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ASYMPTOTIC SOLUTIONS WITH SLOW CONVERGENCE RATE OF HAMILTON-JACOBI EQUATIONS IN EUCLIDEAN n SPACE

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Abstract. In this paper, we study the long-time asymptotics of the Cauchy problem for the Hamilton-Jacobi equation

$$u_t(x,t) + \alpha x \cdot Du(x,t) + H(Du(x,t)) = f(x) \quad \text{in} \quad \mathbb{R}^n \times (0,\infty),$$

where α is a positive constant. In [9], it was shown that there are a constant $c \in \mathbb{R}$ and a viscosity solution v of $c + \alpha x \cdot Dv(x) + H(Dv(x)) = f(x)$ in $\mathbb{R}^n$ such that $u(\cdot,t) - (v(\cdot) + ct) \to 0$ as $t \to \infty$ locally uniformly in $\mathbb{R}^n$. The function $v(x) + ct$ is called the asymptotic solution. Our goal is to give a sufficient condition in order that the set of points where the rate of this convergence is slower than t^{-1} is non-empty. We also give several examples which show that we can not remove, in general, the assumptions in this sufficient condition in order that this set is non-empty. As a result, we clarify crucial factors which cause this slow rate of convergence. They are both a geometrical property of the set of equilibrium points and a lower bound of the initial data.

1. Introduction

Recently, the long-time asymptotics of viscous Hamilton-Jacobi equations and Hamilton-Jacobi equations have been investigated by many authors. We refer to [8, 9, 10] for asymptotic results in a non-compact domain $\mathbb{R}^n$; for asymptotic results in compact domains, see [3, 4, 5, 6, 13, 15, 16]. The Cauchy problem for the Hamilton-Jacobi equation investigated in [9] is the following:

$$u_t(x,t) + \alpha x \cdot Du(x,t) + H(Du(x,t)) = f(x) \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \tag{1.1}$$

$$u(\cdot,0) = u_0 \quad \text{in} \quad \mathbb{R}^n, \tag{1.2}$$

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where α is a positive constant, and H, f and u_0 are given functions in $\mathbb{R}^n$. All (sub/super) solutions in this paper are understood in the viscosity sense.

The Hamilton-Jacobi equation (1.1) is obtained by vanishing the viscosity parameter ϵ in the viscous Hamilton-Jacobi equation with the Ornstein-Uhlenbeck operator

$$u_t(x,t) - \epsilon\Delta u(x,t) + \alpha x \cdot Du(x,t) + H(Du(x,t)) = f(x) \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

(as for this viscous Hamilton-Jacobi equation, see [8]). Here, $\epsilon\Delta - \alpha x \cdot D$ is the Ornstein-Uhlenbeck operator. The first author, jointly with H. Ishii and P. Loreti ([9]), has shown that, under suitable assumptions on H, f and u_0, the Cauchy problem (1.1)–(1.2) admits a unique solution $u \in \mathrm{C}(\mathbb{R}^n \times [0,\infty))$ and that there is a unique constant $c \in \mathbb{R}$ such that

$$\lim_{t\to\infty}(u(\cdot,t) - ct) = v \text{ locally uniformly in } \mathbb{R}^n, \tag{1.3}$$

where $v \in \mathrm{C}(\mathbb{R}^n)$ is a solution of the following PDE of ergodic type:

$$c + \alpha x \cdot Dv(x) + H(Dv(x)) = f(x) \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

In this paper, as the next step in studying the convergence (1.3) more exactly, we are concerned with rates of convergence in (1.3). In particular, we are interested in a slow rate of convergence in (1.3), because crucial factors often hide behind such a rate of convergence. As for rates of convergence in (1.3), we have the following examples. In the following, let

$$B(a,r) = \{x \in \mathbb{R}^n : |x-a| \le r\} \quad \text{for } a \in \mathbb{R}^n,\ r > 0.$$

Example 1. For $\beta > 0$, let

$$H(p) = \tfrac{1}{2}|p|^2 \text{ in } \mathbb{R}^n, \quad f(x) = \tfrac{\beta}{2}|x|^2 \text{ in } \mathbb{R}^n.$$

Assume that $u_0 \in \mathrm{C}(\mathbb{R}^n)$ satisfies the inequality $0 \le u_0(x) \le \varphi(x)$ in $\mathbb{R}^n$, where $0 \le \varphi \in \mathrm{C}^1(\mathbb{R}^n)$ and satisfies $x \cdot D\varphi(x) \ge 2\varphi(x)$ in $\mathbb{R}^n$. As an example of φ, we have

$$\varphi(x) = \sum_{j=1}^{N} a_j |x|^{m_j} \quad \text{in } \mathbb{R}^n,$$

where N, a_j and m_j $(j = 1,2,\ldots,N)$ are constants such that $N \in \mathbb{N}$, $a_j > 0$ $(j = 1,2,\ldots,N)$ and $2 \le m_1 < m_2 < \cdots < m_N$. Then, the Cauchy problem (1.1)–(1.2) admits a unique solution $u \in \mathrm{C}(\mathbb{R}^n \times [0,\infty))$ satisfying

$$\lim_{r\to\infty} \inf\left\{u(x,t) + \tfrac{\alpha}{2}|x|^2 : (x,t) \in (\mathbb{R}^n \setminus B(0,r)) \times [0,T)\right\} = \infty \quad \text{for } T > 0. \tag{1.5}$$

Furthermore, we have

$$-\frac{A}{2}|x|^2 e^{-\theta t} \leq u(x,t) - \frac{A}{2}|x|^2 \leq e^{-\theta t}\varphi(x) \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

where $A = \sqrt{\alpha^2+\beta} - \alpha$ and $\theta = \sqrt{\alpha^2+\beta} + \alpha$. □

Next, for $a, b \in \mathbb{R}$, we use the notations

$$a_+ = \max\{a, 0\}, \qquad a \wedge b = \min\{a, b\}, \quad a \vee b = \max\{a, b\}.$$

Example 2. Let

$$H(p) = \tfrac{1}{2}|p|^2 \text{ in } \mathbb{R}^n, \quad f(x) = \tfrac{\alpha^2}{2}(1-|x|^2)_+ \text{ in } \mathbb{R}^n, \quad u_0(x) = \tfrac{\alpha}{2} \text{ in } \mathbb{R}^n.$$

Then, the Cauchy problem (1.1)–(1.2) admits a unique solution $u \in \mathrm{C}(\mathbb{R}^n \times [0,\infty))$ satisfying (1.5). Furthermore, we have

$$\frac{\alpha}{2(\alpha t+1)}(|x|^2 \wedge 1) \leq u(x,t) - \frac{\alpha^2}{2}t - v(x) \leq \frac{\alpha}{2(\alpha t+1)}|x|^2 \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

where

$$v(x) = \frac{\alpha}{2}(1-|x|^2) + \alpha \int_1^{|x|\vee 1} \sqrt{r^2-1}\, dr \quad \text{for } x \in \mathbb{R}^n. \tag{1.6}$$

We prove both the assertions of Examples 1 and 2 in Section 3 below. These two examples show that, as rates of convergence in (1.3), there are at least two different types. Reflecting them, we define two subsets of $\mathbb{R}^n$. We denote by Γ_{fast} the set of points $x \in \mathbb{R}^n$ such that there are two constants $C_1(x) > 0$ and $\theta(x) > 0$ satisfying

$$|u(x,t) - (v(x) + ct)| \leq C_1(x) e^{-\theta(x)t} \quad \text{for } t \in [0,\infty). \tag{1.7}$$

We also denote by Γ_{slow} the set of points $x \in \mathbb{R}^n$ such that there is a constant $C_2(x) > 0$ satisfying

$$\frac{C_2(x)}{t+1} \leq |u(x,t) - (v(x) + ct)| \quad \text{for } t \in [0,\infty). \tag{1.8}$$

Here, we do not consider other rates of convergence in (1.3) even if they exist. The function $v(x) + ct$ is a solution of (1.1), and is called the asymptotic solution of (1.1)–(1.2).

In Example 1, we have $\Gamma_{\text{fast}} = \mathbb{R}^n$ and $\Gamma_{\text{slow}} = \emptyset$. In Example 2, we have $\Gamma_{\text{fast}} = \{0\}$ and $\Gamma_{\text{slow}} = \mathbb{R}^n \setminus \{0\}$. See [2, 11, 12] for results which treat the case such that $|\hat{u}(x,t) - (\hat{v}(x) + \hat{c}t)| \leq F(x,t)$ in $\mathbb{R}^n \times [0,\infty)$, where $F(x,t) \to 0$ as $t \to \infty$ for each $x \in \mathbb{R}^n$, and $\hat{u}$ and $\hat{v}(\cdot) + \hat{c}t$ are, respectively, a solution and the asymptotic solution of the Cauchy problem for a Hamilton-Jacobi equation. Note that the case $\Gamma_{\text{fast}} = \mathbb{R}^n$ in Example 1 is a typical example of this case.

On the other hand, as far as the authors know, Example 2 is the only result such that $G(x,t) \le |\hat{u}(x,t) - (\hat{v}(x) + \hat{c}t)|$ in $K \times [0,\infty)$, where K is a non-empty subset of $\mathbb{R}^n$ and $G > 0$ and $G(x,t) \to 0$ as $t \to \infty$ for each $x \in K$. Note that the case $\Gamma_{\text{slow}} \neq \emptyset$ is a typical example for $K = \Gamma_{\text{slow}}$. A remarkable example is the case such that $\Gamma_{\text{fast}} = (-\infty, 0]$ and $\Gamma_{\text{slow}} = (0,\infty)$ for $n = 1$ (see Example 8 in Section 4 below).

Our goal of this paper is to give a sufficient condition in order that $\Gamma_{\text{slow}} \neq \emptyset$. We also give several examples which show that we can not remove, in general, the assumptions in our sufficient condition in order that $\Gamma_{\text{slow}} \neq \emptyset$. As a result, we clarify crucial factors which cause $\Gamma_{\text{slow}} \neq \emptyset$. They are both a geometrical property of the set of equilibrium points and a lower bound of the initial data u_0 (for the definition of the set of equilibrium points, see Section 2). This set coincides with the Aubry set in our setting. The Aubry set plays an essential role in the convergence such as (1.3) even when we treat more general Hamilton-Jacobi equations than (1.1) (cf. [5, 6, 9, 10, 15, 16]). On the other hand, letting $t = 0$ in (1.8), we have $C_2(x) \le |u_0(x) - v(x)|$ for $x \in \Gamma_{\text{slow}}$. In order for this to be true, we need some assumption on u_0. In this paper, it is the assumption (2.16) below, which gives a lower bound of u_0.

The contents of this paper are as follows: In Section 2, we state the main results. In Section 3, we prove them. In Section 4, we give several examples.

2. Results

In this section, we state the results. They will be proved in Section 3. First of all, recall results of [9]. Assume that

$$H \text{ is convex on } \mathbb{R}^n, \text{ and } \lim_{|x|\to\infty} H(x)/|x| = \infty. \tag{2.1}$$

We define the Lagrangian L as the convex conjugate of H:

$$L(x) = H^*(x) = \sup\{x \cdot y - H(y) : y \in \mathbb{R}^n\} \quad \text{in } \mathbb{R}^n. \tag{2.2}$$

As is well-known, we have

$$L \text{ is convex on } \mathbb{R}^n, \text{ and } \lim_{|x|\to\infty} L(x)/|x| = \infty. \tag{2.3}$$

Furthermore, assume that $f, u_0 \in \mathrm{C}(\mathbb{R}^n)$ and there is a convex function ℓ on $\mathbb{R}^n$ such that

$$\lim_{|x|\to\infty} (L(x) - \ell(x)) = \infty, \tag{2.4}$$

$$-\infty < \inf \{f(x) + \ell(-\alpha x) : x \in \mathbb{R}^n\}, \tag{2.5}$$

$$-\infty < \inf \Big\{u_0(x) + \tfrac{1}{\alpha}\ell(-\alpha x) : x \in \mathbb{R}^n\Big\}. \tag{2.6}$$

Then, we can define the constant $c \in \mathbb{R}$ by

$$c = \min \{f(x) + L(-\alpha x) : x \in \mathbb{R}^n\}. \tag{2.7}$$

By (2.4) and (2.5), the constant c is well-defined. For $x, y \in \mathbb{R}^n$ and $T > 0$, we denote by $\mathcal{C}(x, y, T)$ and $\mathcal{C}(x, T)$ the set of all absolutely continuous curves $X : [0, T] \mapsto \mathbb{R}^n$ satisfying, respectively, $(X(0), X(T)) = (x, y)$ and $X(0) = x$. We define the functions $d : \mathbb{R}^n \times \mathbb{R}^n \mapsto [-\infty, \infty)$ and $\psi : \mathbb{R}^n \mapsto [-\infty, \infty)$ by

$$d(x,y) = \inf \Big\{ \int_0^T \big[f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t))\big] dt : T > 0, \\ X \in \mathcal{C}(x, y, T)\Big\}, \tag{2.8}$$

$$\psi(x) = \inf \Big\{ \int_0^T \big[f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t))\big] dt + u_0(X(T)) : T > 0, \\ X \in \mathcal{C}(x, T)\Big\}, \tag{2.9}$$

respectively, where $f_c(x) = f(x) - c$ in $\mathbb{R}^n$. Using these functions $d(\cdot,\cdot)$ and $\psi(\cdot)$, we define the function $v : \mathbb{R}^n \mapsto [-\infty, \infty)$ by

$$v(x) = \inf \{d(x, y) + \psi(y) : y \in \mathcal{E}\} \quad \text{in } \mathbb{R}^n, \tag{2.10}$$

where

$$\mathcal{E} = \{x \in \mathbb{R}^n : f(x) + L(-\alpha x) = c\}. \tag{2.11}$$

The set $\mathcal{E}$ is called the set of equilibrium points. This set coincides with the Aubry set in our setting ([10]). By (2.4) and (2.5), $\mathcal{E}$ is a non-empty compact set in $\mathbb{R}^n$. We denote by $\mathrm{C}_L(\mathbb{R}^n \times [0, \infty))$ the set of all functions $u \in \mathrm{C}(\mathbb{R}^n \times [0, \infty))$ satisfying

$$\lim_{r\to\infty} \inf \big\{u(x, t) + \tfrac{1}{\alpha}L(-\alpha x) : (x, t) \in (\mathbb{R}^n \setminus B(0, r)) \times [0, T)\big\} = \infty \tag{2.12}$$
$$\text{for} \quad T > 0.$$

Proposition 1. ([9]) *Assume* (2.1). *Assume also that* $f \in C(\mathbb{R}^n)$ *and there is a convex function* ℓ *on* $\mathbb{R}^n$ *satisfying* (2.4) *and* (2.5). *Let* $u \in C(\mathbb{R}^n \times [0,\infty))$ *and* $v \in C_L(\mathbb{R}^n \times [0,\infty))$ *be, respectively, a subsolution and supersolution of* (1.1). *If* $u(\cdot,0) \le v(\cdot,0)$ *in* $\mathbb{R}^n$, *then we have* $u \le v$ *in* $\mathbb{R}^n \times [0,\infty)$.

Proposition 2. ([9]) *Assume* (2.1). *Assume also that* $f, u_0 \in C(\mathbb{R}^n)$ *and there is a convex function* ℓ *on* $\mathbb{R}^n$ *satisfying* (2.4), (2.5) *and* (2.6). *Then, we have the following:*

(i) *The Cauchy problem* (1.1)–(1.2) *admits a unique solution* u *in* $C_L(\mathbb{R}^n \times [0,\infty))$.

(ii) *The functions* d, ψ *and* v *are, respectively, real-valued continuous functions on* $\mathbb{R}^n \times \mathbb{R}^n$, $\mathbb{R}^n$ *and* $\mathbb{R}^n$. *The function* v *is a solution of* (1.4).

(iii)

$$\lim_{t\to\infty} \max_{x\in B(0,R)} |u(x,t) - (v(x)+ct)| = 0 \quad \textit{for} \quad R > 0. \quad \square \tag{2.13}$$

Now, we state our result in this paper. In the following, we assume the following:

(A1) $H \in \mathrm{C}^1(\mathbb{R}^n)$, H is strictly convex on $\mathbb{R}^n$, $H(x) \ge H(0) = 0$ in $\mathbb{R}^n$, and $\lim_{|x|\to\infty} H(x)/|x| = \infty$.

(A2) There exists a constant $\epsilon > 0$ such that

$$\epsilon(s - s^2)\, x \cdot DH(x) + H\left((1-\epsilon s)x\right) \le (1-\epsilon s^2)H(x) \tag{2.14}$$

$$\text{for } s \in (0,1),\ x \in \mathbb{R}^n.$$

(A3) $f, u_0 \in \mathrm{C}(\mathbb{R}^n)$, and

$$-\infty < \inf\left\{f(x) + (1-\epsilon)L(-\alpha x) : x \in \mathbb{R}^n\right\}, \tag{2.15}$$

$$u_0(0) - \frac{1-\epsilon}{\alpha} L(-\alpha x) \le u_0(x) \ \text{ in } \mathbb{R}^n, \tag{2.16}$$

where ϵ is the constant in (A2).

Remark 1. The assumptions (A1) and (A2) are necessary in order to prove Lemma 5 below. We do not know whether they are necessary in order that $\Gamma_{\text{slow}} \ne \emptyset$. We give typical examples of H satisfying (A1) and (A2) in Section 4 below. Note that (A1) and (A3) imply (2.4), (2.5) and (2.6) for $\ell(x) = (1-\epsilon)L(x)$.

Lemma 1. *Assume* (A1). *Then, we have*

$$L(0) = 0 \textit{ and } L > 0 \textit{ in } \mathbb{R}^n \setminus \{0\}. \tag{2.17}$$

Definition. We define the equivalence law $\sim$ in $\mathcal{E}$ as follows: Let $x, y \in \mathcal{E}$. Then, $x \sim y$ in $\mathcal{E}$, if there exists a Lipschitz-continuous curve $\theta : [0,1] \mapsto \mathbb{R}^n$ such that $\theta(0) = x$, $\theta(1) = y$ and $\theta(t) \in \mathcal{E}$ for all $t \in (0,1)$. For $x \in \mathcal{E}$, we denote by the equivalence class $\mathcal{E}(x)$ the set of all points $y \in \mathcal{E}$ such that $x \sim y$ in $\mathcal{E}$.

Recall the definition of the set Γ_{slow} in Section 1.

Theorem. *Assume* (A1), (A2) *and* (A3). *Let* $u \in C_L(\mathbb{R}^n \times [0,\infty))$ *be a unique solution of the Cauchy problem* (1.1)–(1.2). *If* $\mathcal{E} \ni 0$, *then we have*

$$\frac{\epsilon}{\alpha}(\alpha t + 1)^{-1} L(-\alpha x) \leq u(x,t) - (v(x) + ct) \quad \textit{for} \quad (x,t) \in \mathcal{E}(0) \times [0,\infty), \tag{2.18}$$

where ϵ *is the constant of* (A2). *Hence, if* $\mathcal{E}(0) \setminus \{0\} \neq \emptyset$, *then* $\Gamma_{\text{slow}} \neq \emptyset$ *and* $\Gamma_{\text{slow}} \supset \mathcal{E}(0) \setminus \{0\}$ *because of* (2.13) *and* (2.17).

Remark 2. In Example 1, we see that $\Gamma_{\text{slow}} = \emptyset$, although (A1), (A2) and (A3) are fulfilled, and $\mathcal{E} = \mathcal{E}(0) = \{0\}$. Hence, in order that $\Gamma_{\text{slow}} \neq \emptyset$, we can not remove, in general, the assumption that $\mathcal{E}(0) \setminus \{0\} \neq \emptyset$. In Example 6 of Section 4 below, we give an example such that $\Gamma_{\text{slow}} = \emptyset$, although (A1), (A2) and (A3) are fulfilled, and $\mathcal{E} = \{x \in \mathbb{R}^n : |x| = 1\}$. Hence, in this case, $\mathcal{E}(x_0) \setminus \{x_0\} \neq \emptyset$ for $x_0 \in \mathcal{E}$. This example shows that the point $0 \in \mathbb{R}^n$ in $\mathcal{E}(0) \setminus \{0\} \neq \emptyset$ can not be replaced by other points in $\mathbb{R}^n \setminus \{0\}$ in order that $\Gamma_{\text{slow}} \neq \emptyset$.

Remark 3. The condition (2.16) gives a lower bound of u_0. In order that $\Gamma_{\text{slow}} \neq \emptyset$, we can not remove this condition in general. Indeed, in Example 7 of Section 4 below, we give an example such that even if (A1), (A2), (2.6) and (2.15) are fulfilled for some function ℓ and some constant $\epsilon \in (0,1)$, we have $\mathcal{E} = \mathcal{E}(0) = B(0,1)$ and $\Gamma_{\text{slow}} = \emptyset$ provided that (2.16) is violated. Hence, in order that $\Gamma_{\text{slow}} \neq \emptyset$, (2.16) can not be replaced by (2.6) in general.

Remark 4. Example 2 above and Example 8 below give the case such that $\Gamma_{\text{slow}} \underset{\neq}{\supset} \mathcal{E}(0) \setminus \{0\}$ (in Example 2, $\mathcal{E} = \mathcal{E}(0) = B(0,1)$). In general cases, the authors do not know whether $\Gamma_{\text{slow}} \underset{\neq}{\supset} \mathcal{E}(0) \setminus \{0\}$.

Remark 5. In (1.8), we have no sufficient condition for the case

$$u(x,t) - (v(x) + ct) \leq -\frac{C_2(x)}{t+1} \quad \text{for } t \in [0,\infty).$$

3. Proofs

In this section, we prove the results stated in Sections 1 and 2. First of all, recall the following results about convex analysis:

Proposition 3. [14, Theorems 23.5, 26.6 and Corollary 25.5.1]. *Assume* (A1). *Then we have the following:*

$$L \in C^1(\mathbb{R}^n). \tag{3.1}$$

Both DH *and* DL *are one-to-one mappings from* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$, (3.2)

and $(DH)^{-1} = DL$.

$$x \cdot DL(x) = L(x) + H(DL(x)) \quad \text{in } \mathbb{R}^n. \tag{3.3}$$

$$x \cdot DH(x) = H(x) + L(DH(x)) \quad \text{in } \mathbb{R}^n. \tag{3.4}$$

Proof of Lemma 1. By (A1), we see that $L \geq 0$ in $\mathbb{R}^n$ and $L(0) = 0$. In order to prove $L > 0$ in $\mathbb{R}^n \setminus \{0\}$, we show that

$$DL(x) = 0 \text{ if and only if } x = 0. \tag{3.5}$$

Indeed, by [14, Theorems 23.5 and 25.1], we see that $DH(x) = 0$ if and only if $H(\cdot)$ takes its minimum over $\mathbb{R}^n$ at x. Since H is strictly convex on $\mathbb{R}^n$ and $H(x) \geq H(0)$ in $\mathbb{R}^n$, we see that $DH(x) = 0$ if and only if $x = 0$. Then, by (3.2), we obtain (3.5).

Now, using (3.5) and [14, Theorems 23.5 and 25.1] again, we see that $L(x)$ takes its minimum over $\mathbb{R}^n$ only at $x = 0$. The proof is complete. □

In order to prove Theorem, we give several lemmas.

Lemma 2. *Let* $g \in C^1(\mathbb{R}^n)$, *and* $x, y \in \mathbb{R}^n$. *For a Lipschitz-continuous curve* $\theta : [0,1] \mapsto \mathbb{R}^n$ *such that* $\theta(0) = x$ *and* $\theta(1) = y$, *we define* $X \in \mathcal{C}(x, y, T)$ $(T > 0)$ *by*

$$X(t) = \theta\Big(\frac{t}{T}\Big) \quad \text{for } t \in [0, T].$$

Then, we have

$$\lim_{T\to\infty} \int_0^T \Big[g(-\alpha X(t) - \dot{X}(t)) - g(-\alpha X(t))\Big]\, dt = \frac{1}{\alpha}\left[g(-\alpha y) - g(-\alpha x)\right]. \tag{3.6}$$

Proof. Note that θ is differentiable almost everywhere in $[0,1]$ and $|\dot{\theta}(s)| \leq L_\theta$ almost everywhere in $[0,1]$, where L_θ is the Lipschitz constant of θ. Then, by a change of variables in the integral in (3.6) and the Lebesgue dominated convergence theorem, we have

the left-hand side of (3.6)

$$= \lim_{\delta \searrow 0} \int_0^1 \frac{1}{\delta} \left[g(-\alpha\theta(s) - \delta\dot{\theta}(s)) - g(-\alpha\theta(s)) \right] ds$$
$$= \int_0^1 \frac{d}{d\delta} \left[g(-\alpha\theta(s) - \delta\dot{\theta}(s)) \right]_{\delta=0} ds$$
$$= \int_0^1 Dg(-\alpha\theta(s)) \cdot \left(-\dot{\theta}(s) \right) ds = \frac{1}{\alpha} \int_0^1 \frac{d}{ds} \left[g(-\alpha\theta(s)) \right] ds$$
$$= \frac{1}{\alpha} \left[g(-\alpha y) - g(-\alpha x) \right] = \text{the right-hand side of (3.6).}$$

The proof is complete. □

Lemma 3. *Assume* (A1) *and* (A3). *If* $y \in \mathcal{E}$, *then we have*

$$d(x,y) = \frac{1}{\alpha} \left[L(-\alpha y) - L(-\alpha x) \right] \quad \textit{for } x \in \mathcal{E}(y). \tag{3.7}$$

Proof. 1. Let $y \in \mathcal{E}$ and $x \in \mathcal{E}(y)$. By [9, Lemma 2.2] and (2.7), we have for any $T > 0$ and $X \in \mathcal{C}(x, y, T)$

$$\int_0^T \left[f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t)) \right] dt \geq \frac{1}{\alpha} \left[L(-\alpha y) - L(-\alpha x) \right].$$

Hence, we have

$$d(x,y) \geq \frac{1}{\alpha} \left[L(-\alpha y) - L(-\alpha x) \right].$$

2. Since $x \sim y$ in $\mathcal{E}$, we find a Lipschitz-continuous curve $\theta : [0,1] \mapsto \mathbb{R}^n$ such that $\theta(0) = x$, $\theta(1) = y$ and $\theta(t) \in \mathcal{E}$ for all $t \in (0,1)$. For any $T > 0$, we define $X \in \mathcal{C}(x, y, T)$ by

$$X(t) = \theta\left(\frac{t}{T}\right) \quad \text{for } t \in [0, T].$$

Note that $X(t) \in \mathcal{E}$ for all $t \in [0, T]$. Then, by the definition of $d(x,y)$ and Lemma 2, we have

$$d(x,y) \leq \limsup_{T \to \infty} \int_0^T \left[f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t)) \right] dt$$
$$= \limsup_{T \to \infty} \left\{ \int_0^T \left[f_c(X(t)) + L(-\alpha X(t)) \right] dt + \int_0^T \left[L(-\alpha X(t) - \dot{X}(t)) - L(-\alpha X(t)) \right] dt \right\}$$
$$= \limsup_{T \to \infty} \int_0^T \left[L(-\alpha X(t) - \dot{X}(t)) - L(-\alpha X(t)) \right] dt$$

$$= \frac{1}{\alpha}\left[L(-\alpha y) - L(-\alpha x)\right],$$

because $L \in \mathrm{C}^1(\mathbb{R}^n)$ by (3.1). The proof is complete. □

Lemma 4. *Assume* (A1) *and* (A3). *Then we have*

$$\psi(y) \leq u_0(y) \quad \text{for } y \in \mathbb{R}^n. \tag{3.8}$$

Proof. For $y \in \mathbb{R}^n$ and $T > 0$, choose the curve $X \in \mathcal{C}(y,T)$ so that $X(t) = y$ for all $t \in [0,T]$. Then, we have

$$\begin{aligned}\psi(y) &\leq \int_0^T \left[f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t))\right] dt + u_0(X(T)) \\ &= T\left[f_c(y) + L(-\alpha y)\right] + u_0(y).\end{aligned}$$

Letting $T \searrow 0$, we conclude the lemma. □

Lemma 5. *Assume* (A1), (A2) *and* (A3). *For the constant ϵ of* (A2), *let*

$$\eta(x,t) = \frac{1}{\alpha}\left[\epsilon(\alpha t + 1)^{-1} - 1\right] L(-\alpha x) \quad \text{in } \mathbb{R}^n \times [0,\infty). \tag{3.9}$$

Then $\eta \in C^1(\mathbb{R}^n \times (0,\infty))$ and for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, satisfies

$$\eta_t(x,t) + \alpha x \cdot D\eta(x,t) + H(D\eta(x,t)) + L(-\alpha x) \leq 0. \tag{3.10}$$

Proof. By (3.1), $\eta \in \mathrm{C}^1(\mathbb{R}^n \times (0,\infty))$. Fix $(x,t) \in \mathbb{R}^n \times (0,\infty)$ arbitrarily, and set $s = (\alpha t + 1)^{-1} \in (0,1)$, $y = -\alpha x \in \mathbb{R}^n$. Then, we have

$$\begin{aligned}&M[\eta](x,t) := \eta_t(x,t) + \alpha x \cdot D\eta(x,t) + H(D\eta(x,t)) + L(-\alpha x) \\ &= -\epsilon s^2 L(y) + (\epsilon s - 1)\, y \cdot DL(y) + H\left((1-\epsilon s)DL(y)\right) + L(y).\end{aligned}$$

Set $z = DL(y)$. By (3.2) and (3.4), we obtain

$$M[\eta](x,t) = \epsilon(s - s^2)\, z \cdot DH(z) + H\left((1-\epsilon s)z\right) - (1 - \epsilon s^2)H(z).$$

By (A2), we conclude that $M[\eta](x,t) \leq 0$ in $\mathbb{R}^n \times (0,\infty)$. The proof is complete. □

Proof of Theorem. Let

$$\rho(x,t) = \eta(x,t) + u_0(0) + ct \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

where η is the function of (3.9). By Lemma 5, we see that ρ is a subsolution of (1.1). On the other hand, by (A3), we have

$$\rho(x,0) = u_0(0) - \frac{1-\epsilon}{\alpha} L(-\alpha x) \leq u_0(x) = u(x,0) \quad \text{in } \mathbb{R}^n.$$

By Proposition 1 and (2.7), we obtain $\rho \le u$ in $\mathbb{R}^n \times [0,\infty)$, because u belongs to $\mathrm{C}_L(\mathbb{R}^n \times [0,\infty))$. Therefore, it holds that

$$\frac{\epsilon}{\alpha}(\alpha t+1)^{-1}L(-\alpha x) \le u(x,t)-ct-u_0(0)+\frac{1}{\alpha}L(-\alpha x) \quad \text{for } (x,t) \in \mathbb{R}^n\times[0,\infty).$$

Recall that $\mathcal{E} \ni 0$. By Lemmas 1, 3 and 4, we have

$$\begin{aligned} v(x) : &= \inf\{d(x,y)+\psi(y) : y \in \mathcal{E}\} \le d(x,0)+\psi(0) \\ &\le u_0(0) - \frac{1}{\alpha}L(-\alpha x) \quad \text{for } x \in \mathcal{E}(0). \end{aligned}$$

Putting these results together, we conclude the Theorem. □

Proof of the assertions in Example 1. By Proposition 2, the Cauchy problem (1.1)–(1.2) admits a unique solution $u \in \mathrm{C}(\mathbb{R}^n \times [0,\infty))$ satisfying (1.5). Next, let

$$\begin{aligned} \rho(x,t) &= \frac{A}{2}|x|^2\left(1-e^{-\theta t}\right) \quad \text{in } \mathbb{R}^n \times [0,\infty), \\ \chi(x,t) &= \frac{A}{2}|x|^2 + e^{-\theta t}\varphi(x) \quad \text{in } \mathbb{R}^n \times [0,\infty). \end{aligned}$$

Then, by an elementary calculation, we see that ρ and χ are, respectively, a subsolution and a supersolution of (1.1). Furthermore, we have $\rho(x,0) \le u(x,0) \le \chi(x,0)$ in $\mathbb{R}^n$. Hence, by Proposition 1, we obtain $\rho \le u \le \chi$ in $\mathbb{R}^n \times [0,\infty)$, because u and χ belong to $\mathrm{C}_L(\mathbb{R}^n \times [0,\infty))$. Therefore, we conclude the assertions in Example 1. □

Proof of the assertions in Example 2. By Proposition 2, the Cauchy problem (1.1)–(1.2) admits a unique solution $u \in \mathrm{C}(\mathbb{R}^n \times [0,\infty))$ satisfying (1.5). Next, let

$$\begin{aligned} \rho(x,t) &= \frac{\alpha}{2(\alpha t+1)}(|x|^2 \wedge 1) + \frac{\alpha^2}{2}t + v(x) \quad \text{in } \mathbb{R}^n \times [0,\infty), \\ \chi(x,t) &= \frac{\alpha}{2(\alpha t+1)}|x|^2 + \frac{\alpha^2}{2}t + v(x) \quad \text{in } \mathbb{R}^n \times [0,\infty), \end{aligned}$$

where v is the function of Example 2. Note that if $\varphi(x) = |x|^2 \wedge 1$ in $\mathbb{R}^n$, then $D^+\varphi(x) = \{kx : 0 \le k \le 2\}$ for $|x| = 1$, where $D^+\varphi(x)$ denotes the super-differential set of φ at x. Then, it is easily checked that ρ and χ are, respectively, a subsolution and a supersolution of (1.1). Since $\rho(x,0) \le u(x,0) \le \chi(x,0)$ in $\mathbb{R}^n$, we obtain $\rho \le u \le \chi$ in $\mathbb{R}^n \times [0,\infty)$ by Proposition 1, because u and χ belong to $\mathrm{C}_L(\mathbb{R}^n \times [0,\infty))$. Therefore, we conclude the assertions in Example 2. □

4. Examples

We give several examples which illustrate the results in Section 2. First of all, we give three examples of the Hamiltonians H satisfying (A1) and (A2).

Example 3. Let

$$H(x) = \sum_{j=1}^{N} a_j |x|^{m_j} \quad \text{in } \mathbb{R}^n.$$

Here, we assume the following:

(H1) $N \in \mathbb{N}$.
(H2) $a_j > 0$ for $j = 1, 2, \ldots, N$.
(H3) $1 < m_1 < m_2 < \cdots < m_N$.

Then, the assumptions (A1) and (A2) are fulfilled. Indeed, it is easy to see that (A1) holds. Next, let $0 < \epsilon \le 2/(m_N + 2)$. For $x \in \mathbb{R}^n$, we have

$$g(s) := \epsilon(s - s^2)\, x \cdot DH(x) + H\left((1-\epsilon s)x\right) - (1 - \epsilon s^2)H(x)$$

$$= \sum_{j=1}^{N} a_j |x|^{m_j} \left[\epsilon m_j (s - s^2) + (1-\epsilon s)^{m_j} - (1 - \epsilon s^2)\right] \quad \text{for } s \in (0,1).$$

Then, $g(0+) = 0$. Since

$$\frac{g'(s)}{\epsilon} = \sum_{j=1}^{N} a_j |x|^{m_j} \left[m_j(1-2s) - m_j(1-\epsilon s)^{m_j - 1} + 2s\right] \quad \text{for } s \in (0,1),$$

we have $g'(0+) = 0$. Note that, by (H3), $(1-\epsilon s)^{m_j - 2} \le (1-\epsilon)^{-1}$ for $s \in (0,1)$ and $j = 1, 2, \ldots, N$. Since

$$\frac{g''(s)}{\epsilon} = \sum_{j=1}^{N} a_j (m_j - 1)|x|^{m_j} \left[-2 + \epsilon\, m_j (1-\epsilon s)^{m_j - 2}\right] \quad \text{for } s \in (0,1),$$

we conclude that $g^{''}(s) \le 0$ for $s \in (0,1)$. Therefore, (A2) holds.

Example 4. Let

$$H(x) = \sum_{\ell=1}^{n} h_\ell(x_\ell) \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where

$$h_\ell(t) = \sum_{j=1}^{N_\ell} a_{j,\ell}|t|^{m_{j,\ell}} \quad \text{in } \mathbb{R} \quad (\ell = 1, 2, \ldots, n).$$

Here, we assume the following:

(H1) $N_\ell \in \mathbb{N}$ for $\ell = 1, 2, \ldots, n$.
(H2) $a_{j,\ell} > 0$ for $j = 1, 2, \ldots, N_\ell$, $\ell = 1, 2, \ldots, n$.
(H3) $1 < m_{1,\ell} < m_{2,\ell} < \cdots < m_{N_\ell,\ell}$ for $\ell = 1, 2, \ldots, n$.

Then, the assumptions (A1) and (A2) are fulfilled. Indeed, it is easy to see that $H \in \mathrm{C}^1(\mathbb{R}^n)$, H is strictly convex in $\mathbb{R}^n$ and $H(p) \geq H(0) = 0$ in $\mathbb{R}^n$. It remains to check that $\lim_{|x|\to\infty} H(x)/|x| = \infty$. Let $m = \min\{m_{1,\ell} : 1 \leq \ell \leq n\}$. By Hölder's inequality and (H3), we have

$$\frac{m_{1,\ell}}{m}|t|^m \leq |t|^{m_{1,\ell}} + \frac{m_{1,\ell} - m}{m} \quad \text{for } t \in \mathbb{R},\ \ell = 1, 2, \ldots, n.$$

Hence, we obtain

$$\frac{n}{m}\Big(\min_{1\leq\ell\leq n} a_{1,\ell}m_{1,\ell}\Big)\sum_{\ell=1}^{n}|x_\ell|^m - \frac{1}{m}\sum_{\ell=1}^{n} a_{1,\ell}(m_{1,\ell}-m) \leq \sum_{\ell=1}^{n} a_{1,\ell}|x_\ell|^{m_{1,\ell}} \leq H(x)$$

in $\mathbb{R}^n$. Since

$$|x| \leq n^{(m-1)/m}\Big(\sum_{\ell=1}^{n}|x_\ell|^m\Big)^{1/m} \quad \text{in } \mathbb{R}^n,$$

we see that $\lim_{|x|\to\infty} H(x)/|x| = \infty$. Hence, (A1) holds.

Next, let $0 < \epsilon \leq \min\{2/(m_{N_\ell,\ell} + 2) : 1 \leq \ell \leq n\}$. By the same calculation as that of Example 3, we see that (A2) holds. □

Example 5. For $N \in \mathbb{N}$, let $\{H_j\}_{j=1}^N$ be a sequence of Hamiltonians satisfying (A1). Assume that every H_j $(j = 1, 2, \ldots, N)$ satisfies (A2) with a common constant $\epsilon > 0$. Define the new Hamiltonian H by $H = \sum_{j=1}^N a_j H_j$, where $\{a_j\}_{j=1}^N$ is a sequence such that $a_j > 0$ for $j = 1, 2, \ldots, N$. Then H satisfies (A1) and (A2) with this constant ϵ. Applying this example to Examples 3 and 4, we obtain another example of H satisfying (A1) and (A2). □

Recall that Example 1 shows that we can not remove, in general, the condition $\mathcal{E}(0) \setminus \{0\} \neq \emptyset$ in Theorem in order that $\Gamma_{\text{slow}} \neq \emptyset$ (see Remark 2 in Section 2). Next, we give an example which shows that the point $0 \in \mathbb{R}^n$ in $\mathcal{E}(0) \setminus \{0\} \neq \emptyset$ can not be replaced by other points in $\mathbb{R}^n \setminus \{0\}$ in order that $\Gamma_{\text{slow}} \neq \emptyset$.

Example 6. For $\delta > 0$, let

$$H(p) = \frac{1}{2}|p|^2 \text{ in } \mathbb{R}^n, \qquad f(x) = -\alpha^2\delta|x| \text{ in } \mathbb{R}^n,$$
$$u_0(x) = 0 \text{ in } \mathbb{R}^n.$$

In this case, it is easy to see that (A1), (A2) and (A3) are fulfilled. By Proposition 2 in Section 2, the Cauchy problem (1.1)–(1.2) admits a unique solution u in $\mathrm{C}_L(\mathbb{R}^n \times [0,\infty))$, where $L(v) = |v|^2/2$ in $\mathbb{R}^n$. By an elementary calculation, we see that it is given by

$$u(x,t) = ct + v(x) + \alpha\delta|x|e^{-\alpha t} - \alpha\delta^2\Big(e^{-\alpha t} - \frac{1}{4}e^{-2\alpha t}\Big) \text{ in } \mathbb{R}^n \times [0,\infty),$$

where $c = -\frac{\alpha^2\delta^2}{2}$ and $v(x) = -\alpha\delta|x| + \frac{3\alpha\delta^2}{4}$ are in $\mathbb{R}^n$. In this case, we have $\mathcal{E} = \{x \in \mathbb{R}^n : |x| = \delta\}$. Hence, we have $\Gamma_{\text{fast}} = \mathbb{R}^n$ and $\Gamma_{\text{slow}} = \emptyset$, although $\mathcal{E}(x_0) \setminus \{x_0\} \neq \emptyset$ for any point $x_0 \in \mathcal{E}$. □

Next, we give an example such that even if (A1), (A2), (2.6) and (2.15) for $f, u_0 \in \mathrm{C}(\mathbb{R}^n)$ are fulfilled for some function ℓ and some constant $\epsilon \in (0,1)$, we have $\mathcal{E} = \mathcal{E}(0) = B(0,1)$ and $\Gamma_{\text{slow}} = \emptyset$ provided that (2.16) is violated. This implies that, in order that $\Gamma_{\text{slow}} \neq \emptyset$, (2.16) can not be replaced by (2.6) in general.

Example 7. Let H and f be the functions in Example 2. Let $u_0 = v$ in $\mathbb{R}^n$, where v is the function of (1.6). In this case, it is not difficult to see that $\mathcal{E} = B(0,1)$ and (A1), (A2), (2.6) and (2.15) are fulfilled for $\ell(\cdot) = (1-\epsilon)L(\cdot)$ and $\epsilon \in (0,1)$. However, (2.16) is violated, because

$$u_0(x) = \frac{\alpha}{2}(1 - |x|^2) \text{ in } B(0,1).$$

Note that $v(x) + ct$ is the asymptotic solution of (1.1)–(1.2), where $c = \alpha^2/2$. Then, by Propositions 1 and 2, the Cauchy problem (1.1)–(1.2) admits the unique solution u in $\mathrm{C}_L(\mathbb{R}^n \times [0,\infty))$ such that $u(x,t) = v(x) + ct$ in $\mathbb{R}^n \times [0,\infty)$, where $L(v) = |v|^2/2$ in $\mathbb{R}^n$. Hence, $\Gamma_{\text{fast}} = \mathbb{R}^n$ and $\Gamma_{\text{slow}} = \emptyset$, although $\mathcal{E} = \mathcal{E}(0) = B(0,1)$. □

Finally, we give a remarkable example such that $\mathcal{E} = [0,1]$, $\Gamma_{\text{fast}} = (-\infty, 0]$ and $\Gamma_{\text{slow}} = (0,\infty)$ for $n = 1$. This example is a contrast to Examples 1, 2 and 6.

Example 8. For $n = 1$, let

$$H(p) = \frac{1}{2}p^2 \text{ in } \mathbb{R}, \qquad f(x) = \frac{\alpha^2}{2}\left(1 - (x_+)^2\right)_+ \text{ in } \mathbb{R},$$

$$u_0(x) \quad = \quad \frac{\alpha}{2} + \frac{x^2}{2}\,\chi_{(-\infty,0)}(x) \ \text{ in } \mathbb{R},$$

where χ_E denotes the indicator function of a set E. In this case, we have the following:

(i) (A1), (A2) and (A3) are fulfilled, and the Cauchy problem (1.1)–(1.2) admits a unique solution u in $\mathrm{C}_L(\mathbb{R}\times[0,\infty))$, where $L(v)=v^2/2$ in $\mathbb{R}$.

(ii) $\mathcal{E}=[0,1]$ and

$$\frac{\alpha}{2(\alpha t+1)}(x^2\wedge 1) \le u(x,t) - \frac{\alpha^2}{2}t - v(x) \le \frac{\alpha}{2(\alpha t+1)}x^2 \ \text{ in } [0,\infty)\times[0,\infty),$$

$$u(x,t) - \frac{\alpha^2}{2}t - v(x) = \frac{\alpha x^2}{(2\alpha+1)e^{2\alpha t}-1} \ \text{ in } (-\infty,0)\times[0,\infty),$$

where

$$v(x) = \frac{\alpha}{2}\left(1-(x_+)^2\right) + \alpha\int_1^{(x_+)\vee 1}\sqrt{r^2-1}\,dr \quad \text{for } x\in\mathbb{R}.$$

Hence, $\Gamma_{\text{fast}}=(-\infty,0]$ and $\Gamma_{\text{slow}}=(0,\infty)$ for $n=1$. □

We show the assertions (i) and (ii) above. It is easy to see that (i) holds by an elementary calculation and Proposition 2. In (ii), we have $\mathcal{E}=[0,1]$ easily. Similarly to the proof of the assertions in Example 2 of Section 3, let

$$\rho(x,t) = \begin{cases} \frac{\alpha}{2(\alpha t+1)}(x^2\wedge 1) \ + \ \frac{\alpha^2}{2}t \ + \ v(x) & \text{in } [0,\infty)\times[0,\infty), \\ \frac{\alpha x^2}{(2\alpha+1)e^{2\alpha t}-1} \ + \ \frac{\alpha^2}{2}t \ + \ v(x) & \text{in } (-\infty,0)\times[0,\infty) \end{cases}$$

and

$$\chi(x,t) = \begin{cases} \frac{\alpha}{2(\alpha t+1)}x^2 + \ \frac{\alpha^2}{2}t \ + \ v(x) & \text{in } [0,\infty)\times[0,\infty), \\ \frac{\alpha x^2}{(2\alpha+1)e^{2\alpha t}-1} \ + \ \frac{\alpha^2}{2}t \ + \ v(x) & \text{in } (-\infty,0)\times[0,\infty). \end{cases}$$

Then, it is easily checked that ρ and χ are, respectively, a subsolution and a supersolution of (1.1). Since $\rho(x,0)\le u(x,0)\le\chi(x,0)$ in $\mathbb{R}$, we obtain $\rho\le u\le\chi$ in $\mathbb{R}\times[0,\infty)$ by Proposition 1, because u and χ belong to $\mathrm{C}_L(\mathbb{R}^n\times[0,\infty))$. Hence, (ii) holds. Therefore, we conclude the assertions in Example 8. □

REFERENCES

[1] M. Bardi and I. Capuzzo-Dolcetta, "Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations," Birkhäuser, Boston, 1997.

[2] G. Barles, *Asymptotic behavior of viscosity solutions of first order Hamilton-Jacobi equations*, Ricerche di Matematica., 34 (1985), 227–260.

[3] G. Barles and P.E. Souganidis, *On the large time behavior of solutions of Hamilton-Jacobi equations*, SIAM J. Math. Anal., 31 (2000), 925–939.

[4] G. Barles and P.E. Souganidis, *Some counterexamples on the asymptotic behavior of the solutions of Hamilton-Jacobi equations*, C.R. Acad. Sci. Paris Sér. I Math., 330 (2000), 963–968.

[5] A. Davini and A. Siconolfi, *A generalized dynamical approach to the large time behavior of Hamilton-Jacobi equations*, SIAM J. Math. Anal., 38 (2006), 478–502

[6] A. Fathi, *Sur la convergence du semigroup de Lax-Oleinik*, C.R. Acad. Sci. Paris Sér. I Math., 327 (1998), 267–270.

[7] A. Fathi and A. Siconolfi, *Existence of C^1 critical subsolution of the Hamilton-Jacobi equation*, Invent. Math., 155 (2004), 363–383.

[8] Y. Fujita, H. Ishii and P. Loreti, *Asymptotic solutions of viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator*, Communications in Partial Differential Equations, 31 (2006), 827–848.

[9] Y. Fujita, H. Ishii and P. Loreti, *Asymptotic solutions of Hamilton-Jacobi equations in Euclidean n space*, Indiana Univ. Math. J., 55 (2006), 1671–1700.

[10] H. Ishii, *Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space*, to appear in Annales de l'IHP-Analyse non lineaire.

[11] S.N. Kružkov, *Generalized solutions of nonlinear first order equations with several independent variables.* II, Math. USSR-Sbornic, 1 (1967), 93–116.

[12] P.L. Lions, "Generalized Solutions of Hamilton-Jacobi Equations," Research Notes in Mathematics 69, Pitman Advanced Publishing Program, Boston, 1982.

[13] G. Namah and J-M. Roquejoffre, *Remarks on the long time behaviour of the solution of Hamilton-Jacobi equations*, Commun. in Partial Differential Equations, 24 (1999), 883–893.

[14] R.T. Rockafellar, "Convex Analysis," Princeton Mathematical Series 28, Princeton University Press, Princeton, N.J., 1970.

[15] J-M. Roquejoffre, *Comportement asymptotique des solutions d'équations de Hamilton-Jacobi monodimensionnelles*, C.R. Acad. Sci. Paris, t., 326, Serie I, (1998), 185–189.

[16] J-M. Roquejoffre, *Convergence to steady states or periodic solutions in a class of Hamilton-Jacobi equations*, J. Math. Pures Appl., 80, (2001), 85–104.