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BERNSTEIN PROPERTIES OF SOLUTIONS TO SOME HIGHER-ORDER EQUATIONS

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Abstract. We record some Bernstein properties for entire solutions to higher-order elliptic equations, including certain bi-Laplacian equations, other equations which are second order in the eigenvalues of the Hessian of the solution and some geometrically inspired fourth-order equations. We apply a Bernstein result for second-order elliptic equations on noncompact Riemannian manifolds, whose proof is based on techniques of Yau [20].

1. Introduction

Liouville-type theorems have been extensively studied, on $\mathbb{R}^n$ and on more general manifolds M. For surveys we refer the reader to [17] and [10].

We begin this article with a small generalization of Yau's Bernstein result for harmonic functions on complete, noncompact Riemannian manifolds, Theorem 2.1. We include a proof for the convenience of the reader; it is similar to that in [20], [18], and [14]. Indeed, in our recent work [15] on affine maximal hypersurfaces, we generalized the result in [14] using similar techniques.

In the following brief section of the article we outline the method of obtaining solutions to higher-order equations with the same outer structure as in Theorem 2.1. We also explain how in certain circumstances we obtain nonexistence results.

In the concluding section we give several higher-order applications of Theorem 2.1, providing Bernstein results for smooth entire solutions of other problems. In each case we use Theorem 2.1 to reduce the order of the problem down to a lower-order problem, the form of whose entire solutions is known. We expect more applications will be available in the future as entire solutions to other second-order equations are investigated.

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2. Second-order equations on noncompact Riemannian manifolds

Theorem 2.1. *Let (M, g) be a complete noncompact Riemannian manifold of dimension $n \geq 2$ and with Ricci curvature bounded below by*

$$R_{ij} \geq -(n-1)\, k^2 g_{ij} \tag{2.1}$$

for some constant k. Suppose the function $u : M \to \mathbb{R}^+$ satisfies

$$\Delta u = \eta \frac{\|\nabla u\|_g^2}{u} - f(u), \tag{2.2}$$

where $\eta \neq 1$ is any constant and where the differentiable function f has a root at some $C > 0$ and satisfies the differential inequality

$$f' \leq \left(\frac{n+1-2\eta}{n-1}\right)\frac{f}{u} - (n-1)\, k^2. \tag{2.3}$$

Then u is identically constant.

Here ∇ and Δ denote respectively the covariant derivative and Laplace-Beltrami operator on the Riemannian manifold (M, g). f' is the ordinary derivative of the function f of one variable.

Remarks. 1. An equivalent result is clearly true for a negative function u.

2. When $f \equiv 0$, (2.2) is equivalent to

$$\Delta u^{1-\eta} = 0$$

and is the Euler-Lagrange equation for finding the extrema of the energy functional

$$F[u] = \int_M \left\|\nabla u^{1-\eta}\right\|_g^2 d\mu_g,$$

where $d\mu_g$ is the volume element on M. It is straightforward to verify that extrema are minima. More generally, if

$$f(u) = \frac{-h'(u)\, u^{2\eta}}{2(1-\eta)^2}$$

for some function h, then (2.2) is the Euler-Lagrange equation for finding the extrema of

$$F[u] = \int_M \left\{ \left\| \nabla u^{1-\eta} \right\|_g^2 + h(u) \right\} d\mu_g.$$

Extrema are minima if

$$\eta \frac{h'(u)}{u} + h''(u) \geq 0. \tag{2.4}$$

3. If $k \neq 0$ then, in particular, $f = 0$ does not satisfy condition (2.3). A stronger f is needed to balance the effect of the negative Ricci curvature in order for the Bernstein property to hold.

4. If f satisfies (2.3) but does not have a root at any $C > 0$, then our proof reveals that (2.2) has no smooth entire solution.

5. Of course if condition (2.3) does not hold, there are nontrivial solutions of (2.2). Suppose, for example, we seek solutions of the form $u(x) = g(x_1)$. Then (2.2) reduces to

$$g'' - \eta \frac{(g')^2}{g} = -f(g). \tag{2.5}$$

Suppose further that for any fixed $\eta \neq 1$ we take $f(g) = (\frac{2}{\eta - 1}) g^\eta$. Then a positive, nonconstant, entire solution to (2.5) is $g(x_1) = \left(x_1^2 + 1\right)^{-\frac{1}{\eta-1}}$. It is straightforward to verify that (2.3) does not hold for this f.

6. For $\eta \neq 1$, if we set $v = u^{1-\eta}$, (2.2) can be rewritten as

$$\Delta v = g(v)$$

where $g(v) = (\eta - 1) v^{-\frac{\eta}{1-\eta}} f(v^{\frac{1}{1-\eta}})$. Condition (2.3) is equivalent to

$$g' \geq \left(\frac{n+1}{n-1}\right) \frac{g}{v} + (n-1) k^2.$$

7. If $\eta = 1$ then (2.2) can be rewritten as

$$\Delta \overline{u} = -\overline{f}(\overline{u}), \tag{2.6}$$

where $\overline{u} = \log u$ and $\overline{f}(\overline{u}) = \frac{f(u)}{u}$. This is the Euler-Lagrange equation for finding the extrema of

$$F[u] = \int_M \left\{ \left\| \nabla \log u \right\|_g^2 + h(u) \right\} d\mu_g,$$

where $f(u) = -\frac{u^2}{2} h'(u)$. Extrema are minima if (2.4) holds.

To apply Theorem 2.1 in this case, with $\overline{\eta} = 0$, we need $\overline{u} > 0$ so $u > 1$. We also require $\overline{f}$ to have a root at $\overline{C} > 0$ and to satisfy

$$\overline{f}' \leq \left(\frac{n+1}{n-1}\right) \frac{\overline{f}}{\overline{u}} - (n-1) k^2.$$

This translates to the following slightly different condition on the original $f(u)$:

$$f' \leq \left\{1 + \left(\frac{n+1}{n-1}\right)\frac{1}{\log u}\right\}\frac{f}{u} - (n-1)\,k^2.$$

So if f satisfies the above inequality and has a root at some $C > 1$, we apply the theorem to $\overline{u}$ to conclude that $\overline{u}$ must be identically constant. Hence any strictly greater than 1 smooth entire solution u of (2.2) with $\eta = 1$ and f satisfying the required conditions must be identically constant.

Different behaviour for $\eta = 1$ of solutions of (2.2) is of course also reflected in explicit solutions. For example, if we again seek solutions of the form $u(x) = g(x_1)$, then (2.2) reduces to

$$(\log g)'' = -\frac{f}{g}.$$

If $f = g$ for example, we get the general smooth positive entire solution

$$g(x_1) = \exp\left(-\tfrac{1}{2}x_1^2 + c_1 x_1 + c_2\right).$$

Proof of Theorem 2.1. Beginning with the test function $\Phi = \frac{\|\nabla u\|_g^2}{u^2}$ we compute at $p \in M$ in an orthonormal frame

$$\Phi_{,i} = \frac{2u_{,j}u_{,ji}}{u^2} - \frac{2u_{,j}^2 u_{,i}}{u^3} \tag{2.7}$$

and

$$\Phi_{,ii} = \frac{6u_{,j}^2 u_{,i}^2}{u^4} - \frac{2u_{,j}^2 u_{,ii}}{u^3} - \frac{8u_{,i}u_{,j}u_{,ji}}{u^3} + \frac{2u_{,ji}^2}{u^2} + \frac{2u_{,j}u_{,jii}}{u^2},$$

where "," denotes covariant differentiation, and we sum over j. Hence, summing over i and inserting (2.2),

$$\begin{aligned}\Delta\Phi = 2(3-\eta)\frac{\|\nabla u\|^4}{u^4} &+ \frac{2f\,\|\nabla u\|^2}{u^3} \\ &- \frac{8}{u^3}\sum_{i,j} u_{,i}u_{,j}u_{,ji} + \frac{2}{u^2}\sum_{i,j} u_{,ji}^2 + \frac{2}{u^2}\sum_{i,j} u_{,j}u_{,jii}.\end{aligned}$$

Assuming $\Phi(p) \neq 0$, choose an orthonormal frame field at p such that

$$u_{,1} = \|\nabla u\|\,,\ u_{,i} = 0 \text{ for all } i > 1.$$

Then

$$\Delta\Phi = 2(3-\eta)\frac{u_{,1}^4}{u^4} + \frac{2fu_1^2}{u^3} - \frac{8}{u^3}u_{,1}^2 u_{,11} + \frac{2}{u^2}\sum_{i,j} u_{,ji}^2 + \frac{2}{u^2}u_{,1}\sum_i u_{,1ii}. \tag{2.8}$$

Interchanging covariant derivatives and again using (2.2) we have

$$\sum_i u_{,1ii} = \frac{2\eta u_{,1}u_{,11}}{u} - \frac{\eta u_{,1}^3}{u^2} + \left(R_{11} - f'\right) u_{,1}. \tag{2.9}$$

Applying the Cauchy-Schwarz inequality,

$$\sum_{i,j} u_{,ji}^2 \geq \frac{n}{n-1} u_{,11}^2 + 2\sum_{i>1} u_{,1i}^2 - \frac{2}{n-1}\Big(\eta\frac{u_{,1}^2}{u} - f\Big)u_{,11} + \frac{1}{n-1}\Big(\eta\frac{u_{,1}^2}{u} - f\Big)^2. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8) and using the Ricci curvature bound (2.1) we have

$$\Delta\Phi \geq \frac{2n}{n-1}\frac{u_{,11}^2}{u^2} + \frac{4}{u^2}\sum_{i>1} u_{,1i}^2 + 4\Big\{\Big[\eta\big(\frac{n-2}{n-1}\big) - 2\Big]\frac{u_{,1}^2}{u} + \frac{f}{n-1}\Big\}\frac{u_{,11}}{u^2} \tag{2.11}$$
$$+\Big[6 - 4\eta + \frac{2\eta^2}{n-1}\Big]\frac{u_{,1}^4}{u^4} + \frac{2}{n-1}\frac{f^2}{u^2} - 2\Big[f' + (n-1)k^2 + \big(\frac{2\eta}{n-1} - 1\big)\frac{f}{u}\Big]\frac{u_{,1}^2}{u^2}.$$

Now from (2.7) we compute

$$u_{,11}^2 = \frac{u^4}{4u_{,1}^2}\Big(\sum_i \Phi_{,i}^2 - \frac{4u_{,1}^2}{u^4}\sum_{i>1} u_{,1i}^2 + \frac{8u_{,1}^4 u_{,11}}{u^5} - \frac{4u_{,1}^6}{u^6}\Big)$$

and

$$\frac{u_{,11}u_{,1}^2}{u^2} = \frac{1}{2}\Big(\sum_i \Phi_{,i}u_{,i} + \frac{2u_{,1}^4}{u^3}\Big).$$

Substituting these successively into (2.11) yields

$$\frac{\Delta\Phi}{\Phi} \geq \frac{n}{2\,(n-1)}\frac{\sum_i \Phi_{,i}^2}{\Phi^2} + \frac{2}{n-1}\Big[\,(n-2)\,(\eta-1) + \frac{f}{u\,\Phi}\Big]\frac{\sum_i \Phi_{,i}u_{,i}}{u\,\Phi} \tag{2.12}$$
$$+\frac{2\,(\eta-1)^2}{n-1}\frac{u_{,1}^2}{u^2} + \frac{2}{n-1}\frac{f^2}{u_{,1}^2} + 2\Big\{\Big(\frac{n+1-2\eta}{n-1}\Big)\frac{f}{u} - (n-1)\,k^2 - f'\Big\}.$$

For an upper bound on $\frac{\Delta\Phi}{\Phi}$ we proceed as in [20]. For $p_0 \in M$, define $J : B_a(p_0) \to \mathbb{R}$ by $J(r) = \left(a^2 - r^2\right)^2 \Phi$. Here $r(p) = d(p_0, p)$, the geodesic distance function from p_0 in the metric g. Clearly J is nonnegative on $B_a(p_0)$ and attains its maximum at some interior point p^*. We may assume r^2 is twice differentiable in a neigbourhood of p^* and $\|\nabla u\| > 0$ at p^*. By the local maximum conditions at p^* we have

$$0 = J_{,i} = -2\left(a^2 - r^2\right)\left(r^2\right)_{,i}\Phi + \left(a^2 - r^2\right)^2 \Phi_{,i} \tag{2.13}$$

and

$$0 \geq \Delta J = 2\Phi \left\|\nabla r^2\right\|_g^2 - 2\left(a^2 - r^2\right)\Phi\Delta r^2 - 4\left(a^2 - r^2\right)\left\langle \nabla r^2, \nabla\Phi\right\rangle_g + \left(a^2 - r^2\right)^2 \Delta\Phi.$$

Dividing through by $\left(a^2 - r^2\right)^2 \Phi$ we have at p^*,

$$\frac{\Delta\Phi}{\Phi} \leq \frac{2\Delta r^2}{(a^2 - r^2)} - \frac{2\left\|\nabla r^2\right\|^2}{(a^2 - r^2)^2} + \frac{4}{(a^2 - r^2)\,\Phi}\left\langle \nabla r^2, \nabla\Phi\right\rangle.$$

Combining this upper estimate with (2.12) yields, at p^*,

$$\frac{2(\eta - 1)^2}{(n-1)}\frac{\|\nabla u\|^2}{u^2} \leq \frac{4r\Delta r}{(a^2 - r^2)} + \frac{4\|\nabla r\|^2}{(a^2 - r^2)} - \frac{2r^2\left\|\nabla r^2\right\|^2}{(a^2 - r^2)^2} + \frac{4\left\langle \nabla r^2, \nabla\Phi\right\rangle}{\Phi\,(a^2 - r^2)} - \frac{n}{2(n-1)}\frac{\sum_i \Phi_{,i}^2}{\Phi^2} - \frac{2}{n-1}\left[(n-2)(\eta - 1) + \frac{f}{u\,\Phi}\right]\frac{\sum_i \Phi_{,i} u_{,i}}{u\,\Phi} - \frac{2}{n-1}\frac{f^2}{u_{,1}^2}. \tag{2.14}$$

Here we have neglected the bracketed term with k^2 in (2.11) under the condition that (2.3) holds.

From (2.13) we have at p^*,

$$\frac{\|\nabla\Phi\|^2}{\Phi^2} = \frac{4\left\|\nabla r^2\right\|^2}{(a^2 - r^2)^2}, \quad \frac{\left\langle \nabla u, \nabla\Phi\right\rangle}{\Phi} = \frac{2\left\langle \nabla r^2, \nabla u\right\rangle}{(a^2 - r^2)} \quad \text{and}$$

$$\frac{\left\langle \nabla r^2, \nabla\Phi\right\rangle}{\Phi} = \frac{2\left\|\nabla r^2\right\|^2}{(a^2 - r^2)}.$$

Substituting these into (2.14) yields, at p^*,

$$\frac{2(\eta - 1)^2}{(n-1)}\frac{\|\nabla u\|^2}{u^2} \leq \frac{4r\Delta r}{(a^2 - r^2)} + \frac{4\|\nabla r\|^2}{(a^2 - r^2)} + \left(24 - \frac{8n}{n-1}\right)\frac{r^2\|\nabla r\|^2}{(a^2 - r^2)^2} - \frac{4}{n-1}\left[(n-2)(\eta - 1) + \frac{f}{u\,\Phi}\right]\frac{\left\langle \nabla r^2, \nabla u\right\rangle}{u\,(a^2 - r^2)} - \frac{2}{n-1}\frac{f^2}{u_{,1}^2}.$$

Estimating

$$-\frac{4(n-2)(\eta - 1)}{n-1}\frac{\left\langle \nabla r^2, \nabla u\right\rangle}{u\,(a^2 - r^2)} \leq \frac{(\eta - 1)^2}{n-1}\frac{\|\nabla u\|^2}{u^2} + \frac{4(n-2)^2}{(n-1)}\frac{r^2\|\nabla r\|^2}{(a^2 - r^2)^2}$$

and

$$-\frac{4}{(n-1)}\frac{f}{u\,\Phi}\frac{\left\langle\nabla r^2,\nabla\rho\right\rangle}{(a^2-r^2)}\leq\frac{2}{(n-1)}\frac{f^2}{\|\nabla u\|^2}+\frac{8r^2}{(n-1)}\frac{\|\nabla r\|^2}{(a^2-r^2)^2},$$

we have at p^*,

$$\frac{(\eta-1)^2}{(n-1)}\frac{\|\nabla u\|^2}{u^2}\leq\frac{4r\Delta r}{(a^2-r^2)}+\frac{4\,\|\nabla r\|^2}{(a^2-r^2)}+4n\frac{r^2\,\|\nabla r\|^2}{(a^2-r^2)^2}.$$

Multiplying through by $\left(a^2-r^2\right)^2$ and noting $r\Delta r\leq(n-1)\,(1+kr)$ and $\|\nabla r\|=1$ we have

$$\frac{(\eta-1)^2}{(n-1)}J\Big|_{p^*}\leq c_1\,(n,\eta)\,a^2+c_2\,(n)\,ka^3.$$

Since this inequality holds at the maximum of J, we therefore have at any $p\in B_a\,(p_0)$,

$$\frac{(\eta-1)^2}{(n-1)}J\leq c_1a^2+c_2ka^3.$$

Dividing through by $\left(a^2-r^2\right)^2$ we get

$$\frac{(\eta-1)^2}{(n-1)}\frac{\|\nabla u\|^2}{u^2}\leq\frac{c_1}{a^2(1-\frac{r^2}{a^2})^2}+\frac{c_2\,k}{a(1-\frac{r^2}{a^2})^2},$$

which holds everywhere on $B_a\,(p_0)$. Letting $a\to\infty$ we see that we must have

$$\|\nabla u\|^2\equiv 0$$

and hence $u\equiv C$. For (2.2) to hold we choose C such that $f\,(C)=0$. This completes the proof. □

3. Applications to higher-order equations on noncompact Riemannian manifolds

More generally all we need to apply Theorem 2.1 is some smooth quantity $u=u\,[v]$, which may contain m-th order derivatives of v, but which is signed on M. If u satisfies an $(m+2)$-th order equation of the form (2.2), with $\eta\neq 1$ and f also satisfying the conditions of Theorem 2.1, then the only smooth entire solutions to this PDE are the smooth entire solutions u to the m-th order equation

$$u\equiv C \tag{3.1}$$

for any constant $C>0$.

Remarks. 1. If f satisfies (2.3) but does not have a nonzero root, then there are no signed entire solutions to (2.2). So we have a nonexistence result for such higher-order equations.

2. If (3.1) has no entire solutions for any $C > 0$, then neither does (2.2) for any f's and η's satisfying the conditions of the theorem, another nonexistence result.

3. If (3.1) has entire solutions only for certain $C > 0$, then we must take these C's to get entire solutions to (2.2).

4. If (3.1) has different entire solutions depending on C, then all of these are entire solutions to (2.2).

4. Examples

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth, strictly monotone, onto function. Often in applications h will just be the identity function, but powers and other functions are of course possible.

1. **Third-order equations.** Often first-order PDEs can be solved explicitly, so with Theorem 2.1 we can obtain results for various third-order elliptic equations. For example, if $n = 2$, the first-order linear PDE with constant coefficients,

$$av_{,x} + bv_{,y} + cv = C, \tag{4.1}$$

where $C > 0$ is constant, has general entire solution

$$v(x, y) = \frac{C}{c} + e^{-\frac{c}{b}y}\psi(bx - ay), \tag{4.2}$$

where ψ is an arbitrary entire differentiable function.

Suppose $u = h(av_{,x} + bv_{,y} + cv)$. If u is signed on M, and satisfies (2.2), explicitly

$$\Delta h(av_{,x} + bv_{,y} + cv) = \eta \frac{\|\nabla h(av_{,x} + bv_{,y} + cv)\|_g^2}{h(av_{,x} + bv_{,y} + cv)} - f(h(av_{,x} + bv_{,y} + cv)), \tag{4.3}$$

with some η and f satisfying the conditions of Theorem 2.1, then we conclude that

$$u = h(av_{,x} + bv_{,y} + cv) \equiv h(C)$$

say, for some $C > 0$ constant. Hence v satisfies (4.1). It follows that the only entire solutions v to the third-order equation (4.3) on M are the entire solutions to (4.1), namely those of the form (4.2), for any constant $C > 0$ and any smooth entire function ψ.

Example. Above includes entire solutions on M of third-order PDEs such as

$$v_{,xxx} + v_{,xyy} + v_{,yxx} + v_{,yyy} + c\Delta v = -f\left(v_x + v_y + cv\right),$$

for any function f satisfying (2.3) and any constant c.

2. **Bi-Laplacian equations.** If $u = h(\Delta v - \eta_0 \frac{\|\nabla v\|_g^2}{v} + f_0(v))$ is everywhere positive on M and satisfies an equation of the form

$$\Delta u = \eta_1 \frac{\|\nabla u\|_g^2}{u} - f_1\left(u\right),$$

where η_1 and f_1 satisfy the conditions of Theorem 2.1, then Theorem 2.1 gives that

$$u = h\Big(\Delta v - \eta_0 \frac{\|\nabla v\|_g^2}{v} + f_0\left(v\right)\Big) \equiv h\left(C\right),$$

say, for some positive constant C. Hence

$$\Delta v = \eta_0 \frac{\|\nabla v\|_g^2}{v} - f_0\left(v\right) + C.$$

If now $v > 0$ on M and $f\left(v\right) = f_0\left(v\right) - C$ satisfies condition (2.3), then applying Theorem 2.1 again gives that v is identically constant.

Note that, in terms of f_0 and C, condition (2.3) becomes

$$f_0' \leq \Big(\frac{n+1-2\eta_0}{n-1}\Big)\Big(\frac{f_0 - C}{v}\Big) - (n-1)k^2,$$

which will definitely be true regardless of C if $\eta_0 \geq \frac{1}{2}\left(n+1\right)$ and f_0 itself satisfies (2.3). This is important because for other η_0, even if f_0 satisfies (2.3) itself, $f\left(v\right) = f_0\left(v\right) - C$ might not satisfy (2.3), so there could potentially be other solutions to the fourth-order equation.

Remark. In the case $u = \Delta v$, where u and v are positive on all of M, we can, in particular, obtain the Liouville result for equations of the form

$$\Delta\Delta v - \eta \frac{\left|\nabla \Delta v\right|^2}{\Delta v} = 0,$$

for any constant $\eta \neq 1$, on manifolds M with nonnegative Ricci curvature. This is the Euler-Lagrange equation associated with finding the extrema of the energy functional

$$J\left[v\right] = \int_M \left(\Delta v\right)^{2-\eta} d\mu_g.$$

Moreover, if $\left(\eta - 1\right)\left(\eta - 2\right) > 0$, extrema are minima, while if $\left(\eta - 1\right)\left(\eta - 2\right) < 0$, extrema are maxima.

3. **Fourth-order Hessian equations.** For $1 \leq l \leq n$ we define $S_l\left(D^2 v\right)$ as the l-th elementary symmetric function of the eigenvalues of the Hessian matrix $D^2 v$. In particular, $S_1\left(D^2 v\right) = \Delta v$ and $S_n\left(D^2 v\right) = \det D^2 v$.

In Theorem 3.2 of [1], the authors proved the following result. Suppose u is a smooth, entire, convex solution to the Hessian equation

$$S_l\left(D^2 v\right) = C$$

on $\mathbb{R}^n$, for some $1 \leq l \leq n$. If v also satisfies

$$v(x) \geq b\,|x|^2 - B \tag{4.4}$$

for all $x \in \mathbb{R}^n$, where b and B are positive constants, then v is a quadratic polynomial. In the case $l = 1$, this is a related result to Theorem 2.1; in the case $l = n$ it is the result of Pogorelov for the Monge-Amperé equation [16], and the condition (4.4) is not required.

So if we take $u = h\left(S_l\left(D^2 v\right)\right)$, where $1 \leq l \leq n$, v is convex and v satisfies (4.4) (if $l \neq n$), then if u satisfies an equation of the form (2.2) with η and f also satisfying the conditions of Theorem 2.1, we conclude $u \equiv C > 0$, from which it follows by the above result that v is a quadratic polynomial.

Example. In particular, when $l = n$ and $\eta = 0$ we have a Bernstein property for convex, entire solutions u to equations of the form

$$\Delta \det D^2 v = f\left(\det D^2 v\right),$$

for any differentiable f which has a positive root and which satisfies (2.3).

4. **Fourth-order inverse Hessian equations.** Flanders proved in [6] that a weakly convex entire function v satisfying $\operatorname{tr}\left(I + D^2 v\right)^{-1} = C$, on $\mathbb{R}^n$ for some constant C, is necessarily a quadratic polynomial. So we may apply Theorem 2.1 to $v = h(\operatorname{tr}(I+D^2 v)^{-1})$ satisfying an equation of the form (2.2), with the associated conditions on η and f, to conclude that the only entire, weakly convex solutions to such fourth-order equations are quadratic polynomials.

Flanders also states in [6] a corollary of his result above, that strictly convex entire solutions v to $\operatorname{tr}\left(D^2 v\right)^{-1} = C$ are necessarily quadratic. We therefore have a similar result for the associated fourth-order case.

5. **Fourth-order Hessian quotient-type equations.** We define

$$S_{n,l}\left(D^2 v\right) = \frac{S_n\left(D^2 v\right)}{S_l\left(D^2 v\right)},$$

where $S_l\left(D^2v\right)$ is defined above. In [1], the authors proved that a smooth strictly convex solution v of the Hessian quotient equation

$$S_{n,l}\left(D^2v\right) \equiv C$$

on $\mathbb{R}^n$, which satisfies the bound

$$v\left(x\right) \leq A(1 + |x|^2) \tag{4.5}$$

for some positive constant A, is necessarily a quadratic polynomial. The bound (4.5) is not required in the case $l = n - 1$.

So if $u = h\left(S_{n,l}\left(D^2v\right)\right)$ satisfies an equation of the form (2.2), with η and f satisfying the conditions of Theorem 2.1, we conclude that

$$u = h\left(S_{n,l}\left(D^2v\right)\right) \equiv h\left(C\right)$$

for $C > 0$ constant. If, furthermore, v satisfies (4.5) then by the above result from [1], v is a quadratic polynomial.

6. **Fourth-order Gauss curvature-type equations.** Let M be a hypersurface represented as a convex graph $X = (x, v\left(x\right))$ for $x \in \mathbb{R}^n$. The Gauss curvature of M is given by

$$K = \frac{\det D^2v}{(1 + |Dv|^2)^{\frac{n+2}{2}}}.$$

Within his investigation of self-similar solutions of Gauss curvature flows, Urbas showed in [19] that if M is a complete noncompact embedded convex hypersurface in $\mathbb{R}^{n+1}$ satisfying

$$K = C\left\langle X, \nu\right\rangle^{\beta}, \tag{4.6}$$

where β and C are any positive constants, then M is a hyperplane through the origin. Here $\nu = \frac{(Dv,-1)}{\sqrt{1+|Dv|^2}}$ is the unit normal to M. We may rewrite (4.6) in the form

$$\det D^2v = C\left(1 + |Dv|^2\right)^{\frac{n+2-\beta}{2}}\left(x \cdot Dv - v\right)^{\beta}.$$

So if we set $u = h(\frac{K}{\langle X,\nu\rangle^{\beta}})$, the argument of h is everywhere positive for v convex. If now u satisfies an equation of the form (2.2), where η and f satisfy the conditions of Theorem 2.1, then

$$u = h\Big(\frac{K}{\langle X, \nu\rangle^{\beta}}\Big) \equiv h\left(C\right),$$

say, for some positive constant C, and hence (4.6) holds. Urbas' result now gives that v is a hyperplane through the origin.

7. **Fourth-order Monge-Ampère equations with periodic solutions.** Caffarelli and Li recently generalized in [2] the theorem of Pogorelov [16] about entire convex solutions of the Monge-Ampère equation $\det(D^2 v) = 1$. Let p be a positive, Hölder-continuous, periodic function. So there are constants $a_1, \ldots, a_n > 0$ such that $p(x + a_i e_i) = p(x)$ for all $x \in \mathbb{R}^n$ and each i. Caffarelli and Li show that if v is a convex, entire solution to

$$\det D^2 v = p \tag{4.7}$$

on $\mathbb{R}^n$, then there exist $b \in \mathbb{R}^n$ and a symmetric positive-definite matrix A with $\det(A) = ⨍_{\prod_{1 \leq i \leq n}[0,a_i]} p$, such that

$$w(x) := v(x) - \left[\tfrac{1}{2} Ax \cdot x + b \cdot x\right] \tag{4.8}$$

is a_i-periodic in the ith variable; that is, $w(x + a_i e_i) = w(x)$, for all $x \in \mathbb{R}^n$, $1 \leq i \leq n$. (Because of the affine invariance, the periodicity can actually be in any n linearly independent directions in place of $e_1, \ldots, e_n$.)

So if we set $u = h(\frac{\det D^2 v}{p})$ with p as above and v convex, then if u satisfies an equation of the form (2.2), with η and f satisfying the conditions of Theorem 2.1, then

$$u = h\Big(\frac{\det D^2 v}{p}\Big) \equiv h(C)$$

for some positive constant C, and hence an equation of the form (4.7) holds. Hence by Caffarelli and Li's result we know that the solution v of our fourth-order equation has the form given by (4.8).

8. **Fourth-order special Lagrangian-type equations.** In [21], Yuan proved a Bernstein-type result for the special Lagrangian equation

$$\arctan \lambda_1 + \cdots + \arctan \lambda_n = C, \tag{4.9}$$

where the λ_i's are the eigenvalues of the Hessian matrix $D^2 v$. Specifically, he showed that any entire convex solution v to (4.9) on $\mathbb{R}^n$ must be a quadratic polynomial. This is a generalization of the $n = 2$ result obtained by Fu [7]. The equation (4.9) also corresponds to the Hessian quotient equation when $n = 3$ and $l = 1$.

So suppose we set $u = h(\arctan \lambda_1 + \cdots + \arctan \lambda_n)$; the argument of h is everywhere positive for v convex. If u satisfies an equation of the form (2.2), where η and f satisfy the conditions of Theorem 2.1, then

$$u = h(\arctan \lambda_1 + \cdots + \arctan \lambda_n) \equiv h(C),$$

say, for some positive constant C, and hence (4.9) holds. Yuan's result now gives that v is a quadratic polynomial.

9. **Further fourth-order example when $n = 2$.** In [8], Hang and Wang used complex analysis techniques to prove that if $v : \mathbb{R}^2 \to \mathbb{R}$ is an entire solutions of the equation

$$-e^{-2v}\Delta v = C \tag{4.10}$$

for any constant $C > 0$, with $\int_{\mathbb{R}^2} e^{2u}dx < \infty$, then v necessarily has the form

$$v(x) = \log \frac{2\lambda}{1 + \lambda^2 |x - \xi|^2} - \frac{1}{2}\log C \tag{4.11}$$

for some $\lambda > 0$ and $\xi \in \mathbb{R}^2$. This result was also proved using different techniques in [4] and [5].

So if we take $u = h\left(-e^{-2v}\Delta v\right)$ and know that $-e^{-2v}\Delta v > 0$ and $\int_{\mathbb{R}^2} e^{2v}dx < \infty$, we obtain from Theorem 2.1 that the only solutions to equations of the form (2.2) with η and f as required in the theorem are the solutions of (4.10), namely those of the form (4.11).

For some second-order results in this direction for conformally invariant fully nonlinear equations with $n \geq 3$, to which our theorem may also be applied for fourth-order cases, we refer the reader to work of Li and Li ([11], [12], and [13]). The authors consider the operator on C^2 functions v defined by $A[v] = -\frac{2}{n-2}v^{-\frac{n+2}{n-2}}D^2v + \frac{2n}{(n-2)^2}v^{-\frac{2n}{n-2}}Dv \otimes Dv - \frac{2}{(n-2)^2}v^{-\frac{2n}{n-2}}|Dv|^2 I$, where I is the $n \times n$ identity matrix. For $-\infty < p \leq \frac{n+2}{n-2}$ and F a positive function of the eigenvalues of $A[v]$ the authors show, for example, that positive, superharmonic entire solutions v to the equation

$$F(A) = v^{p - \frac{n+2}{n-2}}$$

are $v \equiv C$ constant for $-\infty < p < \frac{n+2}{n-2}$ or, for $p = \frac{n+2}{n-2}$, there exist some $\xi \in \mathbb{R}^n$ and positive constants a and b satisfying certain conditions related to F such that

$$v(x) = \left(\frac{a}{1 + b^2|x - \xi|^2}\right)^{\frac{n-2}{2}} \quad \text{on } \mathbb{R}^n.$$

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