

**ON THE WELL-POSEDNESS OF THE CAUCHY PROBLEM
FOR DISSIPATIVE MODIFIED
KORTEWEG–DE VRIES EQUATIONS**

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Abstract. In this paper we consider some dissipative versions of the modified Korteweg–de Vries equation $u_t + u_{xxx} + |D_x|^\alpha u + u^2 u_x = 0$ with $0 < \alpha \leq 3$. We prove some well-posedness results on the associated Cauchy problem in the Sobolev spaces $H^s(\mathbb{R})$ for $s > 1/4 - \alpha/4$ on the basis of the $[k; Z]$ -multiplier norm estimate obtained by Tao in [11] for KdV equation.

1. INTRODUCTION

The $X^{s,b}$ spaces, as used by Beals, Bourgain, Kenig–Ponce–Vega, Klainerman–Machedon and others, are fundamental tools to study the low-regularity behavior of nonlinear dispersive equations. The $X^{s,b}$ spaces were introduced by Beals in the context of the semilinear wave equation in [1] and first used by Bourgain to study nonlinear dispersive equations. Kenig, Ponce and Vega in [7] developed the method by using the $X^{s,b}$ spaces to obtain the sharp bilinear and trilinear estimates in these spaces in the case of a generalized KdV equation. Then they deduced the best well-posed result for a KdV equation from the sharp bilinear estimate. But Kenig, Ponce and Vega also pointed out that the method they used there could not improve the well-posed index for the mKdV. The sharp index $\frac{1}{4}$ for the mKdV was deduced in [6] without relying on the use of the $X^{s,b}$ spaces. Now the $X^{s,b}$ spaces have widely been used in the study of nonlinear dispersive equations.

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It is of particular interest to obtain bilinear or multilinear estimates involving these spaces. By Plancherel's theorem and duality, these estimates reduce to estimating a weighted convolution integral in terms of the L^2 norms of the component functions. In [11], Tao systematically studied the weighted convolution estimates on L^2 . In this paper, we will get a bilinear estimate on a new type of $X^{s,b}$ space adapted to the dissipative version of the modified Korteweg–de Vries equation on the basis of the $[k; Z]$ -multiplier norm estimate obtained by Tao in [11] for KdV and obtain new local well-posedness of the associated Cauchy problem. This new type of Bourgain space was first introduced by Molinet and Ribaud in [8]. We defer to introduce Tao's estimates here.

Recently, Molinet and Ribaud consider the Cauchy problem associated with dissipative Korteweg–de Vries equations

$$u_t + u_{xxx} + |D_x|^\alpha u + uu_x = 0, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}, \quad u(0) = \varphi, \quad (1)$$

where $|D_x|^\alpha$ denotes the Fourier multiplier operator with symbol $|\xi|^\alpha$. When $\alpha = 1/2$, equation (1) models the evolution of the free surface for shallow-water waves damped by viscosity. When $\alpha = 2$, equation (1) is the so-called Korteweg–de Vries–Burgers equation, which described the propagation of small-amplitude long waves in some nonlinear dispersive media when dissipative effects occur (see [10]).

In [8], Molinet and Ribaud proved the global well-posedness of (1) for data in $H^s(\mathbb{R})$, $s > -\frac{3}{4}$ for all $\alpha > 0$. In particular, when $\alpha = 2$, they proved the global well-posedness of Korteweg–de Vries–Burgers equation for data in $H^s(\mathbb{R})$, $s > -\frac{3}{4} - \frac{1}{24}$. The surprising part of this result is that the index $s = -\frac{3}{4} - \frac{1}{24}$ is lower than the best known index $s = -3/4$ obtained by Kenig, Ponce and Vega in [7] for the KdV equation and lower than the index $s = -1/2$ of the critical Sobolev space for the dissipative Burgers equation $u_t - u_{xx} + uu_x = 0$ (see [2] and [5]). In [9], Molinet and Ribaud improved this result by introducing a new Bourgain-type space and working in this space. They showed that Korteweg–de Vries–Burgers equation is globally well-posed in $H^s(\mathbb{R})$ for $s > -1$ and in some sense ill-posed in $H^s(\mathbb{R})$ for $s < -1$.

The main purpose of this paper is to consider the Cauchy problem for the following dissipative versions of the mKdV equation on the real line:

$$u_t + u_{xxx} + |D_x|^\alpha u + u^2 u_x = 0, \quad t \in \mathbb{R}_+, \quad u(0) = \varphi \quad (2)$$

with $\alpha \in (0, 3]$, and we prove local well-posedness results on the associated Cauchy problem in the Sobolev spaces $H^s(\mathbb{R})$ for $s > 1/4 - \alpha/4$ on the basis of the $[k; Z]$ -multiplier norm estimate obtained by Tao in [11] for the

KdV equation. We also get the global well-posedness for $1 < \alpha \leq 3$ when $s > \frac{1}{4} - \frac{\alpha}{4}$. It is worth pointing out that the method used to obtain the multilinear estimates by Molinet and Ribaud in [9] does not work well when α is small enough. If we consider the case $0 < \alpha \leq 1$, we can only get that problem (2) is locally well-posed for $s > \frac{1}{2} - \frac{\alpha}{2}$ by running the approach of [9]. The basic ideas of processing the multilinear estimates on $X^{s,b}$ spaces are using the Cauchy–Schwarz inequality and the algebraic smoothing relation in [9] (resonance identity in [11]). In [11], Tao utilizes dyadic decomposition and orthogonality before resorting to Cauchy–Schwarz. The advantages of dyadic decomposition and orthogonality lead to a better estimate when the algebraic smooth relation brings little benefit.

It is clear that we get a better local well-posedness for the dissipative version of the mKdV equation than the local well-posedness of mKdV. This explicates that the dissipative term in (2) plays a key role for the low regularity of the equation. We are interested in finding out the smoothing effect of the dissipative term. From the scaling analysis, the critical index of the mKdV equation is $-\frac{1}{2}$, while the critical index of the equation with only dissipative term is $1 - \frac{\alpha}{2}$. Thus we conjecture that the mean value $\frac{1}{4} - \frac{\alpha}{4}$ of the two critical indices is the critical index for our dissipative version of the mKdV. For instance, we consider the Korteweg–de Vries–Burgers equation, -1 is exactly the mean value of $-\frac{3}{2}$ the critical scaling index of KdV and $-\frac{1}{2}$ the critical scaling index of Burgers equation. Unfortunately, we can not show the ill-posedness below the index $\frac{1}{4} - \frac{\alpha}{4}$. Recently, the first two authors of this paper [4] showed that when $\alpha = 2$, (2) is ill-posed for $s < -\frac{1}{2}$ in some sense. A similar argument can also be used to get the same ill-posedness for our problem.

1.1. Notation. For a Banach space X , we denote by $\|\cdot\|_X$ the norm in X . We will use the Sobolev spaces $H^s(\mathbb{R})$ and their homogeneous versions $\dot{H}^s(\mathbb{R})$ equipped with the norms $\|u\|_{H^s} = \|(1-\Delta)^{s/2}u\|_{L^2}$ and $\|u\|_{\dot{H}^s} = \||D|^{s/2}u\|_{L^2}$. Recall that for $\lambda > 0$,

$$\|f(\lambda t)\|_{H^s} \leq (\lambda^{-1/2} + \lambda^{s-1/2})\|f(t)\|_{H^s}, \quad \|f(\lambda t)\|_{\dot{H}^s} \sim \lambda^{s-1/2}\|f(t)\|_{\dot{H}^s}. \quad (3)$$

We also consider the corresponding space-time Sobolev spaces $H_{x,t}^{s,b}$ endowed with the norm

$$\|u\|_{H^{s,b}}^2 = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} \langle \tau \rangle^{2b} |\hat{u}(\xi, \tau)|^2 d\xi d\tau, \quad (4)$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

Let $U(\cdot)$ be the unitary group in $H^s(\mathbb{R})$, $s \in \mathbb{R}$, which defined the free evolution of the KdV equation, i.e.,

$$U(t) = \exp(itP(D_x)), \tag{5}$$

where $P(D_x)$ is the Fourier multiplier with symbol $P(\xi) = \xi^3$. Since the linear symbol of equation (2) is $i(\tau - \xi^3) + |\xi|^\alpha$, by analogy with the spaces introduced by Bourgain in [3] for purely dispersive equations and by Molinet and Ribaud for the KdV–Burgers equation, we define the function space $X_\alpha^{s,b}$ endowed with the norm

$$\|u\|_{X_\alpha^{s,b}} = \|\langle i(\tau - \xi^3) + |\xi|^\alpha \rangle^b \langle \xi \rangle^s \hat{u}(\xi, \tau)\|_{L^2(\mathbb{R}^2)}. \tag{6}$$

So that

$$\|u\|_{X_\alpha^{s,b}} \sim \|\langle |\tau - \xi^3| + |\xi|^\alpha \rangle^b \langle \xi \rangle^s \hat{u}(\xi, \tau)\|_{L^2(\mathbb{R}^2)}. \tag{7}$$

Note that since $\mathcal{F}(U(-t)u)(\xi, \tau) = \mathcal{F}(u)(\xi, \tau + \xi^3)$, one can re-express the norm of $X_\alpha^{s,b}$ as

$$\begin{aligned} \|u\|_{X_\alpha^{s,b}} &= \|\langle i\tau + |\xi|^\alpha \rangle^b \langle \xi \rangle^s \hat{u}(\xi, \tau + \xi^3)\|_{L^2(\mathbb{R}^2)} \\ &= \|\langle i\tau + |\xi|^\alpha \rangle^b \langle \xi \rangle^s \mathcal{F}(U(-t)u)(\xi, \tau)\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{8}$$

For $T \geq 0$, we consider the localized spaces $X_{\alpha,T}^{s,b}$ endowed with the norm

$$\|u\|_{X_{\alpha,T}^{s,b}} = \inf_{w \in X_\alpha^{s,b}} \{\|w\|_{X_\alpha^{s,b}}, w(t) = u(t) \text{ on } [0, T]\}.$$

Finally, we denote by $W(\cdot)$ the semigroup associated with the free evolution of the equation (2); i.e.,

$$\forall t \geq 0, \mathcal{F}_x(W(t)\varphi)(\xi) = \exp[-t|\xi|^\alpha + it\xi^3]\hat{\varphi}(\xi),$$

and we extend $W(\cdot)$ to a linear operator defined on the whole real axis by setting

$$\forall t \in \mathbb{R}, \mathcal{F}_x(W(t)\varphi)(\xi) = \exp[-|t||\xi|^\alpha + it\xi^3]\hat{\varphi}(\xi). \tag{9}$$

1.2. Main results. We will mainly work on the integral formulation of (2), i.e.,

$$u(t) = W(t)\varphi - \frac{1}{3} \int_0^t W(t-t')\partial_x(u^3(t'))dt', \quad t \geq 0. \tag{10}$$

Actually, to prove the local existence result, we shall apply a fixed-point argument to the following truncated version of (10):

$$u(t) = \psi(t) \left[W(t)\varphi - \frac{\chi_{\mathbb{R}_+}(t)}{3} \int_0^t W(t-t')\partial_x(\psi_T^3(t')u^3(t'))dt' \right], \tag{11}$$

where $t \in \mathbb{R}$ and, in the sequel of this paper, ψ is a time cut-off function satisfying

$$\psi \in C_0^\infty(\mathbb{R}), \quad \text{supp } \psi \subset [-2, 2], \quad \psi \equiv 1 \quad \text{on } [-1, 1],$$

and $\psi_T(\cdot) = \psi(\cdot/T)$. Indeed, if u solves (11) then u is a solution of (10) on $[0, T]$, $T < 1$.

Let us first state our local well-posedness result on the real line. In this paper we restrict ourselves to $0 < \alpha \leq 3$.

Theorem 1. *Let $\varphi \in H^s(\mathbb{R})$, $s > 1/4 - \alpha/4$. Then there exist some $T > 0$ and a unique solution u of (10) in*

$$Z_T = C([0, T], H^s) \cap X_{\alpha, T}^{s, 1/2}. \quad (12)$$

Moreover, the map $\varphi \mapsto u$ is smooth from $H^s(\mathbb{R})$ to Z_T .

If the dissipative effect is strong enough, we can also get the global well-posedness.

Theorem 2. *Let $\varphi \in H^s(\mathbb{R})$, $s > 1/4 - \alpha/4$ and $1 < \alpha \leq 3$. The existence time of the solution of (10) can be extended to infinity. And u belongs to $C((0, +\infty), H^\infty(\mathbb{R}))$.*

Remark. We can not get the global well-posedness for the case $0 < \alpha \leq 1$, since in this case one needs a higher-order a priori estimate than L^2 , while in the case $1 < \alpha \leq 3$, we need only the a priori estimate in L^2 . It will be an interesting question to find out a suitable a priori estimate for the dissipative version of the mKdV and then extend the local well-posedness to the global well-posedness when the dissipative effect is very weak.

This paper is organized as follows. In Section 2 we prove linear estimates in the function space $X_\alpha^{s, 1/2}$, and in Section 3 we introduce Tao's $[k; Z]$ -multiplier norm estimate and derive a trilinear estimate for the nonlinear term $\partial_x(u^3)$ from Tao's estimate. In Section 4, we consider the local well-posedness, while the global well-posedness will be in Section 5.

2. LINEAR ESTIMATES

In this section we study the linear operator $\psi(\cdot)W(\cdot)$ as well as the linear operator L defined by

$$L : f \mapsto \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-t')f(t')dt'. \quad (13)$$

These linear estimates are principally contained in [9].

2.1. **Linear estimate for the free term.**

Lemma 1. *Let $s \in \mathbb{R}$. There exists $C > 0$ such that*

$$\|\psi(t)W(t)\varphi\|_{X_\alpha^{s,1/2}} \leq C\|\varphi\|_{H^s}, \forall \varphi \in H^s(\mathbb{R}). \tag{14}$$

Proof. By definition of $\|\cdot\|_{X_\alpha^{s,1/2}}$,

$$\begin{aligned} & \|\psi(t)W(t)\varphi\|_{X_\alpha^{s,1/2}} \\ &= \left\| \langle \xi \rangle^s \langle i\tau - \xi^3 \rangle + |\xi|^\alpha \mathcal{F}_t \left(\psi(t)e^{-|t||\xi|^\alpha} e^{it\xi^3} \hat{\varphi}(\xi) \right) (\tau) \right\|_{L_{\xi,\tau}^2} \\ &= \left\| \langle \xi \rangle^s \hat{\varphi}(\xi) \right\| \left\| \langle i\tau + |\xi|^\alpha \rangle \mathcal{F}_t \left(\psi(t)e^{-|t||\xi|^\alpha} \right) (\tau) \right\|_{L_\tau^2} \left\| \right\|_{L_\xi^2} \\ &\leq C \left\| \langle \xi \rangle^s \hat{\varphi}(\xi) \right\| \left\| \langle \tau \rangle^{1/2} \mathcal{F}_t \left(\psi(t)e^{-|t||\xi|^\alpha} \right) (\tau) \right\|_{L_\tau^2} \left\| \right\|_{L_\xi^2} \\ &\quad + C \left\| \langle \xi \rangle^{s+\alpha/2} \hat{\varphi}(\xi) \right\| \left\| \mathcal{F}_t \left(\psi(t)e^{-|t||\xi|^\alpha} \right) (\tau) \right\|_{L_\tau^2} \left\| \right\|_{L_\xi^2} \leq C\|\varphi\|_{H^s}, \end{aligned} \tag{15}$$

where we used the fact that

$$\|\psi(t)e^{-|t||\xi|^\alpha}\|_{H_t^b} \leq C\langle \xi \rangle^{\frac{\alpha}{2}(2b-1)}, \forall 0 \leq b \leq 1,$$

which can be obtained from (19) in [9].

2.2. **Linear estimates for the forcing term.**

Lemma 2. *For $w \in \mathcal{S}(\mathbb{R}^2)$ consider k_ξ defined on \mathbb{R} by*

$$k_\xi(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau} - e^{-|t||\xi|^\alpha}}{i\tau + |\xi|^\alpha} \hat{w}(\xi, \tau) d\tau. \tag{16}$$

Then, it holds for all $\xi \in \mathbb{R}$ that

$$\begin{aligned} & \left\| \langle i\tau + |\xi|^\alpha \rangle^{1/2} \mathcal{F}_t(k_\xi) \right\|_{L^2(\mathbb{R})}^2 \\ & \leq C \left[\left(\int_{\mathbb{R}} \frac{|\hat{w}(\xi, \tau)|}{\langle i\tau + |\xi|^\alpha \rangle} d\tau \right)^2 + \left(\int_{\mathbb{R}} \frac{|\hat{w}(\xi, \tau)|^2}{\langle i\tau + |\xi|^\alpha \rangle} d\tau \right) \right]. \end{aligned} \tag{17}$$

By a little modification of Proposition 2 in [9], we can obtain the proof of Lemma 2.

Lemma 3. *Let $s \in \mathbb{R}$.*

a) *There exists $C > 0$ such that for all $v \in \mathcal{S}(\mathbb{R}^2)$,*

$$\left\| \chi_{\mathbb{R}^+}(t)\psi(t) \int_0^t W(t-t')v(t')dt' \right\|_{X_\alpha^{s,1/2}}$$

$$\leq C \left[\|v\|_{X_\alpha^{s, -1/2}} + \left(\int \langle \xi \rangle^{2s} \left(\int \frac{|\hat{v}(\xi, \tau + \xi^3)|}{\langle i\tau + |\xi|^\alpha \rangle} d\tau \right)^2 d\xi \right)^{1/2} \right]. \tag{18}$$

b) For any $0 < \delta < 1/2$ there exists $C_\delta > 0$ such that for all $v \in X_\alpha^{s, -1/2+\delta}$

$$\left\| \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-t')v(t')dt' \right\|_{X_\alpha^{s, 1/2}} \leq C_\delta \|v\|_{X_\alpha^{s, -1/2+\delta}}. \tag{19}$$

Proof. Assume that $v \in \mathcal{S}(\mathbb{R}^2)$. Taking the x -Fourier transform we get

$$\begin{aligned} \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-t')v(t')dt' \\ = U(t) \left[\chi_{\mathbb{R}_+}(t)\psi(t) \int_{\mathbb{R}} e^{ix\xi} \int_0^t e^{-|t-t'|\xi|^\alpha} \mathcal{F}_x(U(-t')v(t'))(\xi) dt' d\xi \right]. \end{aligned}$$

Set $\omega(t') = U(-t')v(t')$. Writing $\mathcal{F}_x(\omega)(\xi, t)$ with the help of its time Fourier transform and using Fubini's theorem one infers that

$$\begin{aligned} \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-t')v(t')dt' \\ = U(t) \left[\chi_{\mathbb{R}_+}(t)\psi(t) \int_{\mathbb{R}^2} e^{ix\xi} e^{-t|\xi|^\alpha} \hat{\omega}(\xi, \tau) \int_0^t e^{it'\tau} e^{t'|\xi|^\alpha} dt' d\xi d\tau \right] \\ = U(t) \left[\chi_{\mathbb{R}_+}(t)\psi(t) \int_{\mathbb{R}^2} e^{ix\xi} \frac{e^{it\tau} - e^{-t|\xi|^\alpha}}{i\tau + |\xi|^\alpha} \hat{\omega}(\xi, \tau) d\xi d\tau \right]. \tag{20} \end{aligned}$$

We set $k_\xi(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau} - e^{-t|\xi|^\alpha}}{i\tau + |\xi|^\alpha} \hat{\omega}(\xi, \tau) d\tau$. Since $\omega(t) = U(-t)v(t) \in \mathcal{S}(\mathbb{R}^2)$, it is clear that for any fixed $\xi \in \mathbb{R}$, k_ξ is continuous on \mathbb{R} and $k_\xi(0) = 0$. Then it is not too hard to derive that $\|\chi_{\mathbb{R}_+}(t)k_\xi(t)\|_{H_t^b} \leq \|k_\xi(t)\|_{H_t^b}$, $0 \leq b \leq 1$, and $b \neq 1/2$. The case $b = 1/2$ follows by Lebesgue's dominated convergence theorem. Thus, in view of (20)

$$\begin{aligned} \left\| \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-t')v(t')dt' \right\|_{X_\alpha^{s, 1/2}} &= \left\| \langle i\tau + |\xi|^\alpha \rangle^{1/2} \langle \xi \rangle^s \right. \\ &\quad \times \mathcal{F}_{x,t} \left(U(-t)\chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-t')v(t')dt' \right) \left. \right\|_{L_{\xi,\tau}^2} \\ &\leq \left\| \langle \xi \rangle^s \right\| \left\| \mathcal{F}_x \left(U(-t)\chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-t')v(t')dt' \right) \right\|_{H_\tau^{1/2}} \left\| \right\|_{L_\xi^2} \\ &\quad + \left\| \langle \xi \rangle^{s+\alpha/2} \right\| \left\| \mathcal{F}_x \left(U(-t)\chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-t')v(t')dt' \right) \right\|_{L_\tau^2} \left\| \right\|_{L_\xi^2} \end{aligned}$$

$$\begin{aligned}
 &\leq \left\| \langle \xi \rangle^s \|\chi_{\mathbb{R}^+}(t)k_\xi(t)\|_{H_t^{1/2}} \right\|_{L_\xi^2} + \left\| \langle \xi \rangle^{s+\alpha/2} \|\chi_{\mathbb{R}^+}(t)k_\xi(t)\|_{L_t^2} \right\|_{L_\xi^2} \\
 &\leq \left\| \langle \xi \rangle^s \|k_\xi(t)\|_{H_t^{1/2}} \right\|_{L_\xi^2} + \left\| \langle \xi \rangle^{s+\alpha/2} \|k_\xi(t)\|_{L_t^2} \right\|_{L_\xi^2} \\
 &\leq C \left\| \langle \xi \rangle^s \left\| \langle i\tau + |\xi|^\alpha \rangle^{1/2} \mathcal{F}_t(k_\xi(t)) \right\|_{L_\tau^2} \right\|_{L_\xi^2}.
 \end{aligned}$$

Then (18) follows directly from Lemma 2 together with the last estimate. To prove (19) we first assume that $v \in \mathcal{S}(\mathbb{R}^2)$. Applying the Cauchy–Schwartz inequality in τ on the second term of the right-hand side of (18), one obtains (19) for $v \in \mathcal{S}(\mathbb{R}^2)$. The result for $v \in X_\alpha^{s, -1/2+\delta}$ follows by density.

3. TAO’S $[k; Z]$ -MULTIPLIER NORM ESTIMATE AND ITS APPLICATION

In this section we introduce Tao’s $[k; Z]$ -multiplier norm estimate and derive the trilinear estimate needed to obtain the local existence result from Tao’s multiplier norm estimate for KdV equation.

Let Z be any abelian additive group with an invariant measure $d\xi$. For any integer $k \geq 2$, we let $\Gamma_k(Z)$ denote the hyperplane

$$\Gamma_k(Z) := \{(\xi_1, \dots, \xi_k) \in Z^k : \xi_1 + \dots + \xi_k = 0\}$$

which is endowed with the measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi_1, \dots, \xi_{k-1}, -\xi_1 - \dots - \xi_{k-1}) d\xi_1 \cdots d\xi_{k-1}.$$

A $[k; Z]$ -multiplier is defined to be any function $m : \Gamma_k(Z) \rightarrow \mathbb{C}$, which was introduced by Tao in [11], and the multiplier norm $\|m\|_{[k; Z]}$ is defined to be the best constant such that the inequality

$$\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^k f_j(\xi_j) \right| \leq \|m\|_{[k; Z]} \prod_{j=1}^k \|f_j\|_{L^2(Z)} \tag{21}$$

holds for all test functions f_j on Z . Tao systematically studied this kind of weighted convolution estimate on L^2 in [11]. To state Tao’s results, we use some notation he used in his paper.

We use $A \lesssim B$ to denote the statement that $A \leq CB$ for some large constant C which may vary from line to line and depend on various parameters, and similarly use $A \ll B$ to denote the statement $A \leq C^{-1}B$. We use $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$.

Any summations over capitalized variables such as N_j , L_j and H are presumed to be dyadic; i.e., these variables range over numbers of the form 2^k for $k \in \mathbb{Z}$. In this paper, we will consider only the $[3; Z]$ -multiplier. Let $N_1, N_2, N_3 > 0$. It will be convenient to define the quantities $N_{max} \geq N_{med} \geq N_{min}$ to be the maximum, median, and minimum of N_1, N_2 and N_3 respectively. Similarly define $L_{max} \geq L_{med} \geq L_{min}$ whenever $L_1, L_2, L_3 > 0$. We also adopt the following summation conventions. Any summation of the form $L_{max} \sim \dots$ is a sum over the three dyadic variables $L_1, L_2, L_3 \gtrsim 1$; thus, for instance

$$\sum_{L_{max} \sim H} := \sum_{L_1, L_2, L_3 \gtrsim 1: L_{max} \sim H} .$$

Similarly, any summation of the form $N_{max} \sim \dots$ sums over the three dyadic variables $N_1, N_2, N_3 > 0$; thus, for instance

$$\sum_{N_{max} \sim N_{med} \sim N} := \sum_{N_1, N_2, N_3 > 0: N_{max} \sim N_{med} \sim N} .$$

If τ, ξ and $h(\cdot)$ are given, we also adopt the convention that λ is shorthand for $\lambda := \tau - h(\xi)$. Similarly, we have

$$\lambda_j := \tau_j - h_j(\xi_j). \tag{22}$$

In this paper, we do not go further on the general framework of Tao’s weighted convolution estimates. We focus our attention to the $[k; Z]$ -multiplier norm estimate for the KdV equation. Throughout the estimate we need the resonance function

$$h(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3 = -\lambda_1 - \lambda_2 - \lambda_3, \tag{23}$$

which measures to what extent the spatial frequencies ξ_1, ξ_2 and ξ_3 can resonate with each other.

By dyadic decomposition of the variables ξ_j and λ_j , as well as the function $h(\xi)$, one is led to consider

$$\|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]}, \tag{24}$$

where $X_{N_1, N_2, N_3; H; L_1, L_2, L_3}$ is the multiplier

$$X_{N_1, N_2, N_3; H; L_1, L_2, L_3}(\xi, \tau) := \chi_{|h(\xi)| \sim H} \prod_{j=1}^3 \chi_{|\xi_j| \sim N_j} \chi_{|\lambda_j| \sim L_j}. \tag{25}$$

From the identities $\xi_1 + \xi_2 + \xi_3 = 0$ and $\lambda_1 + \lambda_2 + \lambda_3 + h(\xi) = 0$ on the support of the multiplier, we see that $X_{N_1, N_2, N_3; H; L_1, L_2, L_3}$ vanishes unless

$$N_{max} \sim N_{med} \tag{26}$$

and

$$L_{max} \sim \max(H, L_{med}). \tag{27}$$

From the resonance identity

$$h(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3 = 3\xi_1\xi_2\xi_3 \tag{28}$$

we see that we may assume that

$$H \sim N_1N_2N_3, \tag{29}$$

since the multiplier in (24) vanishes otherwise.

Now we are in a position to state Tao’s $[k; Z]$ -multiplier norm estimate for the KdV equation in the non-periodic case.

Lemma 4. (See Proposition 6.1 in [11]). *Let $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ satisfy (26), (27) and (29).*

- ((++) Coherence) *If $N_{max} \sim N_{min}$ and $L_{max} \sim H$, then we have*

$$(24) \lesssim L_{min}^{1/2} N_{max}^{-1/4} L_{med}^{1/4}. \tag{30}$$

- ((+-) Coherence) *If $N_2 \sim N_3 \gg N_1$ and $H \sim L_1 \gtrsim L_2, L_3$, then*

$$(24) \lesssim L_{min}^{1/2} N_{max}^{-1} \min(H, \frac{N_{max}}{N_{min}} L_{med})^{1/2}, \tag{31}$$

similarly for permutations.

- *In all other cases, we have*

$$(24) \lesssim L_{min}^{1/2} N_{max}^{-1} \min(H, L_{med})^{1/2}. \tag{32}$$

With Lemma 4 one can now derive a trilinear estimate involving the new Bourgain spaces $X_\alpha^{s,b}$.

Lemma 5. *Let $s > 1/4 - \alpha/4$. For all u_1, u_2 and u_3 on $\mathbb{R} \times \mathbb{R}$ and $0 < \epsilon \ll 1$, we have*

$$\|\partial_x(u_1u_2u_3)\|_{X_\alpha^{s,-1/2+\epsilon}} \lesssim \|u_1\|_{X_\alpha^{s,1/2}} \|u_2\|_{X_\alpha^{s,1/2}} \|u_3\|_{X_\alpha^{s,1/2}}. \tag{33}$$

As seen in [9], Theorem 1 can be reduced to the trilinear estimate Lemma 5 and the linear estimates we obtained in Section 2. What we need to do in the following is to prove the trilinear estimate.

Proof of Lemma 5. By duality and Plancherel it suffices to show that

$$\left\| \frac{(\xi_1 + \xi_2 + \xi_3)\langle \xi_4 \rangle^s}{\langle \tau_4 - \xi_4^3 + i|\xi_4|^\alpha \rangle^{1/2-\epsilon} \prod_{j=1}^3 \langle \xi_j \rangle^s \langle \tau_j - \xi_j^3 + i|\xi_j|^\alpha \rangle^{1/2}} \right\|_{[4,\mathbb{R} \times \mathbb{R}]} \lesssim 1.$$

We estimate $|\xi_1 + \xi_2 + \xi_3|$ by $\langle \xi_4 \rangle$. It follows that

$$\langle \xi_4 \rangle^{s+1} \lesssim \langle \xi_4 \rangle^{1/2} \sum_{j=1}^3 \langle \xi_j \rangle^{s+1/2},$$

where we assume $s > -\frac{1}{2}$, and symmetry to reduce to

$$\left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_3 \rangle^{-s} \langle \xi_2 \rangle^{1/2} \langle \xi_4 \rangle^{1/2}}{\langle \tau_4 - \xi_4^3 + i|\xi_4|^\alpha \rangle^{1/2-\epsilon} \prod_{j=1}^3 \langle \tau_j - \xi_j^3 + i|\xi_j|^\alpha \rangle^{1/2}} \right\|_{[4, \mathbb{R} \times \mathbb{R}]} \lesssim 1.$$

We may replace $\langle \tau_2 - \xi_2^3 + i|\xi_2|^\alpha \rangle^{1/2}$ by $\langle \tau_2 - \xi_2^3 + i|\xi_2|^\alpha \rangle^{1/2-\epsilon}$. By the TT^* identity (see Lemma 3.7, p. 847 in [11]), the estimate is reduced to the following bilinear estimate.

Lemma 6. (*Bilinear estimate*). *Let $s > 1/4 - \alpha/4$. For all u , and v on $\mathbb{R} \times \mathbb{R}$ and $0 < \epsilon \ll 1$, we have*

$$\|uv\|_{L^2(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X_\alpha^{-1/2, 1/2-\epsilon}(\mathbb{R} \times \mathbb{R})} \|v\|_{X_\alpha^{s, 1/2}(\mathbb{R} \times \mathbb{R})}. \tag{34}$$

Proof of the bilinear estimate. By Plancherel it suffices to show that

$$\left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{1/2}}{\langle \tau_1 - \xi_1^3 + i|\xi_1|^\alpha \rangle^{1/2} \langle \tau_2 - \xi_2^3 + i|\xi_2|^\alpha \rangle^{1/2-\epsilon}} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim 1. \tag{35}$$

There are two approaches to computing these quantities and their generalization. One approach proceeds using the Cauchy–Schwarz inequality, thus reducing matters to integrating certain weights on intersections of hypersurfaces $\tau = h(\xi)$; the other utilizes dyadic decomposition and orthogonality before resorting to Cauchy–Schwarz. The advantages of dyadic decomposition are that one can re-use the estimates on dyadic blocks to prove other estimates, and the nature of interactions between different scales of frequency is more apparent.

By dyadic decomposition of the variables ξ_j , λ_j and $h(\xi)$, we may assume that $|\xi_j| \sim N_j$, $|\lambda_j| \sim L_j$ and $|h(\xi)| \sim H$. By the translation invariance of the $[k; Z]$ -multiplier norm, we can always restrict our estimate on $\lambda_j \gtrsim 1$ and $\max(N_1, N_2, N_3) \gtrsim 1$. The comparison principle and orthogonality (see Schur’s test in [11], p. 851) reduce our estimate to show that

$$\begin{aligned} & \sum_{N_{max} \sim N_{med} \sim N} \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{1/2}}{\max(L_1, \langle N_1 \rangle^\alpha)^{1/2} \max(L_2, \langle N_2 \rangle^\alpha)^{1/2-\epsilon}} \\ & \times \|X_{N_1, N_2, N_3; L_{max}; L_1, L_2, L_3}\|_{[3; \mathbb{R} \times \mathbb{R}]} \lesssim 1 \end{aligned} \tag{36}$$

and

$$\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim L_{med}} \sum_{H \ll L_{max}} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{1/2}}{\max(L_1, \langle N_1 \rangle^\alpha)^{1/2} \max(L_2, \langle N_2 \rangle^\alpha)^{1/2-\epsilon}} \times \|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3; \mathbb{R} \times \mathbb{R}]} \lesssim 1 \tag{37}$$

for all $N \gtrsim 1$. This can be accomplished by Tao’s estimate on dyadic blocks for the KdV equation, i.e., Lemma 4 and some concrete summation.

Fix $N \gtrsim 1$. We first prove (37). We may assume (29). By the crude estimate

$$(24) \lesssim L_{min}^{1/2} N_{min}^{1/2},$$

we reduce to

$$\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim L_{med} \gtrsim N_1 N_2 N_3} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{1/2}}{\max(L_1, \langle N_1 \rangle^\alpha)^{1/2} \max(L_2, \langle N_2 \rangle^\alpha)^{1/2-\epsilon}} L_{min}^{1/2} N_{min}^{1/2} \lesssim 1.$$

This high-modulation case is easier to handle. We do not need to consider the effect of the dissipative term, i.e., the part with symbol $|\xi|^\alpha$. Just crudely estimating,

$$\langle N_1 \rangle^{-s} \langle N_2 \rangle^{1/2} \lesssim N^{1/2+\max(0, -s)}$$

and

$$\max(L_1, \langle N_1 \rangle^\alpha)^{1/2} \max(L_2, \langle N_2 \rangle^\alpha)^{1/2-\epsilon} \gtrsim L_{min}^{1/2} L_{med}^{1/2-\epsilon};$$

then performing the L summations, we reduce to

$$\sum_{N_{max} \sim N_{med} \sim N} \frac{N_{min}^\epsilon}{N^{1/2-\max(0, -s)-2\epsilon}} \lesssim 1,$$

which is true if $1/2 - \max(0, -s) > 0$. So, (37) is true if $s > -1/2$.

Now we show the low-modulation case (36). We may assume $L_{max} \sim N_1 N_2 N_3$. We first deal with the contribution where (30) holds. In this case we have $N_1, N_2, N_3 \sim N \gtrsim 1$, so we reduce to

$$\sum_{L_{max} \sim N^3} \frac{N^{-s} N^{1/2}}{\max(L_1, N^\alpha)^{1/2} \max(L_2, N^\alpha)^{1/2-\epsilon}} L_{min}^{1/2} N^{-1/4} L_{med}^{1/4} \lesssim 1. \tag{38}$$

Note that

$$\max(L_1, N^\alpha)^{\frac{1}{2}} \max(L_2, N^\alpha)^{\frac{1}{2}-\epsilon} \gtrsim \max(L_{min}, N^\alpha)^{\frac{1}{2}} \max(L_{med}, N^\alpha)^{\frac{1}{2}-\epsilon}.$$

We may assume that $L_1 \leq L_2 \leq L_3$. We now need consider three subcases: $L_2 \leq N^\alpha, L_1 \leq N^\alpha \leq L_2$ and $N^\alpha \leq L_1$.

If $L_2 \leq N^\alpha$, we reduce to

$$\sum_{L_1 \leq L_2 \leq N^\alpha \lesssim L_3 \sim N^3} \frac{N^{-s} N^{1/4} L_1^{1/2} L_2^{1/4}}{N^{\alpha/2} N^{\alpha(1/2-\epsilon)}} \lesssim \frac{N^{1/4-s} N^{3\alpha/4}}{N^{\alpha-\alpha\epsilon}} \lesssim 1,$$

which is true if $1/4 - s - \alpha/4 < 0$, i.e., $s > 1/4 - \alpha/4$.

If $L_1 \leq N^\alpha \leq L_2$, we reduce to

$$\sum_{L_1 \leq N^\alpha \leq L_2 \leq L_3 \sim N^3} \frac{N^{-s} N^{1/4} L_1^{1/2} L_2^{1/4}}{N^{\alpha/2} L_2^{1/2-\epsilon}} \lesssim \frac{N^{1/4-s} N^{\alpha/2}}{N^{\alpha/2} N^{(1/4-\epsilon)\alpha}} \lesssim 1,$$

which holds if $s > 1/4 - \alpha/4$.

If $N^\alpha \leq L_1$, we reduce to

$$\sum_{N^\alpha \leq L_1 \leq L_2 \leq L_3 \sim N^3} \frac{N^{-s} N^{1/4} L_1^{1/2} L_2^{1/4}}{L_1^{1/2} L_2^{1/2-\epsilon}} \lesssim \frac{N^{1/4-s}}{N^{(1/4-\epsilon)\alpha}} \lesssim 1,$$

which is true if $s > 1/4 - \alpha/4$.

Now we deal with the cases where (31) applies. We do not have perfect symmetry and must consider three cases separately:

$$\begin{aligned} N \sim N_1 \sim N_2 \gg N_3; & \quad H \sim L_3 \gtrsim L_1, L_2 \\ N \sim N_2 \sim N_3 \gg N_1; & \quad H \sim L_1 \gtrsim L_2, L_3 \\ N \sim N_1 \sim N_3 \gg N_2; & \quad H \sim L_2 \gtrsim L_1, L_3. \end{aligned}$$

In the first case we reduce by (31) to

$$\begin{aligned} & \sum_{N_3 \ll N} \sum_{1 \lesssim L_1, L_2 \lesssim N^2 N_3} \frac{N^{1/2-s}}{\max(L_1, N^\alpha)^{1/2} \max(L_2, N^\alpha)^{1/2-\epsilon}} \\ & \times L_{min}^{1/2} N^{-1} \min(N^2 N_3, \frac{N}{N_3} L_{med})^{1/2} \lesssim 1. \end{aligned}$$

Performing the N_3 summation we reduce to

$$\sum_{1 \lesssim L_1, L_2 \lesssim N^3} \frac{N^{1/2-s}}{\max(L_1, N^\alpha)^{1/2} \max(L_2, N^\alpha)^{1/2-\epsilon}} L_{min}^{1/2} N^{-1} N^{3/4} L_{med}^{1/4} \lesssim 1,$$

which is the same as (38), and so it is true if $s > 1/4 - \alpha/4$.

We can deal with the second and third cases in a unified way. By asymmetry it suffices to show the worst case. We simplify using the first half of (31) to

$$\sum_{N_{min} \ll N} \sum_{1 \lesssim L_{min}, L_{med} \ll N^2 N_{min}} \frac{\langle N_{min} \rangle^{-s} N^{1/2}}{L_{min}^{1/2} L_{max}^{1/2-\epsilon}} L_{min}^{1/2} N_{min}^{1/2} \lesssim 1.$$

We may assume $N_{min} \gtrsim N^{-2}$ since the inner sum vanishes otherwise. Performing the L summation we reduce to

$$\sum_{N^{-2} \lesssim N_{min} \ll N} \frac{\langle N_{min} \rangle^{-s} N^{1/2} N_{min}^{1/2}}{(N^2 N_{min})^{1/2-\epsilon}} \lesssim 1,$$

which holds if $s > -1/2$.

To finish the proof of (36) it remains to deal with the cases where (32) holds. This reduces to

$$\begin{aligned} & \sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim N_1 N_2 N_3} \\ & \times \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{1/2}}{\max(L_1, \langle N_1 \rangle^\alpha)^{1/2} \max(L_2, \langle N_2 \rangle^\alpha)^{1/2-\epsilon}} L_{min}^{1/2} N^{-1} L_{med}^{1/2} \lesssim 1. \end{aligned}$$

Performing the L summations, we reduce to

$$\sum_{N_{max} \sim N_{med} \sim N} \langle N_1 \rangle^{-s} \langle N_2 \rangle^{1/2} (N_1 N_2 N_3)^\epsilon N^{-1} \lesssim 1,$$

which is true if $s > -1/2$.

4. LOCAL WELL-POSEDNESS

Let φ in $H^s(\mathbb{R})$, $s > \frac{1}{4} - \frac{\alpha}{4}$. We first prove the existence of a solution u of the integral formulation (10) of the equation (2) on some interval $[0, T]$ for $T \leq 1$ small enough. Clearly, if u is a solution of the integral equation $u = F(u)$ with

$$F(u) = \psi(t) \left[W(t)\varphi - \chi_{\mathbb{R}_+}(t) \int_0^t W(t-t') \partial_x (\psi_T^3(t') u^3(t')) dt' \right], \tag{39}$$

then u is a solution of (10) on $[0, T]$. We are going to solve (39) in the space

$$Z = \{u \in X^{s, 1/2}, \|u\|_Z = \|u\|_{X^{s_c^+, 1/2}} + \nu \|u\|_{X^{s, 1/2}} < +\infty\}, \tag{40}$$

where $s_c^+ \in (\frac{1}{4} - \frac{\alpha}{4}, s)$ is fixed, and where the constant ν is defined for all nontrivial φ by

$$\nu = \frac{\|\varphi\|_{H^{s_c^+}}}{\|\varphi\|_{H^s}}. \tag{41}$$

To get a contractive factor in our argument, we need the following modified trilinear estimates.

Lemma 7. *Given $s > \frac{1}{4} - \frac{\alpha}{4}$, there exist $C, \mu, \delta > 0$ such that for any triple $(u, v, w) \in X^{s, 1/2}$ with compact support in $[-T, T]$*

$$\|\partial_x(uvw)\|_{X^{s, -1/2+\delta}} \leq CT^\mu \|u\|_{X^{s, 1/2}} \|v\|_{X^{s, 1/2}} \|w\|_{X^{s, 1/2}}. \tag{42}$$

The following lemma is a direct consequence of Lemma 7 together with the triangle inequality:

$$\begin{aligned} \forall s \geq s_c^+, \quad \langle \xi \rangle^s &\leq \langle \xi \rangle^{s_c^+} \langle \xi_1 \rangle^{s-s_c^+} + \langle \xi \rangle^{s_c^+} \langle \xi_2 \rangle^{s-s_c^+} \\ &\quad + \langle \xi \rangle^{s_c^+} \langle \xi - \xi_1 - \xi_2 \rangle^{s-s_c^+}. \end{aligned} \tag{43}$$

Lemma 8. *Given $s_c^+ > \frac{1}{4} - \frac{\alpha}{4}$, there exist $C, \mu, \delta > 0$ such that for any $s \geq s_c^+$ and any triple $(u, v, w) \in X^{s, 1/2}$ with compact support in $[-T, T]$*

$$\begin{aligned} \|\partial_x(uvw)\|_{X^{s, -1/2+\delta}} &\leq CT^\mu \left(\|u\|_{X^{s, 1/2}} \|v\|_{X^{s_c^+, 1/2}} \|w\|_{X^{s_c^+, 1/2}} + \|u\|_{X^{s_c^+, 1/2}} \right. \\ &\quad \left. \|v\|_{X^{s, 1/2}} \|w\|_{X^{s_c^+, 1/2}} + \|u\|_{X^{s_c^+, 1/2}} \|v\|_{X^{s_c^+, 1/2}} \|w\|_{X^{s, 1/2}} \right). \end{aligned} \tag{44}$$

The proof of Lemma 7 is exactly the same as that of Lemma 5 except for changing $f_4(\xi_4, \tau_4)$ into $\frac{f_4(\xi_4, \tau_4)}{(|\tau_4 - \xi_4^3| + |\xi_4|^\alpha)^\delta}$ and using the following lemma due to Molinet and Ribaud [9].

Lemma 9. *Let there be f with support in $[-T, T]$ in time. For any $\delta > 0$, there exists $\mu = \mu(\delta) > 0$ such that*

$$\left\| \frac{\hat{f}(\xi, \tau)}{\langle \tau - \xi^3 \rangle^\delta} \right\|_{L_{\xi, \tau}^2} \leq CT^\mu \|f\|_{L_{t, x}^2}.$$

By Lemmas 7 and 8, there exist $\delta, \mu > 0$ depending only on s_c^+ such that

$$\|F(u)\|_{X^{s_c^+, 1/2}} \leq C\|\varphi\|_{H^{s_c^+}} + CT^\mu \|u\|_{X^{s_c^+, 1/2}}^3, \tag{45}$$

$$\|F(u)\|_{X^{s, 1/2}} \leq C\|\varphi\|_{H^s} + CT^\mu \|u\|_{X^{s_c^+, 1/2}}^2 \|u\|_{X^{s, 1/2}}. \tag{46}$$

Gathering the above two estimates, one infers that

$$\|F(u)\|_Z \leq C(\|\varphi\|_{H^{s_c^+}} + \nu\|\varphi\|_{H^s}) + CT^\mu \|u\|_Z^3. \tag{47}$$

Next, since $\partial_x(u^3) - \partial_x(v^3) = \partial_x[(u - v)(u^2 + uv + v^2)]$, we get in the same way that

$$\begin{aligned} \|F(u) - F(v)\|_{X^{s_c^+, 1/2}} &\leq CT^\mu \|u - v\|_{X^{s_c^+, 1/2}} \left(\|u\|_{X^{s_c^+, 1/2}}^2 \right. \\ &\quad \left. + \|u\|_{X^{s_c^+, 1/2}} \|v\|_{X^{s_c^+, 1/2}} + \|v\|_{X^{s_c^+, 1/2}}^2 \right), \end{aligned} \tag{48}$$

and

$$\begin{aligned} \|F(u) - F(v)\|_{X^{s, 1/2}} &\leq CT^\mu \|u - v\|_{X^{s_c^+, 1/2}} \left(\|u\|_{X^{s_c^+, 1/2}} \|u\|_{X^{s, 1/2}} \right. \\ &\quad + \|u\|_{X^{s_c^+, 1/2}} \|v\|_{X^{s, 1/2}} + \|u\|_{X^{s, 1/2}} \|v\|_{X^{s_c^+, 1/2}} \\ &\quad \left. + \|v\|_{X^{s_c^+, 1/2}} \|v\|_{X^{s, 1/2}} \right) + CT^\mu \|u - v\|_{X^{s, 1/2}} \left(\|u\|_{X^{s_c^+, 1/2}}^2 \right. \\ &\quad \left. + \|u\|_{X^{s_c^+, 1/2}} \|v\|_{X^{s_c^+, 1/2}} + \|v\|_{X^{s_c^+, 1/2}}^2 \right). \end{aligned} \tag{49}$$

Combining (48) and (49), we deduce that

$$\|F(u) - F(v)\|_Z \leq CT^\mu \|u - v\|_Z \left(\|u\|_Z + \|v\|_Z \right)^2. \tag{50}$$

Setting $T = (128C^3 \|\varphi\|_{H^{s_c^+}}^2)^{-1/\mu}$, we deduce that from (47) and (50) that F is strictly contractive on the ball of radius $4C\|\varphi\|_{H^{s_c^+}}$ in Z . This proves the existence of a solution $u \in X^{s, 1/2}$ to the equation (2) on the time interval $[0, T]$ with $T = T(\|\varphi\|_{H^{s_c^+}}) > 0$.

The uniqueness of the solution is exactly the same as in [9]. For brevity, we omit the details.

5. GLOBAL WELL-POSEDNESS IN THE STRONG DISSIPATIVE EFFECT.

We now turn to the global well-posedness for the case $1 < \alpha \leq 3$. In this case, we fix $s_c^+ \in (\frac{1}{4} - \frac{\alpha}{4}, 0)$. By an argument similar to the proof of Proposition 2.4 in [9], we can get the following lemma.

Lemma 10. *Let $s \in \mathbb{R}$ and $\delta > 0$. For all $f \in X^{s, -1/2+\delta}$,*

$$t \longmapsto \int_0^t W(t - t') f(t') dt' \in C(\mathbb{R}_+, H^{s+\alpha\delta}). \tag{51}$$

Moreover, if (f_n) is a sequence with $f_n \xrightarrow[n \rightarrow \infty]{} 0$ in $X^{s, -1/2+\delta}$, then

$$\left\| \int_0^t W(t - t') f_n(t') dt' \right\|_{L^\infty(\mathbb{R}_+, H^{s+\alpha\delta})} \xrightarrow[n \rightarrow \infty]{} 0. \tag{52}$$

It is easy to check that $W(\cdot)\varphi \in C([0, +\infty); H^s) \cap C((0, +\infty); H^\infty)$. Then it follows from Lemma 5, Lemma 10 and the local existence of the solution that $u \in C([0, T]; H^s) \cap C((0, T]; H^{s+\alpha\delta})$, for some $T = T(\|\varphi\|_{H^{s_c^+}})$. By induction we have $u \in C((0, T]; H^\infty)$. Since the existence time T of the solution depends only on the norm of data $\|\varphi\|_{H^{s_c^+}}$, to extend the solution into the global sense we need a control on $\|u(t)\|_{H^{s_c^+}}$. Taking the L^2 -scalar

product of (2) with u , we can get that $\|u(t)\|_{L^2}$ is nonincreasing on $(0, T]$. This implies that the solution is global in time.

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