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SOLUTIONS FOR A NONHOMOGENEOUS NONLINEAR SCHROEDINGER EQUATION WITH DOUBLE POWER NONLINEARITY

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Abstract. We consider the problem $-\Delta u+V(x)u = f'(u)+g(x)$ in $\mathbb{R}^N$, under the assumption $\lim_{x\to\infty} V(x) = 0$, and with the nonlinear term f with a *double power* behavior. We prove the existence two solutions when g is sufficiently small and $V < 0$.

1. Perturbation of NSE

We consider the existence of solutions of the following nonhomogeneous problem:

$$-\Delta u + V(x)u = f'(u) + g(x), \quad x \in \mathbb{R}^N; \qquad E_g^V(u) < \infty, \tag{$\mathcal{P}$}$$

where the energy functional is defined by

$$E_g(u) = E_g^V(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2(x)dx - \int_{\mathbb{R}^N} f(u)dx - \int_{\mathbb{R}^N} g(x)u(x)dx.$$

The nonlinearity is given by a function f of *double power* type that is an even function $f \in C^3(\mathbb{R},\mathbb{R})$ with $f(0) = f'(0) = f''(0) = 0$ satisfying the following requirements:

(1) there exist positive numbers c_0, c_2, p, and q with $2 < p < 2^* < q$ such that

$$\begin{cases} c_0|s|^p \le f(s) & \text{for } |s| \ge 1; \\ c_0|s|^q < f(s) & \text{for } |s| \le 1; \end{cases} \tag{f_0}$$

$$\begin{cases} |f''(s)| \le c_2|s|^{p-2} & \text{for } |s| \ge 1; \\ |f''(s)| \le c_2|s|^{q-2} & \text{for } |s| \le 1; \end{cases} \tag{f_2}$$

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(2) there exists $\mu_1 > 2$ and $\mu_2 > 1$ such that, for all $s \neq 0$

$$0 < \mu_1 f(s) \leq f'(s)s, \qquad \mu_2 f'(s)s < f''(s)s^2, \qquad f'''(s)s^3 > 0; \tag{f_μ}$$

(3) for any $u \in \mathcal{D}^{1,2}$ we have

$$f'''(u)u^3 \in L^1. \tag{f_3}$$

For example the required assumptions are satisfied by $f(s) = \frac{|s|^q}{1+|s|^{q-p}}$ with $q-p$ small enough, as shown in the appendix.

We assume $V \in L^{N/2}(\mathbb{R}^N) \cap L^t$, for some $t > N/2$ and

$$\|V\|_{L^{N/2}} < S := \inf_{u \in \mathcal{D}^{1,2}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}}. \tag{1.1}$$

Moreover, we want $V \leq 0$ and $V < 0$ on a set of positive measure.

In [18] the existence of two positive solutions $u_1, u_2 \in H^1(\mathbb{R}^N)$ of the equation $-\Delta u + u = |u|^{p-2}u + g$ is proved when $g \in L^2$ satisfies $0 \leq g \leq C\exp(-(1+\varepsilon)|x|)$, $g \not\equiv 0$.

Recently, in [17], a similar problem for the p-Laplacian is studied. Namely, the author proves, with variational techniques, that the problem $-\Delta_p u + c|u|^{p-2}u = |u|^{p^*-2}u + f(x,u) + h(x)$ in $\mathbb{R}^N$, where $2 \leq p < N$, $c > 0$, $h \in W^{-1,p'}(\mathbb{R}^N)$ and f a is continuous superlinear function such that $f(x,0) = 0$ and $f(x,u) = o(|u|^{p^*-1})$ as $|u| \to \infty$, admits two positive solutions $u_1, u_2 \in H^1(\mathbb{R}^N)$.

The existence of a positive solution of the problem $-\Delta u + u = |u|^{p-1}u + g$ on $\mathbb{R}^N$, $u(x) \to 0$ for $|x| \to \infty$, was proven in [10] when $p > \frac{N}{N-2}$ and $g \in C^{0,\alpha}(\mathbb{R}^N)$, $g \geq 0$, $g \not\equiv 0$ and $g(x) \leq \frac{C}{(1+|x|^2)^{p/p-1}}$ for some $C > 0$. In [3] there is a result of multiplicity for this problem.

In [16] the author shows that the Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^N$ in the critical case $-\Delta u = |u|^{2^*-2}u + g$ has two solutions $u_0, u_1 \in H_0^1(\Omega)$, for g satisfying a suitable condition, and if $g \geq 0$ then $u_0 \geq 0$ and $u_1 \geq 0$.

We are interested in studying the problem with double power nonlinearity.

In pioneering work Berestycki and Lions [7, 8] showed the existence of a positive solution in the case $V \equiv 0$ when $f''(0) = 0$, and f has a supercritical growth near the origin and subcritical at infinity.

More recently, in the papers [2, 5, 6, 14] the double-power growth condition has been used to obtain the existence of positive solutions for different problems of the type ($\mathcal{P}$). In particular, in [5], the authors proved that in the same hypothesis on V the homogeneous problem

$$-\Delta u + Vu = f'(u) \tag{1.2}$$

has a ground-state solution (i.e., least-energy nontrivial solution). Other results on similar problems with the double power nonlinearity can be found in [1, 2, 12].

In this paper we prove the following theorem:

Theorem 1.1. *If $g \in L^{\frac{2N}{N+2}} \cap L^s$, for some $s > \frac{2N}{N+2}$, and if $\|g\|_{\frac{2N}{N+2}}$ is sufficiently small, there exist two solutions of problem $(\mathcal{P})$ in $\mathcal{D}^{1,2}$. The first solution is close to 0; if also $\|g\|_{L^{p'}\cap L^{q'}}$ is small enough, the critical value of the second solution is close to the least energy level m_V of the homogeneous problem* (1.2)*. Furthermore, if $g \geq 0$ the two solutions are nonnegative.*

Remark 1.1. Indeed, the hypothesis on the sign of V is used only to find the second solution, but we prefer a more compact claim for the theorem. Anyway, in the proofs we make it clear when we use any hypothesis.

To get the solutions of $(\mathcal{P})$ we look for critical points of the functional E_g^V constrained on the Nehari manifold

$$\begin{aligned} \mathcal{N}_g^V &= \mathcal{N}_g = \left\{u \in \mathcal{D}^{1,2} : \langle \nabla E_g(u), u\rangle = 0,\, u \neq 0\right\} \\ &= \left\{u \in \mathcal{D}^{1,2} \setminus 0 : \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} Vu^2 - \int_{\mathbb{R}^N} f'(u)u - \int_{\mathbb{R}^N} gu = 0\right\}. \end{aligned}$$

The study of the structure of the Nehari manifold will be a fundamental part of this paper.

This paper is organized as follows: in Section 2, we recall some technical results concerning the appropriate function space required by the growth properties of the nonlinearity f. Moreover, we study the geometry and the properties of the Nehari manifold. In Section 3, we prove a splitting lemma necessary to overcome the lack of compactness. This lemma is a variant of a well-known result of [15]. In Section 4 we prove the existence of two distinct critical points of the functional on the Nehari manifold.

2. Notation and preliminary result

We will use the following notation:

- $\mathcal{D}^{1,2} = \mathcal{D}^{1,2}(\mathbb{R}^N)$ = completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{1/2};$$

- $\|u\|_V^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} Vu^2$; notice that, by (1.1), we have that $\|u\|_V$ is a norm in $\mathcal{D}^{1,2}$ equivalent to the usual one;
- $2^* = \frac{2N}{N-2}$;
- $m_g = \inf\limits_{u\in\mathcal{N}_g^V} E_g^V(u)$;
- $m_{1,g} = \inf\limits_{u\in\mathcal{N}_g^-} E_g^V(u)$;
- $m_0 = \inf\limits_{u\in\mathcal{N}_0^0} E_0^0(u)$; we call ω the minimizer of E_0^0 on $\mathcal{N}_0^0$ radially symmetric;
- $m_V = \inf\limits_{u\in\mathcal{N}_0^V} E_0^V(u)$; we call $\bar{u}$ the minimizer of E_0^V on $\mathcal{N}_0^V$;
- $\Gamma_u = \{x \in \mathbb{R}^N : |u(x)| > 1\}$;
- $|A|$ = the Lebesgue measure of the subset $A \subset \mathbb{R}^N$;
- $B_R = \{x \in \mathbb{R}^N : |x| \leq R\}$;
- $B_R^C = \mathbb{R}^N \smallsetminus B_R$;
- $u_y(x) = u(x+y)$.

In order to study the properties of the functional E_g^V and its Nehari manifold, we consider some suitable Orlicz space $L^p + L^q$, where $2 < p < 2^* < q$, related to the double power growth behavior of the function f. We recall some properties of these spaces to get the smoothness of the functional E_g^V

Given $p \neq q$, we consider the space $L^p + L^q$ made up of the functions $v : \mathbb{R}^N \to \mathbb{R}$ such that

$$v = v_1 + v_2 \quad \text{with} \quad v_1 \in L^p, v_2 \in L^q. \tag{2.1}$$

The space $L^p + L^q$ is a Banach space equipped with the norm

$$\|v\|_{L^p+L^q} = \inf\{\|v_1\|_{L^p} + \|v_2\|_{L^q} : v_1 \in L^p,\ v_2 \in L^q,\ v_1 + v_2 = v\}. \tag{2.2}$$

It is well known (see, for example, [9]) that $L^p + L^q$ coincides with the dual of $L^{p'} \cap L^{q'}$. Then

$$L^p + L^q = \left(L^{p'} \cap L^{q'}\right)' \quad \text{with} \quad p' = \frac{p}{p-1},\ q' = \frac{q}{q-1}, \tag{2.3}$$

and we can introduce the following norm equivalent to the previous one:

$$\|v\|_{L^p+L^q} = \inf_{\varphi\neq 0} \frac{\int v\varphi}{\|\varphi\|_{L^{p'}} + \|\varphi\|_{L^{q'}}}. \tag{2.4}$$

Hereafter we recall some results useful for this paper contained in [4, 6].

Lemma 2.1. *We have the following:*

(1) *If $v \in L^p + L^q$, the following inequalities hold:*

$$\max\left[\|v\|_{L^q(\mathbb{R}^N\smallsetminus\Gamma_v)} - 1, \frac{1}{1+|\Gamma_v|^{\frac{1}{\tau}}}\|v\|_{L^p(\Gamma_v)}\right] \leq \|v\|_{L^p+L^q} \leq \max[\|v\|_{L^q(\mathbb{R}^N\smallsetminus\Gamma_v)}, \|v\|_{L^p(\Gamma_v)}]$$

when $\tau = \frac{pq}{q-p}$.

(2) *Let $\{v_n\} \subset L^p + L^q$. Then $\{v_n\}$ is bounded in $L^p + L^q$ if and only if the sequences $\{|\Gamma_{v_n}|\}$ and $\{\|v\|_{L^q(\mathbb{R}^N\smallsetminus\Gamma_{v_n})} + \|v\|_{L^p(\Gamma_{v_n})}\}$ are bounded.*

(3) *f' is a bounded map from $L^p + L^q$ into $L^{\frac{p}{p-1}} \cap L^{\frac{q}{q-1}}$.*

Remark 2.1. By the previous lemma we have $L^{2^*} \subset L^p + L^q$ when $2 < p < 2^* < q$. Then, by the Sobolev inequality, we get the continuous embedding

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^p + L^q.$$

In order to prove the C^2 regularity of the functional E_g^V, we need the following lemmas proved in [6]:

Lemma 2.2. *If f satisfies the hypotheses (f_0) and (f_2), we have that*

(1) *if θ and u are bounded in L^p+L^q, then $f''(\theta)u$ is bounded in $L^{p'} \cap L^{q'}$;*
(2) *f'' is a bounded map from $L^p + L^q$ into $L^{p/p-2} \cap L^{q/q-2}$;*
(3) *f'' is a continuous map from $L^p + L^q$ into $L^{p/p-2} \cap L^{q/q-2}$;*
(4) *the map $(u, v) \mapsto uv$ from $(L^p + L^q)^2$ in $L^{p/2} + L^{q/2}$ is bounded.*

Lemma 2.3. *The functional E_g^V is of class C^2 and it holds that*

$$E_g'(u)[v] = \langle \nabla E_g^V(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + Vuv - f'(u)v - gv; \quad (2.5)$$

$$E_g''(u)[v,w] = \int_{\mathbb{R}^N} \nabla v \nabla w + Vvw - f''(u)vw. \quad (2.6)$$

Moreover, the Nehari manifold defined as

$$\mathcal{N}_g^V = \left\{u \in \mathcal{D}^{1,2} \smallsetminus 0 : \int_{\mathbb{R}^N} |\nabla u|^2 + Vu^2 - f'(u)u\,dx - gu = 0\right\} \quad (2.7)$$

is of class C^1, and its tangent space at the point u is

$$T_u\mathcal{N}_g^V = \left\{v \in \mathcal{D}^{1,2} : \int_{\mathbb{R}^N} 2\nabla u \nabla v + 2Vuv - f'(u)v\,dx - f''(u)uv - gv = 0\right\}.$$

At last, we introduce the functions

$$\varphi_0^u(t) = \varphi_0(t) := E_0^V(tu) = \int_{\mathbb{R}^N} \frac{t^2}{2}(|\nabla u|^2 + Vu^2) - f(tu); \quad (2.8)$$

$$\varphi_g^u(t) = \varphi_g(t) := E_g^V(tu) = \varphi_0(t) - t\int_{\mathbb{R}^N} gu. \tag{2.9}$$

We have that

$$\varphi_g'(t) = t\|u\|_V^2 - \int_{\mathbb{R}^N} f'(tu)u - \int_{\mathbb{R}^N} gu; \tag{2.10}$$

$$\varphi_g''(t) = \|u\|_V^2 - \int_{\mathbb{R}^N} f''(tu)u^2; \tag{2.11}$$

$$\varphi_g'''(t) = -\int_{\mathbb{R}^N} f'''(tu)u^3. \tag{2.12}$$

Notice that the conditions on f assure that also $\varphi_g'''(t)$ exists. Furthermore, if $\frac{d}{dt}\varphi_g(\bar t) = 0$, then $\langle \nabla E(\bar t u), u\rangle = 0$, so $\bar t u \in \mathcal{N}_g^V$, and *vice versa*, so we want to find the critical points of $\varphi_g(t)$.

To study the manifold $\mathcal{N}_g^V$ it is useful to consider the following manifold:

$$\mathcal{V} = \left\{ w \neq 0 : G(w) := \|w\|_V^2 - \int_{\mathbb{R}^N} f''(w)w^2 = 0 \right\}. \tag{2.13}$$

Lemma 2.4. *We have that for all $u \in \mathcal{D}^{1,2}$ there exists a unique $T_u > 0$ such that $T_u u \in \mathcal{V}$.*

Proof. We have that, using (f_μ) and (f_0),

$$\begin{aligned}\varphi_0'(t) &= t\|u\|_V^2 - \int_{\mathbb{R}^N} f'(tu)u \le t\|u\|_V^2 - \frac{\mu_1}{t}\int_{\mathbb{R}^N} f(tu) \le \\ &\le t\|u\|_V^2 - t^{q-1}c_0\mu_1 \int_{t|u|<1} |u|^q - t^{p-1}c_0\mu_1\int_{t|u|\ge 1}|u|^p \le \\ &\le t\|u\|_V^2 - t^{p-1}c_0\mu_1\int_{|u|\ge 1}|u|^p \to -\infty \text{ when } t\to\infty,\end{aligned} \tag{2.14}$$

because $p > 2$. Furthermore, we have that $\varphi_0'(t)$ is strictly concave when $t \neq 0$, and that $\varphi_0''(0) > 0$, so for every $u \in \mathcal{D}^{1,2}$ there exists a unique maximum point $T_u > 0$ for the function $\varphi_0'(t)$. Thus

$$0 = T_u^2\varphi_0''(T_u) = \|T_u u\|_V^2 - \int_{\mathbb{R}^N} f''(T_u u)(T_u u)^2. \qquad \square$$

Proposition 2.5. *We have that* $\inf_{w\in\mathcal{V}}\|w\|_V^2 > 0$.

Proof. By contradiction, we suppose that there exists a sequence $\{w_n\}_n \subset \mathcal{V}$ such that $\|w_n\|_V^2$ converges to 0. We set $t_n = \|w_n\|_V$; hence, we can write

$w_n = t_n v_n$ where $\|v_n\|_V = 1$. By Remark 2.1, the sequence is bounded in $L^p + L^q$. Since $w_n \in \mathcal{V}$ and t_n converge to 0, we have

$$\begin{aligned}
1 = \|v_n\|_V^2 = \frac{\|w_n\|_V^2}{t_n^2} &= \frac{1}{t_n^2}\int_{\mathbb{R}^N} f''(t_n v_n) v_n^2 \\
&\leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{t_n v_n}} |v_n|^q + c_2 t_n^{p-2} \int_{\Gamma_{t_n v_n}} |v_n|^p \\
&\leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{t_n v_n}} |v_n|^q + c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p \\
&\leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{v_n}} |v_n|^q + c_2 t_n^{q-2} \int_{(\mathbb{R}^N \setminus \Gamma_{t_n v_n}) \cap \Gamma_{v_n}} \frac{|v_n|^p}{t_n^{q-p}} + c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p \\
&\leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{v_n}} |v_n|^q + 2c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p.
\end{aligned}$$

Hence, we get

$$1 \leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{v_n}} |v_n|^q + 2c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p$$

and by claim 2 of Remark 2.1 we get the contradiction. □

Lemma 2.6. *Let $u \in \mathcal{D}^{1,2}$ and let T_u be the unique positive number such that $T_u u \in \mathcal{V}$. Then*

$$L = \inf_{\|u\|_V = 1} T_u - \int_{\mathbb{R}^N} f'(T_u u) u > 0. \tag{2.15}$$

Proof. By contradiction, suppose that there exists a minimizing sequence u_n, with $\|u_n\|_V = 1$ such that $T_{u_n} - \int f'(T_{u_n} u_n) u_n := \sigma_n \to 0$. Let $w_n = T_{u_n} u_n$. We have that

$$T_{u_n}^2 = \|w_n\|_V^2 = \int f''(w_n) w_n^2,$$

because $w_n \in \mathcal{V}$. Furthermore, by hypothesis, we have

$$\|w_n\|_V = \int f'(w_n) \frac{w_n}{\|w_n\|_V} + \sigma_n.$$

Thus, by f_μ,

$$\mu_2 \|w_n\|_V^2 = \mu_2 \int f'(w_n) w_n + \mu_2 \sigma_n \|w_n\|_V$$

$$< \int f''(w_n)w_n^2 + \mu_2\sigma_n\|w_n\|_V = \|w_n\|_V^2 + \mu_2\sigma_n\|w_n\|_V.$$

So, because $\mu_2 > 1$ we have that

$$0 < (\mu_2 - 1)\|w_n\|_V < \mu_2\sigma_n \to 0, \tag{2.16}$$

which is a contradiction. □

Remark 2.2. Obviously, by Lemma 2.5 we have also $B := \inf_{\|u\|_V=1} T_u > 0$, and B does not depend on g.

At last we can give the following characterization of the Nehari manifold.

Proposition 2.7. *Let $\|g\|_{L^{\frac{2N}{N+2}}}$ be sufficiently small, and let $u \in \mathcal{D}^{1,2}$ with $\|u\|_V = 1$. Then*

1. *If $\int gu < 0$, then there exists a unique t_u^1 such that $t_u^1 u \in \mathcal{N}_g^V$ and $t_u^0 < t_u^1$, where t_u^0 is the unique value for which $t_u^0 \in \mathcal{N}_0^V$.*
2. *If $\int gu = 0$, then there exists a unique t_u^1 such that $t_u^1 u \in \mathcal{N}_g^V$ and $t_u^0 = t_u^1$.*
3. *If $\int gu > 0$, then there exist two positive numbers t_u^1 and t_u^2 such that $t_u^j u \in \mathcal{N}_g^V$ and $t_u^2 < T_u < t_u^1 < t_u^0$, where T_u is the unique value for which $T_u u \in \mathcal{V}$.*
4. *t_u^1 and t_u^2 depend C^1 on $g \in L^{\frac{2N}{N+2}}$ and on $u \in \mathcal{D}^{1,2} \setminus \{0\}$. Furthermore, fixing u, we have $t_u^1 \to t_u^0$, when $\|g\|_{L^{\frac{2N}{N+2}}} \to 0$.*

Proof. 1. If $\varphi_g'(\bar{t}) = 0$, with $\bar{t} \neq 0$, by f_μ, we have that

$$\bar{t}^2\varphi_g''(\bar{t}) = \bar{t}\int gu + \int [\bar{t}uf'(\bar{t}u) - \bar{t}^2u^2f''(\bar{t}u)] < 0, \tag{2.17}$$

so $\bar{t}$ is a maximum point for φ_g. Furthermore, we have that $\varphi_g(0) = 0$, $\varphi_g'(0) > 0$ and $\varphi_g''(0) > 0$. Using (f_μ) and (f_0), we have

$$\begin{aligned}
\varphi_g(t) &= \frac{t^2}{2}\|u\|_V^2 - \int_{\mathbb{R}^N} f(tu) - t\int_{\mathbb{R}^N} gu \\
&\le \frac{t^2}{2}\|u\|_V^2 - t\int_{\mathbb{R}^N} gu - c_0t^q\int_{t|u|<1}|u|^q - c_0t^q\int_{t|u|\ge 1}|u|^p \qquad (2.18)\\
&\le \frac{t^2}{2}\|u\|_V^2 - t\int_{\mathbb{R}^N} gu - c_0t^p\int_{|u|\ge 1}|u|^p \to -\infty \text{ when } t \to \infty,
\end{aligned}$$

because $p > 2$. This proves that there is exactly one t_u^1 such that $t_u^1 u \in \mathcal{N}_g$; it is easy to see that $t_u^0 < t_u^1$.

2. In this case, we can prove, as in (2.18), that $\varphi_g(t) \to -\infty$ when $t \to \infty$ and that if $\bar{t} \neq 0$ is a critical point of φ_g then (2.17) holds. Finally, consider that $0 = \varphi_g(0) = \varphi_g'(0) < \varphi_g''(0)$, and so 0 is a local minimum for φ_g, and we can conclude.

3. We have just proved that, for any $u \in \mathcal{D}^{1,2}$, we have a unique maximum point T_u of $\varphi_0'(t)$. So, if we prove that $\int gu < \varphi_0'(T_u)$ we have that there exist two numbers t_u^1 and t_u^2 such that $\varphi_g'(t_u^j) = 0$. Set L as in Lemma 2.6, and consider that

$$\int gu \leq \|g\|_{L^{\frac{2N}{N+2}}} \|u\|_{L^{2^*}} \leq C_1 \|g\|_{L^{\frac{2N}{N+2}}} \|u\|_{\mathcal{D}^{1,2}} \leq C_2 \|g\|_{L^{\frac{2N}{N+2}}} \|u\|_V. \quad (2.19)$$

Recalling that $\|u\|_V = 1$, if $\|g\|_{L^{\frac{2N}{N+2}}}$ is sufficiently small, that is, $C_2 \|g\|_{L^{\frac{2N}{N+2}}} < L$, we have exactly two positive numbers t_u^1 and t_u^2 such that $\varphi_g'(t_u^j) = 0$, and t_u^1 and t_u^2 are respectively the maximum and the minimum points of φ_g.

4. For simplicity we prove only that $t_u^1(g)$ is a C^1 function. The other case is straightforward. Let us define a function $G : \mathbb{R}^+ \times \mathcal{D}^{1,2} \setminus \{0\} \times L^{\frac{2N}{N+2}} \to \mathbb{R}$,

$$G : (t, u, g) \mapsto \frac{d}{dt} \varphi_g^u(t) = t\|u\|_V^2 - \int f'(tu)u - \int gu.$$

We have that G is a C^1 function. Let $\bar{t}$, $\bar{u}$, and $\bar{g}$ be such that $G(\bar{t}, \bar{u}, \bar{g}) = 0$. We know that $\frac{\partial}{\partial t} G(\bar{t}, \bar{u}, \bar{g}) = \frac{d^2}{dt^2} \varphi_{\bar{g}}^{\bar{u}}(\bar{t}) < 0$; thus, by the implicit function theorem there is a C^1 function $t(u, g) = t_u^1(g)$ such that $G(t(u, g), u, g) = 0$. We have then the claimed result. □

The Nehari manifold thus can be described as

$$\mathcal{N}_g^V = \mathcal{N}_{g,V}^+ \cup \mathcal{N}_{g,V}^-, \quad (2.20)$$

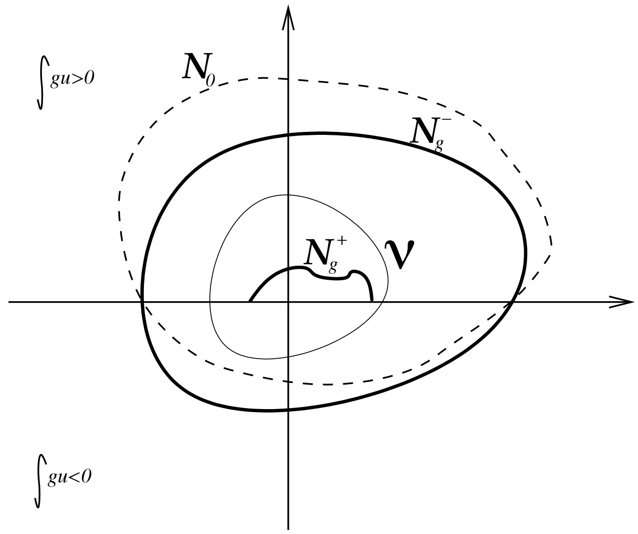
where

$$\begin{aligned} \mathcal{N}_g^+ &= \mathcal{N}_{g,V}^+ := \{u \in \mathcal{N}_g^V : E_g'(u)u = 0,\ E_g''(u)u^2 > 0\}; \\ \mathcal{N}_g^- &= \mathcal{N}_{g,V}^- := \{u \in \mathcal{N}_g^V : E_g'(u)u = 0,\ E_g''(u)u^2 < 0\}. \end{aligned}$$

We have also that $E_g^V > 0$ on $\mathcal{N}_g^-$ and $E_g^V < 0$ on $\mathcal{N}_g^+$. Furthermore, because $\mathcal{N}_0^V$ and $\mathcal{V}$ are bounded away from 0, we have also that $\inf_{u \in \mathcal{N}_g^-} \|u\| > 0$. The geometry of $\mathcal{N}_g^V$ is represented in the following picture.

Remark 2.3. There exists $M > 0$ such that

$$\|u\|_V \leq M \|g\|_{\frac{2N}{N+2}} \text{ for any } u \in \mathcal{N}_g^+; \quad (2.21)$$



indeed, by (f_μ) we have

$$\frac{1}{2}\|u\|_V^2 < \int f(u) + \int gu \leq \frac{1}{\mu_1}\int f'(u)u + \int gu = \frac{1}{\mu_1}\|u\|_V^2 + \Big(1 - \frac{1}{\mu_1}\Big)\int gu,$$

so

$$\Big(\frac{1}{2} - \frac{1}{\mu_1}\Big)\|u\|_V^2 < \Big(1 - \frac{1}{\mu_1}\Big)\int gu.$$

3. The splitting lemma

We recall that a sequence $\{u_n\}_n \in \mathcal{D}^{1,2}$ such that $E_g^V(u_n) \to c$ and $\nabla E_g^V(u_n) \to 0$ is a Palais-Smale sequence at level c for E_g^V. In the same way we say that $\{u_n\}_n \in \mathcal{N}_g^V$ such that $E_g^V(u_n) \to c$, and there exists a sequence $\epsilon_n \to 0$ such that $|\langle \nabla E_g^V(u_n), \varphi\rangle| \leq \epsilon_n\|\varphi\|$, for all $\varphi \in T_{u_n}\mathcal{N}_g^V \cap \mathcal{D}^{1,2}$ is a Palais-Smale sequence at level c for E_g^V restricted to $\mathcal{N}_g^V$.

A functional f satisfies the $(PS)_c$ condition if all the Palais-Smale sequences at level c converge.

Unfortunately, the functional E_g^V on $\mathcal{N}_g^V$ does not satisfy the PS condition in all the energy range. In this section by the splitting lemma we get a description of the PS sequences for the functional E_g^V.

Lemma 3.1. *Let $u_n \in \mathcal{N}_g$ and let $E_g^V(u_n) \to c$. Then $\|u_n\|_V$ is bounded.*

Proof. We have that

$$\|u_n\|_V^2 = \int f'(u_n)u_n + \int gu_n \tag{3.1}$$

because $u_n \in \mathcal{N}_g^V$. Furthermore, for (f_μ) we have

$$\begin{aligned} E_g^V(u_n) &= \frac{1}{2}\|u_n\|_V^2 - \int f(u_n) - \int gu_n \\ &\geq \frac{1}{2}\|u_n\|_V^2 - \frac{1}{\mu_1}\int f'(u_n)u_n - \int gu_n \\ &= \frac{1}{2}\|u_n\|_V^2 - \frac{1}{\mu_1}\|u_n\|_V^2 + \frac{1}{\mu_1}\int gu_n - \int gu_n \\ &= \Big(\frac{1}{2} - \frac{1}{\mu_1}\Big)\|u_n\|_V^2 - \Big(1 - \frac{1}{\mu_1}\Big)\int gu_n \\ &= \|u_n\|_V^2\Big[\Big(\frac{1}{2} - \frac{1}{\mu_1}\Big) - \Big(1 - \frac{1}{\mu_1}\Big)\int g\frac{u_n}{\|u_n\|_V^2}\Big]. \end{aligned} \tag{3.2}$$

If $\|u_n\|_V \to \infty$ we have that

$$\Big|\int g\frac{u_n}{\|u_n\|_V^2}\Big| \leq \|g\|_{L^{\frac{2N}{N+2}}}\frac{\|u\|_{L^{2^*}}}{\|u_n\|_V^2} \leq C\|g\|_{L^{\frac{2N}{N+2}}}\frac{1}{\|u_n\|_V} \to 0. \tag{3.3}$$

So we will have

$$C_1 > E_g^V(u_n) \geq C_2\|u_n\|_V^2 \to \infty, \tag{3.4}$$

which is a contradiction. □

Lemma 3.2. *Let $\{u_n\}_n \subset \mathcal{N}_g$, and let $E_g^V(u_n) \to c$. Then up to a subsequence $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,2}$. Furthermore, setting $\psi_n = u_n - u_0$ we have*

(1) $\|\psi_n\|_V^2 = \|u_n\|_V^2 - \|u_0\|_V^2 + o(1)$;
(2) $E_g^V(\psi_n) = E_g^V(u_n) - E_g^V(u_0) + o(1)$.

Proof. By the previous lemma we have that $\|u_n\|_{\mathcal{D}^{1,2}}$ is bounded. Then $u_n \rightharpoonup u_0$ and we have that

$$\|\psi_n\|_V^2 = \|u_n\|_V^2 - \|u_0\|_V^2 + o(1).$$

Furthermore, we have that

$$\int f(\psi_n) = \int f(u_n) - \int f(u_0) + o(1). \tag{3.5}$$

Indeed, we have the following equation, where $\tau, \theta, \sigma \in (0,1)$:

$$\int f(u_n) - \int f(u_0) - \int f(\psi_n) = \int_{B_R} f(u_0 + \psi_n) - f(u_0)$$

$$
\begin{aligned}
&- \int_{B_R^C} f(u_0) + \int_{B_R^C} f(u_0+\psi_n) - f(\psi_n) - \int_{B_R} f(\psi_n) \\
&= \int_{B_R} f'(u_0+\tau\psi_n)\psi_n - \int_{B_R^C} f(u_0) + \int_{B_R^C} f'(\theta u_0+\psi_n)u_0 - \int_{B_R} f'(\sigma\psi_n)\psi_n.
\end{aligned}
$$

Using Lemma 2.1 we have that the terms in B_R^C are arbitrarily small when R is sufficiently large. Furthermore, since $\psi_n \to 0$ in $L^p(\Omega)$ for all $\Omega \subset \mathbb{R}^N$ bounded and for all $p < 2^*$, we get that

$$\int f(u_n) - \int f(u_0) - \int f(\psi_n) \to 0.$$

The proof follows easily. □

Lemma 3.3. *Suppose that $\psi_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}$. Then we have*

$$\int V\psi_n^2 \to 0 \tag{3.6}$$

$$\int g\psi_n \to 0 \tag{3.7}$$

Proof. Again we use that $\psi_n \to 0$ in $L^p(\Omega)$ for all $\Omega \subset \mathbb{R}^N$ bounded and for all $p < 2^*$. We have that

$$
\begin{aligned}
\int V\psi_n^2 = \int_{B_R} V\psi_n^2 + \int_{\mathbb{R}^N\setminus B_R} V\psi_n^2 &\le \|V\|_{L^t(B_R)}\|\psi_n\|^2_{L^{2t'}(B_R)} \\
&+ \|V\|_{L^{N/2}(\mathbb{R}^N\setminus B_R)}\|\psi_n\|^2_{L^{2^*}(\mathbb{R}^N\setminus B_R)} \to 0,
\end{aligned}
$$

and that

$$
\begin{aligned}
\int g\psi_n = \int_{B_R} g\psi_n + \int_{\mathbb{R}^N\setminus B_R} g\psi_n &\le \|g\|_{L^s(B_R)}\|\psi_n\|_{L^{s'}(B_R)} \\
&+ \|g\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N\setminus B_R)}\|\psi_n\|_{L^{2^*}(\mathbb{R}^N\setminus B_R)} \to 0.
\end{aligned}
$$

□

Lemma 3.4. *Let $\{u_n\}_n$ be a PS sequence at level c for the functional E_g^V restricted to the manifold $\mathcal{N}_g^V$. Then, up to a subsequence, there exist k sequences of points $\{y_n^j\}_n$, $j = 1,\dots,k$, with $|y_n^j| \to \infty$, a solution u^0 of the problem $-\Delta u + Vu = f'(u) + g$, and k solutions u^j, $j = 1,\dots,k$, of the problem $-\Delta u = f'(u)$ such that*

$$u_n(x) \;=\; u^0(x) + \sum_{j=1}^{k} u^j(x-y_n^j) + o(1); \tag{3.8}$$

$$E_g^V(u_n) \quad = \quad E_g^V(u^0) + \sum_{j=1}^{k} E_0^0(u^j) + o(1). \tag{3.9}$$

Proof. Since u_n is a PS sequence for the functional E_g^V restricted to the manifold $\mathcal{N}_g^V$, then u_n is a PS sequence for the functional E_g^V. By Lemma 3.2 we have that u_n converges to u^0 weakly in $\mathcal{D}^{1,2}$ (up to a subsequence), so, given $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int \nabla u_n \nabla \varphi + V u_n \varphi - f'(u_n)\varphi - g\varphi = 0. \tag{3.10}$$

It is easy to see that

$$\int \nabla u_n \nabla \varphi + V u_n \varphi \to \int \nabla u^0 \nabla \varphi + V u^0 \varphi.$$

Arguing as in Step 1 of [5, Lemma 3.3] we get also that, for some $0 < \theta < 1$,

$$\int [f'(u_n) - f'(u^0)]\varphi = \int_{\text{supp } \varphi} f''(\theta u_n + (1-\theta)u^0)(u_n - u^0)\varphi \to 0, \tag{3.11}$$

as $n \to 0$, because $u_n - u^0 \to 0$ in $L^p(\Omega)$, with Ω bounded and $p < 2^*$. So we have proved that u^0 solves $-\Delta u + Vu = f'(u) + g$.

Now we set

$$\psi_n(x) = u_n(x) - u^0(x).$$

Then $\psi_n \rightharpoonup 0$ weakly in $\mathcal{D}^{1,2}$. If $\psi_n \nrightarrow 0$ strongly in $\mathcal{D}^{1,2}$, for Step 3 of [5, Lemma 3.3] we have that there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ with $|y_n| \to \infty$ such that $\psi_n(x + y_n) \to u^1$ in $\mathcal{D}^{1,2}$, and $u^1 \neq 0$.

Because u^0 is a weak solution of $(\mathcal{P})$ and u_n is a PS sequence for E_g^V we have that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int \nabla u_n \nabla \varphi + V u_n \varphi - f'(u_n)\varphi - g\varphi \to 0;$$
$$\int \nabla u^0 \nabla \varphi + V u^0 \varphi - f'(u^0)\varphi - g\varphi = 0.$$

So

$$\int \nabla \psi_n \nabla \varphi + V \psi_n \varphi - f'(u_n) - f'(u^0)\varphi \to 0. \tag{3.12}$$

Using (3.11) we have that ψ_n is a PS sequence for the functional E_0^V. Thus, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int \nabla \psi_n(x + y_n) \nabla \varphi(x) - f'(\psi_n(x + y_n))\varphi(x) dx$$

$$= \int \nabla\psi_n(x)\nabla\varphi(x-y_n) - f'(\psi_n(x))\varphi(x-y_n)dx$$
$$= \int [f'(u_n) - f'(u^0) - f'(\psi_n)]\varphi(x-y_n) - \int V(x)\psi_n(x)\varphi(x-y_n) + o(1).$$

Using the same argument as in Lemma 3.3 we can prove that

$$\int V(x)\psi_n(x)v(x-y_n) \leq C\varepsilon_n\|\varphi\|_{\mathcal{D}^{1,2}}, \text{ with } \varepsilon_n \to 0;$$

furthermore, we have

$$\begin{aligned}
&\int [f'(u_n) - f'(u^0) - f'(\psi_n)]\varphi(x-y_n) \\
= \quad & \int_{B_R} [f'(u_0+\psi_n) - f'(u^0)]\varphi(x-y_n) \\
&+ \int_{B_R^C} [f'(u_0+\psi_n) - f'(\psi_n)]\varphi(x-y_n) \\
&- \int_{B_R^C} f'(u^0)\varphi(x-y_n) + \int_{B_R} f'(\psi_n)\varphi(x-y_n) \\
\leq \quad & \|[f''(u^0+\theta\psi_n) - f''(\theta\psi_n)]\varphi(\cdot-y_n)\|_{L^{p'}(\mathbb{R}^N)}\|\psi_n\|_{L^p(B_R)} \\
&+ \|[f''(\psi_n+\theta u^0) - f''(\theta u^0)]\varphi(\cdot-y_n)\|_{L^{p'}\cap L^{q'}}\|u^0\|_{L^p+L^q(B_R^C)},
\end{aligned}$$

where $0 < \theta < 1$. Because $\|u^0\|_{L^p+L^q(B_R^C)} \to 0$ for $R \to \infty$, and given R $\|\psi_n\|_{L^p(B_R)} \to 0$ as $n \to 0$, we have that

$$\int \nabla\psi_n(x+y_n)\nabla\varphi(x) - f'(\psi_n(x+y_n))\varphi(x)dx \to 0.$$

Finally, it is easy to see that

$$\int \nabla\psi_n(x+y_n)\nabla v(x) - f'(\psi_n(x+y_n))v(x)dx \to \int \nabla u^1\nabla v(x) - f'(u^1)v(x)dx,$$

so we have also proved that u^1 solves the problem $-\Delta u = f(u)$.

Set $\psi_n^2 = \psi(x+y_n) - u^1$. We have that $\psi_n^2 \rightharpoonup 0$; thus,

$$\begin{aligned}
E_g^V(u_n) - E_g^V(u^0) &= E_g^V(\psi_n(x)) + o(1) = E_0^0(\psi_n(x)) + o(1) \\
&= E_0^0(\psi_n(x+y_n)) + o(1) = E_0^0(u^1) + E_0^0(\psi_n^2) + o(1),
\end{aligned}$$

by Lemma 3.2. So,

$$E_g^V(u_n) = E_g^V(u^0) + E_0^0(u^1) + E_0^0(\psi_n^2) + o(1). \tag{3.13}$$

Now, if $\psi_n^2 \to 0$ strongly in $\mathcal{D}^{1,2}$, we have the claim; otherwise, we can proceed by induction and conclude the proof in a finite number of steps. $\square$

4. Main results

We set

$$m_g = \inf_{u \in \mathcal{N}_g^V} E_g^V(u) \quad \text{and} \quad m_{1,g} = \inf_{u \in \mathcal{N}_g^-} E_g^V(u).$$

We show that there exists a solution with critical value m_g and another solution with critical value $m_{1,g}$. We set also

$$m_0 = \inf_{u \in \mathcal{N}_0^0} E_0^0(u), \tag{4.1}$$

and we recall that there exists a positive radially symmetric function $\omega \in \mathcal{N}_0^0$ such that

$$E_0^0(\omega) = m_0 > 0. \tag{4.2}$$

Finally, we set

$$m_V = \inf_{u \in \mathcal{N}_0^V} E_0^V(u). \tag{4.3}$$

We know, by [5], that for any $V \leq 0$ and $V < 0$ on a set of positive measure there exists a function $\bar{u} \in \mathcal{N}_0^V$ such that

$$E_0^V(\bar{u}) = m_V \tag{4.4}$$

and

$$0 < m_V < m_0. \tag{4.5}$$

We prove the following results.

Theorem 4.1. *There exists a $u_g \in \mathcal{N}_g^+$ such that $E_g^V(u_g) = m_g$. Furthermore, when $\|g\|_{L^{\frac{2N}{N+2}}}$ is small, u_g is unique.*

Proof. By definition of $\mathcal{N}_g^+$ we have that $m_g = \inf_{u \in \mathcal{N}_g^+} E_g^V(u)$, and that $m_g < 0$. First we prove that $m_g > -\infty$. For the sake of contradiction, suppose that there exist a sequence $t_n > 0$ and a sequence $\{v_n\}_n \subset \mathcal{D}^{1,2}$ with $\|v_n\|_V = 1$ and $t_n v_n \in \mathcal{N}_g^+$ such that

$$E_g^V(t_n v_n) = \frac{t_n^2}{2} - \int f(t_n v_n) - t_n \int g v_n \to -\infty. \tag{4.6}$$

We have also that

$$t_n^2 - \int f'(t_n v_n) t_n v_n - t_n \int g v_n = 0.$$

So, if t_n is bounded, we have

$$\begin{aligned} E_g^V(t_n v_n) &= -\frac{t_n^2}{2} + \int f'(t_n v_n) t_n v_n - \int f(t_n v_n) \\ &\geq -\frac{t_n^2}{2} + \Big(1 - \frac{1}{\mu_1}\Big) \int f'(t_n v_n) t_n v_n, \end{aligned}$$

which is bounded by Lemma 2.1. Thus we have that, up to a subsequence, $t_n \to +\infty$. Finally, arguing as in (3.2) we have that

$$E_g^V(t_n v_n) \geq \Big(\frac{1}{2} - \frac{1}{\mu_1}\Big) t_n^2 - \Big(1 - \frac{1}{\mu_1}\Big) t_n \int g v_n \to +\infty, \tag{4.7}$$

which is a contradiction.

Now, let u_n be a minimizing sequence. For the Ekeland variational principle, we can suppose u_n to be a PS sequence. For the splitting lemma there exists a $u_g \in \mathcal{N}_g^V$ and k functions u^j, $1 \leq j \leq k$, such that

$$E_g^V(u_n) \to E_g^V(u_g) + \sum_{j=1}^k E_0^0(u^j) = m_g < 0. \tag{4.8}$$

We know that $E_0^0(u^j) \geq m_0 > 0$ for all j. So, if $k > 0$ we will have $E_g^V(u_n) \to m_g + \delta$ for some $\delta > 0$, and this is a contradiction.

So, we have

$$u_n \to u_g \text{ in } \mathcal{D}^{1,2}. \tag{4.9}$$

Furthermore, we have $E_g^V(u_g) = m_g < 0$, so $u_g \in \mathcal{N}_g^+$, and this concludes the proof of the existence.

To prove uniqueness, we argue by contradiction. If u_1 and u_2 are minimizers of E_g^V on $\mathcal{N}_g^+$, both u_1 and u_2 solve ($\mathcal{P}$), so we have

$$\|u_1 - u_2\|_V^2 = \int (f'(u_1) - f'(u_2))(u_1 - u_2) = \int f''(\theta u_1 + (1-\theta) u_2)(u_1 - u_2)^2$$

with $0 < \theta < 1$. So

$$\|u_1 - u_2\|_{L^{2^*}}^2 \leq C \|u_1 - u_2\|_V^2 \leq C \|u_1 - u_2\|_{L^{2^*}}^2 \|f''(\theta u_1 + (1-\theta) u_2)\|_{L^{\frac{2^*}{2^*-2}}}. \tag{4.10}$$

By Remark 2.3, we have that, if $g \to 0$ in $L^{\frac{2N}{N+2}}$, then both u_1 and u_2 are small in $L^p + L^q$, so we have that $f''(\theta u_1 + (1-\theta) u_2) \to 0$ in $L^{p/p-2} \cap L^{q/q-2}$ by Lemma 2.2, and, by interpolation,

$$\|f''(\theta u_1 + (1-\theta) u_2)\|_{L^{\frac{2^*}{2^*-2}}} \to 0,$$

which is a contradiction. $\square$

Proposition 4.2. *Suppose that $g \geq 0$. Then there exists an $u_g \geq 0$ in $\mathcal{N}_g^+$ such that $E_g^V(u_g) = m_g$.*

Proof. Take u_g as in Theorem 4.1. Because $u_g \in \mathcal{N}_g^+$ we have that $\int g u_g > 0$. If u_g changes sign, or u_g is negative, we have that

$$0 < \int g u_g \leq \int g|u_g|. \tag{4.11}$$

So, recalling that f is even we have

$$\begin{aligned} E_g^V(|u_g|) &= \frac{1}{2}\|u_g\|_V^2 - \int f(|u_g|) - \int g|u_g| \\ &\leq \frac{1}{2}\|u_g\|_V^2 - \int f(u_g) - \int g u_g = E_g^V(u_g). \end{aligned}$$

We know that there exists a τ such that $\tau|u_g| \in \mathcal{N}_g^+$. Furthermore, we know by the study of $\varphi_g^{|u_g|}$ that τ is a local minimizer of $\varphi_g^{|u_g|}$; in fact, $\varphi_g^{|u_g|}(\tau) \leq \varphi_g^{|u_g|}(t)$ for all $t \in [0, \tau]$. We have

$$\begin{aligned} \frac{d}{dt}\varphi_g^{|u_g|}(1) &= \frac{d}{dt}E_g^V(t|u_g|)_{|t=1} = \|u_g\|_V^2 - \int f'(|u_g|)|u_g| - \int g|u_g| \\ &\leq \|u_g\|_V^2 - \int f'(u_g)u_g - \int g u_g = \frac{d}{dt}E_g^V(tu_g)_{|t=1} = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{d^2}{dt^2}\varphi_g^{|u_g|}(1) &= \frac{d^2}{dt^2}E_g^V(t|u_g|)_{|t=1} = \|u_g\|_V^2 - \int f''(|u_g|)|u_g|^2 \\ &= \|u_g\|_V^2 - \int f''(u_g)u_g^2 = \frac{d^2}{dt^2}E_g^V(tu_g)_{|t=1} > 0. \end{aligned}$$

Thus, $\tau \geq 1$ and

$$E_g^V(\tau|u_g|) \leq E_g^V(|u_g|) \leq E_g^V(u_g) = m_g, \tag{4.12}$$

which concludes the proof. □

We want to prove that, under suitable hypotheses on g, f, and V, there exists another solution of $\mathcal{P}$, by minimizing the functional E_g^V on $\mathcal{N}_g^-$. In order to prove that a minimizing sequence converges we will show that, for g small,

$$m_{1,g} := \inf_{u \in \mathcal{N}_g^-} E_g^V(u) < m_g + m_0. \tag{4.13}$$

Lemma 4.3. *Suppose that $V \le 0$ and $V < 0$ on a set of positive measure. If $\|g\|_{L^{\frac{2N}{N-2}}}$ sufficiently small, then there exists a $\delta > 0$ such that*

$$m_{1,g} := \inf_{u \in \mathcal{N}_g^-} E_g^V(u) < m_0 - \delta. \tag{4.14}$$

Moreover,

$$\limsup_{\|g\|_{L^{\frac{2N}{N-2}}} \to 0} m_{1,g} \le m_V. \tag{4.15}$$

Proof. By [5, Lemma 4.4(a)] and [5, Theorem 1.1] we know that there exists a $\bar{u} \in \mathcal{N}_0^V$ such that

$$E_0^V(\bar{u}) = \inf_{u \in \mathcal{N}_0^V} E_0^V(u) = m_V < m_0.$$

We set $v = \frac{\bar{u}}{\|\bar{u}\|_V}$, so $\bar{u} = t_0^v v$. We know that there exists $t_1^v = t_1^v(g)$ such that $t_1^v v \in \mathcal{N}_g^-$ by Proposition 2.7. Furthermore, by Proposition 2.7 we have that $t_1^v \to t_0^v$ when $\|g\|_{L^{\frac{2N}{N-2}}} \to 0$, and so

$$m_{1,g} \le E_g^V(t_1^v v) \to E_0^V(\bar{u}) = m_V < m_0 \text{ for } \|g\|_{L^{\frac{2N}{N+2}}} \to 0, \tag{4.16}$$

which concludes the proof. □

Theorem 4.4. *For $\|g\|_{L^{\frac{2N}{N-2}}} \to 0$ there exists $u_{1,g} \in \mathcal{N}_g^-$ a solution of ($\mathcal{P}$). Furthermore, if $g \ge 0$ the solution $u_{1,g}$ can be chosen positive.*

Proof. By the splitting lemma, to obtain the result it is enough to show that $m_{1,g} < m_g + m_0$. In the previous lemma, we have proved that there exists a $\delta > 0$ such that $m_{1,g} < m_0 - \delta$ for $\|g\|_{L^{\frac{2N}{N-2}}}$ sufficiently small. By Remark 2.7 we have also that $m_g \to 0$ when $g \to 0$ in $L^{\frac{2N}{N+2}}$. So there exists $u_{1,g} \in \mathcal{N}_g^V$ a solution of ($\mathcal{P}$). Moreover, $E_g^V(u_{1,g})$ is positive, so $u_{1,g} \in \mathcal{N}_V^-$.

To prove the last claim, consider that $E_g^V(|u_{1,g}|) \le E_g^V(u_{1,g})$. Also, there exists a $\bar{t}$ such that $\bar{t}|u_{1,g}| \in \mathcal{N}_g^-$. Then we have

$$m_{1,g} = E_g^V(u_{1,g}) = \max_t E_g^V(tu_{1,g}) \ge E_g^V(\bar{t}u_{1,g}) \ge E_g^V(\bar{t}|u_{1,g}|). \tag{4.17}$$

So if $u_{1,g}$ is a solution, also $\bar{t}|u_{1,g}| \in \mathcal{N}_g^-$ is a solution of ($\mathcal{P}$). □

Proposition 4.5. *If $\|g\|_{L^{p'} \cap L^{q'}} \to 0$, then $m_{1,g} \to m_V$.*

Proof. We take a sequence of $g_n \to 0$ in $L^{p'} \cap L^{q'}$. We know that for any g_n there exists u_{1,g_n} such that $E^V_{g_n}(u_{1,g_n}) = m_{1,g_n}$. For simplicity we call $u_n = u_{1,g_n}$. Also, we set $v_n = \frac{u_n}{\|u_n\|_{L^p+L^q}}$, and $u_n = t_n v_n$. We have

$$E^V_{g_n}(u_n) = t_n\Big[\frac{1}{2}\int f'(t_n v_n)v_n - \int \frac{f(t_n v_n)}{t_n} - \frac{1}{2}\int g_n v_n\Big], \tag{4.18}$$

and we have that there exist a $\delta > 0$ such that $0 \le E^V_{g_n}(u_n) \le m_v + \delta$. Now, suppose, for the sake of contradiction, that $t_n \to \infty$. Then,

$$\frac{1}{2}\int f'(t_n v_n)v_n - \int \frac{f(t_n v_n)}{t_n} - \frac{1}{2}\int g_n v_n \to 0, \tag{4.19}$$

and so

$$\frac{1}{2}\int f'(t_n v_n)v_n - \int \frac{f(t_n v_n)}{t_n} \to 0. \tag{4.20}$$

By (f_μ), we have that

$$\begin{aligned}&\int f'(t_n v_n)v_n - 2\int \frac{f(t_n v_n)}{t_n}\\ &= \int f'(t_n v_n)v_n - \mu_1 \int \frac{f(t_n v_n)}{t_n} + (\mu_1 - 2)\int \frac{f(t_n v_n)}{t_n}\\ &\ge (\mu_1 - 2)\int \frac{f(t_n v_n)}{t_n}.\end{aligned}$$

So $\int \frac{f(t_n v_n)}{t_n} \to 0$. Now by the hypothesis on f

$$0 \le c_0 t_n^{p-1}\Big[\int_{|v_n>1|} |v_n|^p + \int_{|v_n<1|} |v_n|^q\Big] \le \int \frac{f(t_n v_n)}{t_n} \to 0, \tag{4.21}$$

so we have that both $\int_{|v_n|>1} |v_n|^p$ and $\int_{|v_n|<1} |v_n|^q$ vanish when $n \to \infty$, and so

$$1 = \|v_n\|_{L^p+L^q} \le \max\Big\{\int_{|v_n|>1} |v_n|^p, \int_{|v_n|<1} |v_n|^q\Big\} \to 0 \tag{4.22}$$

which is a contradiction. Furthermore, by Proposition 2.7, we have t_n bounded away from 0. So, we have that there exist two positive constants c_1 and c_2 such that

$$0 < c_1 \le t_n = \|u_n\|_{L^p+L^q} \le c_2 < \infty. \tag{4.23}$$

Now, let τ_n be such that $\tau_n u_n \in \mathcal{N}^V_0$. We can show that $\tau_n \to 1$ when $n \to \infty$. The main idea is that

$$\frac{d}{dt}\varphi^{u_n}_{g_n}(\tau_n) - \frac{d}{dt}\varphi^{u_n}_0(\tau_n) = \int g_n u_n \to 0, \tag{4.24}$$

because $\|u_n\|_{L^p+L^q}$ is bounded and $g_n \to 0$ in $L^{p'} \cap L^{q'}$. The details are omitted for the sake of simplicity. Now we have that

$$E_0^V(\tau_n u_n) - E_0^V(u_n) \to 0. \tag{4.25}$$

We have that $E_{g_n}^V(u_n)$ is bounded, so, up to subsequences, there exists a d such that $E_{g_n}^V(u_n) \to d$ when $n \to \infty$, and, because u_n is bounded in L^p+L^q, also $E_0^V(u_n) \to d$, and, by (4.25), $E_0^V(\tau_n u_n) \to d$.

So, $d \geq m_V$. By Lemma 4.3 we know also that $d \leq m_V$, so we get the claim. □

Proof of Theorem 1. By Theorems 4.1 and 4.4, we have that there exists a $u_g \in \mathcal{N}_g^+$ and $u_{1,g} \in \mathcal{N}_g^-$ that solve $(\mathcal{P})$. Furthermore, by Theorem 4.4 and Proposition 4.2 the solution can be chosen nonnegative. Finally, by Remark 2.3 we have that $u_g \to 0$ in $\mathcal{D}^{1,2}$, and by Proposition 4.5 that $m_{1,g} \to m_V$ when $g \to 0$. □

APPENDIX A. THE HYPOTHESES ON f

We want to prove that there exists a function that satisfies all the conditions required in the Introduction.

We take the function

$$f(s) = \frac{|s|^q}{1+|s|^{q-p}}. \tag{A.1}$$

This function is even, and it satisfies (f_0). We have that, for $s > 0$

$$f'(s) = \frac{qs^{q-1} + ps^{2q-p-1}}{(1+s^{q-p})^2},$$

$$f''(s) = \frac{s^{q-2}}{(1+s^{q-p})^2}\left\{q(q-1)+p(2q-p-1)s^{q-p} - \frac{2(q-p)(q+ps^{q-p})s^{q-p}}{1+s^{q-p}}\right\}.$$

It is easy to see that f satisfies (f_2) and the first part of (f_μ).

We set $\mu_2 = 1+\varepsilon > 1$; then the inequality $(1+\varepsilon)f'(s)s < f''(s)s^2$ becomes

$$(q^2 - 2q - \varepsilon q) + p(2q - p - 2 - \varepsilon)\gamma - \frac{2(q-p)(q+p\gamma)\gamma}{1+\gamma} > 0,$$

where $\gamma = s^{q-p}$. So, we have to prove that

$$q(q-2-\varepsilon) + [p(2q-p-2-\varepsilon) + q(2p-q-2-\varepsilon)]\gamma + p(p-2+\varepsilon)\gamma^2 > 0.$$

Obviously we can choose ε such that $q(q-2-\varepsilon) > 0$ and $p(p-2+\varepsilon) > 0$. Furthermore, we choose $q-p$ sufficiently small such that also $2q-p-2-\varepsilon$ and $2p-q-2-\varepsilon$ are positive, so the second part of (f_μ) is proved.

At last we prove (f_3) and that $f'''(s)s^3 > 0$. We have that, for $s > 0$,

$$\begin{aligned} f'''(s) = & \frac{6(p-q)^3 s^{4q-3p-3}}{(1+s^{q-p})^4} - \frac{6(1+p-2q)(p-q)^2 s^{3q-2p-3}}{(1+s^{q-p})^3} \\ & + \frac{(2p+3p^2+p^3-2q-12pq-6p^2q+9q^2+12pq^2-7q^3)s^{2q-p-3}}{(1+s^{q-p})^2} \\ & + \frac{q(2-3q+q^2)s^{q-3}}{1+s^{q-p}}. \end{aligned}$$

We obtain that

$$f'''(s)s^3 = \frac{As^q}{1+s^{q-p}} + \frac{Bs^{2q-p}}{(1+s^{q-p})^2} + \frac{Cs^{3q-2p}}{(1+s^{q-p})^3} + \frac{Ds^{4q-3p}}{(1+s^{q-p})^4},$$

where

$$\begin{array}{ll} A = q(q-2)(q-1); & B = (p-q)(2+3p+p^2-9q-5pq+7q^2); \\ C = 6(p-q)^2(2q-p-1); & D = 6(p-q)^3. \end{array}$$

We can choose $q-p$ sufficiently small, in order to have $B, C, D << A$. Now, set as above $\gamma = s^{q-p}$; we have

$$f'''(s)s^3 = \frac{s^q\left[A + (3A+B)\gamma + (3A+2B+C)\gamma^2 + (A+B+C+D)\gamma^3\right]}{(1+s^{q-p})^4},$$

which is positive for all $s > 0$. So (f_μ) is completely proved.

Furthermore, we have that

$$\lim_{s\to 0^+} \frac{f'''(s)}{s^{q-3}} = A = q(q-1)(q-2) > 0 \tag{A.2}$$

and

$$\lim_{s\to +\infty} \frac{f'''(s)}{s^{p-3}} = A+B+C+D = p(p-1)(p-2) > 0. \tag{A.3}$$

So, there exists a $c_3 > 0$ such that

$$\left\{ \begin{array}{ll} |f'''(s)| \le c_3|s|^{p-3} & \text{for } |s| \ge 1; \\ |f'''(s)| \le c_3|s|^{q-3} & \text{for } |s| \le 1. \end{array} \right. \tag{A.4}$$

Now, let $\Gamma = \{x \in \mathbb{R}^N : |u(x)| > 1\}$ and $\Delta = \mathbb{R}^N \setminus \Gamma$. We have that

$$\begin{aligned} \int f'''(u)u^3 \; & \le \; \int_\Gamma f'''(u)u^3 + \int_\Delta f'''(u)u^3 \le c_3 \int_\Gamma |u|^p + c_3 \int_\Delta |u|^q \\ & \le \; C_1 + C_2\|u\|_{L^p+L^q} \le C_3 + C_4\|u\|_{\mathcal{D}^{1,2}} < \infty, \end{aligned}$$

and this proves (f_3).

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