. The case $i = 3$ is obtained similarly using (3.11) instead of (3.12). Let $i = 4$; from (3.12) and (3.7) we have

$$\begin{aligned}|k_{\nu\mu}^{(4)}(t)| \le \tilde{C}\|a\|\mu 2^{-\mu}2^{\nu}e^{-\frac{C_0}{2}\int_0^t(\omega(\varepsilon_\nu,\tau)-\omega(\varepsilon_\mu,\tau))d\tau-\beta(\nu-\mu)t}2^{-(\nu-\mu)\theta} \\ \times(\nu+1)^{-\frac{1}{2}}(\mu+1)^{-\frac{1}{2}} \\ \le \tilde{C}\|a\|\mu^{\frac{1}{2}}2^{-\mu}2^{\nu}e^{\frac{1}{16}(\mu-\nu)-\beta(\nu-\mu)t}2^{-(\nu-\mu)\theta}(\nu+1)^{-\frac{1}{2}} \\ \le \tilde{C}\|a\|\mu^{\frac{1}{2}}2^{-\frac{7}{8}\mu}e^{\frac{1}{16}\mu}e^{\beta\mu t}(\nu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\nu}e^{-\frac{1}{16}\nu}e^{-\beta\nu t}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{\nu}|k_{\nu\mu}^{(4)}(t)| = \sum_{\nu=\nu(\delta)}^{\mu-2}|k_{\nu\mu}^{(4)}(t)| \le \tilde{C}\|a\|_{C([0,T],\mathrm{LogLip}(\mathbb{R}))}\,\mu^{\frac{1}{2}}2^{-\frac{7}{8}\mu}e^{\frac{1}{16}\mu}e^{\beta\mu t} \\ \times\sum_{\nu=\nu(\delta)}^{\mu-2}(\nu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\nu}e^{-\frac{1}{16}\nu}e^{-\beta\nu t}\end{aligned}$$

and

$$\begin{aligned}\sum_{\mu}|k_{\nu\mu}^{(4)}(t)| = \sum_{\mu=\nu+2}^{\infty}|k_{\nu\mu}^{(4)}(t)| \le \tilde{C}\|a\|_{C([0,T],\mathrm{LogLip}(\mathbb{R}))}(\nu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\nu}e^{-\frac{1}{16}\nu}e^{-\beta\nu t} \\ \times\sum_{\mu=\nu+2}^{\infty}\mu^{\frac{1}{2}}2^{-\frac{7}{8}\mu}e^{\frac{1}{16}\mu}e^{\beta\mu t}.\end{aligned}$$

Also in this case the conclusion follows considering that

$$\sum_{\nu=\nu(\delta)}^{\mu-2}(\nu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\nu}e^{-\frac{1}{16}\nu}e^{-\beta\nu t} \le C'(\mu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\mu}e^{-\frac{1}{16}\mu}e^{-\beta\mu t}$$

and

$$\sum_{\mu=\nu+2}^{\infty}\mu^{\frac{1}{2}}2^{-\frac{7}{8}\mu}e^{\frac{1}{16}\mu}e^{\beta\mu t} \le C'\nu^{\frac{1}{2}}2^{-\frac{7}{8}\nu}e^{\frac{1}{16}\nu}e^{\beta\nu t}.$$

The other cases are similar, and (3.8) follows. (3.9) is an easy consequence of (3.6). □

Step 6: End of the proof of Theorem 2.1. We choose $\beta = C_\delta/2$; from (3.5) and (3.8) we deduce that

$$E'(u)(t) \le 2\sum_{\nu=0}^{\infty}e^{-C_0\int_0^t\omega(\varepsilon_\nu,\tau)d\tau-2\beta\nu t}2^{-2\nu\theta}\|(Pu)_\nu\|_{L^2}\|\dot{u}_\nu\|_{L^2}$$

$$\leq 2(E(u)(t))^{\frac{1}{2}} \|Pu(t)\|_{H^{-\theta-\beta t}(\mathbb{R})}.$$

From this the conclusion of the proof of Theorem 2.1 follows immediately, taking (3.9) into account. □

Proof of Theorem 2.2. First we mention that the statement of Lemma 3.1 remains true if we replace (2.1) by (2.5). Our strategy is to show that the assumptions for ψ, (2.7) and (2.8) and a special choice of μ and η imply (2.5) from Remark 2.1, (2.3) and (2.4). With a parameter $\tilde{\delta} > 0$, we shall put $\mu(t) = \tilde{\delta} \log 1/t$ and $\eta(t) = \tilde{\delta}\psi(\log 1/t)$ in Theorem 2.1. Evidently, the assumptions for $\mu = \mu(t)$ and $\eta = \eta(t)$ itself are satisfied. Here we use that the condition $-\eta''(t) < \frac{\eta'(t)}{t}$ is equivalent to $-\psi''(-\log t) < 0$. Let us check the assumptions for $\frac{t\mu(t)}{\eta(t)}$. The condition $\lim_{t\to 0+} \frac{t\mu(t)}{\eta(t)} = 0$ follows from $\lim_{t\to\infty} \frac{t}{\psi(t)\exp t} = 0$. The increasing behavior of $\frac{t\mu(t)}{\eta(t)}$ is concluded from the assumption $-\psi'(t) \leq Ct^{-1}\psi(t) < \frac{t-1}{t}\psi(t)$ for sufficiently large t. Consequently, all assumptions for η and μ from Theorem 2.1 are satisfied.

Now let us begin to prove the statement of the lemma. From the condition $-\psi'(t) \leq Ct^{-1}\psi(t)$ with $t = \log(1/\varepsilon)$, for all $\tilde{\delta} > 0$ we have

$$\left|\varepsilon\Big(- \mu'(\varepsilon) + \mu(\varepsilon)\frac{\eta'(\varepsilon)}{\eta(\varepsilon)}\Big)\right| = \left|\tilde{\delta} + \tilde{\delta}\log 1/\varepsilon \frac{-\psi'(\log 1/\varepsilon)}{\psi(\log 1/\varepsilon)}\right| \leq (1+C)\tilde{\delta}.$$

This means that the condition (2.5) from Remark 2.1 is satisfied.

We next change the local condition (2.4) in Theorem 2.1. Since $\eta(t) = \tilde{\delta}\psi(\log 1/t)$, we find that

$$-\frac{d}{dt}\Big\{\tilde{\delta}\log\frac{1}{\eta^{-1}(t)}\Big\} = -\frac{d}{dt}\Big\{\tilde{\delta}\psi^{-1}\Big(\frac{t}{\tilde{\delta}}\Big)\Big\} = -\frac{1}{\psi'\Big(\psi^{-1}\Big(\frac{t}{\tilde{\delta}}\Big)\Big)}. \tag{3.14}$$

Hence, for all $\tilde{\delta} > 0$ there exists a positive constant $r_{\tilde{\delta}} \leq T$ such that

$$\|\partial_t a(t,\cdot)\|_{B^0(\mathbb{R})} \leq -C\frac{d}{dt}(\mu(\eta^{-1}(t))) = -\frac{C}{\psi'\Big(\psi^{-1}\Big(\frac{t}{\tilde{\delta}}\Big)\Big)},$$

which implies the local condition (2.8) in Theorem 2.2. On the other hand, we immediately see that the global condition (2.3) in Theorem 2.1 is equivalent to the global condition (2.7) in Theorem 2.2. □

Proof of Theorem 2.3. The proof of Theorem 2.3 is very similar to that of Theorem 2.1.

Step 1: Estimate for the approximate microlocalized energies. Let $\varepsilon \leq 1$. We set

$$a_\varepsilon(t,x) = \begin{cases} a(\varepsilon/2,x) & \text{if } \; t \leq \varepsilon/2, \\ a(\varepsilon/2,x)\theta(2t/\varepsilon) + a(t,x)(1-\theta(2t/\varepsilon)) & \text{if } \; \varepsilon/2 \leq t \leq \varepsilon, \\ a(t,x) & \text{if } \; t \geq \varepsilon, \end{cases}$$

where θ is a C^∞ function such that $\theta(s) = 1$ if $s \leq 1$, $\theta(s) = 0$ if $s \geq 2$ and for all $s \in \mathbb{R}$, $0 \leq \theta(s) \leq 1$. Using the notations of the previous proofs we define

$$E_{\nu,\varepsilon}(u) = (\dot{u}_\nu, \dot{u}_\nu) + (a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x u_\nu).$$

We have

$$\begin{aligned} \frac{d}{dt}E_{\nu,\varepsilon}(u)(t) &= (\partial_t a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x u_\nu) + 2Re((a_\varepsilon(t,\cdot) - a(t,\cdot))\partial_x u_\nu, \partial_x \dot{u}_\nu) \\ &\quad -2Re([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu) + 2Re((Pu)_\nu, \dot{u}_\nu) \\ &= \Big(\Big(\frac{\partial_t a_\varepsilon(t,\cdot)}{a_\varepsilon(t,\cdot)}\Big) a_\varepsilon(t,\cdot)\partial_x u_\nu, \partial_x u_\nu\Big) \\ &\quad +2Re\Big(\Big(\frac{a_\varepsilon(t,\cdot) - a(t,\cdot)}{\sqrt{a_\varepsilon(t,\cdot)}}\Big)\sqrt{a_\varepsilon(t,\cdot)}\partial_x u_\nu, \partial_x \dot{u}_\nu\Big) \\ &\quad -2Re([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu) + 2Re((Pu)_\nu, \dot{u}_\nu). \end{aligned}$$

We fix $\varepsilon_\nu = 2^{-\nu}$. We obtain

$$\begin{aligned} &\frac{d}{dt}E_{\nu,\varepsilon_\nu}(u)(t) \\ &\leq \omega(\varepsilon_\nu,t)E_{\nu,\varepsilon_\nu}(u)(t) - 2Re([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu) + 2\|(Pu)_\nu\|_{L^2}\|\dot{u}_\nu\|_{L^2}, \end{aligned}$$

where

$$\omega(\varepsilon_\nu, t) = \frac{\chi_{[\varepsilon_\nu/2,T]}(t)}{2t} + C_0 2^\nu \chi_{[0,\varepsilon_\nu]}(t),$$

the function χ_A denotes the characteristic function of the set A and the constant C_0 depends only on $\max|a|$, λ_0 and $\max|\theta'|$.

Step 2: Estimate for total energy.

$$E(u)(t) := \sum_{\nu=0}^{\infty} e^{-\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t} 2^{-2\nu\theta} E_{\nu,\varepsilon_\nu}(u)(t)$$

with β and θ positive constants. We deduce that

$$
\begin{aligned}
E'(u)(t) \leq & -2\beta \sum_{\nu=0}^{\infty} e^{-\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t} 2^{-2\nu\theta} \nu E_{\nu,\varepsilon_\nu}(u)(t) \\
& + \sum_{\nu=0}^{\infty} e^{-\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t} 2^{-2\nu\theta} 2\|(Pu)_\nu\|_{L^2} \|\dot{u}_\nu\|_{L^2} \\
& + \sum_{\nu=0}^{\infty} e^{-\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t} 2^{-2\nu\theta} 2|([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu)|.
\end{aligned}
$$

Step 3: Properties of the weight. We have

$$
\int_0^t \omega(\varepsilon_\nu,\tau)\, d\tau = \begin{cases} C_0 2^\nu t & \text{if } t \leq 2^{-\nu-1}, \\ C_0 2^\nu t + \frac{\log t}{2} + \frac{(\nu+1)\log 2}{2} & \text{if } 2^{-\nu-1} \leq t \leq 2^{-\nu}, \\ C_0 + \frac{\log t}{2} + \frac{(\nu+1)\log 2}{2} & \text{if } t \geq 2^{-\nu}. \end{cases}
$$

Consequently,

$$
\Big| \frac{1}{2} \int_0^t \omega(\varepsilon_\nu,\tau)\, d\tau \Big| \leq \frac{C_0}{2} + (\nu+1)\frac{\log 2}{4}, \tag{3.15}
$$

and

$$
\Big| \frac{1}{2} \int_0^t (\omega(\varepsilon_\nu,\tau) - \omega(\varepsilon_\mu,\tau))\, d\tau \Big| \leq \frac{C_0}{2} + |\nu-\mu|\frac{\log 2}{4}. \tag{3.16}
$$

Step 4: Estimates for the commutator term. Using the notation of the proof of Theorem 2.1 we can write

$$
\begin{aligned}
& \sum_{\nu=0}^{\infty} e^{-\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t} 2^{-2\nu\theta} 2|([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu)| \\
& \leq \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} e^{-\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t} 2^{-2\nu\theta} 2\|[\varphi_\nu, a(t,\cdot)]\psi_\mu\|_{L^2\to L^2} \|\partial_x u_\mu\|_{L^2} 2^\nu \|\dot{u}_\nu\|_{L^2} \\
& \leq \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} 2\|[\varphi_\nu, a(t,\cdot)]\psi_\mu\|_{L^2\to L^2} 2^\nu e^{-\frac{1}{2}\int_0^t (\omega(\varepsilon_\nu,\tau)-\omega(\varepsilon_\mu,\tau))d\tau - \beta(\nu-\mu)t} 2^{-(\nu-\mu)\theta} \\
& \times (\nu+1)^{-\frac{1}{2}} (\mu+1)^{-\frac{1}{2}} e^{-\frac{1}{2}\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - \beta\nu t} 2^{-\nu\theta} (\nu+1)^{\frac{1}{2}} \\
& \times \|\dot{u}_\nu\|_{L^2} e^{-\frac{1}{2}\int_0^t \omega(\varepsilon_\mu,\tau)d\tau - \beta\mu t} 2^{-\mu\theta} (\mu+1)^{\frac{1}{2}} \|\partial_x u_\mu\|_{L^2}.
\end{aligned}
$$

The kernel on $l^2(\mathbb{N})$ is now

$$
k_{\nu\mu}(t) := \|[\varphi_\nu, a(t,\cdot)]\psi_\mu\|_{L^2\to L^2} 2^\nu e^{-\frac{1}{2}\int_0^t (\omega(\varepsilon_\nu,\tau)-\omega(\varepsilon_\mu,\tau))d\tau - \beta(\nu-\mu)t} 2^{-(\nu-\mu)\theta}
$$

$$\times (\nu+1)^{-\frac{1}{2}}(\mu+1)^{-\frac{1}{2}}.$$

We consider the sets

$$A_1 = \{(\nu,\mu) \in \mathbb{N}\times\mathbb{N} \ : \ \mu-2 \geq \nu \geq 0\},$$
$$A_2 = \{(\nu,\mu) \in \mathbb{N}\times\mathbb{N} \ : \ \nu \geq 0, \ \nu+1 \geq \mu \geq 0\}.$$

Denoting by $k^{(i)}_{\nu\mu}(t)$ the function $k_{\nu\mu}(t)\chi_{A_i}(\nu,\mu)$ for $i=1,2$, from (3.12) and (3.16) we deduce that

$$\begin{aligned}
&|k^{(1)}_{\nu\mu}(t)|\\
&\leq \tilde{C}\|a\|\mu 2^{-\mu}2^{\nu}e^{-\frac{1}{2}\int_0^t(\omega(\varepsilon_\nu,\tau)-\omega(\varepsilon_\mu,\tau))d\tau-\beta(\nu-\mu)t}2^{-(\nu-\mu)\theta}(\nu+1)^{-\frac{1}{2}}(\mu+1)^{-\frac{1}{2}}\\
&\leq \tilde{\tilde{C}}\|a\|\mu^{\frac{1}{2}}2^{-\mu}2^{\nu}e^{(\mu-\nu)\frac{\log 2}{4}-\beta(\nu-\mu)t}2^{-(\nu-\mu)\theta}(\nu+1)^{-\frac{1}{2}}\\
&\leq \tilde{\tilde{C}}\|a\|\mu^{\frac{1}{2}}2^{-(1-\theta)\mu}e^{\mu(\frac{\log 2}{4}+\beta t)}2^{(1-\theta)\nu}e^{-\nu(\frac{\log 2}{4}+\beta t)}(\nu+1)^{-\frac{1}{2}}.
\end{aligned}$$

Fixing $\theta = 1/8$ and $T^*_\beta = 1/(8\beta)$ we have

$$\begin{aligned}
&\sum_\nu |k^{(1)}_{\nu\mu}(t)| = \sum_{\nu=0}^{\mu-2}|k^{(1)}_{\nu\mu}(t)|\\
&\leq \tilde{\tilde{C}}\|a\|_{C([0,T],\mathrm{LogLip}(\mathbb{R}))}\mu^{\frac{1}{2}}2^{-\frac{7}{8}\mu}e^{\mu(\frac{\log 2}{4}+\beta t)}\sum_{\nu=0}^{\mu-2}(\nu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\nu}e^{-\nu(\frac{\log 2}{4}+\beta t)} \leq C',
\end{aligned}$$

where C' does not depend on β, t and μ. Moreover,

$$\begin{aligned}
&\sum_\mu |k^{(1)}_{\nu\mu}(t)| = \sum_{\mu=\nu+2}^{\infty}|k^{(1)}_{\nu\mu}(t)|\\
&\leq \tilde{C}\|a\|_{C([0,T],\mathrm{LogLip}(\mathbb{R}))}(\nu+1)^{-\frac{1}{2}}2^{\frac{7}{8}\nu}e^{-\nu(\frac{\log 2}{4}+\beta t)}\sum_{\mu=\nu+2}^{\infty}\mu^{\frac{1}{2}}2^{-\frac{7}{8}\mu}e^{\mu(\frac{\log 2}{4}+\beta t)} \leq C''
\end{aligned}$$

where C'' does not depend on β, t and ν. Hence,

$$\sup_\mu \sum_\nu |k^{(1)}_{\nu\mu}(t)| + \sup_\nu \sum_\mu |k^{(1)}_{\nu\mu}(t)| \leq C,$$

where C does not depend on β and $t \in [0,T^*_\beta]$. Using (3.11) the same is obtained for $i=2$. We conclude that if $\theta = 1/8$ and $t \in [0,T^*_\beta]$, then

$$\sum_{\nu=0}^{\infty} e^{-\int_0^t \omega(\varepsilon_\nu,\tau)d\tau-2\beta\nu t}2^{-2\nu\theta}2|([\varphi_\nu, a(t,\cdot)]\partial_x u, \partial_x \dot{u}_\nu)|$$

$$\leq C_0 \sum_{\nu=0}^{\infty} e^{-\int_0^t \omega(\varepsilon_\nu,\tau)d\tau - 2\beta\nu t} 2^{-2\nu\theta} \nu E_{\nu,\varepsilon_\nu(u)(t)},$$

where C_0 does not depend on β.

Step 5: Conclusion of the proof of Theorem 2.3. It is sufficient now to choose $\beta = C_0/2$; we obtain

$$\frac{d}{dt}E(u)(t) \leq 2(E(u)(t))^{\frac{1}{2}} \|Pu(t)\|_{H^{-\theta}}.$$

Considering (3.15) the conclusion of the proof is easily reached. □

4. Concluding remarks

Remark 4.1. If we compare the statements of Theorems 2.1 and 2.3, then we see that a supposed global Hölder behaviour of $a = a(t,x)$ in t does not improve the local condition related with the supposed global continuity in t (see [6]).

Remark 4.2. There is a gap between the local condition (2.10) and the local condition $|a'(t)| \leq C/t,\ t > 0$, which was proposed in [5] to prove C^∞ well-posedness for the Cauchy problem

$$\partial_t^2 u - a(t)\partial_x^2 u = f(t,x),\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x).$$

It seems to be interesting to determine the optimal regularity H of coefficients in x which allows us to assume instead of (2.10) the local condition $\|\partial_t a(t,\cdot)\|_{H(\mathbb{R})} \leq \frac{C}{t}$.

Remark 4.3. We need the term $\exp(-\beta\nu t)$ in the weight to estimate the commutators. In [4] it is mentioned that if the coefficient $a = a(t,x)$ is from $B^{1+\varepsilon}(\mathbb{R})$ with respect to x with $\varepsilon > 0$ arbitrarily small, then $\theta = 0$; that is, no loss of regularity appears from θ. To estimate the commutators we have only to introduce $\exp(-\beta\nu t)$. But due to the fact that we assume a $C^1(0,T]$ behavior in t, this term produces an arbitrarily small loss δ. Because for $t \in [t_0,T]$ our coefficient is Lipschitz in t and $B^{1+\varepsilon}$ in x, we thus suppose no loss for $t \geq t_0$. Consequently, we expect an energy inequality like

$$\begin{aligned}&\|u(t,\cdot)\|_{H^{1-\delta}(\mathbb{R})} + \|u_t(t,\cdot)\|_{H^{-\delta}(\mathbb{R})}\\ &\leq C_\delta\Big(\|u(0,\cdot)\|_{H^1(\mathbb{R})} + \|u_t(0,\cdot)\|_{L^2(\mathbb{R})} + \int_0^t \|Pu(s)\|_{L^2(\mathbb{R})}ds\Big).\end{aligned}$$

We call such solutions *almost energy solutions*. It would be fine to understand under which precise assumptions we might expect *almost energy solutions*.

Remark 4.4. It seems to be interesting to have a *local uniqueness result* for solutions to (1.3) under the assumptions $a \in C([0,T], B^{1+\varepsilon}(\mathbb{R})) \cap C^1((0,T], B^{1+\varepsilon}(\mathbb{R}))$ with (2.3) and (2.4) and the typical examples for μ and η.

Remark 4.5. In a forthcoming paper we will understand the C^2-theory with respect to t (see [10]) under low regularity assumptions to coefficients with respect to x.

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