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ILL-POSEDNESS AND THE NONEXISTENCE OF STANDING-WAVES SOLUTIONS FOR THE NONLOCAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We establish some properties for the Cauchy problem associated with the nonlocal nonlinear Schrödinger equation $\partial_t u = -i\alpha\partial_x^2 u + \beta u\partial_x(|u|^2) - i\beta u\mathcal{T}_h\partial_x(|u|^2) + i\gamma|u|^2 u$, where $x, t \in \mathbb{R}$, $\mathcal{T}_h$ is the nonlocal operator

$$\mathcal{T}_h u(x) = \frac{1}{2h} p.v. \int_{-\infty}^{\infty} \coth\Big(\frac{\pi(y-x)}{2h}\Big) u(y)dy,$$

with $\alpha > 0$, $\beta \geq 0$, $\gamma \geq 0$, and $h \in (0, +\infty)$. Here $\mathcal{T}_h \longrightarrow \mathcal{H}$ when $h \longrightarrow +\infty$, where $\mathcal{H}$ is the Hilbert transform. We prove rigorously that a Picard interaction scheme can not be applied for solving the Cauchy problem associated with that equation in both the cases $0 < h < \infty$ and $h \to +\infty$, with initial data in Sobolev spaces of negative index. Elsewhere, we study the asymptotic behavior of the solution in relation to a spatial variable, and we also establish the nonexistence of a standing-waves solution for the above equation in several cases.

1. Introduction

Since the decade of 1960 the problem of self-modulation of small-amplitude nonlinear waves has been intensively studied. The cubic nonlinear Schrödinger equation

$$\partial_t u = i\partial_x^2 u + i\gamma|u|^2 u, \tag{1.1}$$

where $u = u(x,t)$ is a complex function and $x, t, \gamma \in \mathbb{R}$, has been used as a universal model for description of the long-term evolution of the envelope

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wave. An important case, the so-called *shallow-deep* limit of a stratified fluid (that is, the upper layer is shallow and the lower one is deep if compared to the length scale of quasi-harmonic wave packets) cannot be described by the cubic NLS equation. So, the problem of deriving a kind of NLS equation modeling quasi-harmonic wave packets arises. D. Pelinovsky [24] gave the first step in solving it, proposing the following equation,

$$i\partial_t u = \partial_x^2 u + iu\,(1 - i\mathcal{T}_h)\,\partial_x(|u|^2), \quad x, t \in \mathbb{R}, \tag{1.2}$$

where $u = u(x,t)$ is a complex function representing the envelope (packet) of the waves, $\mathcal{T}_h$ is a singular integral operator defined by

$$\mathcal{T}_h u(x) = \frac{1}{2h} p.v. \int_{-\infty}^{\infty} \coth\Big(\frac{\pi(y-x)}{2h}\Big) u(y)\,dy, \tag{1.3}$$

with $p.v.$ denoting the principal value of the integral and h denoting a parameter proportional to the depth of the fluid.

Equation (1.2) governs evolution of waves of first order in the shallow-deep limit. It is nonlocal and it has been referred to as the intermediate NLS equation (INLS), intermediate because in the shallow-water limit $h \longrightarrow 0$, equation (1.2) reduces to a defocusing unidimensional NLS

$$i\partial_{\widetilde{t}} u = \partial_{\widetilde{x}}^2 u - \widetilde{u}(|\widetilde{u}|^2 - \rho^2), \qquad \widetilde{u} = \widetilde{u}(\widetilde{x}, \widetilde{t}), \tag{1.4}$$

after redefining x, t and u by the relations $x = \sqrt{h}\widetilde{x}$, $t = h\widetilde{t}$ and $u(x,t) = \widetilde{u}(\widetilde{x}, \widetilde{t})$, respectively. Here the expansion $\mathcal{T}_h \partial_x f = -h^{-1} f + O(h)$ plus the boundary condition $\lim_{|x|\to\infty} |u| = \rho$ was used. In the deep-water limit $h \longrightarrow \infty$ equation (1.2) reduces to the nonlocal NLS equation

$$i\partial_t u = \partial_x^2 u + u(i + \mathcal{H})\partial_x(|u|^2), \quad x, t \in \mathbb{R} \tag{1.5}$$

(cf. [14]) where $u = u(x,t)$ is a complex-valued function, and $\mathcal{H}$ is the Hilbert transform defined by

$$\mathcal{H}u(x,t) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{u(y,t)}{y-x}\,dy. \tag{1.6}$$

Also in [24] it was showed that the INLS equation (1.2) is integrable. Its inverse scattering was constructed in [26].

We note that second-order effects in the long-wave expansion are not described by the INLS equation, while they might influence the behavior of quasi-harmonic internal waves. Taking into account these second-order effects, D. Pelinovsky and R. H. J. Grimshaw (see [25]) proposed a universal evolution equation of the NLS type in the shallow-deep limit of a continuous stratified fluid. The equation established by them is

$$i\partial_t u = \alpha\partial_x^2 u + \beta u\big(i + \mathcal{T}_h\big)\partial_x(|u|^2) - \gamma|u|^2 u, \quad x, t \in \mathbb{R}, \tag{1.7}$$

where the *positive* coefficients α, β, and γ are expressed by the parameter of the fluid stratification (see Appendix A from [25]).

Our aim in this paper is to establish some properties mainly related to local well-posedness for the initial-value problems (IVP) associated with the equations (1.7) when $0 < h < +\infty$, and in the deep-water limit $h \longrightarrow \infty$,

$$i\partial_t u = \alpha\partial_x^2 u + \beta u\big(i + \mathcal{H}\big)\partial_x(|u|^2) - \gamma|u|^2 u, \quad x, t \in \mathbb{R}. \tag{1.8}$$

We also investigate the behavior of the solutions in weighted Sobolev spaces and the existence of standing-waves solutions.

Properties such as obtaining of multi-periodic and multi-soliton solutions, linear stability for dark solitary waves solutions, generation of a dark soliton, the linear spectral problem, asymptotic solutions, and the Cauchy problem via inverse scattering for equation (1.8) (mainly with $\gamma = 0$) have been studied, specifically by Y. Matsuno (cf. [11, 12, 13, 14, 15, 16, 17, 18, 19]) and by D. Pelinovsky/Grimshaw [24, 26]. In [22], based on the smoothing effects produced by the linear Schrödinger equation, we proved local well-posedness for the Cauchy problem associated with equations (1.7) and (1.8) with *small* initial data in $H^s(\mathbb{R})$, $s \geq 1$, and using appropriate conservation laws we were able to prove global well-posednesss for *any* data in $H^1(\mathbb{R})$.

Equation (1.7) (with $\alpha = \beta = 1$ and $0 < h < +\infty$ or $h \to +\infty$) is *gauge equivalent* to the equation

$$\partial_t v = -i\partial_x^2 v + \frac{i}{4}|v|^4 v - iv\mathcal{T}_h\partial_x(|v|^2) + i\gamma|v|^2 v, \tag{1.9}$$

where

$$v = e^{\frac{i}{2}\rho}u \quad \text{and} \quad \rho(x,t) = \int_{-\infty}^{x} |u(y,t)|^2 dy \tag{1.10}$$

(see [22] and [23]). Unfortunately, even after the change of variable (1.10) the problem of loss of regularity persists in equation (1.9); for that reason we do not know how to prove local well-posedness in Sobolev spaces $H^s(\mathbb{R})$ with $s < 1$. Due to the nonlinear term $u\mathcal{T}_h(\bar{u}\partial_x u)$ of (1.9), it seems that the method employed by H. Takaoka is not applicable for solving the well-posedness for the IVP associated to (1.9). But equation (1.9) will help us to prove the nonexistence of standing-waves solutions, because it permits us to obtain a convenient Pohozaev's identity.

The work is organized as follows. In Section 2 we exhibit the main well-posedness results reached up to now for the initial-value problems (IVP) associated with equations (1.7) and (1.8). The smoothing effects determined by the group $\{e^{-it\partial_x^2}\}_{t=-\infty}^{\infty}$, estimates for the maximal function $\sup_{[0,T]}|e^{-it\partial_x^2}\cdot|$ (cf. [7, 8, 9, 10]) and Strichartz estimates (cf. [29]) were our main tools. In Section 3, based on the works of Rial [28], Kato [6] and Iorio [5], we study the asymptotic behavior in relation to the spatial variable of the solution guaranteed by the Theorem 2.1, by solving the equation in Sobolev spaces with weight. Section 4 is devoted to showing that a Picard interaction scheme cannot be applied to the associated integral equation for both the IVP for equations (1.7) and (1.8) (for the cases $\gamma > 0$ and $\gamma = 0$) with data in usual Sobolev spaces $H^s(\mathbb{R})$, $s < 0$, which allow us to see that the flow map cannot be of class C^3 in $H^s(\mathbb{R})$ with $s < 0$. Bourgain was the creator of the idea followed by us (cf. [4]). Similar approaches were made by L. Molinet, J. C. Saut and N. Tzvetkov for the KdV, BO and KP equations (cf. [20, 21]), etc. Scaling arguments suggest that the critical value for the IVP associated with the equation (1.7) with $\gamma = 0$, may be $s = 0$ (cf. [22]). Since the case $\gamma > 0$ does not offer any obstacle, we imagine that we can think the same about the equation (1.7) for any $\gamma \geq 0$. We shall confirm that fact.

The final section is devoted to showing the nonexistence of a classical standing-wave solution $u(x,t) = e^{i\omega t}\varphi(x)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function which decays rapidly to zero at infinity, for equation (1.8) for every $\omega \in \mathbb{R}$, and for equation (1.9) provided that $\omega \leq 0$. The main idea is to show that solutions of equation (1.9) satisfies a convenient Pohozaev's identity. Since the singular integral operator $\mathcal{T}_h$ does not seem to inherit all the Hilbert transform properties (particularly, that from the Lemma 5.1 below), the case $\omega > 0$ for equation (1.9) when $0 < h < +\infty$ is opened.

Before beginning our task we give some notation used in the work. By $\mathcal{F}\{u\}$ or $\hat{u}$ we will denote the Fourier transform of a function with respect to a space variable x. For $1 \leq p \leq \infty$, $p \neq 2$, we denote the norm of $L^p(\mathbb{R})$ by $|\cdot|_p$. The L^2-norm will be written as $\|\cdot\|$ or $\|\cdot\|_0$. $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space. By J^s we denominate the Bessel's potential, $J^s = (1-\partial_x^2)^{s/2}$, and D_x^s denotes the Riesz potential, $D_x^s = (-\partial_x^2)^{s/2}$. $H^s(\mathbb{R})$ denotes the usual Sobolev space with norm $\|\cdot\|_s = \|J^s\cdot\|_0 := \|J^s\cdot\|$. For $1 \leq p, q < \infty$ and $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ we define

$$\|f\|_{L_x^p L_T^q} = \Big(\int_{-\infty}^{\infty}\Big(\int_0^T |f(x,t)|^q dt\Big)^{p/q} dx\Big)^{1/p};$$

$\|f\|_{L^q_T L^p_x}$ is defined of similar manner, and when $p=\infty$ or $q=\infty$, $\|f\|_{L^p_x L^q_T}$ is defined in the natural form. The commutator of an operator K with a function u will be denoted by $[K,u]v := K(uv) - uK(v)$. Given any positives constants B and D, by $C \lesssim D$ (respectively $C \gtrsim D$) we mean that there exists a constant $c>0$ such that $C \le cD$ (respectively $C \ge cD$). We denote $C \lesssim D \lesssim C$ by $C \sim D$. $B(X)$ will represent the space of bounded linear operators on X. We shall denote the set of k-continuously differentiable maps from X to Y by $C^k(X,Y)$, and $C^k_w(X,Y)$ will be the space of that one with k weakly continuous derivatives.

2. Well-posedness results

Consider the IVP for the equations (1.7)–(1.8)

$$\begin{cases} \partial_t u = -i\alpha\partial_x^2 u + \beta u \partial_x(|u|^2) - i\beta u \mathcal{T}_h \partial_x(|u|^2) + i\gamma|u|^2 u, & x,t \in \mathbb{R} \\ u(x,0) = \phi(x). \end{cases} \tag{2.1}$$

The following results related to the IVP (2.1) with $\alpha>0$, $\beta\ge 0$ and $\gamma \ge 0$, for $0<h<\infty$ and $h\to\infty$ (where $\mathcal{H}$ is the Hilbert transform), were proved in [22]:

Theorem 2.1. *There exists a $\delta>0$ such that for any $\phi \in H^s(\mathbb{R})$ with $s\ge 1$ and $\|\phi\|_s < \delta$, there is a unique solution u for the IVP (2.1) in the interval of time $[0,T]$, $T = T(\|\phi\|_s) > 0$ with $T(\|\phi\|_s)\to\infty$ when $\|\phi\|_s \to 0$, satisfying*

$$u \in Z_{s,T} \equiv \{v \in C([0,T], H^s(\mathbb{R})) : \Lambda_T(v) < \infty\},$$

with

$$\Lambda_T(u) = \max\{\lambda_T^i(u) \equiv \lambda_i(u);\ i = 1, 2, \ldots, 5\}, \tag{2.2}$$

where

$$\lambda_1(u) := \sup_{[0,T]} \|u(t)\|_s, \qquad \lambda_2(u) := \|D_x^{s+\frac{1}{2}} u\|_{L^\infty_x L^2_T},$$
$$\lambda_3(u) := (1+T)^{-1}\|u\|_{L^2_x L^\infty_T}, \qquad \lambda_4(u) := \|D_x^s u\|_{L^6_x L^6_T}.$$

Furthermore, for any $T' \in (0,T)$ there exists $\epsilon>0$ such that the map $\varphi \to u$ from $\{\psi \in H^s(\mathbb{R}) : \|\psi-\phi\|_s < \epsilon\}$ into $Z_{s,T}$ is Lipschitz.

Corollary 2.1. *Theorem 2.1 also holds if $\gamma = 0$. In this case, it is enough to consider $\|\phi\| < \delta$ in place of $\|\phi\|_s < \delta$, that is, that a local well-posedness result holds also if we just assume that $\|\phi\|$ is small.*

The next corollary follows immediately from Theorem 2.1.

Corollary 2.2. *Consider the IVP*

$$\begin{cases} \partial_t v = -i\partial_x^2 v + \frac{i}{4}|v|^4 v - iv\mathcal{T}_h\partial_x(|v|^2) + i\gamma|v|^2 v, \quad x,t \in \mathbb{R} \\ v(x,0) = \psi(x) = \exp\{\frac{i}{2}\int_{-\infty}^{x}|\phi(y)|^2 dy\}\,\phi(x). \end{cases} \tag{2.3}$$

Then the IVP (2.3) *is locally well-posed in the sense of Theorem* 2.1; *that is, with small enough initial data in* $H^s(\mathbb{R})$, $s \geq 1$.

Since the proof of Theorem 2.1 was based on the contraction mapping principle, it was possible to extend the result of continuous dependence. The main proof ingredient was the implicit function theorem. Let us see:

Corollary 2.3. *Under the hypotheses of Theorem* 2.1, *we have that there exists a neighborhood* $\widetilde{V}$ *of* $\phi \in H^s(\mathbb{R})$, $s \geq 1$, *such that the map* $\phi \longmapsto u$ *from* $\widetilde{V}$ *into* $Z_{s,T}$ *is smooth.*

Theorem 2.2. *Let* $\phi \in H^1(\mathbb{R})$. *Then the solution* u *asserted by Theorem* 2.1 *can be extended to any time interval* $[0,T]$.

3. The problem with weight

To study the asymptotic behavior of the solutions obtained in Theorem 2.1 in relation to the spatial variable is our purpose here. First we need some definitions and notation.

We start by defining the Hilbert spaces

$$L^2_\omega(\mathbb{R}) := \left\{u : \mathbb{R} \to \mathbb{C} \text{ measurable } : \langle u,v\rangle_{L^2_\omega} := \langle \omega u, \omega v\rangle < \infty\right\}, \tag{3.1}$$

where

$$\langle f,g\rangle := Re\int_{-\infty}^{\infty} f\,\overline{g}\,dx.$$

Definition 3.1. *Let* $s \in \mathbb{R}$ *and let* ω *be an arbitrary weight. We define the Hilbert space* $H_{s,\omega}(\mathbb{R}) \equiv H^s(\mathbb{R}) \cap L^2_\omega(\mathbb{R})$ *with the inner product* $\langle u,v\rangle_{s,\omega} = \langle u,v\rangle_s + \langle u,v\rangle_{L^2_\omega}$.

We will make use of the following set of weights:

$$\mathcal{P} := \left\{\sigma \in C^1(\mathbb{R},\mathbb{R}) : \inf_{x\in\mathbb{R}} \sigma(x) > 0,\ \sup_{x\in\mathbb{R}} |\sigma'(x)| < \infty\right\}. \tag{3.2}$$

With these definitions and notation in hand we are able to enunciate the theorem.

Theorem 3.1. *Let $s > \frac{3}{2}$ and $\omega = \sigma^s$, $\sigma \in \mathcal{P}$. If $\phi \in H_{s,\omega}(\mathbb{R})$ and $u \in C([0,T];\ H^s(\mathbb{R}))$ is the solution of the IVP (2.1) with $u(0) = \phi$ and $0 < h < \infty$ or $h \to \infty$ ($\mathcal{T}_h = \mathcal{H}$), then*

$$u \in C([0,T];\ H_{s,\omega}(\mathbb{R})). \tag{3.3}$$

Furthermore, the map $\phi \longmapsto u$ is continuous.

Remark 3.1. The hypothesis of smallness on the data is not necessary in Theorem 3.1.

As a consequence of this theorem and results which will be proved below, we guarantee that for $k \leq s$

$$\omega^{1-k/s}\partial_x^k u \in C([0,T];\ L^2(\mathbb{R})). \tag{3.4}$$

3.1. **Preliminary results.** The proposition below exhibits a relation between differentiability and decreasing of $H_{s,\omega}$-functions (similar results for other problems can be found in Rial [28], Kato [6] Theorem A.7 and Iorio [5] Theorem A.2).

Proposition 3.1. *For each $s > 0$ and $V \geq 1$ there exists $C = C(s,V) \geq 0$ such that, for any weight satisfying*

$$\omega(y) \leq V\omega(x), \quad x, y \in \mathbb{R}, \quad |x-y| \leq 1 \tag{3.5}$$

we have the following estimates, for $k \in \mathbb{Z}$, $0 \leq k \leq s$:

$$\|\omega^{1-k/s}\partial_x^k u\| \leq c\,\|u\|_{L^2_\omega}^{1-k/s}\,\|u\|_s^{k/s}. \tag{3.6}$$

Proof. Let $\theta = \frac{k}{s}$, $0 \leq k \leq s$. Then

$$\|\omega^{1-\theta}\partial_x^k u\|^2 = \sum_{j\in\mathbb{Z}} \int_j^{j+1} \omega(x)^{2(1-\theta)}|\partial_x^k u|^2 dx. \tag{3.7}$$

(3.5) implies $\omega(x) \leq V\omega(j)$, for $x \in [j, j+1]$; thus, from (3.7) follows that

$$\|\omega^{1-\theta}\partial_x^k u\|^2 \leq V^{2(1-\theta)} \sum_{j\in\mathbb{Z}} \omega(j)^{2(1-\theta)} \int_j^{j+1} |\partial_x^k u|^2 dx. \tag{3.8}$$

Consider a $\psi \in C^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$, with $\text{supp}(\psi) \subset (-1,2)$ and $\psi \equiv 1$ in $[0,1]$. Taking $j \in \mathbb{Z}$, defining $\psi_j(\cdot) = \psi(\cdot - j)$, since $\psi_j \equiv 1$ in $[j, j+1]$, using interpolation between Sobolev spaces, Hölder's inequalities and Proposition 2.3 of [27] (a localized version of the H^s norm), it follows that

$$\|\omega^{1-\theta}\partial_x^k u\|^2 \leq c(s)V^{2(1-\theta)}\Big(\sum_{j\in\mathbb{Z}} \omega(j)^2 \parallel \psi_j u \parallel^2\Big)^{1-\theta} \parallel u \parallel_s^{2\theta}. \tag{3.9}$$

From (3.5) we deduce

$$\sum_{j\in\mathbb{Z}} \omega(j)^2 \int_{-\infty}^{\infty} |\psi_j u|^2 dx \quad \leq \quad 3V^4 \parallel u \parallel_{L^2_\omega}^2 . \tag{3.10}$$

From (3.9)–(3.10) follows the result. □

Now, if $\sigma \in P$, $\sigma \geq \sigma_0 > 0$ (σ_0 is a positive constant), and $|\sigma'| \leq M$, then by the mean value theorem, given $|x-y| \leq 1$ we have

$$\sigma(y) \quad \leq \quad (1 + M\sigma_0^{-1})\sigma(x), \tag{3.11}$$

and so $\omega = \sigma^s$ yields

$$\omega(y) \leq e^{sM/\sigma_0}\omega(x). \tag{3.12}$$

Thus, the weights in the Theorem 3.1 satisfy (3.5).

For controlling the nonlocal term we use the following vector-valued Leibniz rule for fractional derivatives deduced by C. Kenig, G. Ponce and L. Vega in [8], Theorem A.12.

Lemma 3.1. *Let $\alpha \in (0,1)$. Then, for $1 < p < \infty$, it holds that*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p} \leq c\,\|g\|_{L_x^\infty}\,\|D_x^\alpha f\|_{L_x^p} . \tag{3.13}$$

(3.13) *is still valid with $\mathcal{H}D_x^\alpha$ in place of D_x^α, where $\mathcal{H}$ is the Hilbert transform.*

The next lemma is of fundamental importance in the proof of the continuous dependence of a solution in relation to initial data.

Lemma 3.2. *Let $s > \frac{3}{2}$ and $\omega = \sigma^s$ with $\sigma \in \mathcal{P}$, and let $R > 0$. There exists $c = c(s, \sigma, R) > 0$ such that, if $u, v \in B_{s,\omega}(0, R)$, then*

$$\parallel \Gamma_j(u)\nu - \Gamma_j(v)\nu \parallel_{L^2_\omega} \leq c \parallel u - v \parallel_{s,\omega} \parallel \nu \parallel_{s,\omega}, \quad j = 1, \ldots, 4, \tag{3.14}$$

where

$$\Gamma_1(u)v = i|u|^2 v, \qquad \Gamma_2(u)v = |u|^2\partial_x v,$$
$$\Gamma_3(u)v = u^2\partial_x\overline{v}, \qquad \Gamma_4(u)v = -2iu\mathcal{H}\Re(\overline{u}\partial_x v).$$

$\widetilde{\Gamma}_4 = -2iu\mathcal{T}_h\Re(\overline{u}\partial_x v)$ *also satisfies* (3.14), *where $\mathcal{T}_h$ is defined in* (1.3).

Proof. By Hölder's inequality it follows that

$$\begin{aligned} &\parallel \Gamma_4(u)\nu - \Gamma_4(v)\nu \parallel_{L^2_\omega} = 2\|(u\mathcal{H}\Re(\bar{u}\partial_x\nu) - v\mathcal{H}\Re(\bar{v}\partial_x\nu)\|_{L^2_\omega} \qquad (3.15)\\ &\leq 2\Big(\|(u-v)\mathcal{H}\Re(\bar{u}\partial_x\nu)\|_{L^2_\omega} + \|v\mathcal{H}\Re((\bar{u}-\bar{v})\partial_x\nu))\|_{L^2_\omega}\Big)\\ &\leq c\Big(|\mathcal{H}\Re(\bar{u}\partial_x\nu)|_\infty\|u-v\|_{L^2_\omega} + |\mathcal{H}\Re((\bar{u}-\bar{v})\partial_x\nu)|_\infty\|v\|_{L^2_\omega}\Big). \end{aligned}$$

By a Sobolev embedding from $H^{s-1}(\mathbb{R})$ $(2 > s > 3/2)$ in $L^\infty(\mathbb{R})$ and Lemma 3.1, we obtain

$$\begin{aligned}|\mathcal{H}\Re(\bar{u}\partial_x\nu)|_\infty &\lesssim \|D_x^{s-1}\mathcal{H}\Re(\bar{u}\partial_x\nu)\| + \|\bar{u}\partial_x\nu\| = \|D_x^{s-1}(\bar{u}\partial_x\nu)\| + \|\bar{u}\partial_x\nu\| \\ &\lesssim \|D_x^{s-1}(\bar{u}\partial_x\nu) - \bar{u}D_x^{s-1}\partial_x\nu - \partial_x\nu D_x^{s-1}\bar{u}\| + \|\bar{u}D_x^{s-1}\partial_x\nu\| \\ &\qquad + \|\partial_x\nu D_x^{s-1}\bar{u}\| + |u|_\infty\|\partial_x\nu\| \\ &\lesssim |D_x^{s-1}u|_\infty\|D_x\nu\| + |u|_\infty\|D_x^s\nu\| \lesssim \|u\|_s\|D_x^2\nu\|. \end{aligned} \tag{3.16}$$

An inequality similar to (3.16) for the case $s \geqq 2$ follows from the embedding $|\mathcal{H}\Re(\bar{u}\partial_x\nu)|_\infty \lesssim \|D_x^\beta(\bar{u}\partial_x\nu)\| + \|\bar{u}\partial_x\nu\|$, for $\frac{1}{2} < \beta < 1$, and from Lemma 3.1. Therefore the estimates

$$|\mathcal{H}\Re((\bar{u}-\bar{v})\partial_x\nu)|_\infty \lesssim \|u-v\|_s\|\nu\|_s, \tag{3.17}$$

(3.15) and (3.16), imply (3.14) for $j = 4$. The procedure for estimating Γ_1, Γ_2 and Γ_3 is similar and easier. To see that $\widetilde{\Gamma}_4$ satisfies (3.14), it is enough to remark that $\mathcal{T}_h$ is a bounded operator on L^p, $1 < p < \infty$ (see [22], Lemma 3.1), it commutes with derivatives, and therefore (3.15)–(3.16) are also valid for $\widetilde{\Gamma}_4$. □

Let $\sigma \in \mathcal{P}$, $\sigma_0 = \inf\{\sigma(x) : x \in \mathbb{R}\}$ and $M = \sup_{x\in\mathbb{R}} |\sigma'(x)|$. For each $n \in \mathbb{N}$, we have that

$$\sigma_n := \frac{n\sigma}{n+\sigma} \in \mathcal{P} \tag{3.18}$$

and satisfies

$$\inf_{x\in\mathbb{R}} \sigma_n(x) \geq \frac{\sigma_0}{1+\sigma_0}, \tag{3.19}$$

$$\sup_{x\in\mathbb{R}} |\sigma_n'(x)| \leq M. \tag{3.20}$$

Therefore, from (3.12), there exists a $V = V(s, \sigma_0, M) > 0$ such that if $|x - y| \leq 1$, then

$$\omega_n(y) \leq V\omega_n(x), \tag{3.21}$$

where $\omega_n = \sigma_n^s$. We remark that $\omega_n \uparrow \omega$ if $n \to \infty$.

Finally we have all tools in hand to prove the theorem.

3.2. **Proof of Theorem 3.1.** Without loss of generality we can consider $\alpha = \beta = \gamma = 1$ in the IVP (2.1). Let $R > 0$ be such that

$$\sup_{[0,T]} \|u(t)\|_s < R. \tag{3.22}$$

We take a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ in $H^\infty(\mathbb{R})\cap L^2_\omega(\mathbb{R})$ with $\phi_k \to \phi$. By Theorem 2.1 (or the hypothesis of Theorem 3.1) we can suppose that for $k \in \mathbb{N}$, there exists $u_k \in C^1([0,T];H^\infty(\mathbb{R}))$ a solution of (1.8) with $u_k(0)=\phi_k$, such that

$$\sup_{[0,T]} \|u_k(t)\|_s < R. \tag{3.23}$$

Since $\omega_n \le n^s$ we have that $\omega_n u_k \in C^1([0,T];L^2(\mathbb{R}))$, and by integration by parts

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega_n u_k\|^2 &= \langle 2i\partial_x\omega_n\partial_x u_k, \omega_n u_k\rangle + \left\langle \omega_n|u_k|^2\partial_x u_k, \omega_n u_k\right\rangle \\ &\quad + \left\langle \omega_n u_k^2\partial_x \overline{u}_k, \omega_n u_k\right\rangle - 2\left\langle i\omega_n u_k \mathcal{T}_h \Re(\overline{u}_k\partial_x u_k), \omega_n u_k\right\rangle. \end{aligned}\tag{3.24}$$

Since $|\partial_x\omega_n| \le s|\partial_x\sigma_n|\omega_n^{1-\frac{1}{s}}$, (3.20) gives

$$|\partial_x\omega_n| \le c(s,M)\omega_n^{1-\frac{1}{s}}. \tag{3.25}$$

Then, applying Hölder's inequality and (3.25) on (3.24) we obtain

$$\frac{d}{dt}\|\omega_n u_k\|^2 \le c(\|u_k\|_s^2 + \|\omega_n u_k\|^2). \tag{3.26}$$

Integrating (3.26) from t_0 to t_1 $(0 \le t_0 \le t_1 \le T)$ we obtain

$$\left|\, \|\omega_n u_k(t_1)\|^2 - \|\omega_n u_k(t_0)\|^2\right| \le c\int_{t_0}^{t_1}\Big(\|u_k\|_s^2 + \|\omega_n u_k\|^2\Big)d\tau. \tag{3.27}$$

Taking the limit when $k\to\infty$ and by (3.22) it follows that

$$\left|\, \|\omega_n u(t_1)\|^2 - \|\omega_n u(t_0)\|^2\right| \le cR^2(t_1-t_0) + c\int_{t_0}^{t_1}\|\omega_n u(t)\|^2 dt, \tag{3.28}$$

by which applying the Gronwall's inequality we conclude that the functions $\|\omega_n u\|^2$ are uniformly bounded. Then, by the monotone convergence theorem, $\|\omega_n u\|^2 \longrightarrow \|\omega u\|^2$, and so $\|u\|^2_{L^2_\omega} \in C([0,T];\mathbb{R}_+)$. Since

$$u \in C([0,T];L^2)\cap L^\infty([0,T];L^2_\omega(\mathbb{R}))$$

we have that $u \in C_w([0,T];L^2_\omega(\mathbb{R}))$. Thus, $u\in C([0,T];L^2_\omega(\mathbb{R}))$.

It remains only to prove the continuous dependence. We already have the continuous dependence in $H^s(\mathbb{R})$, $s \ge 1$; therefore, it is sufficient to use the norm of $L^2_\omega(\mathbb{R})$. Let $u(t)$ and $v(t)$ be the corresponding solutions of the IVP (1.8) with initial data ϕ and ψ, respectively. Then $\mathbf{w} = u - v$ satisfy

$$\|\mathbf{w}(t)\|^2_{L^2_\omega} \le \|\phi-\psi\|^2_{L^2_\omega} \tag{3.29}$$

$$+ c\sum_{j=1}^{3}\int_0^t (\langle \omega\Gamma_j(u)\mathbf{w}, \omega\mathbf{w}\rangle + \langle \omega(\Gamma_j(u)\mathbf{w} - \Gamma_j(v)\mathbf{w}), \omega\mathbf{w}\rangle)(\tau)d\tau.$$

Taking an $\widetilde{R} > \max\{\sup_{[0,T]} \|u(t)\|_{s,\omega}\,, \ \sup_{[0,T]} \|v(t)\|_{s,\omega}\}$, we have from (3.14) that there exists a $c = c(s, \sigma, \widetilde{R}) > 0$ such that

$$\langle \omega(\Gamma_j(u)\mathbf{w} - \Gamma_j(v)\mathbf{w}), \omega\mathbf{w}\rangle \le c\,\|\mathbf{w}\|_{s,\omega}^2 \tag{3.30}$$

and by the same lemma

$$\langle \omega\Gamma_j(u)\mathbf{w}, \omega\mathbf{w}\rangle \le c\,\|\mathbf{w}\|_{s,\omega}^2\,. \tag{3.31}$$

Substituting (3.30) and (3.31) into (3.29) one obtains

$$\|\mathbf{w}(t)\|_{L^2_\omega}^2 \le \|\phi - \psi\|_{L^2_\omega}^2 + c\int_0^t \|\mathbf{w}(\tau)\|_{s,\omega}^2\, d\tau. \tag{3.32}$$

Now combining the continuous dependence in $H^s(\mathbb{R})$ (with small initial data) with Gronwall's inequality, the continuous dependence in $H_{s,\omega}(\mathbb{R})$ follows from (3.32). □

4. Ill-posedness

We know that the IVP (2.1) is locally well-posed for *small* data in $H^s(\mathbb{R})$, $s \ge 1$ (Section 2); that result was obtained via a contraction method on the integral equation associated to the problem. As a corollary we had that the data-solution map is C^∞-differentiable from $H^s(\mathbb{R})$ to $C([0,T], H^s(\mathbb{R}))$. Then a question arises: Is the method at issue applicable for any Sobolev space? Scaling arguments suggest that the answer is no, for $H^s(\mathbb{R})$ if $s < 0$. In fact, we note that if we make $\gamma = 0$ we have the equation (1.2), and if u is a solution of the IVP associated with it with data ϕ then

$$u_\lambda(x,t) = \lambda^{\frac{1}{2}} u(\lambda x, \lambda^2 t), \quad \forall\ \lambda \in \mathbb{R} - \{0\}, \tag{4.1}$$

with data $u_\lambda(0) = u_\lambda(x,0) = \lambda^{\frac{1}{2}}\phi(\lambda x)$ likewise a solution. So, we get

$$\|D_x^s u_\lambda(0)\| = \lambda^s \|D_x^s \phi\|.$$

So the highest derivative term in the H^s-norm is invariant under the scaling transform (4.1) for $s = 0$; then the critical value for the IVP associated with the equation (1.2) may be $s = 0$. Since equation (1.2) possesses the term $u\mathcal{T}_h\partial_x(|u|^2)$, we can think the same about equations (1.7) and (1.8). We note that a similar scaling argument does not work if $\gamma > 0$. In this section we shall make a rigorous proof of this fact, including the case $\gamma > 0$, when the scaling argument does not work.

4.1. **Ill-posedness for the IVP associated with nonlocal NLS equation (1.8).** Consider the IVP for equation (1.8) (with $\alpha = \beta = 1$, without loss of generality). The integral equation associated is

$$u(t) = U(t)\phi + \int_0^t U(t-\tau)(u\partial_x(|u|^2) - iu\mathcal{H}\partial_x(|u|^2) + i\gamma|u|^2u)(\tau)d\tau. \quad (4.2)$$

We shall establish that the data-solution map for the equation (1.8) is not C^3-differentiable from $H^s(\mathbb{R})$ to $C([0,T], H^s(\mathbb{R}))$, if $s < 0$. It is enough we prove this property at the origin; more precisely, we have the following theorem:

Theorem 4.1. *If $s \in \mathbb{R}$ is such that $s < 0$, then there exists no $T > 0$ such that the IVP* (1.8) *admits a unique local solution defined on the interval $[0,T]$ and such that the map data-solution $\phi \longmapsto u(t)$, $t \in [0,T]$, for* (1.8) *is C^3-differentiable at zero from $H^s(\mathbb{R})$ to $C([0,T], H^s(\mathbb{R}))$.*

Remark 4.1. When $u(t) = U(t)\phi$, we will use the following notation:

$$i\gamma|U(t)\phi|^2U(t)\phi + U(t)\phi\partial_x(|U(t)\phi|^2) - iU(t)\phi\mathcal{H}\partial_x(|U(t)\phi|^2) \equiv F_\gamma(U(t)\phi).$$

An *a priori* condition for the map data-solution being C^3-differentiable at zero from $H^s(\mathbb{R})$ to $C([0,T], H^s(\mathbb{R}))$, is that there exists a metric space Z_T continuously immersed in $C([0,T]; H^s(\mathbb{R}))$ so that, for

$$\|U(t)\phi\|_{Z_T} \leq C\,\|\phi\|_s\,, \qquad \phi \in H^s(\mathbb{R}) \quad (4.3)$$

we have, for $u \in Z_T$,

$$\Big\| \int_0^t U(t-\tau)\Big(i\gamma|u|^2u + u\partial_x(|u|^2) - iu\mathcal{H}\partial_x(|u|^2)(\tau)d\tau\Big)\Big\|_{Z_T} \leq C\,\|u\|^3_{Z_T}. \quad (4.4)$$

By taking $u = U(t)\phi$ in (4.4) and supposing that there exists a Z_T continuously immersed in $C([0,T]; H^s(\mathbb{R}))$, such that the inequalities, and (4.3)–(4.4) are valid, it follows that

$$\Big\| \int_0^t U(t-\tau)F_\gamma(U(t)\phi)d\tau\Big\|_s \leq C\,\|\phi\|^3_s. \quad (4.5)$$

We shall show that there exists a $\phi \in H^s(\mathbb{R})$, $s < 0$, such that (4.5) fails. In short we shall demonstrate the following proposition:

Proposition 4.1. *If $s < 0$, there exists a $\phi \in H^s(\mathbb{R})$ so that* (4.5) *fails.*

Remark 4.2. Here it is important to say that Proposition 4.1 when $\gamma = 0$ does not follow immediately from the case $\gamma > 0$; the case $\gamma = 0$ is slightly more complicated.

Proof of Theorem 4.1. Let us consider Proposition 4.1, which has already been proven. Consider the IVP

$$\begin{cases} \partial_t u = -i\partial_x^2 u + u\partial_x(|u|^2) - iu\mathcal{H}\partial_x(|u|^2) + i\gamma|u|^2 u, & x,t \in \mathbb{R}, \\ u(x,0) = \mu\phi(x), \end{cases} \tag{4.6}$$

and suppose that $u = u(\mu, x, t)$ is a local solution of (4.6) and that the data-solution map is C^3-differentiable at zero from $H^s(\mathbb{R})$ to $C([0,T], H^s(\mathbb{R}))$. The integral equation for (4.6) is

$$u(t) = U(t)\mu\phi + \int_0^t U(t-\tau)(u\partial_x(|u|^2) - iu\mathcal{H}\partial_x(|u|^2) + i\gamma|u|^2 u)(\tau)d\tau.$$

Differentiating u, we have

$$\frac{\partial u}{\partial \mu}\Big|_{\mu=0} = U(t)\phi, \quad \frac{\partial^2 u}{\partial \mu^2}\Big|_{\mu=0} = 0, \text{ and } \frac{\partial^3 u}{\partial \mu^3}\Big|_{\mu=0} = 6\int_0^t U(t-\tau)F_\gamma(U(\tau)\phi)d\tau.$$

By the hypothesis of C^3-differentiability we obtain

$$\Big\| \int_0^t U(t-\tau)F_\gamma(U(\tau)\phi)d\tau \Big\|_s \lesssim \|\phi\|_s^3,$$

but this estimate is (4.5), and by Proposition 4.1 it is false if $s < 0$. □

Proof of Proposition 4.1. The proof is made via a sequence of lemmas. Define the following ϕ by its Fourier transform,

$$\hat{\phi}(\xi) = \eta^{-\frac{1}{2}} N^{-s}\chi_I(\xi), \quad N \gg 1, \text{ and } 0 < \eta \ll 1, \tag{4.7}$$

where $I = [N, N+\eta]$. Notice that $\|\phi\|_s \sim 1$. The lemma below brings an important identity.

Lemma 4.1. *We have the following identities:*

(*i*) *For* $\gamma > 0$,

$$\int_0^t U(t-\tau)F_\gamma(U(\tau)\phi)d\tau \tag{4.8}$$
$$= \frac{-1}{\eta^{3/2}N^{3s}}\int_\Omega \frac{e^{-2it(\xi-\xi_1)(\xi_1+\xi_2)}-1}{2(\xi-\xi_1)(\xi_1+\xi_2)} e^{ix\xi+it\xi^2}\Big(\gamma + \xi - \xi_1 + |\xi-\xi_1|\Big)d\xi\, d\xi_1\, d\xi_2,$$

and so

$$\mathcal{F}\left\{\int_0^t U(t-\tau)F_\gamma(U(\tau)\phi)d\tau\right\} \tag{4.9}$$
$$= \frac{-e^{it\xi^2}}{\eta^{3/2}N^{3s}}\int_{\Omega(\xi)} \frac{e^{-2it(\xi-\xi_1)(\xi_1+\xi_2)}-1}{2(\xi-\xi_1)(\xi_1+\xi_2)}\Big(\gamma + \xi - \xi_1 + |\xi-\xi_1|\Big)d\xi_1\, d\xi_2,$$

with $\Omega(\xi) = \left\{(\xi_1,\xi_2) \in \mathbb{R}^2 : \xi_1 \in I,\, -\xi_2 \in I,\, \xi - \xi_1 - \xi_2 \in I\right\}$ *and* $\Omega = \left\{(\xi_1,\xi_2,\xi) \in \mathbb{R}^3 : (\xi_1,\xi_2) \in \Omega(\xi)\right\}$.

(*ii*) *If* $\gamma = 0$, *then*

$$\int_0^t U(t-\tau)F_0(U(\tau)\phi)d\tau \qquad (4.10)$$
$$= \frac{-2}{\eta^{3/2}N^{3s}} \int_A \frac{e^{-2it(\xi-\xi_1)(\xi_1+\xi_2)}-1}{\xi_1+\xi_2} e^{ix\xi+it\xi^2} d\xi\, d\xi_1\, d\xi_2,$$

and therefore

$$\mathcal{F}\left\{\int_0^t U(t-\tau)\Big(U(\tau)\phi\partial_x(|U(\tau)\phi|^2) - iU(\tau)\phi\mathcal{H}\partial_x(|U(\tau)\phi|^2)\Big)(\tau)d\tau\right\}$$
$$= \frac{-2e^{it\xi^2}}{\eta^{3/2}N^{3s}} \int_{A(\xi)} \frac{e^{-2it(\xi-\xi_1)(\xi_1+\xi_2)}-1}{\xi_1+\xi_2}\, d\xi_1\, d\xi_2, \qquad (4.11)$$

with $A(\xi) = \{(\xi_1,\xi_2) \in \Omega(\xi) : \xi > \xi_1\}$ *and* $A = \{(\xi_1,\xi_2,\xi) \in \mathbb{R}^3 : (\xi_1,\xi_2) \in A(\xi)\}$, *where* $\Omega(\xi)$ *and* Ω *were defined above.*

Proof. For getting (4.8) we must take the inverse Fourier transform with respect to x and to integrate with respect to t:

$$\int_0^t U(t-\tau)F_\gamma(U(\tau)\phi)d\tau = G_1 + G_2 + G_3, \qquad (4.12)$$

where

$$G_1 = \int_0^t \int_{\mathbb{R}} e^{ix\xi+i(t-\tau)\xi^2} \mathcal{F}\left\{i\gamma|U(\tau)\phi|^2U(\tau)\phi\right\}(\xi)d\xi$$
$$= -\frac{\gamma}{\eta^{\frac{3}{2}}N^{3s}} \int_\Omega \frac{e^{-2it(\xi-\xi_1)(\xi_1+\xi_2)}-1}{2(\xi-\xi_1)(\xi_1+\xi_2)}\; e^{ix\xi+it\xi^2} d\xi_1\, d\xi_2\, d\xi,$$
$$G_2 = \int_0^t \int_{\mathbb{R}} e^{ix\xi+i(t-\tau)\xi^2} \mathcal{F}\left\{U(\tau)\phi\; \partial_x(|U(\tau)\phi|^2)\right\}(\xi)d\xi$$
$$= -\frac{1}{\eta^{\frac{3}{2}}N^{3s}} \int_\Omega \frac{e^{-2it(\xi-\xi_1)(\xi_1+\xi_2)}-1}{2(\xi-\xi_1)(\xi_1+\xi_2)}\; (\xi-\xi_1)e^{ix\xi+it\xi^2} d\xi\, d\xi_1\, d\xi_2,$$
$$G_3 = -\int_0^t \int_{\mathbb{R}} e^{ix\xi+i(t-\tau)\xi^2} \mathcal{F}\left\{iU(\tau)\phi\; \mathcal{H}\partial_x(|U(\tau)\phi|^2)\right\}(\xi)d\xi$$
$$= \int_0^t \int_{\mathbb{R}^3} e^{ix\xi+it\xi^2}|\xi-\xi_1|e^{-2i\tau(\xi-\xi_1)(\xi_1+\xi_2)}$$
$$\times\, \widehat{\phi}(\xi-\xi_1-\xi_2)\widehat{\phi}(\xi_1)\widehat{\phi}(-\xi_2)d\xi_2\, d\xi_1\, d\xi\, d\tau$$

$$= -\frac{1}{\eta^{\frac{3}{2}} N^{3s}} \int_{\Omega} \frac{e^{-2i\tau(\xi-\xi_1)(\xi_1+\xi_2)} - 1}{2(\xi-\xi_1)(\xi_1+\xi_2)} |\xi - \xi_1| e^{ix\xi+it\xi^2} d\xi\, d\xi_1\, d\xi_2;$$

thus, adding the terms above we arrive to (4.8). The expression (4.9) is obtained by taking the Fourier transform of (4.8) with relation to ξ. For coming to the identities (4.10) and (4.11) the procedure is exactly as that for obtaining (4.8). □

Lemma 4.2. *Fix* $\xi \in \mathbb{R}$. *Then*

(*a*) *Given* $(\xi_1, \xi_2) \in \Omega(\xi)$ (*or in* $A(\xi)$), *we have*

$$|2(\xi - \xi_1)(\xi_1 + \xi_2)| \leq \eta^2 \tag{4.13}$$

and

$$|\,\gamma + \xi - \xi_1 + |\xi - \xi_1|| \gtrsim 1. \tag{4.14}$$

(*b*) *If* $\xi \in (N, N + \eta)$, *then*

$$|\Omega(\xi)| \gtrsim \eta^2. \tag{4.15}$$

Define

$$B(\xi) = \{(\xi_1, \xi_2) \in A(\xi) : \xi - \xi_1 \geq \frac{\eta}{10}\}.$$

If $\xi \in (N + \frac{\eta}{2}, N + \eta)$, *then*

$$|A(\xi)| \geq |B(\xi)| \gtrsim \eta^2. \tag{4.16}$$

(*c*) *For any* $\gamma \geq 0$, *we have*

$$\operatorname{supp} \mathcal{F} \left\{ \int_0^t U(t-\tau) F_\gamma(U(\tau)\phi) d\tau \right\} \subseteq [N - \eta, N + 2\eta]. \tag{4.17}$$

Proof. (a) is immediate, since $N \leq \xi - \xi_1 - \xi_2 \leq N + \eta$, $|\xi_1 + \xi_2| \leq \eta$ and $|\xi - \xi_1| \leq \eta$.

(b) Fixing $\xi \in (N, N + \eta)$, since $N \leq \xi - \xi_1 - \xi_2 \leq N + \eta$ we obtain

$$-\xi_1 + \xi - N - \eta =: f_1(\xi_1) \leq \xi_2 \leq f_2(\xi_1) := -\xi_1 + \xi - N.$$

Thus, $\Omega(\xi)$ is located between the lines f_1 and f_2, but since $N \leq \xi_1 \leq N + \eta$ and $N \leq -\xi_2 \leq N + \eta$, it follows that $\Omega(\xi)$ is a hexagon composed of the intersection of the region between the lines f_1 and f_2 with the square $I \times -I$, and then

$$|\Omega(\xi)| = \eta^2 - \frac{(\xi - N - \eta)^2 + (\xi - N)^2}{2} \geq \frac{\eta^2}{2}.$$

To prove (4.16) we remark that $B(\xi)$ is the quadrilateral with vertexes $(\xi + \frac{\eta}{10}; -N - \frac{\eta}{10})$, $(\xi + \frac{\eta}{10}; -N - \eta)$, $(N + \eta; -N - \eta)$ and $(N + \eta; -2N - \eta + \xi)$, and therefore

$$|B(\xi)| = (\xi - N)\cdot(N + \frac{9\eta}{10} - \xi) + \frac{(N + \frac{9\eta}{10} - \xi)^2}{2} \gtrsim \eta^2.$$

(c) If $\Omega(\xi)$ is not empty, then

$$\begin{aligned} N - \eta = 2N - N - \eta &\le N + \xi_1 + \xi_2 \le \xi \\ &\le \xi - \xi_1 - \xi_2 + \xi_1 + \xi_2 \le 2N + 2\eta + \xi_2 \le N + 2\eta, \end{aligned}$$

and then $\xi \in [n - \eta, N + 2\eta]$. Thus the lemma is proved. □

Finally, we are ready to conclude the proof. First we consider the case $\gamma > 0$. Let us take $k \in (1, \frac{5}{4})$, $\theta < 0$ (to be chosen later) and $\eta = N^\theta$. An elementary but important proposition says that, given a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a set of finite measure $\Omega \subseteq \mathbb{R}^n$ such that $|f(x)| \ge c_0$ for all $x \in \Omega$, we have

$$\Big|\int_\Omega f(x)dx\Big| \ge c_0|\Omega|. \tag{4.18}$$

Therefore, this statement and Lemma 4.2 give us the following inequality:

$$\begin{aligned} &\Big\|\int_0^t U(t-\tau)F_\gamma(U(\tau\phi))d\tau\Big\|_s \gtrsim \frac{1}{\eta^{\frac{3}{2}}N^{3s}}\Big(\int_{N+(k-1)\eta}^{N+(2-k)\eta}(1+\xi^2)^s \\ &\qquad \times\Big|\int_{\Omega(\xi)}\frac{\sin 2t(\xi-\xi_1)(\xi_1+\xi_2)}{2(\xi-\xi_1)(\xi_1+\xi_2)}(\gamma+\xi-\xi_1+|\xi-\xi_1|)d\xi_1 d\xi_2\Big|^2 d\xi\Big)^{\frac{1}{2}} \\ &\gtrsim \frac{\gamma|t|}{\eta^{\frac{3}{2}}N^{3s}}\Big(\int_{N+(k-1)\eta}^{N+(2-k)\eta}(1+\xi^2)^s|\Omega(\xi)|^2 d\xi\Big)^{1/2} \\ &\gtrsim \frac{\gamma\,\eta|t|}{N^{3s}}\Big(N+2(k-1)\eta\Big)^s \gtrsim \frac{\gamma\,\eta\,|t|}{N^{2s}}, \end{aligned} \tag{4.19}$$

and so by choosing for instance $\theta = s$, we get $1 \sim \|\phi\|_s^3 \ge C|t|N^{-s}$, which is a contradiction for $N \gg 1$ and $s < 0$.

Now we consider the case $\gamma = 0$. By Lemma 4.1 (ii), using the fact that in $A(\xi)$ we have $\xi - \xi_1 > 0$ and by the Lemma 4.2 and (4.18), we obtain

$$\begin{aligned} &\Big\|\int_0^t U(t-\tau)F_0(U(\tau\phi))d\tau\Big\|_s \\ &= \frac{2}{\eta^{\frac{3}{2}}N^{3s}}\Big(\int_{N-\eta}^{N+2\eta}(1+\xi^2)^s\Big|\int_{A(\xi)}\frac{e^{-2it(\xi-\xi_1)(\xi_1+\xi_2)}-1}{2(\xi-\xi_1)(\xi_1+\xi_2)}(\xi-\xi_1)d\xi_1 d\xi_2\Big|^2 d\xi\Big)^{\frac{1}{2}} \\ &\gtrsim \frac{|t|}{\eta^{\frac{3}{2}}N^{3s}}\Big(\int_{N+(k-1)\eta}^{N+(2-k)\eta}(1+\xi^2)^s\Big|\int_{A(\xi)}\frac{\sin 2t(\xi-\xi_1)(\xi_1+\xi_2)}{2t(\xi-\xi_1)(\xi_1+\xi_2)}(\xi-\xi_1)d\xi_1 d\xi_2\Big|^2 d\xi\Big)^{\frac{1}{2}} \end{aligned}$$

$$\gtrsim \frac{|t|}{\eta^{\frac{1}{2}}N^{3s}}\Big(\int_{N+(k-1)\eta}^{N+(2-k)\eta}(1+\xi^2)^s\Big|\int_{B(\xi)}\frac{\sin 2t(\xi-\xi_1)(\xi_1+\xi_2)}{2t(\xi-\xi_1)(\xi_1+\xi_2)}d\xi_1 d\xi_2\Big|^2 d\xi\Big)^{\frac{1}{2}}$$

$$\gtrsim \frac{|t|}{\eta^{\frac{1}{2}}N^{3s}}\Big(\int_{N+(k-1)\eta}^{N+(2-k)\eta}(1+\xi^2)^s|B(\xi)|^2 d\xi\Big)^{1/2} \gtrsim \frac{\eta^2|t|}{N^{2s}} \gtrsim \frac{|t|}{N^s}, \tag{4.20}$$

where we chose $\theta = s/2$. Thus we obtain

$$1 \sim \|\phi\|_s^3 \geq C|t|N^{-s},$$

which is a contradiction for $N \gg 1$ and $s < 0$. □

4.2. **Ill-posedness for the IVP associated with nonlocal NLS equation (1.7).** We shall extend the result of the previous section for the nonlocal INLS equation. We consider the IVP for equation (1.7) (with $\alpha = \beta = 1$, without loss of generality):

$$\begin{cases} \partial_t u = -i\partial_x^2 u + u\partial_x(|u|^2) - iu\mathcal{T}_h\partial_x(|u|^2) + i\gamma|u|^2 u, \\ \qquad\qquad\qquad\qquad\qquad\qquad x,t,\gamma \in \mathbb{R},\ \gamma \geq 0, \\ u(x,0) = \phi(x). \end{cases} \tag{4.21}$$

Theorem 4.2. *If $s \in \mathbb{R}$ is such that $s < 0$, then there is no $T > 0$ so that the IVP (4.21) with $\gamma > 0$ admits a unique local solution defined on the interval $[0,T]$ and such that the data-solution map $\phi \longmapsto u(t)$, $t \in [0,T]$ for (4.21) is C^3-differentiable at zero from $H^s(\mathbb{R})$ to $C([0,T];H^s(\mathbb{R}))$. The same result is valid for the IVP with $\gamma = 0$, if we take sufficiently large $h > 0$ (see the definition of $\mathcal{T}_h$ in (1.3)).*

Proof. As before, when $u(t) = U(t)\phi$, the nonlinear part of the equation will be denoted by

$$i\gamma|U(t)\phi|^2U(t)\phi + U(t)\phi\partial_x(|U(t)\phi|^2) - iU(t)\phi\mathcal{T}_h\partial_x(|U(t)\phi|^2) := \widetilde{F}_\gamma(U(t)\phi).$$

The proof is similar to that of the Theorem 2.1, so let us avoid repeating some arguments. We consider the IVP

$$\begin{cases} \partial_t u = -i\partial_x^2 u + u\partial_x(|u|^2) - iu\mathcal{T}_h\partial_x(|u|^2) + i\gamma|u|^2 u, \quad x,t \in \mathbb{R} \\ u(x,0) = \mu\phi(x), \end{cases} \tag{4.22}$$

where ϕ was defined in (4.7) and $\gamma \geq 0$. We suppose that $u = u(\mu,x,t)$ is a local solution for (4.22) and that the data-solution map is C^3-differentiable at zero from $H^s(\mathbb{R})$ to $C([0,T];H^s(\mathbb{R}))$. The integral equation for (4.22) is

$$u(t) = \mu U(t)\phi + \int_0^t U(t-\tau)(u\partial_x(|u|^2) - iu\mathcal{T}_h\partial_x(|u|^2) + i\gamma|u|^2u)(\tau)d\tau.$$

Differentiating u we obtain

$$\frac{\partial u}{\partial \mu}\Big|_{\mu=0} = U(t)\phi, \quad \frac{\partial^2 u}{\partial \mu^2}\Big|_{\mu=0} = 0, \quad \text{and} \quad \frac{\partial^3 u}{\partial \mu^3}\Big|_{\mu=0} = 6\int_0^t U(t-\tau)\widetilde{F}_\gamma(U(\tau)\phi)d\tau.$$

By the C^3-differentiable hypothesis we obtain

$$\Big\| \int_0^t U(t-\tau)\widetilde{F}_\gamma(U(\tau)\phi)d\tau \Big\|_s \lesssim \|\phi\|_s^3. \tag{4.23}$$

So we will prove that this estimate is false if $s<0$, by taking, for example, the function ϕ defined in (4.7). Therefore we need the following two lemmas.

Lemma 4.3. *For $\gamma \geq 0$, the following identities are valid:*

$$\int_0^t U(t-\tau)\widetilde{F}_\gamma(U(\tau)\phi)d\tau \tag{4.24}$$
$$= \frac{-\eta^{-\frac{3}{2}}}{N^{3s}} \int_\Omega \frac{e^{-2it(\xi-\xi_1)(\xi_1+\xi_2)}-1}{2(\xi-\xi_1)(\xi_1+\xi_2)} e^{ix\xi+it\xi^2}\big(\gamma+f_h(\xi-\xi_1)\big)d\xi\, d\xi_1\, d\xi_2,$$

and thus

$$\mathcal{F}\left\{\int_0^t U(t-\tau)\widetilde{F}_\gamma(U(\tau)\phi)d\tau\right\} \tag{4.25}$$
$$= \frac{-e^{it\xi^2}}{\eta^{3/2}N^{3s}} \int_{\Omega(\xi)} \frac{e^{-2it(\xi-\xi_1)(\xi_1+\xi_2)}-1}{2(\xi-\xi_1)(\xi_1+\xi_2)} \big(\gamma+f_h(\xi-\xi_1)\big)d\xi_1\, d\xi_2,$$

where $f_h(x) = x + x(\coth(hx) - \frac{1}{hx})$, and $\Omega(\xi)$ and Ω are those of Lemma 4.1.

Proof. The polynomial terms of the equation were already treated in Lemma 4.1. Let us see the nonlocal term. Let

$$\widetilde{G}_3 = -\int_0^t U(t-\tau)iU(\tau)\phi\mathcal{T}_h\partial_x(|U(\tau)\phi|^2)d\tau.$$

We take the inverse Fourier transform with relation to x, integrate with relation to t and sum with G_1+G_2 to obtain (4.24). (4.25) is obtained by taking the Fourier transform of (4.24). This proves the lemma. □

The next lemma concentrates the central estimates from our analysis.

Lemma 4.4. *Fix $\xi \in \mathbb{R}$. Then we have the following:*

(i) For $h, \gamma > 0$ fixed and $(\xi_1,\xi_2) \in \Omega(\xi)$, we have $|\gamma + f_h(\xi-\xi_1)| \gtrsim 1$.

(ii) $h>0$ fixed and $(\xi_1,\xi_2) \in B(\xi)$ yield that $f_h(\xi-\xi_1) \gtrsim \eta$, where $B(\xi)$ was defined in Lemma 4.2.

(*iii*) *For all* $\gamma \geq 0$ *we have*

$$\textit{supp}\ \mathcal{F}\left\{\int_0^t U(t-\tau)\widetilde{F}_\gamma(U(\tau)\phi)d\tau\right\} \subseteq [N-\eta, N+2\eta].$$

(*iv*) *For any* $h > 0$ *and* $x \in \mathbb{R}$,

$$-\frac{1}{h} + |x| \lesssim x\coth(hx) \lesssim \frac{1}{h} + |x|.$$

Proof. (*i*) Since the function $\coth(x) - \frac{1}{x}$ ($x \in \mathbb{R}$) is crescent, $\lim_{x\to\infty}(\coth(x) - 1/x) = 1$, and $1 + \coth(x) - \frac{1}{x} > 0$, $\forall x \in \mathbb{R}$, and $\gamma > 0$ is fixed, it follows that

$$|\gamma + f_h(\xi - \xi_1)| \quad \geq \quad \gamma - 2|\xi - \xi_1| \geq \gamma - 2\eta \gtrsim \gamma,$$

where we took $\eta \leq \frac{\gamma}{3}$.

The proof of (*ii*) is very simple: since $\coth(x) - \frac{1}{x} \geq 0$ if $x \geq 0$, and $\xi - \xi_1 \geq \frac{\eta}{10}$, we gain

$$(\xi - \xi_1)\{1 + \coth[h(\xi - \xi_1)] - 1/h(\xi - \xi_1)\} \geq \frac{\eta}{10}.$$

For (*iii*) see the proof of Lemma 4.2 (c).

(*iv*) was proved in [1] (Lemma 4.1). With that the lemma is demonstrated. □

Now we are able to conclude the proof of Theorem 4.2. Let ϕ be that function defined in (4.7). Then, by Lemmas 4.2 (*b*), 4.3, and 4.4 ((*i*) and (*iii*)), we obtain

$$\left\|\int_0^t U(t-\tau)\widetilde{F}_\gamma(U(\tau)\phi)d\tau\right\|_s \gtrsim \frac{\gamma\eta|t|}{N^{2s}}, \tag{4.26}$$

and it is a contradiction if we consider $s < 0$ and $\eta = N^s$, with $N \gg 1$.

Now we treat the case $\gamma = 0$. By the Lemma 4.4 (*iv*)) we have that for $x < 0$, $-2/h \leq f_h(x) < 0$. With this fact and the Lemmas 4.2 (*b*), 4.3, and 4.4 (*ii*) and (*iii*), and taking $s < 0$, we have that

$$\begin{aligned}\left\|\int_0^t U(t-\tau)\widetilde{F}_0(U(\tau)\phi)d\tau\right\|_s &\gtrsim \frac{t\eta^{-\frac{1}{2}}}{N^{3s}}\Big(\int_{N+(k-1)\eta}^{N+(2-k)\eta}(1+\xi^2)^s|B(\xi)|^2d\xi\Big)^{1/2} \\ &\quad - \frac{t\eta^{-\frac{3}{2}}}{hN^{3s}}\Big(\int_{N+(k-1)\eta}^{N+(2-k)\eta}(1+\xi^2)^s|\Omega(\xi)\ \setminus\ A(\xi)|^2d\xi\Big)^{1/2} \\ &\gtrsim \frac{t\eta}{N^{2s}}\Big(\eta - \frac{1}{h}\Big),\end{aligned} \tag{4.27}$$

which is a contradiction if we take $N \gg 1$, $\eta = N^{s/2}$ and (for example) $h \geq 2N^{-s/2}$. With this we conclude the proof of the theorem. $\square$

Remark 4.3. The method developed by H. Biagioni and F. Linares [3] to prove the ill-posedness for the IVP associated with the one-dimensional derivative Schrödinger equation and BO equation, in the sense that the mapping data-solution is not uniformly continuous, cannot be applied here to study the ill-posedness of the IVP associated with equations (1.7) and (1.8), because that method consists basically in the existence of a smooth curve of *solitary-wave solutions,* and we will prove in the next section the nonexistence of solitary-wave solutions for equations (1.7) and (1.8).

5. Nonexistence of standing-waves solutions

We will prove non-existence of standing-wave-type solutions for equation (1.7). That is made via the gauge-equivalent equation (1.9):

$$i\partial_t v = \partial_x^2 v - \frac{1}{4}|v|^4 v + v\mathcal{T}_h \partial_x(|v|^2) - \gamma |v|^2 v. \tag{5.1}$$

The proof is made by contradiction. For that, we shall make use of an appropriate Pohozaev's identity. Standing waves are solutions like

$$v(x,t) = e^{i\omega t}\varphi(x), \tag{5.2}$$

where $x, t \in \mathbb{R}$, ω is a real constant and φ is a smooth real function. Here is our theorem.

Theorem 5.1. *There is no solutions of the type $v(x,t) = e^{i\omega t}\varphi(x)$ for (5.1), when $\gamma \geq 0$, φ is a real smooth function satisfying the asymptotic property as $x\varphi'(x) \longrightarrow 0$ if $|x| \longrightarrow \infty$, and with the following conditions:*

(a) For equation (5.1) with $h < +\infty$, provided that $\omega \leq 0$;

(b) For equation (5.1) with $\mathcal{H}$ in place of $\mathcal{T}_h$ (that is, $h \longrightarrow +\infty$) for all $\omega \in \mathbb{R}$.

The Hilbert transform satisfies the following property:

Lemma 5.1. *For any function $f \in \mathcal{S}(\mathbb{R})$, we have that*

$$\int_{-\infty}^{\infty} x f' \mathcal{H}(f') dx = 0.$$

Proof. Since $\mathcal{H}$ is a skew-adjoint operator, it follows that

$$-\int_{-\infty}^{\infty} x f' \mathcal{H}(f') dx = \int_{-\infty}^{\infty} \mathcal{H}(x f') f' dx. \tag{5.3}$$

By definition of the Hilbert transform, we have

$$\begin{aligned}\mathcal{H}(xf') &= p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yf'}{y-x}dy \\ &= p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(y-x)f'}{y-x}dy + x\,p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f'}{y-x}dy \;=\; x\mathcal{H}(f'). \quad (5.4)\end{aligned}$$

Now it is enough to substitute identity (5.4) in (5.3). □

Proof of Theorem 5.1. Substituting (5.2) into equation (5.1) we obtain

$$-\varphi'' - \omega\varphi + \frac{1}{4}\varphi^5 + \gamma\varphi^3 - \varphi\mathcal{T}_h(\varphi^2)' = 0. \quad (5.5)$$

Multiplying equation (5.5) by φ and integrating by parts, we arrive at the identity

$$\int_{-\infty}^{\infty}(\varphi')^2dx = \omega\int_{-\infty}^{\infty}\varphi^2dx - \frac{1}{4}\int_{-\infty}^{\infty}\varphi^6dx - \gamma\int_{-\infty}^{\infty}\varphi^4dx + \int_{-\infty}^{\infty}\varphi^2\mathcal{T}_h(\varphi^2)'dx. \quad (5.6)$$

Multiplying equation (5.5) by $x\varphi'$ and integrating properly by parts, we see that φ must obey

$$\begin{aligned}&\int_{-\infty}^{\infty}(\varphi')^2dx + \omega\int_{-\infty}^{\infty}\varphi^2dx - \frac{1}{12}\int_{-\infty}^{\infty}\varphi^6dx - \frac{\gamma}{2}\int_{-\infty}^{\infty}\varphi^4dx \quad (5.7)\\ &-\int_{-\infty}^{\infty}x(\varphi^2)'\mathcal{T}_h(\varphi^2)'dx = 0.\end{aligned}$$

From identity (5.7) we have that

$$\begin{aligned}0 < \int_{-\infty}^{\infty}(\varphi')^2dx &= -\omega\int_{-\infty}^{\infty}\varphi^2dx + \frac{1}{12}\int_{-\infty}^{\infty}\varphi^6dx \quad (5.8)\\ &+ \frac{\gamma}{2}\int_{-\infty}^{\infty}\varphi^4dx + \int_{-\infty}^{\infty}x(\varphi^2)'\mathcal{T}_h(\varphi^2)'dx.\end{aligned}$$

Now we apply (5.8) in (5.6) and arrive at the following expression:

$$\begin{aligned}&\omega\int_{-\infty}^{\infty}\varphi^2dx + \frac{1}{2}\int_{-\infty}^{\infty}\varphi^2\mathcal{T}_h(\varphi^2)'dx \quad (5.9)\\ &= \frac{1}{6}\int_{-\infty}^{\infty}\varphi^6dx + \frac{3\gamma}{4}\int_{-\infty}^{\infty}\varphi^4dx + \frac{1}{2}\int_{-\infty}^{\infty}x(\varphi^2)'\mathcal{T}_h(\varphi^2)'dx\end{aligned}$$

By Parseval's identity we see that

$$\int_{-\infty}^{\infty}\varphi^2\mathcal{T}_h(\varphi^2)'dx < 0, \quad (5.10)$$

for both the cases $h < +\infty$ and $h \longrightarrow +\infty$.

(a) Since $\gamma \geq 0$ and by the hypothesis $\omega \leq 0$, it follows from (5.9) and (5.10) that

$$\int_{-\infty}^{\infty} x(\varphi^2)'\mathcal{T}_h(\varphi^2)'dx < 0. \tag{5.11}$$

Identity (5.8) provides

$$\omega \int_{-\infty}^{\infty} \varphi^2 dx < \frac{1}{12}\int_{-\infty}^{\infty} \varphi^6 dx + \frac{\gamma}{2}\int_{-\infty}^{\infty} \varphi^4 dx + \int_{-\infty}^{\infty} x(\varphi^2)'\mathcal{T}_h(\varphi^2)'dx. \tag{5.12}$$

From (5.9), (5.11) and (5.12) we obtain

$$\begin{aligned} &\omega \int_{-\infty}^{\infty} \varphi^2 dx + \frac{1}{2}\int_{-\infty}^{\infty} \varphi^2 \mathcal{T}_h(\varphi^2)'dx \qquad (5.13)\\ &> \omega \int_{-\infty}^{\infty} \varphi^2 dx + \frac{1}{12}\int_{-\infty}^{\infty} \varphi^6 dx + \frac{\gamma}{4}\int_{-\infty}^{\infty} \varphi^4 dx - \frac{1}{2}\int_{-\infty}^{\infty} x(\varphi^2)'\mathcal{T}_h(\varphi^2)'dx; \end{aligned}$$

thus, from (5.10) and (5.13), we obtain

$$\begin{aligned} 0 &> \frac{1}{2}\int_{-\infty}^{\infty} \varphi^2 \mathcal{T}_h(\varphi^2)'dx \qquad (5.14)\\ &> \frac{1}{12}\int_{-\infty}^{\infty} \varphi^6 dx + \frac{\gamma}{4}\int_{-\infty}^{\infty} \varphi^4 dx - \frac{1}{2}\int_{-\infty}^{\infty} x(\varphi^2)'\mathcal{T}_h(\varphi^2)'dx > 0, \end{aligned}$$

which is a contradiction. Therefore, (a) is proven.

(b) In this case, we have the Hilbert transform instead of the operator $\mathcal{T}_h$ in the equation (5.1).

From Lemma 5.1 and identity (5.7) we have that

$$0 < \int_{-\infty}^{\infty} (\varphi')^2 dx = -\omega \int_{-\infty}^{\infty} \varphi^2 + \frac{1}{12}\int_{-\infty}^{\infty} \varphi^6 dx + \frac{\gamma}{2}\int_{-\infty}^{\infty} \varphi^4 dx. \tag{5.15}$$

Now we apply (5.15) in (5.6) and arrive at the following expression:

$$\begin{aligned} &\omega \int_{-\infty}^{\infty} \varphi^2 dx + \frac{1}{2}\int_{-\infty}^{\infty} \varphi^2 \mathcal{T}_h(\varphi^2)'dx \qquad (5.16)\\ &= \frac{1}{12}\int_{-\infty}^{\infty} \varphi^6 dx + \frac{\gamma}{2}\int_{-\infty}^{\infty} \varphi^4 dx + \frac{1}{12}\int_{-\infty}^{\infty} \varphi^6 dx + \frac{\gamma}{4}\int_{-\infty}^{\infty} \varphi^4 dx\\ &> \omega \int_{-\infty}^{\infty} \varphi^2 dx + \frac{1}{12}\int_{-\infty}^{\infty} \varphi^6 dx + \frac{\gamma}{4}\int_{-\infty}^{\infty} \varphi^4 dx, \end{aligned}$$

so from (5.16) we obtain an inequality similar to inequality (5.14):

$$0 > \frac{1}{2}\int_{-\infty}^{\infty} \varphi^2 \mathcal{T}_h(\varphi^2)'dx > \frac{1}{12}\int_{-\infty}^{\infty} \varphi^6 dx + \frac{\gamma}{4}\int_{-\infty}^{\infty} \varphi^4 dx > 0, \tag{5.17}$$

and we have again a contradiction. Therefore, (b) is proven. □

Remark 5.1. For the time being, we do not know how to prove case (a) in Theorem 5.1 provided that $\omega > 0$. Actually, based on the works of Abdelouhab *et al.* ([1]) and Bona *et al.* ([2]), we are working on the problem of the convergence of the solution of (4.21) to the solution of the limit equation (1.8) when $h \to \infty$, into an appropriate subspace of $C([0,T], H^1(\mathbb{R}))$ for any $T > 0$. So proving that fact and using (b) in Theorem 5.1 we expect to obtain the nonexistence of standing-wave solutions when $\omega > 0$ for equation (5.1).

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