Differential and Integral Equations Volume 20, Number 10 (2007), 1081–1106

ON NONNEGATIVE SOLUTIONS OF SINGULAR BOUNDARY-VALUE PROBLEMS FOR EMDEN–FOWLER-TYPE DIFFERENTIAL SYSTEMS

Mariella Cecchi
Department of Electronics and Telecommunications, University of Florence
Via S. Marta 3, 50139 Firenze, Italy

Zuzana Došlá[1]
Department of Mathematics and Statistics, Masaryk University
Janáčkovo nám. 2a, 60200 Brno, Czech Republic

Ivan Kiguradze
A. Razmadze Mathematical Institute, 1 Aleksidze St., 0193 Tbilisi, Georgia

Mauro Marini
Department of Electronics and Telecommunications, University of Florence
Via S. Marta 3, 50139 Firenze, Italy

(Submitted by: Jean Mawhin)

Abstract. We investigate some boundary-value problems for an Emden–Fowler-type differential system

$$u_1' = g_1(t)u_2^{\lambda_1}, \quad u_2' = g_2(t)u_1^{\lambda_2}$$

on a finite or infinite interval $I = [a, b)$, where $g_i : I \to [0, \infty)$ $(i = 1, 2)$ are locally integrable functions. We give the optimal, in a certain sense, sufficient conditions that guarantee the existence of a unique (at least of one) nonnegative solution, satisfying one of the two following boundary conditions:

$$\text{i) } u_1(a) = c_0, \quad \lim_{t\to b} u_1(t) = c; \quad \text{ii) } u_2(a) = c_0, \quad \lim_{t\to b} u_1(t) = c,$$

in case $0 \le c_0 < c < +\infty$ (in case $c_0 \ge 0$, $c = +\infty$ and $\lambda_1\lambda_2 > 1$). Moreover, the global two-sided estimations of the above-mentioned solutions are obtained together with applications to differential equations with p-Laplacian.

Accepted for publication: June 2007.
AMS Subject Classifications: 34B16, 34B18, 34B40, 34C11.
[1]Supported by the Research Project 0021622409 of the Ministery of Education of the Czech Republic

1. Introduction

Let $-\infty < a < b \le +\infty$ and $I = [a, b)$. Consider the system

$$u_1' = g_1(t)u_2^{\lambda_1}, \quad u_2' = g_2(t)u_1^{\lambda_2}, \tag{1.1}$$

on the interval I (finite or infinite), where λ_1 and λ_2 are positive numbers, and the functions $g_i : I \to \mathbb{R}_+ = [0, +\infty)$ $(i = 1, 2)$ are locally Lebesgue integrable on I, i.e., integrable on $[a, t_0]$ for any $t_0 \in (a, b)$. By a *nonnegative solution* of the system (1.1), defined on I, we mean a locally absolutely continuous vector function $(u_1, u_2) : I \to \mathbb{R}_+^2$ which satisfies (1.1) almost everywhere on I.

We study the following problems on nonnegative solutions of the system (1.1), defined on I: (1.2), (1.3), (1.4), (1.5)

$$u_1(a) = c_0, \quad \lim_{t\to b} u_1(t) = c_1; \tag{1.2}$$

$$u_2(a) = c_0, \quad \lim_{t\to b} u_1(t) = c_1; \tag{1.3}$$

$$u_1(a) = c_0, \quad \lim_{t\to b} u_1(t) = +\infty; \tag{1.4}$$

$$u_2(a) = c_0, \quad \lim_{t\to b} u_1(t) = +\infty, \tag{1.5}$$

where c_0 is a nonnegative and c_1 is a positive constant. Observe that the functions g_i in (1.1) need not be integrable on I; in this sense, the problems (1.1), (1.k) $(k = 2, 3, 4, 5)$ are singular. As for the problems (1.1), (1.4) and (1.1), (1.5), they are also singular if the functions g_i are integrable on I, and in this case we get the blow-up phenomena.

The system (1.1) includes the differential equation with the p -Laplacian operator

$$\left(\left(\frac{u'}{h_0(t)}\right)^p\right)' = h(t)u^\ell, \tag{1.6}$$

where p and ℓ are positive constants, $h_0 : I \to (0, +\infty)$ and $h : I \to \mathbb{R}_+$ are locally Lebesgue-integrable functions.

A locally absolutely continuous nondecreasing function $u : I \to \mathbb{R}_+$ is said to be a *nonnegative nondecreasing solution* of the equation (1.6) if there exists a locally absolutely continuous function $v : I \to \mathbb{R}_+$ such that

$$\left(\frac{u'(t)}{h_0(t)}\right)^p = v(t), \quad v'(t) = h(t)v^\ell(t) \ \text{ a.e. on } I.$$

From what was said above we can see that a locally absolutely continuous nondecreasing function $u : I \to \mathbb{R}_+$ is a solution of the equation (1.6) if and only if the vector function (u_1, u_2) with the components

$$u_1(t) = \Big(\frac{u'(t)}{h_0(t)}\Big)^p, \quad u_2(t) = u(t) \tag{1.7$_1$}$$

is a solution of the system (1.1), where

$$\lambda_1 = \ell, \quad \lambda_2 = \frac{1}{p}, \quad g_1(t) = h(t), \quad g_2(t) = h_0(t). \tag{1.8$_1$}$$

Analogously, the equalities

$$u_1(t) = u(t), \quad u_2(t) = \Big(\frac{u'(t)}{h_0(t)}\Big)^p \tag{1.7$_2$}$$

establish one-to-one correspondence between the set of nonnegative nondecreasing solutions of the equation (1.6) and that of nonnegative solutions of the system (1.1), where

$$\lambda_1 = \frac{1}{p}, \quad \lambda_2 = \ell, \quad g_1(t) = h_0(t), \quad g_2(t) = h(t). \tag{1.8$_2$}$$

The above-stated problems on nonnegative solutions of the system (1.1) correspond to the following problems on nonnegative nondecreasing solutions of the equation (1.6):

$$\lim_{t\to a} \frac{u'(t)}{h_0(t)} = c_0, \quad \lim_{t\to b} \frac{u'(t)}{h_0(t)} = c_1; \tag{1.9$_1$}$$

$$u(a) = c_0, \quad \lim_{t\to b} u(t) = c_1; \tag{1.9$_2$}$$

$$u(a) = c_0, \quad \lim_{t\to b} \frac{u'(t)}{h_0(t)} = c_1; \tag{1.10$_1$}$$

$$\lim_{t\to a} \frac{u'(t)}{h_0(t)} = c_0, \quad \lim_{t\to b} u(t) = c_1; \tag{1.10$_2$}$$

$$\lim_{t\to a} \frac{u'(t)}{h_0(t)} = c_0, \quad \lim_{t\to b} \frac{u'(t)}{h_0(t)} = +\infty; \tag{1.11$_1$}$$

$$u(a) = c_0, \quad \lim_{t\to b} u(t) = +\infty; \tag{1.11$_2$}$$

$$u(a) = c_0, \quad \lim_{t\to b} \frac{u'(t)}{h_0(t)} = +\infty; \tag{1.12$_1$}$$

$$\lim_{t\to a} \frac{u'(t)}{h_0(t)} = c_0, \quad \lim_{t\to b} u(t) = +\infty. \tag{1.12$_2$}$$

Asymptotic properties of solutions of differential system (1.1) and equation (1.6) have been widely investigated in the literature (see, e.g., [1–21] and references therein), where the following terminology has been used.

If $b = +\infty$, then a positive solution (u_1, u_2) of the system (1.1) (positive, nondecreasing solution u of equation (1.6)), defined in some neighborhood $+\infty$, is called *proper*. The proper solution (u_1, u_2) (solution u) is called *weakly increasing*, if

$$\lim_{t\to+\infty} u_1(t) < +\infty \quad \Big(\lim_{t\to+\infty} \frac{u'(t)}{h_0(t)} < +\infty \Big)$$

and *strongly increasing*, if

$$\lim_{t\to+\infty} u_1(t) = +\infty \quad \Big(\lim_{t\to+\infty} \frac{u'(t)}{h_0(t)} = +\infty \Big).$$

A positive solution (u_1, u_2) of the system (1.1) (positive, nondecreasing solution u of equation (1.6)), defined in some finite interval $[t_*, t^*)$ and satisfying the condition

$$\lim_{t\to t^*} u_1(t) = +\infty \quad \big(\lim_{t\to t^*} u'(t) = +\infty \big)$$

is called *blow-up*, or by the terminology adopted in [14], a *singular solution of the second kind*.

The problems on the existence of positive, nondecreasing proper and blow-up solutions of different types have been investigated in details. However, every such problem can be reduced to one of the above-formulated singular boundary-value problems. Therefore a complete description of sets of proper and blow-up solutions of the system (1.1) and equation (1.6) can be achieved by solving the problems (1.1), (1.2)–(1.1), (1.5) and (1.6), (1.9_1)–(1.6) (1.12_2). Nevertheless, these problems remain still little studied. In our work we will make an attempt to fill in this gap.

In Section 2 we establish optimal, in a certain sense, sufficient conditions which guarantee the existence of a unique nonnegative solution of the problems (1.1), (1.2) and (1.1), (1.3). Similarly, in Section 3 we give optimal sufficient conditions for the existence of at least one nonnegative solution of the problems (1.1), (1.4) and (1.1), (1.5). Moreover, for the solutions of the problems (1.1), (1.4) and (1.1), (1.5) we obtain two-sided global estimations. In Section 4, we apply our results to the problems (1.6), (1.9_1)–(1.6), (1.12_2).

In particular, when $b = +\infty$ ($b < +\infty$) our theorems improve the earlier obtained results on the existence of weakly and strongly increasing proper solutions (blow-up solutions) of the differential equation (1.6) and the system (1.1), see [1–6], [13–19] and [21].

Throughout the paper the following notation will be used: $[x]_+ = (x + |x|)/2$ for a real number x; $L(I;\mathbb{R}_+)$ is the set of Lebesgue-integrable functions; $L_{\mathrm{loc}}(I;\mathbb{R}_+)$ is the set of functions that are Lebesgue integrable in the interval $[a,t_0]$ for an arbitrary $t_0 \in I$.

2. PROBLEMS (1.1), (1.2) AND (1.1), (1.3)

We study the problems (1.1), (1.2) and (1.1), (1.3) in the case when

$$g_1 \in L(I;\mathbb{R}_+), \quad g_2 \in L_{\mathrm{loc}}(I;\mathbb{R}_+), \quad G_0 \in L(I;\mathbb{R}_+) \tag{2.1}$$

and

$$\operatorname{meas}\{t \in I : \ g_1(t) > 0\} > 0, \tag{2.2}$$

where

$$G_0(t) = g_1(t)\Big(\int_a^t g_2(s)\,ds\Big)^{\lambda_1}. \tag{2.3}$$

For any $x > 0$ put

$$\varphi(x,\lambda) = \begin{cases} \Big[x^{1-\lambda} - (1-\lambda)\displaystyle\int_a^b G_0(s)\,ds\Big]_+^{\frac{1}{1-\lambda}} & \text{for } \ \lambda < 1 \\ \exp\Big(-\displaystyle\int_a^b G_0(s)\,ds\Big)x & \text{for } \ \lambda = 1 \\ \Big(x^{1-\lambda} + (\lambda-1)\displaystyle\int_a^b G_0(s)\,ds\Big)^{\frac{1}{1-\lambda}} & \text{for } \ \lambda > 1 \end{cases}. \tag{2.4}$$

Theorem 2.1. *Assume* (2.1), (2.2) *and*

$$0 \le c_0 \le \varphi(c_1,\lambda), \tag{2.5}$$

where $\lambda = \lambda_1\lambda_2$. Then the problem (1.1), (1.2) *has a unique nonnegative solution.*

For the proof of Theorem 2.1 the following two lemmas will be needed.

Lemma 2.1. *Assume* (2.1), (2.2) *and*

$$0 < c_0 \le \varphi(c_1,\lambda), \tag{2.6}$$

where $\lambda = \lambda_1\lambda_2$. Let $(v_1,v_2) : I \to \mathbb{R}_+^2$ be a solution of the system of differential inequalities

$$0 \le v_1'(t) \le g_1(t)v_2^{\lambda_1}(t), \quad 0 \le v_2'(t) \le g_2(t)v_1^{\lambda_2}(t), \tag{2.7}$$

satisfying the initial conditions

$$v_1(a) = c_0, \quad v_2(a) = 0. \tag{2.8}$$

Then

$$v_1(b-) = \lim_{t\to b} v_1(t) \le c_1. \tag{2.9}$$

Proof. In view of (2.3), it follows from (2.7) and (2.8) that

$$0 \le v_2(t) \le \int_a^t g_2(s) v_1^{\lambda_2}(s)\,ds \le v_1^{\lambda_2}(t) \int_a^t g_2(s)\,ds$$

and $0 \le v_1'(t) \le G_0(t) v_1^{\lambda}(t)$. From this inequality it follows that $v_1(b-) \le v(b-)$, where v is a solution of the Cauchy problem

$$v'(t) = G_0(t) v^{\lambda}(t), \quad v(a) = c_0.$$

Since

$$v(t) = \begin{cases} \Big(c_0^{1-\lambda} + (1-\lambda)\displaystyle\int_a^t G_0(\xi)\,d\xi\Big)^{\frac{1}{1-\lambda}} & \text{for } \ \lambda \ne 1 \\ c_0 \exp\Big(\displaystyle\int_a^t G_0(\xi)\,d\xi\Big) & \text{for } \ \lambda = 1, \end{cases}$$

summarizing (2.4) and (2.6), we obtain (2.9). □

Lemma 2.2. *Let $\omega_i : I \times \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 0, 1, 2)$ be functions satisfying the conditions*

$$\omega_i(t, 0) = 0 \ \text{ a.e. on } I \ \ (i = 0, 1, 2), \tag{2.10}$$

$$\omega_0(\cdot, x) \in L(I; \mathbb{R}_+), \quad \int_a^b \omega_0(s, x)\,ds > 0 \ \text{ for } \ x \in (0, +\infty). \tag{2.11}$$

Let $(v_1, v_2) : I \to \mathbb{R}^2$ be a solution of the system of differential inequalities

$$v_1'(t) \operatorname{sgn} v_2(t) \ge \omega_0(t, |v_2(t)|), \quad v_2'(t) \operatorname{sgn} v_1(t) \ge 0, \tag{2.12$_1$}$$

$$|v_1'(t)| \le \omega_1(t, |v_2(t)|), \quad |v_2'(t)| \le \omega_2(t, |v_1(t)|) \tag{2.12$_2$}$$

such that

$$v_1(a) v_2(a) = 0, \quad v_1(b-) = 0. \tag{2.13}$$

Then $v_i(t) \equiv 0$ $(i = 1, 2)$.

Proof. In view of (2.12$_1$) and (2.13)we have

$$\frac{d}{dt}\,(v_1(t) v_2(t)) \ge 0 \ \text{ a.e. on } I, \tag{2.14}$$

$$v_1(t) v_2(t) \ge 0 \ \text{ for } \ t \in I. \tag{2.15}$$

First, we show that

$$v_1(t) v_2(t) \equiv 0. \tag{2.16}$$

Assume the contrary. In view of (2.14) and (2.15), there exists $t_0 \in (a,b)$ such that $v_1(t)v_2(t) > 0$ for $t_0 \le t < b$. From here and (2.12_1) we get

$$|v_1(t)|' = v_1'(t)\operatorname{sgn} v_1(t) = v_1'(t)\operatorname{sgn} v_2(t) \ge 0 \ \text{ a.e. on } t \in (t_0,b)$$

and $|v_1(b-)| \ge |v_1(t_0)| > 0$, which contradicts the equality $v_1(b-) = 0$. Therefore (2.16) is true.

Now we will prove that

$$v_2(t) \equiv 0. \tag{2.17}$$

Assume the contrary. Then there exists an interval $(a_1,b_1) \subset (a,b)$ such that

$$v_2(t) \ne 0 \ \text{ for } \ a_1 < t < b_1 \tag{2.18}$$

and

$$\text{either } \ a_1 = a,\ b_1 = b \ \text{ or } \ a_1 > a,\ v_2(a_1) = 0 \ \big(b_1 < b,\ v_2(b_1) = 0\big). \tag{2.19}$$

In accordance with (2.16) and (2.18), we have $v_1(t) = 0$ for $a_1 < t < b_1$. So, from (2.10) and (2.12_2) we get $v_2'(t) = 0$ almost everywhere on $t \in (a_1,b_1)$. In view of this equality, it follows from (2.18) and (2.19) that $a_1 = a$, $b_1 = b$ and $v_2(t) \equiv c$, where $c = v_2(a) \ne 0$. Hence, by (2.12_1) and (2.13) we obtain

$$v_1'(t)\operatorname{sgn} c \ge \omega_0(t,|c|) \ \text{ a.e. on } I, \quad v_1(a) = v_1(b-) = 0.$$

Consequently,

$$0 = \big(v_1(b-) - v_1(a)\big)\operatorname{sgn} c \ge \int_a^b \omega_0(s,|c|)\,ds,$$

which contradicts (2.11). Hence (2.17) holds.

In view of (2.10) and (2.17), from (2.12_2) we get $v_1'(t) \equiv 0$ a.e. on $t \in I$. From here and from the equality $v_1(b-) = 0$ it follows that $v_1(t) \equiv 0$. □

Proof of Theorem 2.1. First we will prove the assertion in the case where $b < +\infty$. For any natural k put

$$\delta_k = \frac{b-a}{2k}, \quad \tau_k(t) = \begin{cases} a & \text{for } \ a \le t \le a+\delta_k \\ t-\delta_k & \text{for } \ a+\delta_k < t \le b \end{cases} \tag{2.20}$$

and consider the delayed differential system

$$u_1'(t) = g_1(t)u_2^{\lambda_1}(\tau_k(t)), \quad u_2'(t) = g_2(t)u_1^{\lambda_2}(\tau_k(t)) \tag{2.21}$$

with the initial conditions

$$u_1(a) = c_0, \quad u_2(a) = \gamma. \tag{2.22}$$

In accordance with (2.1) and (2.20), for every $\gamma \in \mathbb{R}_+$ the problem (2.21), (2.22) has the unique nonnegative solution $(u_{1k}(\cdot\,,\gamma), u_{2k}(\cdot\,,\gamma))$ which is continuous with respect to the parameter γ. Moreover, $u_{1k}(\cdot\,,\gamma)$ has the finite limit $u_{1k}(b-,\gamma)$, and $u_{1k}(b-,\cdot\,) : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

If $c_0 = 0$, then $u_{ik}(t,0) \equiv 0$ $(i = 1,2)$ and, consequently,

$$u_{1k}(b-,0) \le c_1. \tag{2.23}$$

We will show that this inequality holds also for $c_0 > 0$. Indeed, it follows from (2.20) and (2.21) that the vector function (v_1, v_2), where $v_i(t) = u_{ik}(t,0)$ $(i = 1,2)$, is a solution of the problem (2.7), (2.8) and so, according to the condition (2.6) and Lemma 2.1, we get (2.23).

By (2.4) we have $\varphi(x,\lambda) \le x$ for $x > 0$. Therefore, from (2.5) we have $c_0 \le c_1$. In accordance with (2.2), there exists $\gamma_0 \ge 0$ such that

$$\gamma_0^{\lambda_1} = (c_1 - c_0)\Big(\int_a^b g_1(s)\,ds\Big)^{-1}.$$

Hence, from (2.21) and (2.22) we get $u_{2k}(t,\gamma_0) \ge \gamma_0$ for $t \in I$ and

$$u_{1k}(b-,\gamma_0) \ge c_0 + \gamma_0^{\lambda_1}\int_a^b g_1(s)\,ds \ge c_1. \tag{2.24}$$

In view of the continuity of $u_{1k}(b-,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, the existence of a number $\gamma_k \in [0,\gamma_0]$ such that $u_{1k}(b-,\gamma_k) = c_1$ follows from (2.23) and (2.24). Put

$$\overline{u}_{ik}(t) = u_{ik}(t,\gamma_k) \ \text{ for } \ t \in I \ \ (i = 1,2).$$

Obviously, $(\overline{u}_{1k}, \overline{u}_{2k})$ is a solution of the system (2.21) satisfying the boundary conditions

$$\overline{u}_{1k}(a) = c_0, \quad \overline{u}_{1k}(b-) = c_1.$$

Moreover,

$$0 \le \overline{u}_{1k}(t) \le c_1, \ \ 0 \le \overline{u}_{2k}(t) \le \gamma_0 + c_1^{\lambda_2}\int_a^t g_2(s)\,ds \ \text{ for } \ t \in I, \tag{2.25}$$

$$0 \le \overline{u}_{1k}{}'(t) \le G(t), \ \ 0 \le \overline{u}_{2k}{}'(t) \le c_1^{\lambda_2} g_2(t) \ \text{ a.e. on } \ I, \tag{2.26}$$

and

$$0 \le c_1 - \overline{u}_{1k}(t) \le \int_t^b G(s)\,ds \ \text{ for } \ t \in I, \tag{2.27}$$

where

$$G(t) = g_1(t)\Big(\gamma_0 + c_1^{\lambda_2}\int_0^t g_2(s)\,ds\Big)^{\lambda_1}$$

and, as follows from (2.1),

$$G \in L(I;\mathbb{R}_+). \tag{2.28}$$

In accordance with (2.1) and (2.26), the inequalities (2.25), (2.26) yield that the sequences $(\overline{u}_{ik})_{k=1}^{\infty}$ $(i = 1, 2)$ are uniformly bounded and equicontinuous on $[a, b-\varepsilon]$ for any small $\varepsilon > 0$. Hence by the Ascoli-Arzelá lemma there exists a sequence of natural integers $(k_m)_{m=1}^{\infty}$ for which $(\overline{u}_{ik_m})_{m=1}^{\infty}$ $(i = 1, 2)$ converge uniformly on each interval $[a, b-\varepsilon]$ for any small $\varepsilon > 0$. Set

$$u_i(t) = \lim_{m\to\infty} \overline{u}_{ik_m}(t) \ \ (i = 1, 2).$$

Passing to the limit as $m \to \infty$ in the equalities

$$\overline{u}_{1k_m}(t) = c_0 + \int_a^t g_1(s)\overline{u}_{2k_m}^{\lambda_1}(\tau_{k_m}(s))\, ds,$$
$$\overline{u}_{2k_m}(t) = \overline{u}_{2k_m}(a) + \int_a^t g_2(s)\overline{u}_{1k_m}^{\lambda_2}(\tau_{k_m}(s))\, ds \ \text{ for } \ a \le t < b,$$

by the Lebesgue dominated convergence theorem and by (2.20) we obtain

$$u_1(t) = c_0 + \int_a^t g_1(s)u_2^{\lambda_1}(s)\, ds,$$
$$u_2(t) = u_2(a) + \int_a^t g_2(s)u_1^{\lambda_2}(s)\, ds \quad \text{for } \ t \in I.$$

On the other hand, in light of (2.27), it holds that

$$0 \le c_1 - u_1(t) \le \int_t^b G(s)\, ds \ \text{ for } \ t \in I.$$

Consequently, (u_1, u_2) is a solution of the problem (1.1), (1.2).

Let us now prove that the problem (1.1), (1.2) does not have solutions different from (u_1, u_2). Let $(\overline{u}_1, \overline{u}_2)$ be an arbitrary solution of (1.1), (1.2). Put

$$v_i(t) = \overline{u}_i(t) - u_i(t) \ \ (i = 1, 2).$$

Then

$$v_1'(t) = g_1(t)\big(\overline{u}_2^{\lambda_1}(t) - u_2^{\lambda_1}(t)\big), \quad v_2'(t) = g_2(t)\big(\overline{u}_1^{\lambda_2}(t) - u_1^{\lambda_2}(t)\big). \tag{2.29}$$

Moreover, v_1 and v_2 satisfy (2.13).

It is known that if $\alpha \ge 1$, then for any $x > 0$, $\overline{x} > 0$ it holds that

$$(\overline{x}^{\,\alpha} - x^{\alpha})\operatorname{sgn}(\overline{x} - x) \ge |\overline{x} - x|^{\alpha}, \ \ |\overline{x}^{\,\alpha} - x^{\alpha}| \le \alpha z^{\alpha-1}|\overline{x} - x|,$$

and if $0 < \alpha < 1$, then

$$(\overline{x}^{\,\alpha} - x^{\alpha})\operatorname{sgn}(\overline{x} - x) \ge \alpha z^{\alpha-1}|\overline{x} - x|, \quad |\overline{x}^{\,\alpha} - x^{\alpha}| \le |\overline{x} - x|^{\alpha},$$

where $z = \max\{x, \overline{x}\}$. Obviously, $z \le x + \overline{x} < 1 + x + \overline{x}$ for $x > 0$ and $\overline{x} > 0$. Therefore, (2.29) yields the inequalities (2.12_1) and (2.12_2), where

$$\omega_0(t,x) = \begin{cases} g_1(t)x^{\lambda_1} & \text{for } \lambda_1 \ge 1 \\ \lambda_1 g_1(t)\big(1 + \overline{u}_2(t) + u_2(t)\big)^{\lambda_1 - 1} x & \text{for } 0 < \lambda_1 < 1 \end{cases}$$

and for $i \in \{1,2\}$,

$$\omega_i(t,x) = \begin{cases} \lambda_i g_i(t)\big(1 + \overline{u}_{3-i}(t) + u_{3-i}(t)\big)^{\lambda_i - 1} x & \text{for } \lambda_i \ge 1 \\ g_i(t)x^{\lambda_i} & \text{for } 0 < \lambda_i < 1. \end{cases}$$

On the other hand, taking into account (2.1) and (2.2), it is obvious that the functions $\omega_i : I \times \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 0,1,2)$ satisfy the conditions (2.10) and (2.11). Applying Lemma 2.2, we get $v_i(t) \equiv 0$ $(i = 1,2)$; i.e., $\overline{u}_i(t) \equiv u_i(t)$ $(i = 1,2)$.

It remains to consider the case where $b = +\infty$. In this case, by means of the transformation

$$x = 1 - (1 + t - a)^{-1}, \quad u_i(t) = w_i(x) \;\; (i = 1,2),$$

the problem (1.1), (1.2) is reduced to the problem

$$w_1' = h_1(x)w_2^{\lambda_1}, \quad w_2' = h_2(x)w_2^{\lambda_2}, \tag{2.30}$$

$$w_1(0) = c_0, \quad \lim_{x\to 1} w_1(x) = c_1, \tag{2.31}$$

where clearly, the symbol $'$ denotes the derivative with respect to x, and

$$h_i(x) = (1-x)^{-2} g_i\big(a + (1-x)^{-1} - 1\big) \;\; (i = 1,2).$$

On the other hand, by the conditions (2.1) and (2.2) we have

$$h_1 \in L([0,1);\mathbb{R}_+),\ h_2 \in L_{\text{loc}}([0,1);\mathbb{R}_+),\ \text{meas}\big\{x \in [0,1) : h_1(x) > 0\big\} > 0, \tag{2.32}$$

$$\int_0^1 H_0(x)\,dx = \int_a^{+\infty} G_0(t)\,dt < +\infty, \tag{2.33}$$

where

$$H_0(x) = h_1(x)\Big(\int_0^x h_2(\xi)\,d\xi\Big)^{\lambda_1}.$$

However, according to the above-proved result, the conditions (2.32), (2.33) and (2.5) guarantee the unique solvability of the problem (2.30), (2.31). Consequently, the problem (1.1), (1.2) is uniquely solvable as well. □

Remark 2.1. The condition (2.2) is necessary for the unique solvability of the problem (1.1), (1.2). In fact, if $g_1(t) \equiv 0$, then for $c_0 \ne c_1$ the problem

(1.1), (1.2) does not have a solution, and for $c_0 = c_1$ it has an infinite set of solutions.

Remark 2.2. In Theorem 2.1 the condition (2.5) is optimal and it cannot be replaced by the condition

$$0 \le c_0 \le \varphi(c_1, \lambda) + \varepsilon \tag{2.34}$$

for any small $\varepsilon > 0$. Indeed, if $g_1 \in L(I; \mathbb{R}_+)$ is any function satisfying condition (2.2),

$$g_2(t) \equiv 0 \text{ and } c_0 = c_1 + \varepsilon,$$

the problem (1.1), (1.2) does not have a nonnegative solution even if all conditions of Theorem 2.1 are fulfilled except (2.5), which is replaced by (2.34).

Now we consider the problem (1.1), (1.3). Suppose

$$G_1(t) = g_1(t)\Big(1 + \int_a^t g_2(s)\,ds\Big)^{\lambda_1}.$$

For any $x > 0$ and $y > 0$, put

$$\psi(x, y, \lambda) = \begin{cases} (x^{\frac{1}{\lambda_2}} + y)^{1-\lambda} - x^{\frac{1-\lambda}{\lambda_2}} - (1-\lambda)\displaystyle\int_a^b G_1(s)\,ds & \text{for } \lambda < 1 \\ x^{\frac{1}{\lambda_2}} + y - \exp\Big(\displaystyle\int_a^b G_1(s)\,ds\Big) x^{\frac{1}{\lambda_2}} & \text{for } \lambda = 1 \\ x^{\frac{1-\lambda}{\lambda_2}} - (x^{\frac{1}{\lambda_2}} + y)^{1-\lambda} - (\lambda - 1)\displaystyle\int_a^b G_1(s)\,ds & \text{for } \lambda > 1 \end{cases}.$$

Theorem 2.2. *Let the conditions* (2.1) *hold and, moreover, either* $c_0 = 0$ *or*

$$c_0 > 0 \text{ and } \psi(c_0, c_1, \lambda) \ge 0. \tag{2.35}$$

Then the problem (1.1), (1.3) *has a unique nonnegative solution.*

To prove the theorem, we need the following lemma.

Lemma 2.3. *Assume* (2.1) *and* (2.35). *In addition, let the system of differential inequalities* (2.7) *have a solution* (v_1, v_2) *satisfying the conditions*

$$v_1(a) = 0, \quad v_2(a) = c_0. \tag{2.36}$$

Then the inequality (2.9) *is valid.*

Proof. Put

$$v(s) = c_0^{\frac{1}{\lambda_2}} + v_1(s).$$

Then we have from (2.7) and (2.36)

$$v(a+) = c_0^{\frac{1}{\lambda_2}}, \quad v(b-) = c_0^{\frac{1}{\lambda_2}} + v_1(b-), \quad v(t) \geq c_0^{\frac{1}{\lambda_2}},$$

$$0 < v_2(t) \leq c_0 + \int_0^t g_2(s) v_1^{\lambda_2}(s)\, ds \leq$$

$$\leq c_0 + v_1^{\lambda_2}(t) \int_0^t g_2(s)\, ds \leq v^{\lambda_2}(t)\Big(1 + \int_0^t g_2(s)\, ds\Big)$$

and

$$\frac{v'(t)}{v^{\lambda}(t)} = \frac{1}{v^{\lambda}(t)} \int_0^t g_1(s) v_2^{\lambda_1}(s)\, ds \leq G_1(t) \ \text{ for } \ t \in I.$$

Integrating this inequality from a to b, we get

$$\psi(c_0, v_1(b-), \lambda) \leq 0.$$

Since ψ is an increasing function with respect to the second argument, from here and (2.35) the inequality (2.9) follows. □

Proof of Theorem 2.2**.** If $g_1(s) \equiv 0$, then the unique solvability of the problem (1.1), (1.3) is obvious. Hence, in what follows, we assume that the condition (2.2) is satisfied. The proof of the existence of a nonnegative solution of the investigated problem will be omitted because it is similar to that of the problem (1.1), (1.2). The only difference is that one must use Lemma 2.3 instead of Lemma 2.1.

Now we prove that the problem (1.1), (1.3) has no more than one nonnegative solution. Assume to the contrary that there exist two different nonnegative solutions (u_1, u_2) and $(\overline{u}_1, \overline{u}_2)$.

If $\overline{u}_1(s) = u_1(s)$ for some $s \in I$, then by Theorem 2.1, $\overline{u}_1(t) = u_1(t)$ for $s < t < b$. Thus, because (u_1, u_2) and $(\overline{u}_1, \overline{u}_2)$ are different solutions, there exists $t_0 \in I$ such that $\overline{u}_1(t) \neq u_1(t)$ for $a < t < t_0$ and

$$\overline{u}_1(t_0) = u_1(t_0). \tag{2.37}$$

Without loss of generality we can assume that

$$\overline{u}_1(t) > u_1(t) \ \text{ for } \ a < t < t_0. \tag{2.38}$$

Then

$$\overline{u}_2(t) - u_2(t) = \int_a^t g_2(s)\big(\overline{u}_1^{\lambda_2}(s) - u_1^{\lambda_2}(s)\big)\, ds \geq 0 \ \text{ for } \ a < t < t_0$$

and

$$\big(\overline{u}_1(t) - u_1(t)\big)' = g_1(t)\big(\overline{u}_2^{\lambda_2}(t) - u_2^{\lambda_2}(t)\big) \geq 0 \ \text{ a.e. on } \ (a, t_0).$$

The last inequality gives a contradiction with the conditions (2.37) and (2.38). If $\overline{u}_1(s) \neq u_1(s)$ for any $s \in I$, the argument is similar and so the proof is complete. □

Remark 2.3. It can be easily seen that if $g_1 \in L(I;\mathbb{R}_+)$, $g_2 \in L_{\rm loc}(I;\mathbb{R}_+)$ and $G_0 \notin L(I;\mathbb{R}_+)$, then for any $c_0 \geq 0$ and $c_1 \in (0,+\infty)$ the problems $(1.1),(1.2)$ and $(1.1),(1.3)$ do not have solutions. Consequently, in Theorems 2.1 and 2.2 the condition $G_0 \in L(I;\mathbb{R}_+)$ is necessary.

3. Problems (1.1), (1.4) and (1.1), (1.5)

We study the problems $(1.1),(1.4)$ and $(1.1),(1.5)$ in the case when (2.1) holds but the assumption (2.2) is replaced by the stronger one

$$\text{meas}\Big\{s \in (t,b):\ g_1(s)g_2(s) > 0\Big\} > 0 \ \text{ for }\ a < t < b. \tag{3.1}$$

Below we will use the following notation:

$$\lambda = \lambda_1\lambda_2, \quad \mu = \frac{(1+\lambda_1)(1+\lambda_2)}{\lambda_1+\lambda_2+2}, \quad \mu_i = \frac{1+\lambda_i}{\mu} \ \ (i=1,2), \tag{3.2}$$

$$g(t) = \Big(\mu_1 g_1(t)\Big)^{\frac{1}{\mu_1}}\Big(\mu_2 g_2(t)\Big)^{\frac{1}{\mu_2}}, \tag{3.3}$$

$$G_\rho(t) = g_1(t)\Big(\rho + \int_a^t g_2(s)\,ds\Big)^{\lambda_1} \quad (\rho > 0). \tag{3.4}$$

Obviously, when $G_0 \in L(I;\mathbb{R}_+)$, we have $G_\rho \in L(I;\mathbb{R}_+)$. Put

$$\begin{aligned} r_1(t;\rho) &= \bigg[\Big((\lambda-1)\int_t^b G_\rho(s)\,ds\Big)^{\frac{1}{1-\lambda}} - 1\bigg]_+, \\ r_2(t;\rho) &= \int_a^t g_2(s) r_1^{\lambda_2}(s;\rho)\,ds, \end{aligned} \tag{3.5}$$

and for $g \in L(I;\mathbb{R}_+)$ put

$$r_i^*(t;\rho) = \rho + \Big((1+\lambda_i)\int_a^t g_i(\tau)\Big((\mu-1)\int_\tau^b g(s)\,ds\Big)^{\frac{\lambda_i}{1-\mu}} d\tau\Big)^{\frac{1}{1+\lambda_i}} \ \ (i=1,2). \tag{3.6}$$

Theorem 3.1. *Assume* (2.1), (3.1) *and*

$$\lambda > 1, \ \ 0 \leq c_0 < \Big((\lambda-1)\int_a^b G_0(s)\,ds\Big)^{\frac{1}{1-\lambda}}. \tag{3.7}$$

Then the problem (1.1), (1.4) *has at least one nonnegative solution* (u_1,u_2), $g \in L(I;\mathbb{R}_+)$ *and there exists a positive number* ρ *such that* (u_1,u_2) *satisfies*

$$r_i(t;\rho) \leq u_i(t) \leq r_i^*(t;\rho) \ \ \textit{for}\ \ t \in I \ \ (i=1,2). \tag{3.8}$$

To prove this theorem, we need the four lemmas below.

Lemma 3.1. *Let*

$$g_i \in L_{loc}(I;\mathbb{R}_+) \quad (i=1,2), \quad \lambda > 1, \tag{3.9}$$

and the condition (3.1) *be satisfied. In addition, let the system* (1.1) *have at least one nontrivial nonnegative solution* (u_1,u_2) *on* I. *Then*

$$\int_a^b g(s)\,ds < \infty, \tag{3.10}$$

and (u_1,u_2) *satisfies*

$$u_1(t)u_2(t) \le \Big((\mu-1)\int_t^b g(s)\,ds\Big)^{\frac{1}{1-\mu}} \quad \textit{for } \; t \in I. \tag{3.11}$$

Proof. In view of (3.1) and the fact that (u_1,u_2) is a nontrivial solution of (1.1), there exists $a_0 \in I$ such that

$$v(t) = u_1(t)u_2(t) > 0 \quad \text{for } \; a_0 < t < b. \tag{3.12}$$

Moreover,

$$\text{either } \; a_0 = a, \;\; \text{or } \; a_0 > a \;\; \text{and } \; v(t) = 0 \;\; \text{for } \; a < t < a_0. \tag{3.13}$$

It is clear that

$$v'(t) = g_1(t)u_2^{\lambda_1+1}(t) + g_2(t)u_1^{\lambda_2+1}(t). \tag{3.14}$$

Since $\lambda > 1$, it follows from (3.2) that

$$\mu > 1, \quad \mu_i > 1 \;\; (i=1,2), \quad \frac{1}{\mu_1} + \frac{1}{\mu_2} = 1.$$

Using this and the Young inequality, we get

$$g(t)v^{\mu}(t) = \Big((\mu_1 g_1(t))^{\frac{1}{\mu_1}} u_1^{\mu}(t)\Big)\Big((\mu_2 g_2(t))^{\frac{1}{\mu_2}} u_2^{\mu}(t)\Big) \le$$
$$\le g_1(t)u_2^{\lambda_1+1}(t) + g_2(t)u_1^{\lambda_2+1}(t) \quad \text{for } \; a_0 < t < b.$$

Hence it follows from (3.12) and (3.14) that

$$\frac{v'(t)}{v^{\mu}(t)} \ge g(t) \quad \text{a.e. on } (a_0, b).$$

Integrating this inequality from t to s, $a_0 < t < s < b$, we have

$$\frac{1}{v^{\mu-1}(t)} \ge \frac{1}{v^{\mu-1}(s)} + (\mu-1)\int_t^s g(\tau)\,d\tau > (\mu-1)\int_t^s g(\tau)\,d\tau,$$

and, consequently, as $s \to b$,

$$\infty > \frac{1}{v^{\mu-1}(t)} \geq (\mu - 1) \int_t^b g(\tau)\, d\tau \quad \text{for } a_0 < t < b.$$

From here and the fact that $g \in L_{\text{loc}}(I; \mathbb{R}_+)$, the inequalities (3.10) and (3.11) follow. □

Lemma 3.2. *Let the assumptions of Lemma* 3.1 *be fulfilled. Then there exists a positive constant* ρ *such that any solution* $(u_1, u_2) : I \to \mathbb{R}_+^2$ *of the system* (1.1) *satisfies the estimation*

$$u_i(t) \leq r_i^*(t; \rho) \ \text{ for } \ t \in I \ \ (i = 1, 2). \tag{3.15}$$

Proof. In view of (3.1), there exists $a_0 \in (a, b)$ such that

$$\int_a^{a_0} g_i(s)\, ds > 0 \ \ (i = 1, 2). \tag{3.16}$$

On the other hand, according to Lemma 3.1, the condition (3.10) is satisfied. Since $\lambda > 1$, we have $\mu > 1$. Put

$$\rho_1 = \Big((\mu - 1) \int_a^b g(s)\, ds \Big)^{\frac{1}{(1-\mu)\lambda_2}} \Big(\int_a^{a_0} g_2(s)\, ds \Big)^{-\frac{1}{\lambda_2}},$$
$$\rho_2 = \Big((\mu - 1) \int_a^b g(s)\, ds \Big)^{\frac{1}{(1-\mu)\lambda_1}} \Big(\int_a^{a_0} g_1(s)\, ds \Big)^{-\frac{1}{\lambda_1}},$$

and $\rho = \max\{1, \rho_1, \rho_2\}$. Let $(u_1, u_2) : I \to \mathbb{R}_+^2$ be a solution of the system (1.1). According to Lemma 3.1, the estimation (3.11) holds. If $u_1(a) \leq 1$, then, obviously,

$$u_1(a) \leq \rho. \tag{3.17}$$

We will show that this estimation holds also when $u_1(a) > 1$. In this case we have $u_1(a_0) > u_1(a) > 1$, and from here and (3.11) we get

$$u_2(a_0) < \Big((\mu - 1) \int_{a_0}^b g(s)\, ds \Big)^{\frac{1}{1-\mu}}.$$

Taking into account this and the condition (3.16), we have from (1.1)

$$u_2(a_0) = u_2(a) + \int_a^{a_0} g_2(s) u_1^{\lambda_2}(s)\, ds > u_1^{\lambda_2}(a) \int_a^{a_0} g_2(s)\, ds$$

and so

$$u_1(a) < u_2^{\frac{1}{\lambda_2}}(a_0) \Big(\int_a^{a_0} g_2(s)\, ds \Big)^{-\frac{1}{\lambda_2}} < \rho_1.$$

Hence, (3.17) holds.

From (1.1) we have

$$\big(u_1^{1+\lambda_1}(t)\big)' = (1+\lambda_1)g_1(t)\big(u_1(t)u_2(t)\big)^{\lambda_1}.$$

From here and the inequalities (3.11) and (3.17) we have

$$\big(u_1^{1+\lambda_1}(t)\big)' \le (1+\lambda_1)g_1(t)\Big((\mu-1)\int_t^b g(s)\,ds\Big)^{\frac{\lambda_1}{1-\mu}}$$

and by integration

$$u_1^{1+\lambda_1}(t) \le \rho^{1+\lambda_1} + (1+\lambda_1)\int_a^t g_1(\tau)\Big((\mu-1)\int_\tau^b g(s)\,ds\Big)^{\frac{\lambda_1}{1-\mu}} d\tau \ \text{ for } \ t \in I.$$

Consequently, using (3.2) and the inequality $(x+y)^q \le x^q + y^q$ for $x>0$, $y>0$ $(0<q<1)$, we have

$$u_1(t) \le r_1^*(t;\rho) \ \text{ for } \ t\in I.$$

Analogously, the second estimation (3.15) can be proved. □

Lemma 3.3. *Let $\lambda > 1$ and the conditions* (2.1) *and* (3.1) *be satisfied. In addition, let the system* (1.1) *have at least one nontrivial nonnegative solution (u_1,u_2) on I. Then there exists a positive number ρ such that u_1 satisfies the estimation*

$$1+u_1(t) \ge \Big(\delta + (\lambda-1)\int_t^b G_\rho(s)\,ds\Big)^{\frac{1}{1-\lambda}} \ \textit{ for } \ t\in I, \tag{3.18}$$

where

$$\delta = \lim_{t\to b}(1+u_1(t))^{1-\lambda}. \tag{3.19}$$

Proof. By Lemma 3.2, there exists $\rho>0$ such that any solution $(u_1,u_2):I\to\mathbb{R}_+^2$ of the system (1.1) satisfies the estimation $u_i(a)\le\rho$ $(i=1,2)$. Taking into account this and (3.4), from (1.1) we have

$$u_2(t) = u_2(a) + \int_a^t g_2(s)u_1^{\lambda_2}(s)\,ds \le \rho + \Big(\int_a^t g_2(s)\,ds\Big)u_1^{\lambda_2}(t) \le$$
$$\le \Big(\rho + \int_a^t g_2(s)\,ds\Big)(1+u_1(t))^{\lambda_2} \ \text{ for } \ t\in I$$

and

$$\frac{(\lambda-1)u_1'(t)}{(1+u_1(t))^\lambda} \le (\lambda-1)G_\rho(t) \ \text{ for } \ t\in I.$$

Integrating this inequality from t to b, we get

$$(1+u_1(t))^{1-\lambda} \le \delta + (\lambda-1)\int_t^b G_\rho(s)\,ds \ \text{ for } \ t\in I.$$

From here, the estimation (3.18) follows. □

In the sequel, we will also use the following obvious lemma.

Lemma 3.4. *Let* $g_i \in L_{loc}(I; \mathbb{R}_+)$ $(i = 1, 2)$ *and* $(u_{1k}, u_{2k}) : I \to \mathbb{R}^2_+$ $(k = 1, 2, \dots)$ *be a sequence of solutions of the system* (1.1) *such that*

$$\limsup_{k\to+\infty} u_{ik}(t) < +\infty \ \ \textit{for} \ \ t \in I \ \ (i = 1, 2).$$

Then there exists a solution $(u_1, u_2) : I \to \mathbb{R}^2_+$ *of the system* (1.1) *and a subsequence of the given sequence* (u_{1k_m}, u_{2k_m}) $(m = 1, 2, \dots)$ *such that for any* $b_0 \in (a, b)$,

$$\lim_{k\to\infty} u_{ik_m}(t) = u_i(t) \ \ (i = 1, 2)$$

uniformly on $[a, b_0]$.

Proof of Theorem 3.1**.** In accordance with (2.4) and (3.7), there exists $c > 0$ such that $c_0 < \varphi(c + k, \lambda)$ $(k = 1, 2, \dots)$. By Theorem 2.1, for any integer k the system (1.1) has a solution $(u_{1k}, u_{2k}) : I \to \mathbb{R}^2_+$ satisfying the boundary conditions

$$u_{1k}(a+) = c_0, \quad u_{1k}(b-) = c + k. \tag{3.20}$$

On the other hand, by Lemmas 3.1–3.3, the condition (3.10) is satisfied and there exists a positive constant ρ such that for any integer k there are satisfied the following inequalities:

$$u_{ik}(t) \le r_i^*(t; \rho) \ \ \text{for} \ \ t \in I \ \ (i = 1, 2) \tag{3.21}$$

and

$$1 + u_{1k}(t) \ge \left(\delta_k + (\lambda - 1)\int_t^b G_\rho(s)\,ds\right)^{\frac{1}{1-\lambda}} \ \ \text{for} \ \ t \in I \ \ (i = 1, 2), \tag{3.22}$$

where $\delta_k = (c + k)^{1-\lambda}$. By Lemma 3.4 and the condition (3.21) without loss of generality we can suppose that for any $b_0 \in (a, b)$ the sequence (u_{1k}, u_{2k}) $(k = 1, 2, \dots)$ is uniformly convergent on (a, b_0) to some solution $(u_1, u_2) : I \to \mathbb{R}^2_+$ of the system (1.1). Then in view of (3.5) and (3.20)–(3.22) we have that (u_1, u_2) is a solution of the problem (1.1), (1.4) satisfying the inequalities (3.8). □

Theorem 3.2. *Assume* (2.1), (3.1) *and*

$$\lambda > 1, \ \ 0 \le c_0 < \left((\lambda - 1)\int_a^b G_1(s)\,ds\right)^{\frac{\lambda_2}{1-\lambda}}. \tag{3.23}$$

Then the problem (1.1), (1.4) *has at least one nonnegative solution* (u_1, u_2) *and there exists a positive number* ρ *such that* (u_1, u_2) *satisfies* (3.8).

Theorem 3.2 can be proved analogously to Theorem 3.1. The only difference is in using Theorem 2.2 instead of Theorem 2.1. □

4. Problems for the equation (1.6)

For an arbitrary $\rho \geq 0$, we set

$$H_{1\rho}(t) = h(t)\Big(\rho + \int_0^t h_0(s)\,ds\Big)^{\ell}, \quad H_{2\rho}(t) = h_0(t)\Big(\rho + \int_0^t h(s)\,ds\Big)^{1/p}. \quad (4.1)$$

We study the problems $(1.6), (1.9_i)$ and $(1.6), (1.10_i)$ in the cases when

$$h \in L(I;\mathbb{R}_+), \quad h_0 \in L_{\mathrm{loc}}(I;(0,+\infty)), \quad H_{10} \in L(I;\mathbb{R}_+); \qquad (4.2_1)$$

$$h_0 \in L(I;(0,+\infty)), \quad h \in L_{\mathrm{loc}}(I;\mathbb{R}_+), \quad H_{20} \in L(I;\mathbb{R}_+) \qquad (4.2_2)$$

and

$$\mathrm{meas}\,\big\{t \in I:\ h(t) > 0\big\} > 0. \qquad (4.3)$$

When investigating the problems $(1.6), (1.11_i)$ and $(1.6), (1.12_i)$, instead of (4.3) we assume that

$$\mathrm{meas}\,\big\{s \in (t,b):\ h(s) > 0\big\} > 0 \ \text{ for } \ t \in I. \qquad (4.4)$$

If for some $i \in \{1,2\}$ the condition (4.2_i) is fulfilled, then for arbitrary $x > 0$ and $y > 0$ we define

$$\varphi_i(x,\ell,p) = \begin{cases} \Big[x^{\frac{p-\ell}{p}} - \dfrac{p-\ell}{p}\displaystyle\int_a^b H_{i0}(s)\,ds\Big]_+^{\frac{p}{p-\ell}} & \text{for } \ \ell < p \\ \exp\Big(-\displaystyle\int_a^b H_{i0}(s)\,ds\Big)x & \text{for } \ \ell = p\,, \\ \Big(x^{\frac{p-\ell}{p}} + \dfrac{\ell-p}{p}\displaystyle\int_a^b H_{i0}(s)\,ds\Big)^{\frac{p}{p-\ell}} & \text{for } \ \ell > p \end{cases} \qquad (4.5_i)$$

$$\psi_1(x,y,\ell,p) = \begin{cases} (x^p+y)^{\frac{p-\ell}{p}} - x^{p-\ell} - \dfrac{p-\ell}{p}\displaystyle\int_a^b H_{11}(s)\,ds & \text{for } \ \ell < p \\ x^p + y - \exp\Big(\displaystyle\int_a^b H_{11}(s)\,ds\Big)x^p & \text{for } \ \ell = p\,, \\ x^{p-\ell} - (x^p+y)^{1-\frac{\ell}{p}} - \dfrac{\ell-p}{p}\displaystyle\int_a^b H_{11}(s)\,ds & \text{for } \ \ell > p \end{cases}$$

$$\psi_2(x,y,\ell,p)=\begin{cases}(x^{1/\ell}+y)^{\frac{p-\ell}{p}}-x^{\frac{p-\ell}{\ell p}}-\dfrac{p-\ell}{p}\displaystyle\int_a^b H_{21}(s)\,ds & \text{for } \ell<p\\ x^{1/\ell}+y-\exp\Big(\displaystyle\int_a^b H_{21}(s)\,ds\Big)x^{1/\ell} & \text{for } \ell=p\,.\\ x^{\frac{p-\ell}{\ell p}}-(x^{1/\ell}+y)^{\frac{p-\ell}{p}}-\dfrac{\ell-p}{p}\displaystyle\int_a^b H_{21}(s)\,ds & \text{for } \ell>p\end{cases}$$

Theorem 4.1. *Let $i\in\{1,2\}$. Assume (4.2_i), (4.3) and the condition*

$$0\le c_0\le\varphi_i(c_1,\ell,p) \tag{4.6_i}$$

hold. Then the problem (1.6), (1.9_i) has a unique nonnegative, nondecreasing solution.

Proof. As mentioned in Introduction, formulas (1.7_i) establish a one-to-one correspondence between the set of nonnegative, nondecreasing solutions of the problem $(1.6),(1.9_i)$ and that of nonnegative solutions of the problem $(1.1),(1.2)$, where the numbers λ_1, λ_2 and the functions g_1, g_2 are given by the equalities (1.8_i). On the other hand, from the equalities $(4.1),(4.5_i)$ and the conditions (4.2_i), (4.3) and (4.6_i) follow the conditions (2.1), (2.2) and (2.5). However, by Theorem 2.1, the last three conditions guarantee the existence of a unique nonnegative solution of the problem $(1.1),(1.2)$. Consequently, the problem $(1.6),(1.9_i)$ has likewise a unique nonnegative, nondecreasing solution. □

Analogously, from Theorem 2.2 we have the following.

Theorem 4.2. *Let $i\in\{1,2\}$ and the condition (4.2_i) hold. Moreover, either $c_0=0$, or $c_0>0$ and $\psi_i(c_0,c_1,\ell,p)\ge 0$. Then the problem $(1.6),(1.10_i)$ has a unique nonnegative, nondecreasing solution.*

Let ν, ν_1, and ν_2 be the numbers and h_1 the function defined by

$$\nu=\frac{(1+\ell)(1+p)}{\ell p+2p+1}\,,\quad \nu_1=\frac{\ell p+2p+1}{1+p}\,,\quad \nu_2=\frac{\ell p+2p+1}{p(1+\ell)}\,; \tag{4.7}$$

$$h_1(t)=(\nu_1 h(t))^{1/\nu_1}(\nu_2 h(t))^{1/\nu_2}. \tag{4.8}$$

If $\ell>p$, then for an arbitrary $\rho>0$ we put

$$w_0(t;\rho)=\Big[\Big(\frac{\ell-p}{p}\int_t^b H_{1\rho}(s)\,ds\Big)^{\frac{p}{p-\ell}}-1\Big]_+^{1/p},\ \ w_1(t;\rho)=\int_a^t h_0(s)w_0(s;\rho)\,ds; \tag{4.9}$$

$$w_2(t;\rho)=\Big[\Big(\frac{\ell-p}{p}\int_t^b H_{2\rho}(s)\,ds\Big)^{\frac{p}{p-\ell}}-1\Big]_+; \tag{4.10}$$

$$w^*(t;\rho)=\rho+\Big(\Big(1+\frac{1}{p}\Big)\int_a^t h_0(\tau)\Big((\nu-1)\int_\tau^b h_1(s)\,ds\Big)^{\frac{1}{p(1-\nu)}}\,d\tau\Big)^{\frac{p}{1+p}}. \quad (4.11)$$

We study the problem $(1.6), (1.11_i)$ in the case when

$$\ell > p, \quad 0 \le c_0 < \Big(\frac{\ell - p}{p}\int_a^b H_{i0}(s)\,ds\Big)^{\frac{p}{p-\ell}}, \quad (4.12_i)$$

and the problems $(1.6), (1.12_1)$ and $(1.6), (1.12_2)$ in the cases when

$$\ell > p, \quad 0 \le c_0 < \Big(\frac{\ell - p}{p}\int_a^b H_{11}(s)\,ds\Big)^{\frac{1}{p-\ell}}, \quad (4.13_1)$$

and

$$\ell > p, \quad 0 \le c_0 < \Big(\frac{\ell - p}{p}\int_a^b H_{21}(s)\,ds\Big)^{\frac{\ell p}{p-\ell}}, \quad (4.13_2)$$

respectively.

Theorem 4.3. *Let $i \in \{1,2\}$ and the conditions (4.2_i), (4.4) and (4.12_i) hold. Then the problem $(1.6), (1.11_i)$ has at least one nonnegative, nondecreasing solution u and there exists a positive constant ρ, independent of c_0, such that u satisfies the estimation*

$$w_i(t;\rho) \le u(t) \le w^*(t;\rho) \ \text{ for } \ t \in I. \quad (4.14)$$

Theorem 4.4. *Let $i \in \{1,2\}$ and the conditions (4.2_i), (4.4) and (4.13_i) hold. Then the problem (1.6), (1.12_i) has at least one nonnegative, nondecreasing solution u and there exists a positive constant ρ, independent of c_0, such that u satisfies* (4.14).

Theorem 4.3 (Theorem 4.4) is proved similarly to Theorem 4.1. The only difference is that instead of Theorem 2.1 we use Theorem 3.1 (Theorem 3.2).

As an example, let us consider the case when $b = +\infty$,

$$a > 0, \quad \lim_{t\to a} h_0(t) = 1 \quad (4.15)$$

and there exist constants $\gamma_1 > 0$, $\gamma_2 > \gamma_1$, $\sigma \in \mathbb{R}$ and $\sigma_0 \in \mathbb{R}$ such that

$$\gamma_1 t^\sigma \le h(t) \le \gamma_2 t^\sigma, \quad \gamma_1 t^{\sigma_0} \le h_0(t) \le \gamma_2 t^{\sigma_0} \ \text{ a.e. on } \ (a,+\infty). \quad (4.16)$$

In this case, the boundary conditions (1.11_i) and (1.12_i) $(i = 1,2)$ take, respectively, the form

$$u'(a) = c_0, \quad \lim_{t\to+\infty}\big(t^{-\sigma_0}u'(t)\big) = +\infty; \quad (4.17_1)$$

$$u(a) = c_0, \quad \lim_{t\to+\infty} u(t) = +\infty; \quad (4.17_2)$$

$$u(a) = c_0, \quad \lim_{t\to+\infty}\big(t^{-\sigma_0}u'(t)\big) = +\infty; \quad (4.18_1)$$

$$u'(a) = c_0, \quad \lim_{t\to+\infty} u(t) = +\infty, \tag{4.18$_2$}$$

where $u'(a) = \lim_{t\to a} u'(t)$.

Along with (1.6), (4.17$_i$) and (1.6), (4.18$_i$) ($i = 1, 2$) we consider also for every $t_0 \in (a, +\infty)$ the problem on the existence of a blow-up solution u of the equation (1.6) which satisfies one of the following two conditions:

$$u'(a) = c_0, \quad \lim_{t\to t_0} u(t) = +\infty; \tag{4.19$_1$}$$

$$u(a) = c_0, \quad \lim_{t\to t_0} u(t) = +\infty. \tag{4.19$_2$}$$

In view of (4.1) and (4.16), to fulfil (4.2$_1$) it is necessary and sufficient that

$$\text{either } \sigma_0 > -1, \ \sigma < -1 - \ell(\sigma_0 + 1), \quad \text{or } \sigma_0 \le -1, \ \sigma < -1, \tag{4.20$_1$}$$

and to fulfil (4.2$_2$) it is necessary and sufficient that

$$\text{either } \sigma > -1, \ \sigma_0 < -1 - \frac{\sigma+1}{p}, \quad \text{or } \sigma \le -1, \ \sigma_0 < -1. \tag{4.20$_2$}$$

We consider the problems (1.6), (4.17$_i$) and (1.6), (4.19$_i$) in the case when

$$\ell > p, \quad 0 \le c_0 < \Big(\frac{\ell - p}{p}\int_a^{+\infty} H_{i0}(s)\,ds\Big)^{\frac{p}{p-\ell}}, \tag{4.21$_i$}$$

and the problems (1.6), (4.18$_1$) and (1.6), (4.18$_2$), respectively, in the cases when

$$\ell > p, \quad 0 \le c_0 < \Big(\frac{\ell - p}{p}\int_a^{+\infty} H_{11}(s)\,ds\Big)^{\frac{1}{p-\ell}}, \tag{4.22$_1$}$$

and

$$\ell > p, \quad 0 \le c_0 < \Big(\frac{\ell - p}{p}\int_a^{+\infty} H_{21}(s)\,ds\Big)^{\frac{\ell p}{p-\ell}}. \tag{4.22$_2$}$$

Corollary 4.1. *Let $i \in \{1, 2\}$ and assume* (4.15), (4.16), (4.20$_i$) *and* (4.21$_i$). *Then the problem* (1.6), (4.17$_i$) *has at least one nonnegative, nondecreasing solution u and there exist constants $\rho_1 > 0$ and $\rho_2 > \rho_1$, independent of c_0, such that u satisfies the inequalities*

$$\rho_1 \le \liminf_{t\to+\infty}(t^{-\alpha}u(t)) \le \limsup_{t\to+\infty}(t^{-\alpha}u(t)) \le \rho_2, \tag{4.23}$$

where

$$\alpha = \frac{1 + \sigma + p(1 + \sigma_0)}{p - \ell}\,. \tag{4.24}$$

Proof. According to the above, if along with (4.16) the condition (4.20_i) is fulfilled, then the condition (4.2_i) is likewise fulfilled. On the other hand, (4.16) and (4.21_i) guarantee the fulfilment of the conditions (4.4) and (4.12_i). Moreover, (4.15) shows that the boundary conditions (1.11_i) and (4.17_i) are equivalent. Consequently, all the conditions of Theorem 4.3 are fulfilled. Therefore the problem $(1.6), (1.17_i)$ has at least one nonnegative, nondecreasing solution u and every such solution satisfies (4.14), where ρ is a positive constant, independent of c_0. If $i = 1$, $\sigma_0 > -1$, $\sigma < -1 - \ell(\sigma_0 + 1)$ ($i = 2$, $\sigma > -1$, and $\sigma_0 < -1 - \frac{\sigma+1}{p}$), then in view of (4.15) and (4.16), it follows from (4.1), (4.9) and (4.10) that

$$w_i(t;\rho) \geq \rho_1 t^\alpha \quad \text{for} \quad t \geq a_0,$$

where $a_0 \geq a$ and $\rho_1 > 0$ are independent of c_0 and u. Therefore from (4.14) we find that

$$\liminf_{t\to+\infty} \big(t^{-\alpha} u(t)\big) \geq \rho_1. \tag{4.25}$$

Let us now show that the above inequality holds even in the case when $i = 1$, $\sigma_0 \leq -1$, $\sigma < -1$ ($i = 2$, $\sigma \leq -1$, and $\sigma_0 < -1$). Let $v : [a, +\infty) \to \mathbb{R}$ be a locally absolutely continuous function such that

$$v(t) = \Big(\frac{u'(t)}{h_0(t)}\Big)^p, \quad v'(t) = h(t)u^\ell(t) \;\text{ a.e. on }\; (a, +\infty). \tag{4.26}$$

Then by virtue of (4.16), almost everywhere on $(a, +\infty)$ we have

$$v^{1/p}(t)v'(t) \geq \frac{\gamma_1}{\gamma_2}\, t^{\sigma-\sigma_0} u^\ell(t)u'(t), \tag{4.27}$$

$$v^{1/p}(t)v'(t) \leq \frac{\gamma_2}{\gamma_1}\, t^{\sigma-\sigma_0} u^\ell(t)u'(t). \tag{4.28}$$

Consider first the case when $\sigma < \sigma_0$. We choose $t_0 > a$ and $t_1 > t_0$ so that $v(t_0) > 0$ and

$$\beta_1 t^{\sigma-\sigma_0} u^{\ell+1}(t_0) < v^{\frac{p+1}{p}}(t_0) \;\text{ for }\; t \geq t_1,$$

where $\beta_1 = \frac{(1+p)\gamma_1}{(1+\ell)p\gamma_2}$. Integrating (4.27) on (t_0, t), we obtain

$$\begin{aligned} v^{\frac{p+1}{p}}(t) &\geq v^{\frac{p+1}{p}}(t_0) + \frac{(p+1)\gamma_1}{p\gamma_2}\int_{t_0}^t s^{\sigma-\sigma_0} u^\ell(s)u'(s)\,ds \\ &> v^{\frac{p+1}{p}}(t_0) + \beta_1 t^{\sigma-\sigma_0}\big(u^{\ell+1}(t) - u^{\ell+1}(t_0)\big) \\ &> \beta_1 t^{\sigma-\sigma_0} u^{\ell+1}(t) \text{ for } t \geq t_1. \end{aligned}$$

Hence it follows from (4.26) that

$$v'(t)v^{-\frac{\ell(p+1)}{p(\ell+1)}}(t) < \beta_2 t^{\sigma+\frac{\ell(\sigma_0-\sigma)}{\ell+1}} \ \text{ a.e. on } \ (t_1,+\infty),$$

where $\beta_2 = \gamma_2\beta_1^{-\frac{\ell}{\ell+1}}$. The integration of this inequality on $(t,+\infty)$ yields

$$v^{\frac{p-\ell}{p(\ell+1)}}(t) < \beta_3^{\frac{p-\ell}{\ell+1}} t^{\frac{\sigma+1+\ell(\sigma_0+1)}{\ell+1}} \ \text{ for } \ t \geq t_1,$$

where

$$\beta_3 = \Big(\frac{(\ell-p)\beta_2}{p(1+\sigma+(1+\sigma_0)\ell)}\Big)^{\frac{\ell+1}{p-\ell}}.$$

Taking into account (4.16), we get from the last inequality

$$u'(t) > \gamma_1\beta_3 t^{\frac{\sigma+1+\ell(\sigma_0+1)}{p-\ell}+\sigma_0} \ \text{ for } \ t \geq t_1$$

and

$$u(t) > \rho_1 t^{\alpha} \ \text{ for } \ t \geq t_1,$$

where $\rho_1 = (\gamma_1\beta_3)/\alpha$. Consequently, the inequality (4.25) is valid.

The validity of the inequality (4.25) for $\sigma \geq \sigma_0$ can be proved analogously. In this case instead of (4.27) we have to apply the inequality (4.28).

To complete the proof, it remains to show that

$$\limsup_{t\to+\infty} \big(t^{-\alpha}u(t)\big) \leq \rho_2, \tag{4.29}$$

where ρ_2 is a positive constant, not depending on c_0 and u.

Since, along with the condition (4.20_i), the inequality $\ell > p$ is likewise fulfilled, then from (4.7) and (4.24) we obtain

$$\frac{\sigma}{\nu_1} + \frac{\sigma_0}{\nu_2} + 1 = \frac{(1+p)(1+\sigma)+p(1+\ell)(1+\sigma_0)}{\ell p+2p+1} < 0,$$
$$\Big(1+\sigma_0+\Big(\frac{\sigma}{\nu_1}+\frac{\sigma_0}{\nu_2}+1\Big)\frac{1}{p(1-\nu)}\Big)\frac{p}{1+p} = \alpha.$$

By virtue of these conditions and the inequalities (4.16), from (4.8) and (4.11) we have

$$w^*(t;\rho) \leq \rho_2 t^{\alpha} \ \text{ for } \ t \geq a,$$

where ρ_2 is a positive constant, depending only on a, p, ℓ, γ_1, γ_2, σ_0, and σ. Taking into account the above estimation, from (4.14) we obtain the inequality (4.29). □

Reasoning analogously, from Theorem 4.4 we obtain the following.

Corollary 4.2. *Let $i \in \{1,2\}$ and assume* (4.15), (4.16), (4.20$_i$) *and* (4.22$_i$). *Then the problem* (1.6), (1.18$_i$) *has at least one nonnegative, nondecreasing solution u and there exist constants $\rho_1 > 0$ and $\rho_2 > \rho_1$, independent of c_0, such that u satisfies the inequalities* (4.23), *where α is given by* (4.24).

Remark 4.1. According to Corollaries 4.1 and 4.2, it is clear that in Theorems 3.1 and 3.2 (in Theorems 4.3 and 4.4) the two-sided estimations (3.8) and (4.14) are optimal in the sense that they cannot be replaced by the estimations

$$\eta_1(t)r_i(t;\rho) \le u_i(t) \le \eta_2(t)r_i^*(t;\rho) \ \text{ for } \ t \in I \ \ (i=1,2)$$

and

$$\eta_1(t)w_i(t;\rho) \le u(t) \le \eta_2(t)w^*(t;\rho) \ \text{ for } \ t \in I,$$

where $\eta_i : [a,+\infty) \to (0,+\infty)$ $(i=1,2)$ are continuous functions such that either

$$\lim_{t\to b}\eta_1(t) = +\infty, \ \text{ or } \ \lim_{t\to b}\eta_2(t) = 0.$$

Corollary 4.3. *Let $i \in \{1,2\}$ and assume* (4.15), (4.16), (4.20$_i$) *and* (4.21$_i$) *are fulfilled. Then for an arbitrary $t_0 \in (a,+\infty)$ the problem* (1.6), (4.19$_i$) *has at least one nonnegative, nondecreasing solution u and there exist constants $\rho_1(t_0) > 0$ and $\rho_2(t_0) > \rho_1(t_0)$, independent of c_0, such that u satisfies the inequalities*

$$\rho_1(t_0) \le \liminf_{t\to t_0}\Big((t-t_0)^{\frac{p+1}{\ell-p}}u(t)\Big) \le \limsup_{t\to t_0}\Big((t-t_0)^{\frac{p+1}{\ell-p}}u(t)\Big) \le \rho_2(t_0). \quad (4.30)$$

Proof. First, notice that the boundary conditions (4.19$_1$) are equivalent to the conditions

$$u'(a) = c_0, \quad \lim_{t\to t_0} u'(t) = +\infty, \qquad (4.19_1')$$

because the functions h_0 and h are bounded in the interval (a,t_0).

Let $a_0 = t_0 - a$. Using the transformation

$$x = \frac{a_0^2}{t_0 - t}, \quad v(x) = u(t) \ \text{ for } \ a < t < t_0, \qquad (4.31)$$

the equation (1.6) is reduced to the equation

$$\bigg(\Big(\frac{v'}{f_0(x)}\Big)^p\bigg)' = f(x)v^\ell, \qquad (4.32)$$

where, clearly, the symbol $'$ denotes the derivative with respect to x, and

$$f_0(x) = \Big(\frac{a_0}{x}\Big)^2 h_0\Big(t_0 - \frac{a_0^2}{x}\Big), \quad f(x) = \Big(\frac{a_0}{x}\Big)^2 h\Big(t_0 - \frac{a_0^2}{x}\Big). \qquad (4.33)$$

As for the boundary conditions $(4.19'_1)$ and (4.19_2), they take, respectively, the form

$$v'(a_0) = c_0, \quad \lim_{x\to+\infty} (x^2 v'(x)) = +\infty \tag{4.34$_1$}$$

and

$$v(a_0) = c_0, \quad \lim_{x\to+\infty} v(x) = +\infty. \tag{4.34$_2$}$$

By virtue of the conditions (4.15), (4.16) and (4.21_i), it follows from (4.33) that

$$\begin{gathered}\lim_{x\to a_0} f_0(x) = 1,\\ \delta_1(t_0) \le x^2 f_0(x) \le \delta_2(t_0), \quad \delta_1(t_0) \le x^2 f(x) \le \delta_2(t_0) \ \text{ for } \ x \ge a_0\end{gathered} \tag{4.35}$$

and

$$\ell > p, \quad 0 \le c_0 < \Big(\frac{\ell - p}{p} \int_a^{+\infty} F_{i0}(s)\,ds\Big), \tag{4.36$_i$}$$

where $\delta_1(t_0)$ and $\delta_2(t_0)$ are positive constants, depending on t_0,

$$F_{10}(x) = f(x)\Big(\int_{a_0}^x f_0(s)\,ds\Big)^{\ell}, \quad F_{20}(x) = f_0(x)\Big(\int_{a_0}^x f_0(s)\,ds\Big)^{1/p}.$$

By Corollary 4.1, under the conditions (4.35) and (4.36_i) the problem $(4.32), (4.34_i)$ has at least one nonnegative, nondecreasing solution v and there exist constants $\eta_1(t_0) > 0$ and $\eta_2(t_0) > \eta_1(t_0)$, independent of c_0, such that v satisfies the inequalities

$$\eta_1(t_0) \le \liminf_{x\to+\infty} \big(x^{\frac{p+1}{p-\ell}} v(x)\big) \le \limsup_{x\to+\infty} \big(x^{\frac{p+1}{p-\ell}} v(x)\big) \le \eta_2(t_0).$$

On the other hand, according to the above, the transformation (4.31) establishes a one-to-one correspondence between the set of nonnegative, nondecreasing solutions of the problems $(1.6), (1.19_i)$ and $(4.32), (4.34_i)$. Therefore it is clear that the problem $(1.6), (1.19_i)$ has at least one nonnegative, nondecreasing solution and every such solution satisfies the inequalities (4.30), where

$$\rho_k(t_0) = a_0^{\frac{2(p+1)}{p-\ell}} \eta_k(t_0) \ \ (k = 1, 2). \qquad \square$$

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