Differential and Integral Equations Volume 20, Number 10 (2007), 1167–1184

# EXISTENCE, UNIQUENESS AND MULTIPLICITY OF ROTATING FLUXON WAVES IN ANNULAR JOSEPHSON JUNCTIONS

Guy Katriel
Einstein Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, 91904, Israel

(Submitted by: Matania Ben-Artzi)

**Abstract.** We prove that the equation modelling an annular Josephson junction has a rotating fluxon wave solution for all values of the parameters. We also obtain results on uniqueness of the rotating fluxon wave in some parameter regimes, and on multiplicity of rotating fluxon waves in other parameter regimes.

## 1. Introduction

The equation modelling a long Josephson junction is [7]

$$\phi_{tt} + \alpha\phi_t + \sin(\phi) = \phi_{xx} + \beta\phi_{xxt} + \gamma, \tag{1.1}$$

where $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ ($\alpha$ and $\beta$ describe dissipative effects, while $\gamma$ describes the applied bias current). In the case of annular geometry the periodicity condition

$$\phi(x + L, t) = \phi(x, t) + 2\pi, \tag{1.2}$$

is imposed, where $L$ is the circumference of the junction.

**Definition 1.** *A rotating fluxon wave* is a solution of (1.1) and (1.2) of the form

$$\phi(x, t) = \theta(kx + \omega t), \tag{1.3}$$

where the function $\theta(z)$ satisfies

$$\theta(z + 2\pi) = \theta(z) + 2\pi \quad \forall z \in \mathbb{R}. \tag{1.4}$$

---

Partially supported by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).
Accepted for publication: August 2007.
AMS Subject Classifications: 35B10, 35Q53, 82D55, 47J05.

Clearly, the assumption that (1.3) satisfies (1.2) implies that

$$k = \frac{2\pi}{L}, \tag{1.5}$$

while the frequency $\omega$ is an unknown, to be found as part of the solution (in Proposition 5 we show that necessarily $\omega > 0$). We note that the time period $T$ of rotation of the wave around the annulus is given by $T = \frac{2\pi}{\omega}$.

Substituting (1.3) into (1.1) we obtain

$$\beta\omega k^2\theta'''(z) + (k^2 - \omega^2)\theta''(z) - \alpha\omega\theta'(z) - \sin(\theta(z)) + \gamma = 0, \tag{1.6}$$

so that rotating fluxon waves are given by solutions $(\omega, \theta(z))$ of (1.6), with $\theta(z)$ satisfying (1.4).

Our aim in this paper is to study the questions of existence and uniqueness/multiplicity of rotating fluxon waves. We shall obtain existence of a rotating fluxon wave for all values of the parameters:

**Theorem 2.** *For any $L > 0$, $\alpha > 0$, $\beta > 0$, and $\gamma > 0$,* (1.1) *and* (1.2) *has a rotating fluxon wave solution* (1.3), *with $\omega > 0$.*

To prove Theorem 2 and our other results, we use methods of nonlinear functional analysis, reducing the problem to a fixed-point equation depending on a parameter ($\omega$), to be solved by means of Leray-Schauder theory, coupled with an auxiliary scalar-valued equation, which we analyze. We have used a similar technique to prove existence of travelling waves of the *discrete* damped and dc-driven sine-Gordon equation [5]. The details of the analysis differ, and in the discrete sine-Gordon case we could not prove existence for all parameter values (which reflects the phenomenon of "pinning" in the discrete case)—while here we are able to do so.

Several researchers have studied long Josephson junctions, and annular ones in particular, experimentally in [2, 8, 9], numerically in [1, 6], and analytically in [3, 4, 6, 7].

Note that in the case $\beta = 0$ (known as the damped dc-driven sine-Gordon equation), (1.6) reduces to the damped dc-driven pendulum equation, which is amenable to phase-plane analysis and is thus well understood, and the existence of rotating fluxon waves for all $\alpha > 0$ and $\gamma > 0$ is known (see [10], Section 2.2). The situation for $\beta > 0$ is different. Most existing analytical studies assume that either $\alpha$, $\beta$ and $\gamma$ are small, so that the equation is treated as a (singular) perturbation of the sine-Gordon equation (the case $\alpha = 0$, $\beta = 0$, $\gamma = 0$ of (1.1)), or that $\beta$ is small so that the equation is a perturbation of the damped dc-driven sine-Gordon equation. An exception is [6], which studies travelling fluxon waves of kink type on an infinite line, that is, solutions of (1.1) of the form (1.3) with (1.4) replaced by $\theta(+\infty) - \theta(-\infty) = 2\pi$. Existence results for such waves are proved by a geometric analysis of the corresponding three-dimensional phase-space, and a very intricate structure of solutions is revealed. Here, as we noted, our approach is functional-analytic rather than geometric, leading to the general existence result of Theorem 2.

Our method also allows us to derive some further information about the set of rotating fluxon waves. In the physical literature, a common and useful way to describe the behavior of a system is the "I-V characteristic"—in this case the $\gamma - \omega$ (bias current-frequency) characteristic:

**Definition 3.** The *bias-frequency characteristic* is the set $\chi \subset (0,\infty) \times (0,\infty)$ of all pairs $(\omega,\gamma)$ for which (1.1) and (1.2) has a rotating fluxon wave (1.3), with frequency $\omega$.

The set $\chi$ depends, of course, on the parameters $L$, $\alpha$, and $\beta$. The following theorem shows that the set $\chi$ is "large."

**Theorem 4.** *For any $L > 0$, $\alpha > 0$, and $\beta > 0$, the bias-frequency characteristic $\chi$ has a subset $\chi_0 \subset \chi$ which has the following properties*:

(i) $\chi_0$ *is connected.*
(ii) *For any $\omega > 0$, there exists $\gamma > 0$ with $(\omega,\gamma) \in \chi_0$.*
(iii) *For any $\gamma > 0$, there exists $\omega > 0$ with $(\omega,\gamma) \in \chi_0$.*
(iv) $(0,0) \in \overline{\chi_0}$.
(v) $\chi \setminus \chi_0$ *is a bounded set.*

Note that Theorem 2 follows at once from part (iii) of Theorem 4.

We shall also obtain some bounds on the frequency $\omega$ of rotating fluxon waves (see Section 2):

**Proposition 5.** *For* **any** *rotating fluxon wave of* (1.1) *and* (1.2), *we have $\omega > 0$ and*

$$\frac{\gamma - 1}{\alpha} \leq \omega \leq \frac{\gamma}{\alpha}. \tag{1.7}$$

We note that since we are assured that $\omega > 0$, the lower bound in (1.7) is nontrivial only when $\gamma > 1$.

The above results are completely general in that they hold for all values of the parameters. We also obtain some further results under some restrictions on the parameters. While we do not expect our conditions on the parameters, which are imposed by our methods of proof, to be sharp, these more restricted results expose some of the phenomena which can occur and point the way to further investigation (analytical or numerical) by raising various questions which remain unresolved.

We do not know whether in general we can take $\chi_0 = \chi$ in Theorem 4, or in other words whether the characteristic $\chi$ is connected. In Section 3 we shall provide a positive answer to this question, and also obtain further information about the bias-frequency characteristic $\chi$, for a certain subset of the space of parameters $L$, $\alpha$, and $\beta$:

**Theorem 6.** *Assume $L > 0$, $\alpha > 0$, and $\beta > 0$. If*

$$k = \frac{2\pi}{L} > 1 \tag{1.8}$$

*and*

$$\alpha + \beta k^2 \geq \sqrt{2}k, \tag{1.9}$$

*then the bias-frequency characteristic $\chi$ can be represented in the form*

$$\chi = \{(\omega, \overline{\gamma}(\omega)) : \omega > 0\} \tag{1.10}$$

*where $\overline{\gamma} : [0, \infty) \to [0, \infty)$ is continuous and real-analytic in $(0, \infty)$, and satisfies*

$$\overline{\gamma}(0) = 0, \tag{1.11}$$

$$\lim_{\omega \to +\infty} \overline{\gamma}(\omega) = +\infty. \tag{1.12}$$

*In particular, $\chi$ is connected, and for a given bias $\gamma > 0$ the frequencies of rotating fluxon waves are the solutions $\omega$ of the equation*

$$\overline{\gamma}(\omega) = \gamma. \tag{1.13}$$

Let us note that in the case $\beta = 0$, part of the above theorem is *not* true: while $\chi$ is connected and can be parametrized as in (1.10) (for any $k > 0$ and $\alpha > 0$; see [10], Section 2.2), the function $\overline{\gamma}$ is *not* analytic in the case $\beta = 0$; in fact, it has a cusp at $\omega = k$ (see [10], Figure 2), and the value $\omega = k$ separates the "low-voltage" and "high-voltage" regions. Theorem 6 shows that the term $\phi_{xxt}$ in (1.1) has a smoothing effect on the curve $\chi$ (at least in the case that (1.8) and (1.9) hold). When we prove Theorem 6 we will point out why the assumption $\beta > 0$ is crucial, which leads to this difference between the damped dc-forced sine-Gordon equation and (1.1).

A natural question is whether the rotating fluxon wave, whose existence is guaranteed by Theorem 2, is unique (for given $L$, $\alpha$, $\beta$, and $\gamma$). In Section 4 we obtain a uniqueness result in the case that $\gamma$ is sufficiently large.

**Theorem 7.** *Fix any $L > 0$, $\alpha > 0$, and $\beta > 0$. Then for sufficiently large $\gamma > 0$, there is a* **unique** *rotating fluxon wave.*

On the other hand, in Section 5 we show that uniqueness does not hold in general, by proving that for certain parameter ranges there exist at least three rotating fluxon waves with different frequencies.

**Theorem 8.** *Assume* (1.8), *and replace* (1.9) *by the stronger assumption*

$$\beta k \geq \sqrt{2}. \tag{1.14}$$

*Then there exists a value* $\gamma_0 > 0$ *such that given any* $0 < \epsilon < \frac{1}{2}\gamma_0$*, there exists* $\alpha_0 > 0$ *so that for any* $\alpha \in (0, \alpha_0)$ *and any* $\gamma \in (\epsilon, \gamma_0 - \epsilon)$*,* (1.1) *and* (1.2) *has* **at least** *three rotating fluxon solutions with distinct frequencies.*

We can expect this multiplicity of rotating fluxon waves to lead to bistability, jump and hysteresis phenomena as the bias $\gamma$ is varied. We do not know whether the result of Theorem 8 is valid for all values of the parameters $L$, $\alpha$, and $\beta$, or whether, on the contrary, there are some values of these parameters for which there is a *unique* rotating fluxon wave for *all* $\gamma > 0$.

We mention here that all our results and proofs in this paper remain valid, with minor changes, if the nonlinearity $\sin(\phi)$ in (1.1) is replaced by $p(\phi)$, where $p$ is an arbitrary $C^1$ $2\pi$-periodic function, satisfying $\int_0^{2\pi} p(\theta)d\theta = 0$.

An important issue which we do not address here is the question of stability. For example: is it true that for any value of $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ there exists at least one rotating fluxon wave of (1.1) which is at least locally asymptotically stable?

## 2. Existence of rotating fluxon waves

In this section we prove Theorem 4, which in particular implies Theorem 2. To do so we first recast our problem into a functional-analytic form, so that we can apply Leray-Schauder theory.

Multiplying both sides of (1.6) by $\theta'(z)$ and integrating over $[0, 2\pi]$, taking (1.4) into account, we obtain

$$\omega\Big[\beta k^2 \int_0^{2\pi} (\theta''(z))^2 dz + \alpha \int_0^{2\pi} (\theta'(z))^2 dz\Big] = 2\pi\gamma.$$

By our assumption that $\alpha > 0$, $\beta > 0$, and $\gamma > 0$, this implies that $\omega > 0$, as stated in Proposition 5, and thus we may set

$$\lambda = \frac{1}{\omega}. \tag{2.1}$$

We also set

$$\theta(z) = z + u(z), \tag{2.2}$$

with $u(z)$ satisfying

$$u(z+2\pi)=u(z)\quad \forall z\in\mathbb{R}, \tag{2.3}$$

so that we can rewrite (1.6) as

$$\beta k^2 u'''(z)+\Big(\lambda k^2-\frac{1}{\lambda}\Big)u''(z)-\alpha u'(z)-\lambda\sin(z+u(z))+\lambda\gamma-\alpha=0. \tag{2.4}$$

Rotating fluxon solutions of (1.1) and (1.2) thus correspond to pairs $(\lambda, u(z))$ with $\lambda > 0$ and $u(z)$ satisfying (2.3) and (2.4).

By integrating (2.4) over $[0, 2\pi]$, taking (2.3) into account, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(s+u(s))ds=\gamma-\frac{\alpha}{\lambda}, \tag{2.5}$$

so that we can rewrite (2.4) as

$$\begin{aligned}\beta k^2 u'''(z) \quad &+ \quad \Big(\lambda k^2-\frac{1}{\lambda}\Big)u''(z)-\alpha u'(z)\\ &- \quad \lambda\sin(z+u(z))+\lambda\frac{1}{2\pi}\int_0^{2\pi}\sin(s+u(s))ds=0.\end{aligned} \tag{2.6}$$

We note that if $(\lambda, u(z))$ is a solution to our problem, then so is $(\lambda, u(z+c)+c)$ for any constant $c$ (this of course corresponds to the time-invariance of (1.1)); hence, by adjusting $c$ we may assume that

$$\int_0^{2\pi}u(s)ds=0. \tag{2.7}$$

We thus seek pairs $(\lambda, u(z))$, satisfying (2.3), (2.5), (2.6) and (2.7).

Multiplying (2.6) by $1+u'(z)$ and integrating over $[0, 2\pi]$ we obtain

**Lemma 9.** *Any solution* $(\lambda, u)$ *of* (2.3), (2.6) *and* (2.7) *satisfies*

$$\lambda\int_0^{2\pi}\sin(s+u(s))ds=\beta k^2\int_0^{2\pi}(u''(s))^2ds+\alpha\int_0^{2\pi}(u'(s))^2ds. \tag{2.8}$$

In particular, together with (2.5), (2.8) implies that

$$\lambda\gamma\geq\alpha,$$

which gives the upper bound of (1.7). Also, noting that the value of the integral on the left-hand side of (2.5) is at most 1, we have

$$\gamma-1\leq\frac{\alpha}{\lambda}, \tag{2.9}$$

which implies the lower bound in (1.7). We have thus proven Proposition 5.

We denote by $X$, $Y$, and $Z$ the Banach spaces of real-valued functions

$$X = \left\{ u \in H^3[0,2\pi] : u^{(i)}(0) = u^{(i)}(2\pi),\ \ 0 \le i \le 2,\ \ \int_0^{2\pi} u(s)ds = 0 \right\},$$

$$Y = \left\{ u \in H^1[0,2\pi] : \int_0^{2\pi} u(s)ds = 0 \right\},$$

$$Z = \left\{ u \in L^2[0,2\pi] : \int_0^{2\pi} u(s)ds = 0 \right\},$$

with the norms

$$\|u\|_X = \left( \frac{1}{2\pi} \int_0^{2\pi} (u'''(s))^2 ds \right)^{\frac{1}{2}},$$

$$\|u\|_Y = \left( \frac{1}{2\pi} \int_0^{2\pi} (u'(s))^2 ds \right)^{\frac{1}{2}}, \qquad \|u\|_Z = \left( \frac{1}{2\pi} \int_0^{2\pi} (u(s))^2 ds \right)^{\frac{1}{2}}.$$

Assuming $v \in H^1[0,2\pi]$ with $\int_0^{2\pi} v(s)ds = 0$, developing $v$ in a Fourier series and using the Parseval identity one obtains

$$\int_0^{2\pi} (v(s))^2 ds \le \int_0^{2\pi} (v'(s))^2 ds, \tag{2.10}$$

known as Wirtinger's inequality, implying the following inequalities, which we will often use:

$$\|u\|_Y \le \|u\|_X \ \ \forall u \in X, \quad \|u\|_Z \le \|u\|_Y \ \ \forall u \in Y.$$

We define a family of linear mappings $L_\lambda : X \to Z$ ($\lambda \ne 0$) by

$$L_\lambda(u) = \beta k^2 u'''(z) + \left( \lambda k^2 - \frac{1}{\lambda} \right) u''(z) - \alpha u'(z) \tag{2.11}$$

and a nonlinear mapping $N : Z \to Z$ by

$$N(u) = \sin(z + u(z)) - \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s))ds. \tag{2.12}$$

Solving (2.3), (2.6), and (2.7) is equivalent to solving

$$L_\lambda(u) = \lambda N(u), \quad u \in X, \tag{2.13}$$

so that our problem reduces to finding $(\lambda, u) \in (0,\infty) \times X$ such that (2.13) and (2.5) hold (we note that, by a simple bootstrap argument, any solution $u \in X$ of (2.13) is in fact $C^\infty$). In other words, defining the nonlinear functional $\Phi : (0,\infty) \times Z \to \mathbb{R}$ by

$$\Phi(\lambda, u) = \frac{\alpha}{\lambda} + \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s))ds, \tag{2.14}$$

we need to solve the pair of equations (2.13) and

$$\Phi(\lambda, u) = \gamma, \tag{2.15}$$

for the unknowns $(\lambda, u) \in (0, \infty) \times X$.

We derive some estimates on $L_\lambda$ which will be useful. Let $u \in X$ and write $u$ as a Fourier series

$$u(z) = \sum_{l \neq 0} a_l e^{ilz}.$$

Noting that

$$L_\lambda(e^{ilz}) = -\Big[(l\alpha + \beta k^2 l^3)i + l^2(\lambda k^2 - \lambda^{-1})\Big]e^{ilz},$$

so that

$$L_\lambda^{-1}(e^{ilz}) = -\Big[(l\alpha + \beta k^2 l^3)i + l^2(\lambda k^2 - \lambda^{-1})\Big]^{-1} e^{ilz},$$

we obtain

$$\begin{aligned}
\|L_\lambda^{-1}(u)\|_Y &= \|\sum_{l \neq 0} a_l \Big[(l\alpha + \beta k^2 l^3)i + l^2(\lambda k^2 - \lambda^{-1})\Big]^{-1} e^{ilz}\|_Y \\
&= \Big(\sum_{l \neq 0} l^2 |a_l|^2 \Big[(l\alpha + \beta k^2 l^3)^2 + l^4(\lambda k^2 - \lambda^{-1})^2\Big]^{-1}\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{l \neq 0} |a_l|^2 \Big[(\alpha + \beta k^2 l^2)^2 + l^2(\lambda k^2 - \lambda^{-1})^2\Big]^{-1}\Big)^{\frac{1}{2}} \\
&\leq \Big[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2\Big]^{-\frac{1}{2}} \Big[\sum_{l \neq 0} |a_l|^2\Big]^{\frac{1}{2}} \\
&= \Big[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2\Big]^{-\frac{1}{2}} \|u\|_Z.
\end{aligned}$$

so that

$$\|L_\lambda^{-1}\|_{Z,Y} \leq \Big[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2\Big]^{-\frac{1}{2}}. \tag{2.16}$$

In particular (dropping the first term on the right-hand side of (2.16)),

$$\|L_\lambda^{-1}\|_{Z,Y} \leq \frac{\lambda}{|\lambda^2 k^2 - 1|}. \tag{2.17}$$

We can also obtain

$$\|L_\lambda^{-1}(u)\|_X = \Big(\sum_{l \neq 0} l^6 |a_l|^2 \Big[(l\alpha + \beta k^2 l^3)^2 + l^4(\lambda k^2 - \lambda^{-1})^2\Big]^{-1}\Big)^{\frac{1}{2}}$$

$$\leq \quad \frac{\lambda}{|\lambda^2 k^2 - 1|}\Big(\sum_{l\neq 0} l^2 |a_l|^2\Big)^{\frac{1}{2}} = \frac{\lambda}{|\lambda^2 k^2 - 1|}\|u\|_Y,$$

so that

$$\|L_\lambda^{-1}\|_{Y,X} \leq \frac{\lambda}{|\lambda^2 k^2 - 1|}. \tag{2.18}$$

Defining $F : (0, \infty) \times Z \to Z$ by

$$F(\lambda, u) = \lambda L_\lambda^{-1} \circ N(u), \tag{2.19}$$

we may rewrite (2.13) as

$$u = F(\lambda, u), \quad u \in Z. \tag{2.20}$$

We note that since $L_\lambda^{-1}$ maps $Z$ onto $X$, any solution $u \in Z$ of (2.20) is in fact in $X$.

Since $\|\sin(z + u(z))\|_{L^2} \leq 1$ and $N(u)$ is the orthogonal projection of $\sin(z + u(z)) \in L^2[0, 2\pi]$ into $Z$, we have

$$\|N(u)\|_Z \leq 1 \quad \forall u \in Z. \tag{2.21}$$

Using (2.21) and (2.16), we have the bound

$$\|F(\lambda, u)\|_Z \leq \|F(\lambda, u)\|_Y \leq \lambda\Big[(\alpha+\beta k^2)^2+(\lambda k^2-\lambda^{-1})^2\Big]^{-\frac{1}{2}} \quad \forall u \in Z,\ \lambda > 0. \tag{2.22}$$

We extend the definition of $F(\lambda, u)$ by setting $F(0, u) = 0$ for all $u \in Z$ (this is necessary because $L_\lambda$ is undefined for $\lambda = 0$). The bound (2.22) shows that the extension is continuous, and moreover that the mapping $F$ takes bounded sets in $[0, \infty) \times Z$ into bounded sets in $Y$, hence (since the embedding of $Y$ into $Z$ is compact) into compact sets in $Z$, so that $F : [0, \infty) \times Z \to Z$ is a compact mapping.

We define

$$\Sigma = \{(\lambda, u) \in (0, \infty) \times Z : u = F(\lambda, u)\}.$$

The solutions of (2.13) and (2.15) are the solutions of the equation

$$\Phi(\lambda, u) = \gamma, \quad (\lambda, u) \in \Sigma. \tag{2.23}$$

In other terms, recalling (2.1), the bias-frequency characteristic $\chi$ defined in the introduction is given by

$$\chi = \{(\omega, \Phi(\omega^{-1}, u)) : (\omega^{-1}, u) \in \Sigma\ \}. \tag{2.24}$$

We will now apply the following version of the Leray-Schauder continuation principle (see, *e.g.*, [11], Theorem 14.D):

**Theorem 10.** *Let $Z$ be a Banach space and let $F : [0,\infty) \times Z \to Z$ be a continuous and compact mapping, with*

$$F(0,u) = 0 \quad \forall u \in Z.$$

*Define $\Sigma \subset (0,\infty) \times Z$ by*

$$\Sigma = \{(\lambda, u) \in (0,\infty) \times Z : u = F(\lambda, u)\}.$$

*Then there exists an unbounded connected set $\Sigma_0 \subset \Sigma$ with $(0,0) \in \overline{\Sigma}_0$.*

Since the mapping $F$ that we have defined satisfies the hypotheses of Theorem 10, we have a set $\Sigma_0$ as in the theorem.

We define $\chi_0 \subset \chi$ by

$$\chi_0 = \{(\omega, \Phi(\omega^{-1}, u)) : \omega > 0,\ (\omega^{-1}, u) \in \Sigma_0\}, \tag{2.25}$$

and we will show that $\chi_0$ has all the properties claimed in Theorem 4.

Since $\Phi$ is continuous and $\Sigma_0$ is connected, $\chi_0$ is connected, so we have (i) of Theorem 4.

Part (ii) of Theorem 4 follows from the following:

**Lemma 11.** *Let $\Sigma_0$ be as in Theorem* 10. *For any $\lambda > 0$ there exists $u \in Z$ such that $(\lambda, u) \in \Sigma_0$.*

**Proof.** Since from (2.20) and (2.22) we have

$$(\lambda, u) \in \Sigma \Rightarrow \|u\|_Z \le \lambda \frac{1}{\beta k^2 + \alpha}, \tag{2.26}$$

we conclude that the unboundedness of $\Sigma_0$ is in the $\lambda$-direction; that is, for any $M > 0$ there exists $(\lambda, u) \in \Sigma_0$ with $\lambda > M$. In other words, denoting by $\Lambda \subset (0,\infty)$ the projection of $\Sigma_0$ to the $\lambda$-axis,

$$\Lambda = \{\lambda > 0 : \exists u \in Z \text{ such that } (\lambda, u) \in \Sigma_0\},$$

we have that $\Lambda$ contains arbitrarily large values. Since $(0,0) \in \overline{\Sigma}_0$, $\Lambda$ also contains arbitrarily small positive values. Since $\Sigma_0$, hence $\Lambda$, is connected, we conclude that $\Lambda = (0,\infty)$. □

To prove that $\chi_0$ satisfies property (iii) of Theorem 4, we need to show that

$$(0,\infty) \subset \Phi(\Sigma_0). \tag{2.27}$$

Since $\Sigma_0$ is connected and $\Phi$ is continuous, $\Phi(\Sigma_0)$ is also connected, so that to prove (2.27) it suffices to show that $\Phi$ takes arbitrarily large and arbitrarily small positive values on $\Sigma_0$. Since the integral in the definition (2.14) of $\Phi$ is bounded by 1, it is immediate that

$$\lim_{\lambda \to 0+,\ (\lambda,u)\in\Sigma} \Phi(\lambda, u) = +\infty, \tag{2.28}$$

so that $\Phi$ takes arbitrarily large values on $\Sigma_0$. We shall prove

**Lemma 12.**

$$\lim_{\lambda\to+\infty,\ (\lambda,u)\in\Sigma} \Phi(\lambda,u) = 0. \tag{2.29}$$

This implies that $\Phi$ takes arbitrarily small positive values on $\Sigma_0$, completing the proof of (2.27), and thus part (iii) of Theorem 4 holds.

**Proof of Lemma 12.** Applying Wirtinger's inequality (2.10) with $v = u'$, Lemma 9 implies

$$(\lambda,u)\in\Sigma \ \Rightarrow\ \lambda\frac{1}{2\pi}\Big|\int_0^{2\pi}\sin(s+u(s))ds\Big| \le (\beta k^2+\alpha)\frac{1}{2\pi}\int_0^{2\pi}(u'(s))^2 ds. \tag{2.30}$$

From (2.20) and (2.17) we obtain

$$(\lambda,u)\in\Sigma \ \Rightarrow\ \|u\|_Y \le \frac{\lambda^2}{|\lambda^2k^2-1|}. \tag{2.31}$$

(2.30) together with (2.31) implies

$$(\lambda,u)\in\Sigma \ \Rightarrow\ \Big|\frac{1}{2\pi}\int_0^{2\pi}\sin(s+u(s))ds\Big| \le \frac{(\beta k^2+\alpha)\lambda^3}{(\lambda^2k^2-1)^2},$$

which implies that

$$\lim_{\lambda\to+\infty,\ (\lambda,u)\in\Sigma} \frac{1}{2\pi}\int_0^{2\pi}\sin(s+u(s))ds = 0; \tag{2.32}$$

hence, we have (2.29). □

To prove claim (iv) of Theorem 4, let $\omega_n = \frac{1}{n}$ $(n \ge 1)$, and note that by part (ii) of the Theorem, there exists, for each $n$, $\gamma_n > 0$ such that $(\omega_n,\gamma_n)\in\chi_0$. By the definition of $\chi_0$ there exists $u_n\in\Sigma_0$ such that $\gamma_n = \Phi(\omega_n^{-1},u_n)$. But by Lemma 12 we have $\gamma_n\to 0$ as $n\to\infty$. Hence we have $(\omega_n,\gamma_n)\to(0,0)$, so we have $(0,0)\in\overline{\chi_0}$.

To prove part (v) of Theorem 4, we will use the following two lemmas, which will also be used in the following sections.

**Lemma 13.** *If*

$$\lambda\Big[(\alpha+\beta k^2)^2+(\lambda k^2-\lambda^{-1})^2\Big]^{-\frac{1}{2}} < 1, \tag{2.33}$$

*then $F(\lambda,\cdot): Z\to Y$ (hence also $F(\lambda,\cdot): Z\to Z$) is a contraction.*

**Proof.** Computing the Frechét derivative of $N : Z \to Z$ we have

$$N'(u)(v) = \cos(z + u(z))v(z) - \frac{1}{2\pi}\int_0^{2\pi} \cos(s + u(s))v(s)ds,$$

so that $N'(u)$ is the mapping from $Z$ to $L^2[0, 2\pi]$ given by $v(z) \to \cos(z + u(z))v(z)$ followed by an orthogonal projection into the subspace $Z$, and this implies

$$\|N'(u)\|_{Z,Z} \le 1 \quad \forall u \in Z. \tag{2.34}$$

From (2.16), (2.19) and (2.34) we conclude that

$$\|D_u F(\lambda, u)\|_{Z,Y} = \lambda \|L_\lambda^{-1} N'(u)\|_{Z,Y} \le \lambda \Big[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2\Big]^{-\frac{1}{2}} \tag{2.35}$$

$\forall\, u \in Z$, which, by (2.33), completes the proof. □

**Lemma 14.** *Fixing $\alpha \ge 0$ and $\beta > 0$, there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0)$ the equation* (2.20) *has a* **unique** *solution, which we denote by $u_\lambda$. The mapping $\lambda \to u_\lambda$ from $[0, \lambda_0)$ into $Z$ is continuous and is real-analytic for $\lambda \in (0, \lambda_0)$.*

**Proof.** It is easy to see that we can find $\lambda_0 > 0$ such that (2.33) holds for $\lambda \in [0, \lambda_0)$, and thus by Lemma 13, for any $\lambda \in [0, \lambda_0)$ the mapping $F(\lambda, .) : Z \to Z$ is a contraction, so that, by Banach's fixed-point theorem, equation (2.20) has a **unique** solution, which we denote by $u_\lambda$. Real-analyticity of the mapping $\lambda \to u_\lambda$ follows from the real-analytic implicit function theorem. □

Returning to the proof of part (v) of Theorem 4, we note that since, by Lemma 11, for each $\lambda$ there exists $u$ such that $(\lambda, u) \in \Sigma_0$, and since the uniqueness in Lemma 14 implies that when $\lambda < \lambda_0$ such a $u$ is unique in $\Sigma$, we must have

$$0 < \lambda < \lambda_0,\ (\lambda, u) \in \Sigma \ \Rightarrow\ (\lambda, u) \in \Sigma_0. \tag{2.36}$$

Thus, if we set $\omega_0 = \frac{1}{\lambda_0}$, we have by (2.36), (2.24), and (2.25)

$$\omega > \omega_0,\ (\omega, \gamma) \in \chi \ \Rightarrow\ (\omega, \gamma) \in \chi_0. \tag{2.37}$$

Setting $\gamma_0 = \alpha\omega_0 + 1$ and using Proposition 5 and (2.37) we obtain

$$\gamma > \gamma_0,\ (\omega, \gamma) \in \chi \ \Rightarrow\ \omega > \omega_0,\ (\omega, \gamma) \in \chi \ \Rightarrow\ (\omega, \gamma) \in \chi_0, \tag{2.38}$$

and (2.37) and (2.38) mean that

$$(\omega, \gamma) \in \chi \setminus \chi_0 \ \Rightarrow\ \omega \le \omega_0,\ \gamma \le \gamma_0,$$

so that $\chi \setminus \chi_0$ is bounded.

We note for later use that (2.8) implies the following:

**Lemma 15.**

$$(\lambda, u) \in \Sigma, \quad \lambda > 0 \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \sin(s + u(s))ds > 0.$$

The strictness in the above inequality follows from the fact that, by (2.8), if we had equality this would imply $u = 0$, but it is easy to check that $(\lambda, 0) \notin \Sigma$ for $\lambda > 0$.

## 3. More on the bias-frequency characteristic

In this section we prove Theorem 6, which shows that for a certain open subset of the space of parameters $L$, $\alpha$, and $\beta$, the bias-frequency characteristic $\chi$ is a connected, real analytic curve, and can be represented as the graph of a function: $\gamma = \overline{\gamma}(\omega)$.

**Lemma 16.** *Assume that* $\alpha \geq 0$, $\beta > 0$ *and the inequalities* (1.8) *and* (1.9) *hold. Then for all* $\lambda \geq 0$, *the equation* (2.20) *has a* **unique** *solution, which we denote by* $u_\lambda$, *and therefore*

$$\Sigma = \Sigma_0 = \{(\lambda, u_\lambda) : \lambda \geq 0\}. \tag{3.1}$$

*The mapping* $\lambda \to u_\lambda$ *from* $[0, \infty)$ *into* $Y$ *is continuous and real-analytic on* $(0, \infty)$.

**Proof.** We use Banach's fixed point theorem. By Lemma 13, to show that $F(\lambda, \cdot)$ is a contraction from $Z$ to itself for all $\lambda \geq 0$ it suffices to show that (2.33) holds for all $\lambda \geq 0$. After some manipulations, and setting $\mu = \lambda^2$, we see that this inequality is equivalent to

$$p(\mu) \equiv (k^4 - 1)\mu^2 + [(\alpha + \beta k^2)^2 - 2k^2]\mu + 1 > 0 \quad \forall \mu \geq 0. \tag{3.2}$$

The condition $k > 1$ ensures that the quadratic function $p(\mu)$ has a minimum, and we have $p(0) = 1 > 0$. Moreover, the condition (1.9) ensures that $p'(0) \geq 0$, which implies $p$ is increasing on $[0, \infty)$, so that (3.2) holds. We thus obtain a unique solution $u_\lambda$ to (2.20). The fact that it is real-analytic in $\lambda \in (0, \infty)$ follows from the analytic implicit function theorem and the fact that the function $\lambda \to L_\lambda^{-1}$ from $(0, \infty)$ to the space $B(Z)$ of bounded linear operators from $Z$ to itself is real-analytic. □

Let us note here that it is precisely the last statement in the above proof—the fact that $\lambda \to L_\lambda^{-1}$ is real-analytic—that breaks down when $\beta = 0$, and is responsible for the fact that in that case (*i.e.*, the case of the damped, dc-forced sine-Gordon equation) the curve $\chi$ is *not* smooth (see [10], Figure 2). The reason that, when $\beta = 0$, $\lambda \to L_\lambda^{-1}$ is not real-analytic can be gleaned by recalling the definition (2.11): if $\beta = 0$ then at $\lambda = \frac{1}{k}$ the coefficient of the highest derivative in the operator $L_\lambda$ vanishes.

Thus, under the assumptions of Lemma 16, we have

$$\chi = \chi_0 = \{(\omega, \Psi(\omega^{-1})) : \omega > 0\},$$

where $\Psi : (0,\infty) \to \mathbb{R}$ is defined by

$$\Psi(\lambda) = \Phi(\lambda, u_\lambda), \tag{3.3}$$

so defining $\overline{\gamma}(\omega) = \Psi(\omega^{-1})$ we have (1.10). The fact that $\overline{\gamma}(\omega) > 0$ for $\omega > 0$ follows from Lemma 15. (1.11) follows from Lemma 12, and (1.12) follows from (2.28). This concludes the proof of Theorem 6.

## 4. Uniqueness of rotating fluxon waves for large bias current

We now prove Theorem 7. By Lemma 14, defining $\lambda_0$ as in Lemma 14, for any $\lambda \in [0, \lambda_0)$ the equation (2.20) has a **unique** solution, which we denote by $u_\lambda$. We define the function $\Psi : (0, \lambda_0) \to \mathbb{R}$ by (3.3), where $\Phi$ is the functional defined by (2.14), so that solutions of (2.23) with $\lambda \in (0, \lambda_0)$ correspond to solutions of

$$\Psi(\lambda) = \gamma. \tag{4.1}$$

We show below that there exists some $\lambda_1 \in (0, \lambda_0)$ such that

$$\lambda \in (0, \lambda_1) \quad \Rightarrow \quad \Psi'(\lambda) < 0. \tag{4.2}$$

We note that if $\gamma$ is sufficiently large then, by (1.7), we have $\lambda \in (0, \lambda_1)$ for any solution of (4.1). Therefore (4.2) implies uniqueness of solutions of (4.1), hence uniqueness of the rotating fluxon wave, for sufficiently large $\gamma$, which is the content of Theorem 7.

To prove claim (4.2) we note that, since

$$\Psi(\lambda) = \frac{\alpha}{\lambda} + \frac{1}{2\pi}\int_0^{2\pi} \sin(s + u_\lambda(s))ds,$$

we have

$$\Psi'(\lambda) = -\frac{\alpha}{\lambda^2} + \frac{1}{2\pi}\int_0^{2\pi} \cos(s + u_\lambda(s))D_\lambda u_\lambda(s)ds,$$

and using Cauchy's inequality

$$\Psi'(\lambda) \le -\frac{\alpha}{\lambda^2} + \|D_\lambda u_\lambda\|_Z, \tag{4.3}$$

so to prove (4.2) it suffices to show that $\|D_\lambda u_\lambda\|_Z = o(\frac{1}{\lambda^2})$ as $\lambda \to 0+$. In fact we shall prove a much stronger statement:

**Lemma 17.** *We have*

$$\|D_\lambda u_\lambda\|_Z = O(\lambda) \quad as\ \lambda \to 0+\,. \tag{4.4}$$

**Proof.** We differentiate the identity

$$L_\lambda(u_\lambda) = \lambda N(u_\lambda)$$

with respect to $\lambda$, obtaining

$$(D_\lambda L_\lambda)(u_\lambda) + L_\lambda(D_\lambda u_\lambda) = N(u_\lambda) + \lambda N'(u_\lambda)(D_\lambda u_\lambda). \tag{4.5}$$

We note that, because we assume $\lambda \in (0, \lambda_0)$, where $\lambda_0$ is defined as in Lemma 14, $\lambda L_\lambda^{-1} N'(u_\lambda)$ is a contraction from $Z$ to itself, so the inverse of $I - \lambda L_\lambda^{-1} N'(u_\lambda)$ is well defined as a mapping from $Z$ to itself. Moreover we have, using (2.35),

$$\|[I - \lambda L_\lambda^{-1} N'(u_\lambda)]^{-1}\|_{Z,Z} \le \frac{1}{1 - \lambda \|L_\lambda^{-1} N'(u_\lambda)\|_{Z,Z}} = O(1) \quad as\ \lambda \to 0+\,. \tag{4.6}$$

We can rewrite (4.5) in the form

$$D_\lambda u_\lambda = [I - \lambda L_\lambda^{-1} N'(u_\lambda)]^{-1} L_\lambda^{-1}[N(u_\lambda) - (D_\lambda L_\lambda)(u_\lambda)],$$

which implies

$$\begin{aligned} &\|D_\lambda u_\lambda\|_Z \\ \le\ &\|[I - \lambda L_\lambda^{-1} N'(u_\lambda)]^{-1}\|_{Z,Z}\Big[\|L_\lambda^{-1} N(u_\lambda)\|_Z + \|L_\lambda^{-1}(D_\lambda L_\lambda)(u_\lambda)\|_Z\Big]. \end{aligned} \tag{4.7}$$

Using (2.31) we obtain

$$\begin{aligned} \|N(u_\lambda)\|_Y &= \Big(\frac{1}{2\pi}\int_0^{2\pi} \cos^2(s + u_\lambda(s))(1 + u_\lambda'(s))^2 ds\Big)^{\frac{1}{2}} \\ \le\ &\Big(\frac{1}{2\pi}\int_0^{2\pi} (1 + u_\lambda'(s))^2 ds\Big)^{\frac{1}{2}} = [1 + \|u_\lambda\|_Y^2]^{\frac{1}{2}} \le \Big[1 + \Big(\frac{\lambda^2}{\lambda^2 k^2 - 1}\Big)^2\Big]^{\frac{1}{2}}, \end{aligned}$$

so that

$$\|N(u_\lambda)\|_Y = O(1) \quad as\ \lambda \to 0+\,. \tag{4.8}$$

From (2.18) and (4.8) we have

$$\|L_\lambda^{-1} N(u_\lambda)\|_Z \le \|L_\lambda^{-1} N(u_\lambda)\|_X \le \|L_\lambda^{-1}\|_{Y,X} \|N(u_\lambda)\|_Y = O(\lambda) \quad as\ \lambda \to 0+. \tag{4.9}$$

From the definition (2.11) of $L_\lambda$ we have, for all $u \in X$,

$$(D_\lambda L_\lambda)(u) = \Big(k^2 + \frac{1}{\lambda^2}\Big) u''(z);$$

hence,

$$\|(D_\lambda L_\lambda)(u)\|_Y = \Big(k^2 + \frac{1}{\lambda^2}\Big)\|u\|_X \quad \forall u \in X. \tag{4.10}$$

From (4.9) we have

$$\|u_\lambda\|_X = \lambda \|L_\lambda^{-1} N(u_\lambda)\|_X = O(\lambda^2) \text{ as } \lambda \to 0+, \tag{4.11}$$

so using (4.10) and (4.11)

$$\|(D_\lambda L_\lambda)(u_\lambda)\|_Y = O(1) \text{ as } \lambda \to 0+, \tag{4.12}$$

and together with (2.18)

$$\|L_\lambda^{-1}(D_\lambda L_\lambda)(u_\lambda)\|_Z \le \|L_\lambda^{-1}(D_\lambda L_\lambda)(u_\lambda)\|_X = O(\lambda) \text{ as } \lambda \to 0+. \tag{4.13}$$

Using (4.6), (4.7), (4.9) and (4.13) we obtain (4.4). □

## 5. Multiplicity of rotating fluxon waves

In this section we prove Theorem 8, which shows that for some values of the parameters $L$ and $\beta$, if $\alpha > 0$ is sufficiently small, then there exists an interval of $\gamma$'s for which there exist at least three rotating fluxon waves.

**Lemma 18.** *Assume that $\beta > 0$ and that* (1.8) *and* (1.14) *hold. Then for any $\alpha \ge 0$ and $\lambda \ge 0$, the equation* (2.20) *has a* **unique** *solution, which we denote by $u_{\alpha,\lambda}$. The mapping $(\alpha,\lambda) \to u_{\alpha,\lambda}$ from $[0,\infty)\times[0,\infty)$ into $Z$ is continuous.*

As in the proof of Lemma 16, this follows from the Banach fixed-point theorem applied to (2.20). Note that (1.14) implies that (1.9) holds for all $\alpha > 0$.

Since in the following argument the dependence on $\alpha$ is important, we display it in the notation (of course $u_{\alpha,\lambda}$ also depends on the parameters $k$ and $\beta$, but since we shall regard these as fixed we can suppress this dependence).

**Proof of Theorem 8.** Under the conditions of Lemma 18 we can define $\Psi_\alpha(\lambda)$ as in (3.3), now defined for all $\lambda \in [0,\infty)$, so that rotating fluxon waves correspond to solutions of

$$\Psi_\alpha(\lambda) = \gamma, \quad \lambda > 0. \tag{5.1}$$

We write $\Psi_\alpha(\lambda)$ in the form (see (2.14))

$$\Psi_\alpha(\lambda) = \frac{\alpha}{\lambda} + \Upsilon_\alpha(\lambda), \tag{5.2}$$

where

$$\Upsilon_\alpha(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \sin(s + u_{\alpha,\lambda}(s))ds.$$

We note some properties of the function $\Upsilon_\alpha(\lambda)$ (valid for any $\alpha \geq 0$). Since $u_{\alpha,0} = 0$ we have

$$\Upsilon_\alpha(0) = 0. \tag{5.3}$$

From (2.32) we obtain

$$\lim_{\lambda \to +\infty} \Upsilon_\alpha(\lambda) = 0. \tag{5.4}$$

Lemma 15 implies

$$\Upsilon_\alpha(\lambda) > 0 \quad \forall \lambda > 0. \tag{5.5}$$

(5.3), (5.4), and (5.5), imply that $\Upsilon_\alpha(\lambda)$ attains a positive global maximum at some $\lambda > 0$. In particular, this is true for $\alpha = 0$, and we denote the global maximum of $\Upsilon_0(\lambda)$ by $\gamma_0$, and assume that it is attained at $\lambda_0 > 0$:

$$\Upsilon_0(\lambda_0) = \gamma_0 = \max_{\lambda > 0} \Upsilon_0(\lambda). \tag{5.6}$$

We now fix an arbitrary $0 < \epsilon < \frac{1}{2}\gamma_0$. By (5.3) and (5.4) we can choose $0 < \lambda_1 < \lambda_0$ and $\lambda_2 > \lambda_0$ so that

$$\Upsilon_0(\lambda_1) < \frac{\epsilon}{4}, \quad \Upsilon_0(\lambda_2) < \frac{\epsilon}{4}. \tag{5.7}$$

Since the dependence of $\Upsilon_\alpha(\lambda)$ on $\alpha$ is continuous, we can choose $\alpha_0 > 0$ sufficiently small so that

$$|\Upsilon_\alpha(\lambda) - \Upsilon_0(\lambda)| < \frac{\epsilon}{2} \quad \forall \alpha \in [0, \alpha_0],\ \lambda \in [0, \lambda_2]. \tag{5.8}$$

From (5.6), (5.7), and (5.8) we obtain, for all $\alpha \in [0, \alpha_0]$,

$$\Upsilon_\alpha(\lambda_0) > \gamma_0 - \frac{\epsilon}{2}, \tag{5.9}$$

$$\Upsilon_\alpha(\lambda_1) < \frac{3}{4}\epsilon, \quad \Upsilon_\alpha(\lambda_2) < \frac{3}{4}\epsilon. \tag{5.10}$$

Moreover we can choose $\alpha_0 > 0$ sufficiently small so that

$$\frac{\alpha_0}{\lambda_1} < \frac{\epsilon}{2}. \tag{5.11}$$

From (5.2) and (5.11) we then obtain

$$|\Psi_\alpha(\lambda) - \Upsilon_\alpha(\lambda)| < \frac{\epsilon}{2} \quad \forall \alpha \in [0, \alpha_0],\ \lambda \geq \lambda_1.$$

Together with (5.9) and (5.10) this gives, for all $0 < \alpha < \alpha_0$,

$$\Psi_\alpha(\lambda_0) > \gamma_0 - \epsilon, \tag{5.12}$$

$$\Psi_\alpha(\lambda_1) < \epsilon, \tag{5.13}$$

$$\Psi_\alpha(\lambda_2) < \epsilon, \tag{5.14}$$

and by (5.2) we also have

$$\lim_{\lambda \to 0+} \Psi_\alpha(\lambda) = +\infty. \tag{5.15}$$

Thus, by continuity of $\Psi_\alpha$ on $(0, \infty)$, if we take $\gamma \in (\epsilon, \gamma_0 - \epsilon)$, then by (5.15) and (5.13) the equation (5.1) has a solution $\lambda \in (0, \lambda_1)$, by (5.13) and (5.12) the equation (5.1) has a solution $\lambda \in (\lambda_1, \lambda_0)$, and by (5.12) and (5.14) the equation (5.1) has a solution $\lambda \in (\lambda_1, \lambda_2)$. We have thus found, under the stated assumptions, three solutions of (5.1), corresponding to three rotating fluxon waves with different frequencies. □

## References


[1] D.L. Brown, M.G. Forest, B.J. Miller, and N.A. Petersson, *Computation and stability of fluxons in a singularly perturbed sine-Gordon model of the Josephson junction*, SIAM J. Appl. Math., 54 (1994), 1048–1066.

[2] A. Davidson, B. Dueholm, B. Kryger, and N.F. Pedersen, *Experimental investigation of trapped sine-Gordon solitons*, Phys. Rev. Lett., 55 (1985), 2059–2062.

[3] G. Derks, A. Doelman, S.A van Gils, and T. Visser, *Travelling waves in a singularly perturbed sine-Gordon equation*, Physica D, 180 (2003), 40–70.

[4] W. Hauck, *Kinks and rotations in long Josephson junctions*, Math. Meth. Appl. Sci., 24 (2001), 1189–1217.

[5] G. Katriel, *Existence of travelling waves in discrete sine-Gordon rings*, SIAM J. Math. Anal., 36 (2005), 1434–1443.

[6] A.G. Maksimov, V.I. Nekorkin, and M.I. Rabinovich, *Soliton trains and I-V characteristics of long Josephson junctions*, Int. J. Bifurcation & Chaos, 5 (1995), 491–505.

[7] D.W. McLaughlin, and A.C. Scott, *Perturbation analysis of fluxon dynamics*, Phys. Rev. A, 18 (1978), 1652–1680.

[8] A.V. Ustinov, T. Doderer, R.P. Huebener, N.F. Pedersen, B. Mayer, and V.A. Oboznov, *Dynamics of sine-Gordon solitons in the annular Josephson junctions*, Phys. Rev. Lett., 69 (1992), 1815–1818.

[9] A.V. Ustinov, *Solitons in Josephson junctions*, Physica D, 123 (1998), 315–329.

[10] S. Watanabe, H.S.J. van der Zant, S. Strogatz, and T.P. Orlando, *Dynamics of circular arrays of Josephson junctions and the discrete sine-Gordon equation*, Physica D, 97 (1996), 429–470.

[11] E. Zeidler, "Nonlinear Functional Analysis and its Applications I," Springer-Verlag, New York, 1993.