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SECOND-ORDER PARABOLIC EQUATIONS WITH UNBOUNDED COEFFICIENTS IN EXTERIOR DOMAINS

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Abstract. In this paper, we consider elliptic and parabolic equations with unbounded coefficients in smooth exterior domains $\Omega \subset \mathbb{R}^N$, subject to Dirichlet or Neumann boundary conditions. Under suitable assumptions on the growth of the coefficients, the solution of the parabolic problem is governed by a semigroup $\{T(t)\}$ on $L^p(\Omega)$ for $1 < p < \infty$ and on $C_b(\overline{\Omega})$. Furthermore, uniform- and L^p-estimates for higher-order spatial derivatives of $\{T(t)\}$ are obtained. They imply optimal Schauder estimates for the solution of the corresponding elliptic and parabolic problems.

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1. Introduction

Parabolic equations with unbounded coefficients were investigated in great detail in the last years; see e.g., the references [8], [21], [24], [23], [22], [2], [10], [5], [13], [6] and [12]. One reason for this is various applications of these equations to stochastic analysis.

Analytically, it is also interesting to consider parabolic equations with unbounded coefficients in the spaces $C_b(\overline{\Omega})$ and $L^p(\Omega)$ when Ω is an exterior domain. For the special case of the Ornstein-Uhlenbeck operator in exterior domains, subject to Dirichlet conditions, we refer to [12].

In this paper, we consider more general situations. More precisely, we consider the Cauchy problem

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + g(t,x), & t>0, \quad x\in\Omega,\\ \mathcal{B}u(t,x) = 0, & t>0, \quad x\in\partial\Omega,\\ u(0,x) = f(x), & \qquad\quad x\in\overline{\Omega}, \end{cases} \tag{1.1}$$

where

$$\mathcal{A}u = \sum_{i,j=1}^{N} q_{ij}(\cdot)D_{ij}u + \sum_{i=1}^{N} b_i(\cdot)D_i u \tag{1.2}$$

is an elliptic differential operator with unbounded coefficients satisfying certain conditions and subject to Dirichlet or Neumann boundary conditions, i.e., $\mathcal{B}u = u$ or $\mathcal{B}u = \frac{\partial u}{\partial \nu}$ (and also more general boundary conditions; see Remark 3.11).

In the first part of the paper, we consider equation (1.1) in a space of continuous functions. Whereas existence and uniqueness results of classical solutions to problem (1.1) as well as regularity properties are known for the case of $\mathbb{R}^N$ (for the latest results see [6]), this seems not to be the case for general unbounded domains Ω. In particular, when $g\equiv 0$, estimates for second- and third-order spatial derivatives of u seem to be unknown in this situation. An important motivation for the study of such estimates lies in the fact that they imply Schauder estimates for both the solutions of problem (1.1) when $g\not\equiv 0$ and the solution of the elliptic equation

$$\lambda u - \mathcal{A}u = f.$$

We prove that, under appropriate assumptions on the coefficients of $\mathcal{A}$ (see Section 2), the realization $\mathcal{A}_{\mathcal{B}}$ of (1.2) in $C_b(\overline{\Omega})$ generates a positive semigroup $\{T(t)\}$ in $C_b(\overline{\Omega})$ satisfying the estimate

$$\|T(t)f\|_{C_b^{\beta}(\overline{\Omega})} \le Ct^{-\frac{\beta-\alpha}{2}}\|f\|_{C_{\mathcal{B}}^{\alpha}(\overline{\Omega})}, \qquad t\in(0,T),$$

for $f \in C^\alpha_{\mathcal{B}}(\overline{\Omega})$, $\alpha \in [0,2]$ and $\beta \in [0,3]$ with $\alpha \le \beta$, and $T > 0$.

These estimates yield some interesting consequences: in fact, they allow a partial characterization of the domain of the "weak generator" A of the semigroup $\{T(t)\}$. In particular, we show that $D(A)$ coincides with the maximal domain of the realization of $\mathcal{A}$, subject to the boundary operator $\mathcal{B}$ in $C_b(\overline{\Omega})$, and that $C^\theta_b(\Omega)$ is an intermediate space of index $\theta/2$ between $C_b(\overline{\Omega})$ and $D(A)$.

The second part of the paper considers similar questions in the L^p-context. We deal with uniformly elliptic operators in divergence form, perturbed by a drift term, whose coefficients do not grow faster than $|x|\log|x|$ at infinity. In contrast to the C_b-case, we have to assume more restrictive conditions on the coefficients of $\mathcal{A}$. In particular, we will assume that the diffusion coefficients are bounded. Note that these restrictions seem to be necessary for our approach: indeed, the operator

$$\mathcal{A}u(x) = u''(x) - \mathrm{sign}(x)|x|^{1+\varepsilon}u'(x), \qquad x \in \mathbb{R},$$

does not generate a C_0-semigroup in $L^p(\mathbb{R})$ for any $p \in (1,\infty)$; see [26].

Under such assumptions, we prove that we can associate a strongly continuous semigroup (denoted by $\{T_\Omega(t)\}$) with the operator $\mathcal{A}$, subject to Dirichlet or Neumann boundary conditions. L^p-estimates for the first-, second- and third-order spatial derivatives of $T_\Omega(t)f$, for $f \in L^p(\Omega)$, are the main result of this second part of the paper. Furthermore, as a byproduct, we also prove L^p–L^q smoothing estimates for $\{T_\Omega(t)\}$.

We also note here that, in the case of $\mathbb{R}^N$, the fundamental solution of parabolic equations with certain unbounded coefficients has been investigated many years earlier by several authors; see e.g., [15, 16, 17, 18, 7, 1].

Some words about our notation are in order. For any open set $\mathcal{O} \subset \mathbb{R}^N$, we denote by $C_b(\overline{\mathcal{O}})$ the space of all bounded and continuous functions $f : \overline{\mathcal{O}} \to \mathbb{R}$. For $k > 0$, $C^k_b(\overline{\mathcal{O}})$ is the subspace of $C_b(\overline{\mathcal{O}})$ consisting of all functions which are continuously differentiable up to the $[k]$-th order in $\overline{\mathcal{O}}$, such that all the derivatives are bounded and those of maximal order are $(k-[k])$-Hölder continuous in $\overline{\mathcal{O}}$. The space $C^k_b(\overline{\mathcal{O}})$ is endowed with the norm

$$\|u\|_{C^k_b(\overline{\mathcal{O}})} = \sum_{|\alpha|\le[k]} \|D^\alpha u\|_{C_b(\overline{\mathcal{O}})} + \sum_{|\alpha|=[k]} [D^\alpha u]_{C^{k-[k]}_b(\overline{\mathcal{O}})}.$$

When $\mathcal{O}$ is bounded, we omit the subscript "b."

If $\alpha \in [1,4)$, we denote by $C^\alpha_{\mathcal{B}}(\overline{\mathcal{O}})$ the subset of $C^\alpha_b(\overline{\mathcal{O}})$ defined as follows: $C^\alpha_{\mathcal{B}}(\overline{\mathcal{O}}) = \{f \in C^\alpha_b(\overline{\mathcal{O}}) : \mathcal{B}f \equiv \epsilon(\alpha)\mathcal{B}\mathcal{A}f \equiv 0 \text{ on } \partial\Omega\}$, where $\epsilon(\alpha) = 0$ if $\alpha < 2$ and $\epsilon(\alpha) = 1$ if $\alpha \ge 2$ in the case of Dirichlet boundary conditions.

Similarly, $\epsilon(\alpha) = 0$ if $1 \le \alpha < 3$ and $\epsilon(\alpha) = 0$ if $\alpha \ge 3$ for Neumann boundary conditions. If $\alpha \in (0,1)$, $C^\alpha_{\mathcal{B}}(\overline{\mathcal{O}}) = C^\alpha_b(\overline{\mathcal{O}})$ for Neumann boundary conditions and $C^\alpha_{\mathcal{B}}(\overline{\mathcal{O}}) = \{f \in C^\alpha_b(\overline{\mathcal{O}}) : f \equiv 0 \text{ on } \partial\Omega\}$ in the case of Dirichlet boundary conditions. We also use the notation $\mathcal{O}_T$ and $\partial_0\mathcal{O}$ ($T \in (0,\infty]$) to denote the sets $\mathcal{O}_T := (0,T) \times \mathcal{O}$ and $\partial_0\mathcal{O} := \{0\} \times \partial\mathcal{O}$, respectively.

2. The case $C_b(\overline{\Omega})$: Assumptions and preliminaries

In this section we deal with elliptic operators $\mathcal{A}$ of the form (1.2) whose coefficients are defined in an exterior domain $\Omega \subseteq \mathbb{R}^N$ with boundary of class $C^{3+\gamma}$ for some $\gamma \in (0,1)$. The assumptions on the coefficients of $\mathcal{A}$ are the following ones.

Hypotheses $(HCb)_\Omega$**.**

(i) $q_{ij}, b_j \in C^{3+\delta}_{\rm loc}(\overline{\Omega})$, $q_{ij}(x) = q_{ji}(x)$ for $x \in \Omega$, $i,j = 1,\dots,N$ and some $\delta \in (0,1)$. Moreover,

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \ge \mu(x)|\xi|^2, \qquad \xi \in \mathbb{R}^N,\ x \in \Omega, \tag{2.1}$$

for some function $\mu : \overline{\Omega} \to \mathbb{R}$ such that $\inf_{x\in\overline{\Omega}} \mu(x) = \mu_0 > 0$;

(ii) there exist $\lambda_0 > 0$ and a function $\varphi \in C^2(\overline{\Omega})$ such that

$$\lim_{\substack{|x|\to\infty\\ x\in\Omega}} \varphi(x) = \infty \qquad \text{and} \qquad \sup_{x\in\Omega} (\mathcal{A}\varphi(x) - \lambda_0\varphi(x)) < \infty;$$

(iii) there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$\max\left\{\Big|\sum_{j=1}^N q_{ij}(x)x_j\Big|,\ \sum_{j=1}^N q_{jj}(x),\ \sum_{j=1}^N b_j(x)x_j\right\} \le \kappa_1(1+|x|^2)\mu(x),$$
$$x \in \Omega,\ \ i = 1,\dots,N,$$

$$|D^\beta q_{ij}(x)| \le \kappa_2\mu(x), \quad x \in \Omega,\ \ i,j = 1,\dots,N,\ \ |\beta| = 1,3;$$

(iv) there exist constants $\kappa_3, \kappa_4 > 0, \kappa_5 \in \mathbb{R}$ and a function $r : \Omega \to \mathbb{R}$ satisfying

$$\sup_{x\in\Omega} \frac{r(x)}{\mu(x)} < \infty,$$

and

$$\sum_{i,j=1}^N D_i b_j(x)\xi_i\xi_j \ \ \le\ \ r(x)|\xi|^2, \quad \xi \in \mathbb{R}^N,\ x \in \Omega, \tag{2.2}$$

$$\begin{aligned} r(x)+\kappa_3|D^\beta b(x)| &\leq \kappa_4\mu(x), \quad x\in\Omega,\ \ |\beta|=2,3,\\ \sum_{i,j,h,k=1}^N D_{hk}q_{ij}(x)m_{ij}m_{hk} &\leq \kappa_5\mu(x)\sum_{h,k=1}^N m_{hk}^2, \quad x\in\Omega, \end{aligned}$$

for any symmetric matrix $M=(m_{hk})$.

Remark 2.1. i) When $\mathcal{A}$ is an elliptic operator with unbounded coefficients, maximum principles, in general, fail to hold without any additional assumption. Hence, elliptic and parabolic problems associated with such operators may admit more than a unique solution. In the one-dimensional case this fact was established by Feller (see [9]) who gave a precise characterization of the situations in which the elliptic equation $\lambda u - qu'' - bu' = f \in C_b(\mathbb{R})$ admits a unique bounded solution (see also [6, Chapters 1 and 2] for further details).

Hypothesis $(HCb)_\Omega$-(ii) is a typical assumption used to derive a maximum principle for elliptic operators with unbounded coefficients and has been already considered in several papers; see e.g. [2, 3, 4, 10, 21].

ii) The estimate (2.2) is a dissipativity condition. A counterexample given in [2, Example 5.3] shows that this kind of assumption is necessary in order to obtain gradient estimates (see (2.5) below).

Examples 2.2. Take $R>0$ and $\Omega=\mathbb{R}^N\setminus B(R)$.

(i) Let us consider the operator

$$\mathcal{A}f(x)=(1+|x|^2)^p\Delta f(x)-|x|^{2m}\langle x, Df(x)\rangle, \quad x\in\Omega,$$

for smooth functions f, $m>1$ and $0\leq p<m+1$. Then, $\mathcal{A}$ satisfies Hypotheses $(HCb)_\Omega$. In fact, Hypothesis $(HCb)_\Omega$-(ii) is satisfied by taking $\varphi(x)=|x|^2$ and, as λ_0, any positive number. Hypotheses $(HCb)_\Omega$-(i) and -(iii) are trivially satisfied, and Hypothesis $(HCb)_\Omega$-(iv) holds with $r(x)=-|x|^{2m}$ for $x\in\Omega$, any $\kappa_3>0$, and suitable constants κ_4 (depending on the choice of κ_3) and κ_5.

(ii) We can consider also situations in which the maximum and the minimum eigenvalues of the matrix $Q(x)$ (say $\lambda_{\min}(x)$ and $\lambda_{\max}(x)$) exhibit a different behaviour at ∞. This is the case when they satisfy the following conditions:

$$(a) \qquad \limsup_{\substack{|x|\to\infty\\ x\in\Omega}} \frac{\lambda_{\max}(x)}{|x|\lambda_{\min}(x)} < +\infty,$$

$$(b) \qquad \limsup_{\substack{|x|\to\infty\\ x\in\Omega}} \{N\lambda_{\max}(x)|x|^{-2}-|x|^{2m}\} < +\infty,$$

the moduli of the derivatives of q_{ij} $(i,j=1,\ldots,N)$, up to the third order, can be controlled from above at infinity in terms of $\lambda_{\min}$, and the drift coefficients are as in (i).

Let us now state several results related to elliptic and parabolic problems, associated with the operator $\mathcal{A}$ on $\Omega=\mathbb{R}^N$, that we need in what follows. To this end, for $f\in C_b(\mathbb{R}^N)$ consider the following problems set in the whole of $\mathbb{R}^N$:

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x), & t>0, \quad x\in\mathbb{R}^N, \\ u(0,x) = f(x), & x\in\mathbb{R}^N, \end{cases} \tag{2.3}$$

$$\lambda u - \mathcal{A}u = f. \tag{2.4}$$

Then, the following holds.

Proposition 2.3 ([5]). *Let Hypotheses $(HCb)_{\mathbb{R}^N}$ be satisfied, and let $f\in C_b(\mathbb{R}^N)$. Then we have the following:*

(a) *Equation (2.3) has a unique solution $u\in C_b([0,\infty)\times\mathbb{R}^N)\cap C^{1,2}((0,\infty)\times\mathbb{R}^N)$.*

(b) *For $t>0$, set $T(t)f:=u(t,\cdot)$. Then, the family $\{T(t)\}$ defines a semigroup of positive contractions on $C_b(\mathbb{R}^N)$.*

(c) *For $k,l\in[0,3]$ with $k\le l$ and $\widehat{\omega}>0$, there exists a positive constant $C=C(\widehat{\omega})$ such that*

$$\|T(t)f\|_{C_b^l(\mathbb{R}^N)} \le Ce^{\widehat{\omega}t}t^{-\frac{l-k}{2}}\|f\|_{C_b^k(\mathbb{R}^N)}, \qquad f\in C_b^k(\mathbb{R}^N). \tag{2.5}$$

(d) *For any $\lambda>\lambda_0$, equation (2.4) has a unique solution $u\in D_{\max}(\mathcal{A})$, where*

$$D_{\max}(\mathcal{A}) = \Big\{u\in C_b(\mathbb{R}^N)\cap \bigcap_{1\le p<\infty} W^{2,p}_{\rm loc}(\mathbb{R}^N) : \mathcal{A}u\in C_b(\mathbb{R}^N)\Big\}. \tag{2.6}$$

(e) *Let $\theta\in(0,2)$. Then $D_{\max}(\mathcal{A})\subset C_b^\theta(\mathbb{R}^N)$ and there exist constants $C=C(\theta)>0$ and $\tilde{\lambda}\in\mathbb{R}$ such that*

$$\|u\|_{C_b^\theta(\mathbb{R}^N)} \le C\|u\|^{1-\frac{\theta}{2}}_{C_b(\mathbb{R}^N)}\|(\tilde{\lambda}-\mathcal{A})u\|^{\frac{\theta}{2}}_{C_b(\mathbb{R}^N)}, \qquad u\in D_{\max}(\mathcal{A}). \tag{2.7}$$

3. UNIFORM ESTIMATES AND CONSEQUENCES

We start this section stating some maximum principles which yield uniqueness results for the elliptic problem

$$\begin{cases} \lambda u - \mathcal{A}u = f, & \text{in } \Omega, \\ \mathcal{B}u = 0, & \text{in } \partial\Omega. \end{cases} \tag{3.1}$$

To this end, we define

$$D_\Omega := \Big\{u \in C_b(\overline{\Omega}) \cap \bigcap_{1\le p<\infty} W^{2,p}_{\rm loc}(\Omega) : \mathcal{A}u \in C_b(\overline{\Omega}),\ \mathcal{B}u = 0 \text{ on } \partial\Omega\Big\},$$

and we endow it with the graph norm, i.e., $\|u\|_{D_\Omega} = \|u\|_\infty + \|\mathcal{A}u\|_\infty$, $u \in D_\Omega$. The space D_Ω turns out to be the domain of the maximal realization of the operator $\mathcal{A}$ in $C_b(\overline{\Omega})$, subject to Dirichlet or Neumann boundary conditions.

Proposition 3.1. *Suppose that Hypotheses* $(HCb)_\Omega$ *are satisfied. Then, the following properties hold.*

(i) *There exists* $\lambda_1 \ge \lambda_0$ *such that for* $\lambda \ge \lambda_1$, *the problem* (3.1) *admits at most one solution* $u \in D_\Omega$.

(ii) *Let* $T > 0$. *Assume that* $u : \overline{\Omega}_T \to \mathbb{R}$ *is bounded, belongs to* $C^{1,2}(\Omega_T)$ *and satisfies*

$$\begin{cases} D_t u(t,x) & \le \mathcal{A}u(t,x), \quad t>0, \quad x \in \Omega, \\ \mathcal{B}u(t,x) & \le 0, \qquad\qquad t>0, \quad x \in \partial\Omega, \\ u(0,x) & \le 0, \qquad\qquad\qquad\qquad x \in \overline{\Omega}. \end{cases}$$

Further, assume that

– u *is continuous in* $\overline{\Omega}_T$ *and* $D_j u \in C((0,T] \times \overline{\Omega})$ *if* $\mathcal{B}u = \frac{\partial u}{\partial \nu}$;

– $u \in C(\overline{\Omega}_T \setminus \partial_0\Omega)$ *if* $\mathcal{B}u = u$.

Then, $u \le 0$. *In particular, for* $f \in C_b(\overline{\Omega})$, *the Cauchy problem* (1.1) (*with* $g \equiv 0$) *admits at most a classical solution* u *satisfying*

$$\sup_{(t,x)\in(0,T)\times\Omega} |u(t,x)| \le \|f\|_{C_b(\overline{\Omega})}. \tag{3.2}$$

Proof. When $\mathcal{A}$ is subject to homogeneous Dirichlet boundary conditions, assertion (i) can be obtained by adapting the arguments in the proof of Proposition A.1 (see the appendix). Assertion (ii) in that case is due to [10, Theorem A.2 and Corollary A.3]. When $\mathcal{A}$ is subject to Neumann boundary conditions, the assertions are due to [2, Propositions 2.1 and 2.3] (see also [6, Propositions 10.1.3, 11.2.4 and Theorem 11.1.5]). □

3.1. **The elliptic case.** We now state the first main result of this section related to the elliptic problem (3.1).

Theorem 3.2. *There exists* $\lambda_\star \ge \lambda_0$ *such that for* $\lambda \ge \lambda_\star$ *and* $f \in C_b(\overline{\Omega})$, *the elliptic problem* (3.1) *admits a unique solution* $u \in D_\Omega$.

Proof. We adapt to our situation the technique developed in the proof of [12, Theorem 2.5]. As a first step, we extend the operator $\mathcal{A}$ to the whole of

$\mathbb{R}^N$ by a uniformly elliptic operator $\widehat{A}$ satisfying Hypotheses $(HCb)_{\mathbb{R}^N}$. To this end, note that, since the coefficients of $\mathcal{A}$ belong to $C^{3+\delta}_{\rm loc}(\overline{\Omega})$ for some $\delta \in (0,1)$, they can be extended to functions $\widehat{q}_{ij}$ and $\widehat{b}_j$ $(i,j=1,\ldots,N)$ belonging to $C^{3+\delta}_{\rm loc}(\mathbb{R}^N)$ (see e.g. [27, Chapter VI, Section 2]). By continuity, we may determine ε sufficiently small such that the coefficients $\widehat{q}_{ij}(x)$ $(i,j=1,\ldots,N)$ satisfy the ellipticity condition (2.1), with $\mu_0/2$ instead of μ_0, for any $x \in \Omega \cup \Gamma_\varepsilon$, where $\Gamma_\varepsilon = \{x \in \mathbb{R}^N \setminus \Omega : \mathrm{dist}(x,\partial\Omega) \le \varepsilon\}$. Let ϱ be a smooth function such that

$$0 \le \varrho \le 1, \quad \varrho \equiv 0 \ \text{ in } \left\{x \in \mathbb{R}^N \setminus \Omega : \mathrm{dist}(x,\partial\Omega) \ge \tfrac{\varepsilon}{2}\right\}, \quad \varrho \equiv 1 \text{ in } \Omega$$

and let $\widehat{A}$ be the elliptic operator defined on smooth functions $\phi : \mathbb{R}^N \to \mathbb{R}$ by

$$\widehat{A}\phi = (1-\varrho)\Delta\phi + \varrho\mathcal{A}\phi.$$

Then, $\widehat{A}$ satisfies $(HCb)_{\mathbb{R}^N}$.

Fix $R > 0$ such that $\mathbb{R}^N \setminus \Omega \subset B(R)$. Moreover, set

$$\Omega_R := \{x \in \Omega : |x| \le R+3\}.$$

Further, choose three smooth functions ψ, ζ and χ with the following properties:

$$0 \le \psi \le 1, \quad \psi \equiv 0 \text{ in } B(R+1), \quad \psi \equiv 1 \text{ in } \mathbb{R}^N \setminus B(R+2), \tag{3.3}$$

$$0 \le \zeta \le 1, \quad \zeta \equiv 0 \text{ in } B(R+1/2), \quad \zeta \equiv 1 \text{ in } \mathbb{R}^N \setminus B(R+1), \tag{3.4}$$

$$0 \le \chi \le 1, \quad \chi \equiv 1 \text{ in } B(R+2), \quad \chi \equiv 0 \text{ in } \mathbb{R}^N \setminus B(R+5/2). \tag{3.5}$$

Finally, set

$$\widehat{f} := \zeta f, \qquad \widetilde{f} := \chi f. \tag{3.6}$$

Of course, $\widehat{f} \in C_b(\mathbb{R}^N)$ and $\widetilde{f} \in C_b(\overline{\Omega}_R)$. From Proposition 2.3 we deduce that, for any $\lambda > \lambda_0$, the equation $\lambda u - \widehat{A}u = \widehat{f}$ admits a unique solution $u := R(\lambda,\widehat{A})\widehat{f}$ in $D(\widehat{A})$, where $D(\widehat{A})$ is given by (2.6) with $\widehat{A}$ instead of $\mathcal{A}$. Moreover, by classical results for elliptic equations in bounded smooth domains (see e.g. [19, Theorem 3.1.24(ii)]), the realization A_{Ω_R} of the operator $\mathcal{A}$ in $C(\overline{\Omega}_R)$, with domain

$$D(A_{\Omega_R}) = \left\{u \in C(\overline{\Omega}_R) \cap \bigcap_{1\le p<\infty} W^{2,p}_{\rm loc}(\Omega_R) : \mathcal{A}u \in C(\overline{\Omega}_R), \mathcal{B}_R u \equiv 0 \text{ on } \partial\Omega\right\},$$

is the generator of an analytic semigroup and its resolvent set contains the half-line $(\tilde{\lambda}_0, \infty)$ for some $\tilde{\lambda}_0 \in \mathbb{R}$. Here,

$$\mathcal{B}_R u = \begin{cases} \mathcal{B}u, & \text{on } \partial\Omega, \\ u, & \text{on } \partial B(R+3). \end{cases} \tag{3.7}$$

Therefore, for any $\lambda > \tilde{\lambda}_0$, the equation $\lambda u - \mathcal{A}u = \widetilde{f}$ admits a unique solution $u = R(\lambda, A_{\Omega_R})\widetilde{f}$ in $D(A_{\Omega_R})$. We set

$$U_\lambda f := \psi R(\lambda, \widehat{A})\widehat{f} + (1-\psi)R(\lambda, A_{\Omega_R})\widetilde{f}, \tag{3.8}$$

for any $\lambda > \overline{\lambda}_0 := \max\{\lambda_0, \tilde{\lambda}_0\}$. Since $U_\lambda f = R(\lambda, A_{\Omega_R})\widetilde{f}$ in $\Omega \cap B(R+1)$, then $\mathcal{B}U_\lambda f = 0$ on $\partial\Omega$. Furthermore, $U_\lambda f \in C_b(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $p \in [1, \infty)$. Moreover,

$$\lambda U_\lambda f - \mathcal{A}U_\lambda f = f + T_\lambda f,$$

where

$$\begin{aligned}(T_\lambda f)(x) := &-2\langle Q(x)D\psi(x), (DR(\lambda, \widehat{A})\widehat{f})(x) - (DR(\lambda, A_{\Omega_R})\widetilde{f})(x)\rangle \\ &- (\mathcal{A}\psi)(x)\big((R(\lambda, \widehat{A})\widehat{f})(x) - (R(\lambda, A_{\Omega_R})\widetilde{f})(x)\big), \quad x \in \Omega.\end{aligned} \tag{3.9}$$

Using Proposition 2.3(e) and [19, Corollary 3.1.24(ii)], we deduce that $R(\lambda, \widehat{A})\widehat{f}$ and $R(\lambda, A_{\Omega_R})\widetilde{f}$ belong to $C^1_b(\mathbb{R}^N)$ and $C^1(\overline{\Omega}_R)$, respectively. Up to replacing $\overline{\lambda}_0$ by a larger value, if necessary, by (2.7) and [19, Theorem 3.1.19] we get

$$\|DR(\lambda, \widehat{A})\widehat{f}\|_{C_b(\mathbb{R}^N)} \le \frac{C}{\sqrt{\lambda}}\|\widehat{f}\|_{C_b(\mathbb{R}^N)} \le \frac{C}{\sqrt{\lambda}}\|f\|_{C_b(\overline{\Omega})} \tag{3.10}$$

and

$$\|DR(\lambda, A_{\Omega_R})\widetilde{f}\|_{C(\overline{\Omega}_R)} \le \frac{C}{\sqrt{\lambda}}\|\widetilde{f}\|_{C(\overline{\Omega}_R)} \le \frac{C}{\sqrt{\lambda}}\|f\|_{C_b(\overline{\Omega})}, \tag{3.11}$$

for any $\lambda > \overline{\lambda}_0$. Therefore, from (3.9), (3.10) and (3.11) we get

$$\begin{aligned}|(T_\lambda f)(x)| \le \frac{2C}{\sqrt{\lambda}}\Big(&2\sup_{R+1\le|x|\le R+2}|Q(x)D\psi(x)| \\ &+ \frac{1}{\sqrt{\lambda}}\sup_{R+1\le|x|\le R+2}|(\mathcal{A}\psi)(x)|\Big)\|f\|_{C_b(\overline{\Omega})}, \qquad x \in \Omega.\end{aligned}$$

The right-hand side of the above estimate tends to 0 as λ tends to ∞. This implies that there exists $\lambda_\star$ such that $\|T_\lambda\|_{\mathcal{L}(C_b(\overline{\Omega}))} \le 1/2$ for any $\lambda \ge \lambda_\star$. Hence, for such values of λ, problem (3.1) is solvable in D_Ω and its solution (which is unique by Proposition 3.1(i), if we assume that $\lambda_\star \geq \lambda_1$) is given by

$$R(\lambda, \mathcal{A})f := U_\lambda(I + T_\lambda)^{-1} f. \tag{3.12}$$

The proof is now complete. □

3.2. **The parabolic case.** In this subsection we show that the solution of the parabolic boundary-value problem (1.1) (with $g \equiv 0$) is governed by a semigroup $\{T(t)\}$ on $C_b(\overline{\Omega})$. Furthermore, we obtain estimates for spatial derivatives of $T(t)f$ up to the third order, for $t > 0$ and $f \in C_b(\overline{\Omega})$, which are similar to the estimates known for $\mathbb{R}^N$.

Let us start with the following lemma that, in the L^p-setting, has been proved in [13, Lemma 4.6]. The proof in the C_b-setting can be obtained from this one with minor changes.

Lemma 3.3. *Let X and Y be Banach spaces contained in $C_b(\overline{\mathcal{O}})$, where $\mathcal{O} \subset \mathbb{R}^N$ is an open set. Assume that $R : (0,\infty) \to \mathcal{L}(Y, X)$ and $S : (0,\infty) \to \mathcal{L}(Y)$ are functions such that $t \mapsto (R(t)f)(x)$ and $t \mapsto S(t)f$ are continuous in $(0,\infty)$ for $f \in Y$ and $x \in \mathcal{O}$. Further, assume that*

$$(a)\quad \|R(t)\|_{\mathcal{L}(Y,X)} \leq C_0 t^\alpha e^{\omega t}, \quad (b)\quad \|S(t)\|_{\mathcal{L}(Y)} \leq C_0 t^\theta e^{\omega t}, \quad t > 0, \tag{3.13}$$

for some constants $C_0, \omega \geq 0$ and some $\alpha, \theta > -1$. For $f \in Y$ and $t > 0$, set $R_0(t)f := R(t)f$ and

$$(R_n(t)f)(x) := \int_0^t (R_{n-1}(t-s)S(s)f)(x)\, ds, \quad n \in \mathbb{N},\ t > 0,\ x \in \overline{\Omega}.$$

Then, there exists a constant $C > 0$ such that

$$\sum_{n=0}^\infty \|R_n(t)f\|_X \leq C t^\alpha e^{\omega t} \|f\|_Y, \qquad t > 0,\ f \in Y. \tag{3.14}$$

Moreover, if $\alpha \geq 0$, the convergence of the series in (3.14) is uniform on bounded subsets of $[0,\infty)$.

Consider the family of bounded operators on $C_b(\overline{\Omega})$ defined by

$$T_0(t)f := \psi \widehat{T}(t)\widehat{f} + (1-\psi) T_{\Omega_R}(t)\widetilde{f}, \qquad t > 0, \tag{3.15}$$

$$(T_n(t)f)(x) := \int_0^t (T_{n-1}(t-s)S(s)f)(x)\, ds, \qquad t > 0,\ x \in \overline{\Omega}, \tag{3.16}$$

where

$$\begin{aligned}(S(t)f)(x) := &-2\langle Q(x)D\psi(x), (D\widehat{T}(t)\widehat{f})(x) - (DT_{\Omega_R}(t)\widetilde{f})(x)\rangle \\ &- (\mathcal{A}\psi)(x)\big((\widehat{T}(t)\widehat{f})(x) - (T_{\Omega_R}(t)\widetilde{f})(x)\big),\end{aligned} \tag{3.17}$$

for $t > 0$, $x \in \overline{\Omega}$ and $f \in C_b(\overline{\Omega})$. Set

$$T(t)f := \sum_{n=0}^{\infty} T_n(t)f, \qquad t > 0,\ \ f \in C_b(\overline{\Omega}). \tag{3.18}$$

Here, $\widehat{T}(t)$ and $T_{\Omega_R}(t)$ denote the semigroups generated by $\widehat{A}$ and A_{Ω_R}, respectively, and the functions ψ, $\widehat{f}$ and $\widetilde{f}$ are defined as in (3.3) and in (3.6).

We now prove the convergence of the series in (3.18) and show some of its smoothing properties. For this purpose, we need the following lemma, which is a classical result for elliptic operators with smooth and bounded coefficients. A proof of it can be obtained from the results in [19, Chapter 2] using simple tools from interpolation theory.

Lemma 3.4. *Assume that Hypotheses* $(HCb)_\Omega$ *hold. Then, for any* $\omega > 0$ *and any* $\alpha, \beta \in [0, 3]$ *such that* $\alpha \le \beta$, *there exists* $C > 0$ *such that*

$$\|T_{\Omega_R}(t)g\|_{C^\beta_{\mathcal{B}_R}(\overline{\Omega}_R)} \le Ct^{-\frac{\beta-\alpha}{2}} e^{\omega t} \|g\|_{C^\alpha_{\mathcal{B}_R}(\overline{\Omega}_R)}, \qquad t > 0, \tag{3.19}$$

for any $g \in C^\alpha_{\mathcal{B}_R}(\overline{\Omega})$. *Here,* $\mathcal{B}_R$ *is the boundary operator defined in* (3.7).

We can now show the convergence of the series (3.18) in $C_b(\overline{\Omega})$ and prove uniform estimates for its spatial derivatives.

Proposition 3.5. *Assume that Hypotheses* $(HCb)_\Omega$ *hold. Then, the series defined in* (3.18) *converges in* $C_b(\overline{\Omega})$. *Moreover, for any* $\alpha \in [0, 2]$ *and any* $\beta \in [0, 3]$ *such that* $0 \le \beta - \alpha < 3$, *and any* $f \in C^\alpha_{\mathcal{B}}(\overline{\Omega})$, $T(t)f \in C^\beta_{\mathcal{B}}(\overline{\Omega})$ *and for any* $\omega > 0$, *there exists* $C > 0$ *such that*

$$\|T(t)f\|_{C^\beta_b(\overline{\Omega})} \le Ct^{-\frac{\beta-\alpha}{2}} e^{\omega t} \|f\|_{C^\alpha_b(\overline{\Omega})}, \qquad t > 0,\ \ f \in C^\alpha_b(\overline{\Omega}). \tag{3.20}$$

Proof. Let us observe that the function $\widetilde{f}$ vanishes on a neighborhood of $\partial B(R + 3)$. Therefore, if $f \in C^\alpha_{\mathcal{B}}(\overline{\Omega})$ for some $\alpha \in [0, 2]$, then the function $\widetilde{f}$ belongs to $C^\alpha_{\mathcal{B}_R}(\overline{\Omega})$, where $\mathcal{B}_R$ is the boundary operator defined in (3.7). Moreover,

$$\|\widetilde{f}\|_{C^\alpha(\overline{\Omega}_R)} \le C\|f\|_{C^\alpha_b(\overline{\Omega})}. \tag{3.21}$$

Using (2.5), (3.19), (3.21) and formula (3.6), we easily see that the function $T_0(t)f$ in (3.15) belongs to $C^\theta_{\mathcal{B}}(\overline{\Omega})$ for any $t > 0$ and any $\theta \le 3$ and

$$\|T_0(t)f\|_{C^\theta_{\mathcal{B}}(\overline{\Omega})} \le Ct^{-\frac{\theta-\delta}{2}} e^{\omega t} \|f\|_{C^\delta_{\mathcal{B}}(\overline{\Omega})}, \qquad t > 0, \tag{3.22}$$

for any $0 \le \delta \le \theta \le 3$.

Fix $\beta \in [0, 3]$. We now split the rest of the proof into two steps.

Step 1). Here, we assume that $0 \le \beta - \alpha < 2$. As is immediately seen, for any $t > 0$ the function $S(t)f$ (see (3.17)) vanishes in $B(R+1)$ and in $\mathbb{R}^N \setminus B(R+2)$, so that it vanishes in a neighborhood of the boundary of Ω_R. Hence, $\mathcal{B}_R S(t)f \equiv 0$ for $t > 0$. Moreover, $S(t)f \in C_b^\alpha(\overline{\Omega})$, for any $t > 0$ and, due to (2.5) and (3.19), we get

$$\begin{aligned}\|S(t)f\|_{C_b^\alpha(\overline{\Omega})} &\le \||QD\psi|\|_{C_b^\alpha(\overline{\Omega})}\big(\|D\widehat{T}(t)\widehat{f}\|_{C_b^\alpha(\mathbb{R}^N)} + \|DT_{\Omega_R}(t)\widetilde{f}\|_{C^\alpha(\overline{\Omega}_R)}\big)\\ &\quad + \|\mathcal{A}\psi\|_{C_b^\alpha(\overline{\Omega})}\big(\|\widehat{T}(t)\widehat{f}\|_{C_b^\alpha(\mathbb{R}^N)} + \|T_{\Omega_R}(t)\widetilde{f}\|_{C^\alpha(\overline{\Omega}_R)}\big)\\ &\le C\big(\|\widehat{T}(t)\widehat{f}\|_{C_b^{1+\alpha}(\mathbb{R}^N)} + \|T_{\Omega_R}(t)\widetilde{f}\|_{C^{1+\alpha}(\overline{\Omega}_R)}\big)\\ &\le Ct^{-\frac{1}{2}}e^{\omega t}\|f\|_{C_b^\alpha(\overline{\Omega})}, \end{aligned} \tag{3.23}$$

for $t > 0$ and $f \in C_b^\alpha(\overline{\Omega})$. Hence, the operator $S(t)$ satisfies condition (3.13)(b) with $\theta = 1/2$ and $Y = C_{\mathcal{B}}^\alpha(\overline{\Omega})$. Therefore, we can apply Lemma 3.3 with $X = C_{\mathcal{B}}^\beta(\overline{\Omega})$, which yields estimate (3.20).

Step 2). To conclude the proof, we have to check estimate (3.20) in the case when $\beta - \alpha \ge 2$. In this situation, we cannot apply directly Lemma 3.3 with $R = T_0$, since the function $\|T_0(t)f\|_{C_b^\beta(\overline{\Omega})}$ may be not integrable in a right-neighborhood of $t = 0$ (see (3.22)). To overcome this difficulty, it suffices to show that the previous lemma can be applied with $R = T_1$. For this purpose, we take advantage of the smoothing properties of the operator $S(t)$. Fix $\rho \in (\beta - 2, 1)$ and observe that, using (2.5) and (3.19), we can easily show that $S(t)$ maps $C_{\mathcal{B}}^\alpha(\overline{\Omega})$ into $C_{\mathcal{B}}^{\alpha+\rho}(\overline{\Omega})$ and

$$\|S(t)f\|_{C_{\mathcal{B}}^{\alpha+\rho}(\overline{\Omega})} \le Ct^{-\frac{1+\rho}{2}}e^{\omega t}\|f\|_{C_{\mathcal{B}}^\alpha(\overline{\Omega})},$$

for any $t > 0$. Hence, by (3.22), we get

$$\begin{aligned}\|T_0(t-s)S(s)f\|_{C_{\mathcal{B}}^\beta(\overline{\Omega})} &\le \|T_0(t-s)\|_{\mathcal{L}(C_{\mathcal{B}}^{\alpha+\rho}(\overline{\Omega}),C_{\mathcal{B}}^\beta(\overline{\Omega}))}\|S(s)f\|_{C_{\mathcal{B}}^{\alpha+\rho}(\overline{\Omega})}\\ &\le C(t-s)^{-\frac{\beta-\alpha-\rho}{2}}s^{-\frac{1+\rho}{2}}e^{\omega t}\|f\|_{C_{\mathcal{B}}^\alpha(\overline{\Omega})},\end{aligned}$$

for $0 < s < t$. As a byproduct, it follows that the function $T_1(t)f$ belongs to $C_{\mathcal{B}}^\beta(\overline{\Omega})$ for any $t > 0$ and

$$\begin{aligned}\|T_1(t)f\|_{C_{\mathcal{B}}^\beta(\overline{\Omega})} &\le Ce^{\omega t}\|f\|_{C_{\mathcal{B}}^\alpha(\overline{\Omega})}\int_0^t (t-s)^{-\frac{\beta-\alpha-\rho}{2}}s^{-\frac{1+\rho}{2}}ds\\ &\le Ct^{-\frac{\beta-\alpha-1}{2}}e^{\omega t}\|f\|_{C_{\mathcal{B}}^\alpha(\overline{\Omega})}.\end{aligned}$$

Hence, we can apply Lemma 3.3 to the series $\sum_{n=1}^\infty T_n(t)$ with the same choices of the spaces X and Y as in Step 1. It follows that the previous series belongs to $C^\beta_{\mathcal{B}}(\overline{\Omega})$ and there exists a positive constant C such that

$$\Big\|\sum_{n=1}^{\infty} T_n(t)\Big\|_{C^\beta_{\mathcal{B}}(\overline{\Omega})} \le Ct^{-\frac{\beta-\alpha-1}{2}}e^{\omega t}\|f\|_{C^\alpha_{\mathcal{B}}(\overline{\Omega})}, \qquad t>0.$$

To conclude the proof of this step, it suffices to apply estimate (3.22) with $(\delta,\theta)=(\alpha,\beta)$. □

We can now prove the main result of this section.

Theorem 3.6. *Assume that Hypotheses* $(HCb)_\Omega$ *hold and let* $f \in C_b(\overline{\Omega})$*. Then,*

(i) *there exists a positive semigroup* $\{T(t)\}$ *of contractions in* $C_b(\overline{\Omega})$ *such that* $u = T(\cdot)f$ *is the unique solution of the Cauchy problem* (1.1) (*with* $g \equiv 0$)*;*
(ii) $T(t)f \in C^3_b(\overline{\Omega})$ *for* $t>0$ *and, for* $\alpha \in [0,2]$ *and* $\beta \in (0,3]$ *satisfying* $\alpha \le \beta$*, estimate* (3.20) *holds true for any* $\omega > 0$*.*

Proof. By (3.12) one can see that

$$\int_0^\infty e^{-\lambda t}(T(t)f)(x)\,dt = \sum_{n=0}^{\infty}\int_0^\infty e^{-\lambda t}(T_n(t)f)(x)\,dt = (R(\lambda,\mathcal{A})f)(x)$$

holds for $x \in \overline{\Omega}$ and λ large enough, where $\{T(t)\}$ is the family defined in (3.18). Thus, $\{T(t)\}$ is a semigroup on $C_b(\overline{\Omega})$ and, hence, also on $C_{\mathcal{B}}(\overline{\Omega})$. Furthermore, $\lim_{t\to 0^+} T(t)f = f$ uniformly in the compact sets of $\overline{\Omega}$ for any $f \in C_{\mathcal{B}}(\overline{\Omega})$, since from the proof of Proposition 3.5, the series defining $T(\cdot)f$ converges uniformly in $[0,T]\times\overline{\Omega}$ (for any $T>0$) and the functions T_n (see (3.15) and (3.16)) are continuous on $\{0\}\times\Omega$, if $\mathcal{B}$ is the Dirichlet boundary operator, and in $\{0\}\times\overline{\Omega}$, if $\mathcal{B}$ is the Neumann boundary operator. This implies that $\{T(t)\}$ is a weak semigroup on $C_{\mathcal{B}}(\overline{\Omega})$ with weak generator $\mathcal{A}$ in the sense of [24]. Thus,

$$\Big(\mathcal{A}\int_0^t (T(s)f)(\cdot)\,ds\Big)(x) = (T(t)f)(x) - f(x), \quad t \ge 0,\ x \in \overline{\Omega},\ f \in C_{\mathcal{B}}(\overline{\Omega})$$

holds true.

The estimate (3.20) with $\alpha \in [0,2]$ and $\beta \in [0,3]$, such that $0 \le \beta-\alpha < 3$, follows from Proposition 3.5. The remaining case $(\alpha,\beta)=(0,3)$ can be obtained using the semigroup law and (3.20). In fact, for $t>0$, $\delta \in [0,2]$ and $f \in C_b(\overline{\Omega})$ we have

$$\|T(t)f\|_{C^3_b(\overline{\Omega})} \le \|T(t/2)\|_{L(C^{1+\alpha}_b(\overline{\Omega}),C^3_b(\overline{\Omega}))}\|T(t/2)f\|_{C^{1+\alpha}_b(\overline{\Omega})}$$

$$\leq Ce^{\omega t}\left(\frac{t}{2}\right)^{-1+\frac{\alpha}{2}}\left(\frac{t}{2}\right)^{-\frac{1+\alpha}{2}}\|f\|_{C_b(\overline{\Omega})} = 2\sqrt{2}Ce^{\omega t}t^{-\frac{3}{2}}\|f\|_{C_b(\overline{\Omega})},$$

for some $C > 0$, independent of f.

It remains to show that, for $f \in C_b(\overline{\Omega})$, the function $T(\cdot)f$ is the classical solution of the Cauchy problem (1.1) (with $g \equiv 0$) and that $\{T(t)\}$ is a positive semigroup of contractions. As a first step, we notice that, by Proposition 3.5, $T(t)f \in C^1_{\mathcal{B}}(\overline{\Omega})$ for $t > 0$. Therefore, the function $T(\cdot)f$ satisfies the boundary condition $\mathcal{B}T(t)f = 0$ on $\partial\Omega$ for any $t > 0$.

Next, we claim that, for $t > 0$ and $x \in \Omega$, the function $t \mapsto (T(t)f)(x)$ is continuously differentiable in $(0,\infty)$. To prove the claim, we show that all the functions $(T_n(\cdot)f)(x)$ are differentiable in $(0,\infty)$ and the series $\sum_{n=0}^{\infty}\|D_tT_n(t)f\|_{C(K)}$ converges for any compact set $K \subset \Omega$. Observe first that the function $(T_0(\cdot)f)(x)$ is differentiable in $(0,\infty)$ for $f \in C_b(\overline{\Omega})$ and $x \in \Omega$, and that

$$D_t(T_0(t)f)(x) = \psi(x)(\mathcal{A}\widehat{T}(t)\widehat{f})(x) + (1-\psi(x))(\mathcal{A}T_{\Omega_R}(t)\widetilde{f})(x),$$

$t > 0$, $x \in \Omega$, where $\widetilde{f}$ and $\widehat{f}$ are defined according to (3.6). Let $g \in C^{\theta}_{\mathcal{B}}(\overline{\Omega})$ for some $\theta \in (0,1)$. By the proof of Proposition 3.5 it follows that, for any compact set $K \subset \Omega$, there exist two positive constants C and ω such that

$$\|D_tT_0(t)g\|_{C(K)} \leq Ct^{-1+\frac{\theta}{2}}e^{\omega t}\|g\|_{C^{\theta}_b(\overline{\Omega})}, \qquad t > 0.$$

From (3.23) it follows that the function $t \mapsto (T_1(t)f)(x)$ is differentiable in $(0,\infty)$ and that

$$(D_tT_1(t)f)(x) = \int_0^t D_t(T_0(t-s)S(s)f)(x)\,ds + (S(t)f)(x),$$

with $t > 0$ and $x \in \Omega$. Therefore,

$$\begin{aligned}\|D_tT_1(t)f\|_{C(K)} &\leq C\|f\|_{C_b(\overline{\Omega})}\int_0^t (t-s)^{-1+\frac{\theta}{2}}s^{-\frac{1+\theta}{2}}\,ds + Ct^{-\frac{1}{2}}\|f\|_{C_b(\overline{\Omega})}\\ &= C\|f\|_{C_b(\overline{\Omega})}t^{-\frac{1}{2}}\left(\int_0^1 (1-s)^{-1+\frac{\theta}{2}}s^{-\frac{1+\theta}{2}}\,ds + 1\right).\end{aligned}$$

Hence, $t \mapsto (T_2(t)f)(x)$ is differentiable in $(0,\infty)$ and

$$(D_tT_2(t)f)(x) = \int_0^t D_t(T_1(t-s)S(s)f)(x)\,ds, \qquad t > 0,\ x \in \Omega.$$

Since $T_{n-1}(0) = 0$ for $n \geq 2$, we obtain the recursive formula

$$(D_tT_n(t)f)(x) = \int_0^t D_t(T_{n-1}(t-s)S(s)f)(x)\,ds, \qquad t > 0,\ x \in \Omega.$$

Applying Lemma 3.3 with $R(t) = D_tT_1(t)$ and $S(t)$ being defined as in (3.17), the claim follows.

In the following step we show that $D_tT(t)f = \mathcal{A}T(t)f$ in Ω for any $t > 0$. Note that, for $f \in C_b(\overline{\Omega})$,

$$\Big(\mathcal{A}\int_0^\delta (T(\tau+t)f)(\cdot)\,d\tau\Big)(x) = \int_0^\delta (\mathcal{A}T(\tau+t)f)(x)\,d\tau, \quad \delta, t > 0,\ \ x \in \Omega.$$

Hence,

$$\begin{aligned}\frac{1}{\delta}\int_0^\delta (D_tT(\tau+t)f)(x)\,d\tau &= \frac{1}{\delta}\{(T(\delta+t)f)(x) - (T(t)f)(x)\}\\ = \frac{1}{\delta}\Big(\mathcal{A}\int_0^\delta (T(\tau+t)f)(\cdot)\,d\tau\Big)(x) &= \frac{1}{\delta}\int_0^\delta (\mathcal{A}T(\tau+t)f)(x)\,d\tau.\end{aligned}$$

Letting δ tend to 0^+, we obtain

$$(D_tT(t)f)(x) = (\mathcal{A}T(t)f)(x), \qquad t > 0,\ \ x \in \Omega.$$

Further, $T(\cdot)f$ is continuous on $\{0\}\times\overline{\Omega}$ provided $\mathcal{B}$ is the Neumann boundary operator, and in $\{0\}\times\Omega$ provided $\mathcal{B}u = u$. Thus, $T(\cdot)f$ is the unique classical solution of the Cauchy problem (1.1) (with $g \equiv 0$).

Finally, $\{T(t)\}$ is a positive semigroup due to the maximum principle in Proposition 3.1(ii). In particular, estimate (3.2) implies that $T(t)$ is a contraction, for any $t > 0$. The remaining assertions follow from Proposition 3.5. $\square$

3.3. **Consequences.** In this subsection we state optimal Schauder estimates for the classical solution of the nonhomogeneous Cauchy problem (1.1). They can be proved adapting the proofs given in [21] (see also [6, Chapter 6]), with slight modifications. For this reason, we leave most of the proofs to the reader.

One of the main tools to prove our results is the following lemma.

Lemma 3.7. *Let $\theta \in (0,3)\backslash\{1,2\}$, $I \subset \mathbb{R}$ be an interval, and $\varphi : I \to C_b^\theta(\overline{\Omega})$ be a function such that, for $x \in \overline{\Omega}$, the real function $t \mapsto \varphi(t)(x)$ is continuous in I and*

$$\|\varphi(t)\|_{C_b^\theta(\overline{\Omega})} \le c(t), \qquad t \in I,$$

for some function $c \in L^1(I)$. Then, the function f, defined by

$$f(x) := \int_I \varphi(t)(x)\,dt, \qquad x \in \overline{\Omega},$$

belongs to $C_b^\theta(\overline{\Omega})$ *and* $\|f\|_{C_b^\theta(\overline{\Omega})} \le M\|c\|_{L^1(I)}$*, for some constant* $M > 0$ *independent of* f*. Further, if* $\mathcal{B}\varphi(t) \equiv 0$ *on* $\partial\Omega$ *for any* $t \in I$*, then* $\mathcal{B}f$ *vanishes identically on* $\partial\Omega$ *as well.*

A result of this type has been proved in [20, Section 3] for $\Omega = \mathbb{R}^N$. The generalization to exterior domains is based on a suitable extension operator.

Theorem 3.8. *Assume that Hypotheses* $(HCb)_\Omega$ *hold. Then, for* $\lambda > 0$, $\alpha \in (0,1)$ *and* $f \in C_{\mathcal{B}}^\alpha(\overline{\Omega})$*, there exists a unique solution* $u \in C_b^{2+\alpha}(\overline{\Omega})$ *of the elliptic problem* (3.1)*. Moreover, there exists a constant* $C > 0$*, independent of* f*, such that*

$$\|u\|_{C_b^{2+\alpha}(\overline{\Omega})} \le C\|f\|_{C_b^\alpha(\overline{\Omega})}.$$

Furthermore, if $u \in D_\Omega$ *satisfies* $\mathcal{A}u \in C_{\mathcal{B}}^\alpha(\overline{\Omega})$*, then* $u \in C_b^{2+\alpha}(\overline{\Omega})$*.*

Finally, for any $\theta \in (0,2)$*,* $C_b^\theta(\Omega)$ *is an intermediate space of class* $J_{\theta/2}$*, between* $C_b(\overline{\Omega})$ *and* D_Ω*; i.e., there exists a positive constant* C *such that*

$$\|u\|_{C_b^\theta(\overline{\Omega})} \le C\|u\|_{C_b(\overline{\Omega})}^{1-\frac{\theta}{2}}\|u\|_{D_\Omega}^{\frac{\theta}{2}}, \qquad u \in D_\Omega.$$

Proof. Using Theorems 3.2 and 3.6, one can prove that for any $\lambda > 0$ the function u, defined by $u(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)dt =: (R(\lambda, \mathcal{A})f)(x)$ for any $x \in \mathbb{R}^N$, is a solution of the elliptic problem (3.1), and it is its unique solution if $\lambda \ge \lambda_1$, by Proposition 3.1. On the other hand, if $\lambda < \lambda_1$ any solution v to (3.1) is a fixed point of the operator $\Gamma \in \mathcal{L}(C_b(\overline{\Omega}))$ defined by $\Gamma(w) = (\lambda_1 - \lambda)R(\lambda_1, \mathcal{A})w + R(\lambda_1, \mathcal{A})f$ for any $w \in C_b(\overline{\Omega})$. As is immediately seen, Γ is a contraction if $\lambda \in [\lambda_1/2, \lambda_1]$. Hence, for such λ's $R(\lambda, \mathcal{A})f$ is still the unique solution to problem (3.1). Iterating this argument, we can easily show that, for any $\lambda > 0$ and any $f \in C_{\mathcal{B}}^\alpha(\overline{\Omega})$, $R(\lambda, \mathcal{A})f$ is the unique solution to (3.1).

The remaining assertions can be shown arguing as in the proof of [21] and taking Lemma 3.7 into account. See also [6, Proposition 6.2.6]. □

We finally consider the nonhomogeneous Cauchy problem (1.1). Clearly, the candidate solution of (1.1) is the function

$$u(t,x) = (T(t)f)(x) + \int_0^t (T(t-s)g(s,\cdot))(x)\,ds, \quad t > 0,\ x \in \overline{\Omega}. \tag{3.24}$$

Here, the integrals are meant pointwise, since $t \mapsto T(t)\psi$ may fail to be integrable in $(0,T)$ when $\psi \in C_b(\overline{\Omega})$ (see [6, Chapter 1]). For a function g which may blow up at $t = 0$ the following results hold true.

Proposition 3.9. *Let* $\alpha, \theta \in (0,1)$, $\beta \in (\alpha, 2+\alpha]$ *and* $T > 0$. *Further, let* $f \in C_b(\overline{\Omega})$ *and let* g *be a continuous function in* $(0,T] \times \overline{\Omega}$ *such that* $g(t,\cdot) \in C^{\beta}_{\mathcal{B}}(\overline{\Omega})$ *for* $t \in (0,T]$ *and*

$$\sup_{0<t\leq T} t^{\theta} \|g(t,\cdot)\|_{C^{\beta}_b(\overline{\Omega})} < \infty.$$

Finally, let u *be defined as in* (3.24). *Then,* $u \in C([0,T]\times\overline{\Omega}) \cap C^{1,2}((0,T]\times\Omega)$ *and it is the unique bounded classical solution to problem* (1.1). *Moreover,* $u(t,\cdot) \in C^{2+\alpha}_b(\Omega)$ *for* $t \in (0,T]$ *and there exists a constant* $C > 0$, *independent of* u, f *and* g, *such that for* $t \in (0,T]$,

$$\|u(t,\cdot)\|_{C_b(\overline{\Omega})} + t^{1+\frac{\alpha}{2}} \|u(t,\cdot)\|_{C^{2+\alpha}_b(\overline{\Omega})} \leq C(\|f\|_{C_b(\overline{\Omega})} + \sup_{0<t\leq T} t^{\theta} \|g(t,\cdot)\|_{C^{\beta}_b(\overline{\Omega})}).$$

The uniqueness of the bounded classical solution to the problem (1.1) follows by Proposition 3.1(ii). The proof that u is a bounded classical solution of (1.1) is similar to that in [21] (see also [6, Proposition 5.2.4]).

We finally state optimal Schauder estimates for the parabolic problem (1.1). More precisely, by using Proposition 3.9 and adapting to our situation the arguments in [21], one obtains the following result.

Theorem 3.10. *Suppose that Hypotheses* $(HCb)_{\Omega}$ *hold. Let* $T > 0$, $\alpha \in (0,1)$ *and* $g : [0,T] \times \overline{\Omega} \to \mathbb{R}$ *be a bounded and continuous function such that* $g(t,\cdot) \in C^{\alpha}_{\mathcal{B}}(\overline{\Omega})$ *for* $t \in [0,T]$ *and*

$$\sup_{t\in[0,T]} \|g(t,\cdot)\|_{C^{\alpha}_b(\overline{\Omega})} < \infty.$$

Further, let $f \in C^{2+\alpha}_{\mathcal{B}}(\overline{\Omega})$. *Then, the function* u *defined by* (3.24) *is the unique strict solution to* (1.1) *and there exists a constant* $C > 0$, *independent of* u, f *and* g, *such that*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{C^{2+\alpha}_b(\overline{\Omega})} \leq C(\|f\|_{C^{2+\alpha}_b(\overline{\Omega})} + \sup_{t\in[0,T]} \|g(t,\cdot)\|_{C^{\alpha}_b(\overline{\Omega})}).$$

Remark 3.11. Taking into account Proposition A.1 (see the appendix), it is possible to show that all the previous results hold if one replaces Dirichlet or Neumann boundary conditions by the more general boundary conditions induced by the operator $\mathcal{B}$ in (A.1), provided that c_0 is nonnegative on $\partial\Omega$.

In the general case when c_0 is not everywhere nonnegative on $\partial\Omega$, we can still construct the semigroup $\{T(t)\}$, which is not necessarily contractive. As a consequence, in this situation we can not allow ω in the estimates (3.20) to be any positive real number, but we can still prove these estimates for suitable positive constants ω and C. All the other results remain still true.

4. ASSUMPTIONS AND PRELIMINARIES IN $L^p(\Omega)$

In this section we consider again the parabolic problem (1.1) (with $g \equiv 0$), however this time in the setting of L^p-spaces. Our main result, Theorem 5.3, corresponds to Theorem 3.6 for the case of continuous functions.

The assumptions on the coefficients of the operator $\mathcal{A}$ are slightly different from the ones given in Section 2 in the setting of continuous functions. In fact, we assume that $\mathcal{A}$ is in divergence form. Note that $\mathcal{A}$ may be rewritten in divergence form provided the coefficients are smooth enough. More precisely, setting $F_j = b_j - \sum_{i=1}^N D_i q_{ij}$, $j = 1, \ldots, N$, the operator $\mathcal{A}$ has the form

$$\mathcal{A}\varphi = \mathrm{div}(QD\varphi) + F \cdot D\varphi. \tag{4.1}$$

In the remaining part of this paper we always assume that $\mathcal{A}$ is given by (4.1). Moreover, we assume that Ω is an exterior domain with a boundary of class $C^{3+\gamma}$ for some $\gamma \in (0,1)$. As far as the coefficients of the operator $\mathcal{A}$ are concerned, we make the following assumptions.

Hypotheses $(HLp)_\Omega$**.**

(i) $q_{ij} = q_{ij} \in C_b^4(\Omega) \cap C_{\rm loc}^{4+\delta}(\overline{\Omega})$, $F_j \in C_{\rm loc}^{3+\delta}(\overline{\Omega})$ for some $\delta \in (0,1)$ and any $i,j = 1, \ldots, N$. Moreover,

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2, \qquad \xi \in \mathbb{R}^N,\ x \in \Omega,$$

for some constant $\mu > 0$;

(ii) (a) $|\langle DF(x)Q(x)\xi, \xi\rangle| \leq \beta_1|\xi|^2$,

(b) $|\langle DF(x)\xi, \xi\rangle| \leq \beta_2|\xi|^2$,

(c) $\left|\sum_{k=1}^N F_k(x)D_k q_{ij}(x)\right| \leq \beta_3, \quad i,j = 1, \ldots, N,$

for any $x \in \Omega$, $\xi \in \mathbb{R}^N$ and some constants $\beta_1, \beta_2, \beta_3 > 0$;

(iii) $D^\alpha F_j$ is bounded for $|\alpha| = 2, 3$ and $j = 1, \ldots, N$.

Here, $Q(\cdot) := (q_{ij}(\cdot))$.

Remark 4.1. (i). Hypothesis $(HLp)_\Omega$-(ii)-(b) implies that Hypothesis $(HCb)_\Omega$-(ii) is satisfied with $\varphi(x) = \log|x|$ provided $|x|$ is large.

(ii). There is no relationship between Hypotheses $(HLp)_\Omega$-(ii)-(a) and $(HLp)_\Omega$-(ii)-(b). Indeed, in $\mathbb{R}^2$ take

$$Q(x,y) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad F(x,y) = \log(1 + x^2 + y^2)\begin{pmatrix} -y \\ x \end{pmatrix}, \qquad (x,y) \in \mathbb{R}^2,$$

where α and β are two positive constants. Then, F satisfies Hypothesis $(HLp)_{\mathbb{R}^2}$-(ii)-(b) but not Hypothesis $(HLp)_{\mathbb{R}^2}$-(ii)-(a). On the other hand, take now

$$F(x,y) = \log(1+x^2+y^2)\begin{pmatrix}-y/\beta\\ x/\alpha\end{pmatrix}, \qquad (x,y)\in\mathbb{R}^2.$$

Then, F satisfies Hypothesis $(HLp)_{\mathbb{R}^2}$-(ii)-(a) but not Hypothesis $(HLp)_{\mathbb{R}^2}$-(ii)-(b).

(iii). Observe that Hypotheses $(HLp)_\Omega$-(ii)-(a) and $(HLp)_\Omega$-(ii)-(c), the boundedness of the diffusion coefficients and their gradients, and, finally, the uniform ellipticity condition are needed in order to prove that the operator

$$\widehat{A}_p u = \operatorname{div}(QDu) + F\cdot Du,$$

with domain

$$D(\widehat{A}_p) = \{u \in W^{2,p}(\mathbb{R}^N) : F\cdot Du \in L^p(\mathbb{R}^N)\}, \tag{4.2}$$

generates a C_0-semigroup $\{\widehat{T}_p(t)\}$ on $L^p(\mathbb{R}^N)$ for any $p\in(1,\infty)$ (see [26, Theorem 2.4]).

(iv). Regularity assumptions on Q and F, and Hypotheses $(HLp)_\Omega$-(ii)-(b) and $(HLp)_\Omega$-(iii), are needed to have pointwise estimates for the first-, second- and third-order spatial derivatives of $\widehat{T}_p(\cdot)f$ (see [5, Theorem 4.7 and Corollary 4.8]).

Examples 4.2. Take $Q = I$ and $\Omega = \mathbb{R}^N$.

(i) Any smooth Lipschitz function $F:\mathbb{R}^N\to\mathbb{R}^N$ with bounded second- and third-order derivatives satisfies $(HLp)_{\mathbb{R}^N}$.

(ii) Consider a matrix M with $\langle Mx, x\rangle = 0$ for any $x\in\mathbb{R}^N$ and an arbitrary matrix B. Then, the function F, defined by

$$F(x) := \log(1+|x|^2)Mx + Bx$$

for $x\in\mathbb{R}^N$, satisfies $(HLp)_{\mathbb{R}^N}$.

Due to the pointwise gradient estimates in [5, Theorem 4.7 and Corollary 4.8]) and an interpolation argument, and since $\widehat{A}_p$ generates a C_0-semigroup on $L^p(\mathbb{R}^N)$ (see [26, Theorem 2.4]), we obtain the following result.

Proposition 4.3. *Let $p\in(1,\infty)$ and assume that Hypotheses $(HLp)_{\mathbb{R}^N}$ are satisfied. Then, the operator $\widehat{A}_p$ generates a C_0-semigroup $\{\widehat{T}_p(t)\}$ on $L^p(\mathbb{R}^N)$ which satisfies*

$$\|\widehat{T}_p(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N))} \le e^{\omega_p t}, \qquad t\ge 0,$$

for some $\omega_p \in \mathbb{R}$. *Moreover, for* $k, l \in [0,3]$ *with* $k \leq l$, *there exist constants* $M > 0$ *and* $\widehat{\omega}_p \in \mathbb{R}$ *such that*

$$\|\widehat{T}_p(t)f\|_{W^{l,p}(\mathbb{R}^N)} \leq Mt^{-\frac{l-k}{2}} e^{\widehat{\omega}_p t}\|f\|_{W^{k,p}(\mathbb{R}^N)}, \quad t > 0,\ f \in W^{k,p}(\mathbb{R}^N). \tag{4.3}$$

Remark 4.4. Adapting the proof given in [20] in the C_b-case and taking advantage of the estimate (4.3), with $k = 0$ and $l = 1$, one can show that

$$\|Du\|_{L^p(\mathbb{R}^N)} \leq C\|u\|^{\frac{1}{2}}_{L^p(\mathbb{R}^N)}\|(\widehat{A}_p - \omega)u\|^{\frac{1}{2}}_{L^p(\mathbb{R}^N)}, \qquad u \in D(\widehat{A}_p),$$

for some constants $C, \omega > 0$.

5. L^p ESTIMATES AND CONSEQUENCES

We now consider operators of the above form in exterior domains subject to certain boundary conditions. More precisely, set

$$A_\Omega u \ := \ \mathrm{div}(QDu) + F \cdot Du,$$

with domain

$$D(A_\Omega) \ := \ \{u \in W^{2,p}_{\mathcal{B}}(\Omega) : F \cdot Du \in L^p(\mathbb{R}^N)\},$$

where $W^{2,p}_{\mathcal{B}}(\Omega) := \{u \in W^{2,p}(\Omega) : \mathcal{B}u = 0 \text{ on } \partial\Omega\}$. As in the previous sections, we only consider $\mathcal{B}u = u$ or $\mathcal{B}u = \frac{\partial u}{\partial \nu}$, ν being the outward unit normal vector to $\partial\Omega$.

Let us start by showing that, under rather general assumptions, $\mathcal{A}$ generates a C_0-semigroup on $L^p(\Omega)$. This, in particular, generalizes [12, Theorem 2.5] where the Ornstein-Uhlenbeck semigroup in exterior domains was considered for the first time.

Theorem 5.1. *Let* $p \in (1, \infty)$ *and* $\Omega \subset \mathbb{R}^N$ *be an exterior domain with* $C^{2+\gamma}$ *boundary for some* $\gamma \in (0,1)$. *Assume that* Q *satisfies the ellipticity condition in* $(HLp)_\Omega$-*(i) and*

(i) $q_{ij} \in C^{2+\delta}_{\mathrm{loc}}(\overline{\Omega})$, $F_i \in C^{1+\delta}_{\mathrm{loc}}(\overline{\Omega})$ *for some* $\delta \in (0,1)$ *and* $i, j = 1, \ldots, N$;

(ii) *Hypotheses* $(HLp)_\Omega$-*(ii) hold.*

Then, the operator $(A_\Omega, D(A_\Omega))$ *generates a* C_0-*semigroup* $\{T_\Omega(t)\}$ *on* $L^p(\Omega)$ *satisfying*

$$\|T_\Omega(t)\|_{\mathcal{L}(L^p(\Omega))} \leq M_p e^{\omega_p t}, \qquad t \geq 0,$$

for some $\omega_p \in \mathbb{R}$ *and* $M_p \geq 1$. *Moreover, there exist constants* M *and* ω *such that*

$$\|T_\Omega(t)f\|_{L^\infty(\Omega)} \leq Mt^{-\frac{N}{2}} e^{\omega t}\|f\|_{L^1(\Omega)}, \qquad f \in L^1(\Omega),\ t > 0. \tag{5.1}$$

Proof. Being rather long, we split the proof into two steps.
Step 1). Here, we prove that the operator A_Ω is quasi dissipative. We limit ourselves to dealing with the case when Ω is equipped with homogeneous Neumann conditions since this case is a bit more tricky than the other one. Indeed, integrating by parts, it is not immediate to get rid of the integral term on the boundary of Ω.

We recall first that, due the smoothness of $\partial\Omega$, the function $d(x) = \mathrm{dist}(x,\overline{\Omega})$, $x \in \mathbb{R}^N$, is in $C^2(\Gamma(\rho))$ for some $\rho > 0$, where $\Gamma(\rho) = \{x \in \overline{\Omega} : d(x) \le \rho\}$. Moreover, $Dd(x) = -\nu(x)$, $x \in \partial\Omega$ (cf. [14, Section 14.6]). Now, let us consider the skew-symmetric matrix $P(x)$ defined by

$$P(x)\xi = \vartheta(x)\left\{\langle \xi, Q(x)Dd(x)\rangle Dd(x) - \langle \xi, Dd(x)\rangle Q(x)Dd(x)\right\},$$

$x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$, where ϑ is a cut-off function supported in $\Gamma(\rho/2)$ with $\vartheta \equiv 1$ on $\partial\Omega$. Set $Q_1 = Q + P$. Then,

$$Q_1(x)\nu(x) = \langle \nu(x), Q(x)\nu(x)\rangle \nu(x), \qquad x \in \partial\Omega.$$

This implies that the conormal derivative (with respect to the matrix Q_1) satisfies

$$\langle Q_1(x)\nu(x), Du(x)\rangle = \langle \nu(x), Q(x)\nu(x)\rangle \frac{\partial u}{\partial \nu}(x),$$

at $x \in \partial\Omega$, for any (smooth) function u.

Since $P(x)$ is skew-symmetric, an immediate computation shows that

$$\mathcal{A}\varphi = \mathrm{div}(Q_1 D\varphi) + F\cdot D\varphi - \sum_{i,j=1}^N D_i P_{ij} D_j\varphi := \mathrm{div}(Q_1 D\varphi) + F'\cdot D\varphi,$$

on smooth functions φ.

Let us first consider the case when $p \ge 2$ and set $\lambda u - A_\Omega u = f$ for $u \in D(A_\Omega)$. Multiplying the previous equality by $u|u|^{p-2}\varrho_n$ (where ϱ_n is any smooth function such that $\chi_{B(n)} \le \varrho_n \le \chi_{B(2n)}$ and $\|D\varrho_n\|_\infty \le Kn^{-1}$ for any $n \in \mathbb{N}$ and some positive constant K, independent of n), integrating by parts, and then, letting n go to $+\infty$, gives

$$\begin{aligned}\int_\Omega fu|u|^{p-2}dx = \lambda \int_\Omega |u|^p dx + (p-1)\int_\Omega \langle QDu, Du\rangle\, |u|^{p-2}dx \\ - \int_\Omega \langle F', Du\rangle u|u|^{p-2}dx. \qquad (5.2)\end{aligned}$$

We now rewrite in a more convenient way the last integral term in the right-hand side of (5.2). For this purpose, we split

$$F_j' = (1-\vartheta)F_j + \vartheta F_j + \sum_{i=1}^{N} D_i P_{ij}, \qquad j = 1, \ldots, N,$$

where ϑ is as above. Since both ϑF_j and $\sum_{i=1}^{N} D_i P_{ij}$ are bounded functions, we can estimate

$$\begin{aligned}
&\Big| \int_\Omega \Big(\langle \vartheta F, Du \rangle + \sum_{i,j=1}^{N} D_i P_{ij} D_j u \Big) u|u|^{p-2} dx \Big| \\
&\le C_1 \int_\Omega |u|^{p-1}|Du| dx \le C_2 \Big(\int_\Omega |u|^{p-2}|Du|^2 dx \Big)^{\frac{1}{2}} \Big(\int_\Omega |u|^p dx \Big)^{\frac{1}{2}} \\
&\le \varepsilon \int_\Omega |u|^{p-2}|Du|^2 dx + C_\varepsilon \int_\Omega |u|^p dx, \qquad (5.3)
\end{aligned}$$

for any $\varepsilon > 0$ and some positive constants C_1, C_2 and C_ε.

To estimate the integral term containing $(1-\vartheta)F$, we observe that, by [26, Lemma 2.2], $C_c^1(\mathbb{R}^N)$ is dense in the space $\{u \in W^{1,p}(\mathbb{R}^N) : F \cdot Du \in L^p(\mathbb{R}^N)\}$, endowed with the norm $u \mapsto \|u\|_{L^p(\mathbb{R}^N)} + \|F \cdot Du\|_{L^p(\mathbb{R}^N)}$. Hence, adapting this result to our situation, we can infer that there exists a sequence $\{u_n\} \in C_c^1(\overline{\Omega})$ such that $u_n \to u$ and $F \cdot Du_n \to F \cdot Du$ in $L^p(\Omega)$ as $n \to +\infty$.

Integrating by parts the term $\int_\Omega (1-\vartheta)\langle F, Du_n\rangle\, u_n|u_n|^{p-2}dx$ and noting that the function $(1-\vartheta)F$ vanishes on $\partial\Omega$, we get

$$\begin{aligned}
&\int_\Omega (1-\vartheta)\langle F, Du_n\rangle\, u_n|u_n|^{p-2}dx = -\frac{1}{p}\int_\Omega \operatorname{div}((1-\vartheta)F)|u_n|^p dx \\
&= -\frac{1}{p}\int_\Omega (1-\vartheta)(\operatorname{div} F)|u_n|^p dx + \frac{1}{p}\int_\Omega \langle D\vartheta, F\rangle\, |u_n|^p dx.
\end{aligned}$$

Hence, since, by assumptions, $\operatorname{div} F$ is bounded in Ω, we can estimate

$$\Big| \int_\Omega (1-\vartheta)\langle F, Du_n\rangle\, u_n|u_n|^{p-2}dx \Big| \le C_3 \int_\Omega |u_n|^p dx,$$

for some positive constant C_3, independent of n. Letting n go to $+\infty$ yields

$$\Big| \int_\Omega (1-\vartheta)\langle F, Du\rangle\, u|u|^{p-2}dx \Big| \le C_3 \int_\Omega |u|^p dx. \qquad (5.4)$$

Summing up, from (5.2), (5.3) and (5.4), we get

$$\|f\|_{L^p(\Omega)}\|u\|_{L^p(\Omega)}^{p-1} \ge (\lambda - C_\varepsilon - C_3)\|u\|_{L^p(\Omega)}^p + ((p-1)\mu - \varepsilon)\int_\Omega |u|^{p-2}|Du|^2 dx,$$

from which we immediately deduce that A_Ω is quasi dissipative, by taking $\varepsilon = (p-1)\mu$ and $\lambda > C_{(p-1)\mu} + C_3$.

In the case when $p \in (1,2)$, we replace the function $|u|^{p-2}u$, previously considered, by the function $u(u^2+\sigma)^{(p-2)/2}$, where σ is a positive parameter. Arguing as above, we get

$$\int_\Omega fu(u^2+\sigma)^{\frac{p-2}{2}}dx = \lambda \int_\Omega u^2(u^2+\sigma)^{\frac{p-2}{2}}dx \tag{5.5}$$
$$+\int_\Omega \langle QDu, Du\rangle (u^2+\sigma)^{\frac{p-4}{2}} \left((p-1)u^2+\sigma\right) dx - \int_\Omega \langle F', Du\rangle u(u^2+\sigma)^{\frac{p-2}{2}}dx.$$

Next, letting σ go to 0^+ we get

$$\begin{aligned}
&\|f\|_{L^p(\Omega)}\|u\|^{p-1}_{L^p(\Omega)} \geq \lambda \|u\|^p_{L^p(\Omega)}\\
&+\limsup_{\sigma\to 0^+}\int_\Omega \langle QDu, Du\rangle\, (u^2+\sigma)^{\frac{p-4}{2}}((p-1)u^2+\sigma)dx - \int_\Omega \langle F', Du\rangle u|u|^{p-2}dx\\
&\geq \lambda\|u\|^p_{L^p(\Omega)} + (p-1)\mu \limsup_{\sigma\to 0^+}\int_\Omega |Du|^2(u^2+\sigma)^{\frac{p-4}{2}}((p-1)u^2+\sigma)dx\\
&-\int_\Omega \langle F', Du\rangle u|u|^{p-2}dx\\
&= \lambda\|u\|^p_{L^p(\Omega)} + (p-1)\mu\int_\Omega |Du|^2|u|^{p-2}dx - \int_\Omega \langle F', Du\rangle u|u|^{p-2}dx,
\end{aligned}$$

where we have taken advantage of the monotone convergence theorem. As in the case when $p \geq 2$, we have to estimate the last integral term in (5.5). For this purpose, we use the same splitting as above, and integrate by parts, first approximating u with a sequence of compactly supported functions. We get

$$\begin{aligned}
&\int_\Omega (1-\vartheta)\langle F, Du\rangle\, u(u^2+\sigma)^{\frac{p-2}{2}}dx\\
&= \int_\Omega \langle F, D\vartheta\rangle\, u^2(u^2+\sigma)^{\frac{p-2}{2}}dx - \int_\Omega (1-\vartheta)(\operatorname{div} F)u^2(u^2+\sigma)^{\frac{p-2}{2}}dx\\
&- \int_\Omega (1-\vartheta)\langle F, Du\rangle\, u\left((p-1)u^2+\sigma\right)(u^2+\sigma)^{\frac{p-4}{2}}dx,
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\int_\Omega (1-\vartheta)\langle F, Du\rangle\, (u^2+\sigma)^{\frac{p-4}{2}}u(pu^2+2\sigma)dx\\
=\;& \int_\Omega \langle F, D\vartheta\rangle\, u^2(u^2+\sigma)^{\frac{p-2}{2}}dx - \int_\Omega (1-\vartheta)(\operatorname{div} F)u^2(u^2+\sigma)^{\frac{p-2}{2}}dx.
\end{aligned}$$

Letting σ go to 0^+, we get

$$\int_\Omega (1-\vartheta)\langle F, Du\rangle\, u|u|^{p-2}dx = \frac{1}{p}\int_\Omega \langle F, D\vartheta\rangle\, |u|^p dx - \frac{1}{p}\int_\Omega (1-\vartheta)(\mathrm{div}\, F)|u|^p dx. \tag{5.6}$$

From (5.3), (5.5) and (5.6), it follows that A_Ω is quasi dissipative also in this situation.

Step 2). As in the proof of Theorem 3.2, we extend A_Ω to a uniformly elliptic operator $\widehat{A}$ whose coefficients are defined in the whole of $\mathbb{R}^N$ and satisfy Hypotheses $(HLp)_{\mathbb{R}^N}$. Fix $f \in L^p(\Omega)$ and consider, as in (3.6), the functions

$$\widehat{f} = \zeta f \quad\text{and}\quad \widetilde{f} = \chi f.$$

Then, $\widehat{f} \in L^p(\mathbb{R}^N)$ and $\widetilde{f} \in L^p(\Omega_R)$, where we recall that $\Omega_R = \{x \in \Omega : |x| < R+3\}$. By Proposition 4.3, the operator $\widehat{A}_p$, with domain given by (4.2), generates a quasi-contractive C_0-semigroup in $L^p(\mathbb{R}^N)$. On the other hand, the realization A_{Ω_R} of $\mathcal{A}$ in $L^p(\Omega_R)$ with domain $D(A_{\Omega_R}) = \{u \in W^{2,p}(\Omega_R) : \mathcal{B}u = 0 \text{ on } \partial\Omega_R\}$ generates a contraction analytic semigroup by [19, Theorems 2.1.11, 3.1.2, and 3.1.3]). As in (3.8), we define

$$U_\lambda f := \psi R(\lambda, \widehat{A}_p)\widehat{f} + (1-\psi)R(\lambda, A_{\Omega_R})\widetilde{f}.$$

Since $U_\lambda f = R(\lambda, A_{\Omega_R})\widetilde{f}$ in $\Omega \cap B(R+1)$, we obtain $\mathcal{B}U_\lambda f = 0$ on $\partial\Omega$. Hence, $U_\lambda f \in W^{2,p}(\Omega)$ and, by the proof of Theorem 3.2, we have

$$(\lambda - A_\Omega)U_\lambda f = f + T_\lambda f,$$

where T_λ is given by (3.9). Since

$$R(\lambda, \widehat{A})\widehat{f} = \int_0^{+\infty} e^{-\lambda t}\widehat{T}(t)\widehat{f}\, dt, \qquad \lambda > \widehat{\omega}_p,$$

from (4.3) we get

$$\begin{aligned} \|DR(\lambda, \widehat{A})f\|_{L^p(\mathbb{R}^N)} &\le C\|\widehat{f}\|_{L^p(\mathbb{R}^N)} \int_0^{+\infty} t^{-\frac{1}{2}} e^{-(\lambda-\widehat{\omega}_p)t} dt \\ &\le \frac{C}{\sqrt{\lambda - \widehat{\omega}_p}}\|f\|_{L^p(\Omega)}, \qquad \lambda > \widehat{\omega}_p. \end{aligned} \tag{5.7}$$

Hence, from (5.7) and [19, Theorem 3.1.3] we get

$$\|T_\lambda f\|_{L^p(\Omega)} \le \frac{C}{\sqrt{\lambda}}\|f\|_{L^p(\Omega)},$$

for large $\lambda > 0$. Thus, there exists $\lambda_1 > 0$ such that $u = U_\lambda(I + T_\lambda)^{-1} f \in D(A_\Omega)$ and

$$\lambda u - A_\Omega u = f, \qquad \lambda > \lambda_1.$$

From Steps 1 and 2 and the Lumer-Phillips theorem, the first assertion now follows.

Step 3. In order to prove (5.1), fix $f \in L^1(\Omega) \cap L^p(\Omega)$ and let $t > 0$. Upper Gaussian estimates for $\{T_{\Omega_R}(t)\}$ imply (5.1). By [13, Lemma 4.6] (which is the L^p-version of Lemma 3.3) and (5.9) below, it suffices to show (5.1) for $\Omega = \mathbb{R}^N$. To this end, note that $\widehat{T}_p(t)\widehat{f}$ is given by the Trotter product formula

$$\widehat{T}_p(t)\widehat{f} = \lim_{n\to\infty} [S_1(t/n)S_2(t/n)]^n \, \widehat{f},$$

where $\{S_1(t)\}$ is the semigroup generated by $\operatorname{div}(QDu)$ and $(S_2(t)\widehat{f})(x) = \widehat{f}(\phi(t,x))$, ϕ being the flow solving the ordinary differential equation

$$\partial_t \phi = F(\phi), \qquad \phi(0,x) = x$$

(for details, see [26]). Since $\| \, [S_1(t/n)S_2(t/n)]^n \, \widehat{f}\|_{L^1(\mathbb{R}^N)} \le e^{\omega t}\|\widehat{f}\|_{L^1(\mathbb{R}^N)}$, for some $\omega \in \mathbb{R}$. Fatou's lemma implies, for $t > 0$,

$$\|\widehat{T}_p(t)\widehat{f}\|_{L^1(\mathbb{R}^N)} \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left| ([S_1(t/n)S_2(t/n)]^n \, \widehat{f})(x) \right| dx \le e^{\omega t}\|\widehat{f}\|_{L^1(\mathbb{R}^N)}.$$

Hence, $\widehat{T}_p(t)$ may be extended to a bounded operator on $L^1(\mathbb{R}^N)$. Since $C_c^1(\mathbb{R}^N)$ is a core of $\{S_2(t)\}$ (see again [26, Lemma 2.2]), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N} u(\lambda u - \widehat{A}_p u)\, dx &\ge \int_{\mathbb{R}^N} \Big\{\mu|Du|^2 + \Big(\lambda + \frac{1}{2}\operatorname{div}F\Big)|u|^2\Big\} dx \\ &\ge \inf\{\mu, 1\}\|u\|_{W^{1,2}(\mathbb{R}^N)},\end{aligned}$$

for $\lambda > 1 + \frac{1}{2}\|\operatorname{div}F\|_\infty$ and $u \in D(\widehat{A}_2)$. Using Nash's inequality, we deduce, as in the proof of [22, Proposition 5.4], that

$$\|\widehat{T}_p(t)\widehat{f}\|_{L^\infty(\mathbb{R}^N)} \le Mt^{-\frac{N}{2}} e^{\omega t}\|\widehat{f}\|_{L^1(\mathbb{R}^N)}, \qquad t > 0.$$

This proves the second statement. □

L^p–L^q smoothing properties of $\{T_\Omega(t)\}$ now follow by interpolation, as the following corollary shows.

Corollary 5.2. *Let $1 < p \le q < \infty$. Under the assumptions of Theorem* 5.1, *there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that*

$$\|T_\Omega(t)f\|_{L^q(\Omega)} \le Me^{\omega t}t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}, \qquad t > 0, \ \ f \in L^p(\Omega).$$

Estimates for higher-order spatial derivatives of $\{T_\Omega(t)f\}$ are given in the following theorem.

Theorem 5.3. *Let $f \in L^p(\Omega)$, $t > 0$ and assume that Hypotheses $(HLp)_\Omega$ hold. Then, $T_\Omega(t)f \in W^{3,p}(\Omega)$ and, for $\beta \in (0,3)$, there exist positive constants M and ω_p such that*

$$\|T_\Omega(t)f\|_{W^{\beta,p}(\Omega)} \le Mt^{-\frac{\beta}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)}, \qquad t > 0,\ f \in L^p(\Omega). \tag{5.8}$$

Proof. Throughout the proof, we denote by C_p and ω_p positive constants, independent of f and t, which may vary from line to line.

By the proof of Theorem 5.1 we have

$$R(\lambda, A_\Omega)f = U_\lambda(I + T_\lambda)^{-1}f, \qquad f \in L^p(\Omega),$$

for any $\lambda > \lambda_1$ and some $\lambda_1 > 0$. We claim that the series

$$\sum_{n=0}^{\infty} T_n(t)f$$

converges for $t > 0$ and $f \in L^p(\Omega)$, and that it coincides with $T_\Omega(t)$ for all $t > 0$. Here,

$$T_0(t)f = \psi\widehat{T}_p(t)\widehat{f} + (1-\psi)T_{\Omega_R}(t)\widetilde{f},$$

$$S(t)f = -2\langle QD\psi, D(\widehat{T}_p(t)\widehat{f} - T_{\Omega_R}(t)\widetilde{f})\rangle - (\widehat{T}_p(t)\widehat{f} - T_{\Omega_R}(t)\widetilde{f})\mathcal{A}\psi$$

and

$$T_n(t)f = \int_0^t T_{n-1}(t-s)S(s)f ds,$$

for any $t > 0$ and any $f \in L^p(\Omega)$. In fact, by (4.3) and the analyticity of $\{T_{\Omega_R}(t)\}$

$$\|D\widehat{T}_p(t)f\|_{L^p(\mathbb{R}^N)} \le C_p t^{-\frac{1}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)}, \qquad t > 0$$

and

$$\|DT_{\Omega_R}(t)f\|_{L^p(\Omega_R)} \le C_p t^{-\frac{1}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)}, \quad t > 0,$$

for $f \in L^p(\Omega)$. Therefore, by [13, Lemma 4.6] with $R_0 = T_0$, $\alpha = 0$, and $\beta = -\frac{1}{2}$ the series $\sum_{n=0}^{\infty} T_n(t)f$ converges uniformly on bounded sets of $(0,\infty)$. Taking Laplace transforms of $\sum_{n=0}^{\infty} T_n(\cdot)f$, we deduce that

$$T_\Omega(t)f = \sum_{n=0}^{\infty} T_n(t)f, \qquad t > 0,\ f \in L^p(\Omega). \tag{5.9}$$

Proposition 4.3 implies that

$$\|\widehat{T}_p(t)\widehat{f}\|_{W^{1+\theta,p}(\mathbb{R}^N)} \leq C_p t^{-\frac{1+\theta}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)}, \qquad t>0,\ \ f\in L^p(\Omega), \tag{5.10}$$

for $\theta \in (0,1)$. The same estimate is true for $\{T_{\Omega_R}(t)\}$. Thus,

$$\|T_0(t)f\|_{W^{1+\theta,p}(\Omega)} \leq C_p t^{-\frac{1+\theta}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)}, \qquad t>0,\ \ f\in L^p(\Omega),$$

for $\theta \in (0,1)$. As above, we apply Lemma [13, Lemma 4.6] to obtain

$$\|T_\Omega(t)f\|_{W^{1+\theta,p}(\Omega)} \leq C_p t^{-\frac{1+\theta}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)}, \qquad t>0,\ \ f\in L^p(\Omega). \tag{5.11}$$

On the other hand, by (5.10)

$$\begin{aligned}\|S(t)f\|_{W^{\theta,p}(\Omega)} &\leq C_p\left(\|\widehat{T}_p(t)\widehat{f}\|_{W^{1+\theta,p}(\mathbb{R}^N)} + \|T_{\Omega_R}(t)\widetilde{f}\|_{W^{1+\theta,p}(\Omega_R)}\right)\\ &\leq C_p t^{-\frac{1+\theta}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)},\end{aligned}$$

for $t>0$, $f\in L^p(\Omega)$ and $\theta\in(0,1)$. Note that

$$\|T_{\Omega_R}(t)f\|_{W^{\theta,p}(\Omega_R)} \leq C_p t^{-\frac{\theta-\delta}{2}} e^{\omega_p t}\|f\|_{W^{\delta,p}_{\mathcal{B}}(\Omega_R)}, \tag{5.12}$$

where $W^{\delta,p}_{\mathcal{B}_R}(\Omega_R)$ denotes the subset of $W^{\delta,p}(\Omega_R)$ consisting of functions u such that $\mathcal{B}_R u \equiv 0$ on $\partial\Omega_R$, if this condition makes sense. Estimates (5.12) are classical results for elliptic operators with bounded and smooth coefficients.

Now, arguing as in the proof of Proposition 3.5, from (4.3) and (5.12) we obtain

$$\|D^2T_0(t-s)S(s)f\|_{W^{\alpha,p}(\Omega)} \leq C_p e^{\omega_p t}(t-s)^{-1+\frac{\theta-\alpha}{2}} s^{-\frac{1+\theta}{2}}\|f\|_{L^p(\Omega)},$$

$0<s<t$, $0<\alpha<\theta<1$. Hence, $T_1(t)$ is well defined for any $t>0$ and

$$\|D^2T_1(t)f\|_{W^{\alpha,p}(\Omega)} \leq C_p t^{-\frac{1+\alpha}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)}, \qquad t>0,\ \ f\in L^p(\Omega),$$

since $1+\frac{\theta-\alpha}{2}<1$ and $\frac{1+\theta}{2}<1$. Applying [13, Lemma 4.6] again with $R_0 = D_{ij}T_1$ and S as above, yields that the series $\sum_{n=1}^\infty D_{ij}T_n(t)f$ converges in $W^{\alpha,p}(\Omega)$ and

$$\sum_{n=1}^\infty \|D_{ij}T_n(t)f\|_{W^{\alpha,p}(\Omega)} \leq C_p t^{-\frac{1+\alpha}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)},$$

$i,j=1,\ldots,N,\ \ t>0,\ \ f\in L^p(\Omega)$.

On the other hand, from (4.3) and (5.12) and an interpolation argument, we also get

$$\|D^2T_0(t)f\|_{W^{\alpha,p}(\Omega)} \leq C_p t^{-1-\frac{\alpha}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)}, \quad t>0,\ \ f\in L^p(\Omega),$$

which proves that

$$\|D^2T_\Omega(t)f\|_{W^{\alpha,p}(\Omega)} \le C_p t^{-1-\frac{\alpha}{2}} e^{\omega_p t}\|f\|_{L^p(\Omega)}, \quad t>0,\ f\in L^p(\Omega). \tag{5.13}$$

Hence, (5.8) follows from (4.3), (5.11) and (5.13). □

APPENDIX A

In this appendix we present the details of the proof of Proposition 3.1 in the case when the elliptic operator $\mathcal{A}$, subject to more general boundary conditions of the type

$$(\mathcal{B}u)(x) = c_0(x)u(x) + \sum_{j=1}^{N} c_j(x)D_ju(x), \qquad x\in\partial\Omega, \tag{A.1}$$

is considered. More precisely, we give a proof of the maximum principle in the case when $\mathcal{B}$ is a first-order boundary operator.

Here, we suppose that c_j, $j=0,\ldots,N$, are continuously differentiable in a neighborhood $\Gamma_\varepsilon = \{x\in\Omega : \mathrm{dist}(x,\partial\Omega)\le\varepsilon\}$ of $\partial\Omega$ for some $\varepsilon>0$, and the pair $(\mathcal{A},\mathcal{B})$ satisfies the complementing condition. Furthermore, we assume that $\mathcal{B}$ is a first-order boundary operator. Finally, without loss of generality, we may assume that

$$\kappa := \inf_{x\in\partial\Omega}\Big(\sum_{j=1}^{N} c_j(x)\nu_j(x)\Big) > 0, \tag{A.2}$$

since $\mathcal{B}$ satisfies the complementing condition. We recall that $\nu(x)$ denotes the outward unit normal at $x\in\partial\Omega$.

Proposition A.1. *Suppose that Hypotheses $(HCb)_\Omega$ are satisfied. Then, the following properties hold.*

(i) *There exists $\lambda_1\ge\lambda_0$ such that for $\lambda\ge\lambda_1$, the elliptic problem* (3.1) *admits at most one solution $u\in D_\Omega$.*

(ii) *Let $T>0$. Assume that $u:\overline{\Omega}_T\to\mathbb{R}$ is bounded, belongs to $C^{1,2}(\Omega_T)$ and satisfies*

$$\begin{cases} D_tu(t,x) & \le \mathcal{A}u(t,x), \quad t>0,\quad x\in\Omega,\\ \mathcal{B}u(t,x) & \le 0, \qquad\qquad t>0,\quad x\in\partial\Omega,\\ u(0,x) & \le 0, \qquad\qquad\qquad\quad x\in\overline{\Omega}. \end{cases} \tag{A.3}$$

Further, assume that u is continuous in $\overline{\Omega}_T$, and $D_ju\in C((0,T]\times\overline{\Omega})$ if $\mathcal{B}$ is a first-order operator. Then, $u\le 0$. In particular, for $f\in$

$C_b(\overline{\Omega})$, the Cauchy problem (1.1) (*with $g \equiv 0$*) *admits at most a classical solution u satisfying*

$$\sup_{(t,x)\in(0,T)\times\Omega} |u(t,x)| \le K_T \|f\|_{C_b(\overline{\Omega})}, \tag{A.4}$$

for some positive constant K_T. In the case when c_0 is everywhere nonnegative in Ω we can take $K_T = 1$.

Proof. Let us start with the proof of property (i). Arguing as in [6, Lemma 3.1.2], it can be checked that, if $u \in W^{2,p}_{\rm loc}(\Omega)$, for any $p \in [1,\infty)$, is such that $\mathcal{A}u \in C_b(\overline{\Omega})$ and u has a maximum at some point $x_0 \in \Omega$, then $\mathcal{A}u(x_0) \le 0$.

So, let us assume that $u \in D_\Omega$ solves the equation $\lambda u - \mathcal{A}u = 0$. We will show that, if λ is sufficiently large, then $u \le 0$ in Ω. Applying the same argument to the function $-u$, we then will obtain $-u \le 0$ in Ω. Hence, $u \equiv 0$. Being rather long, we split the rest of the proof of property (i) into two steps.

Step 1). Assume first that $c_0 \ge 0$ on $\partial\Omega$ (see (A.1)). Define the function $\widehat{\varphi}$ as $\widehat{\varphi} := \zeta\varphi$, where φ is as in $(HCb)_\Omega$-(ii) and $\zeta \in C^\infty(B(2R))$ is a suitable smooth function such that $\zeta \equiv 1$ in $\mathbb{R}^N \setminus B(R) \subset \Omega$. It is immediate to check that $\widehat{\varphi}$ satisfies Hypothesis $(HCb)_\Omega$-(ii) and $\mathcal{B}\widehat{\varphi} = 0$. Moreover, replacing $\widehat{\varphi}$ by $\widehat{\varphi}+C$ for a suitable constant C, we may also assume that (i) $\mathcal{A}\widehat{\varphi}-\lambda_0\widehat{\varphi} \le 0$, (ii) $\widehat{\varphi} \ge 0$ in $\overline{\Omega}$, and (iii) $\mathcal{B}\widehat{\varphi} \ge 0$ on $\partial\Omega$.

Let $\lambda \ge \lambda_0$ and take $u \in D_\Omega$. For $n \in \mathbb{N}$, define the function $u_n : \overline{\Omega} \to \mathbb{R}$ by

$$u_n(x) := u(x) - \frac{1}{n}\widehat{\varphi}(x), \qquad x \in \overline{\Omega}.$$

We are going to prove that $u_n \le 0$ for any $x \in \overline{\Omega}$. Letting n go to ∞, we will then obtain that u is nonpositive in Ω. As is immediately seen, u_n satisfies

$$\begin{cases} \lambda u_n(x) - \mathcal{A}u_n(x) & \le 0, \quad x \in \Omega, \\ \mathcal{B}u_n(x) & \le 0, \quad x \in \partial\Omega. \end{cases} \tag{A.5}$$

Since u is bounded and $\widehat{\varphi}(x)$ tends to ∞ as x tends to ∞, the function u_n admits, for $n \in \mathbb{N}$, a global maximum at some point $x_n \in \overline{\Omega}$. In order to obtain a contradiction, assume that $u_n(x_n) > 0$. By the above results, $x_n \notin \Omega$. Otherwise $(\mathcal{A}u_n)(x_n) \le 0$ and this would contradict (A.5). Thus, $x_n \in \partial\Omega$. However, [25, Theorem 2.12]) and (A.2) imply that

$$\sum_{j=1}^N c_j(x_n) D_j u_n(x_n) > 0.$$

This yields the desired contradiction.

Step 2). Let us now consider the case when c_0 assumes negative values in $\partial\Omega$. Observe that, since $\partial\Omega$ is smooth, the distance function $d(x) = \mathrm{dist}(x, \partial\Omega)$ belongs to $C^2(\Gamma_\rho)$ provided ρ is sufficiently small. Here, $\Gamma_\rho = \{x \in \overline{\Omega} : d(x) < \rho\}$. Moreover, $Dd(x) = -\nu(x)$ for $x \in \partial\Omega$ by [10, Proposition B.3] and [14, Section 14.6]. By standard techniques, we find $\psi \in C_b^2(\mathbb{R}^N)$ such that $\psi = d$ in $\Gamma_{\rho/2}$, ψ is everywhere nonnegative, $\psi = 0$ in $\mathbb{R}^N \setminus B(M)$ and $\|\psi\|_{C_b(\mathbb{R}^N)} \le 2\rho$, provided M is sufficiently large so that $\Gamma_\rho \subset B(M)$. We set $w(x) := \exp(-C\psi(x))$ for $x \in \mathbb{R}^N$, where C is a positive constant satisfying $C\kappa \ge \|c_0\|_{C(\partial\Omega)}$ and κ is defined in (A.2). Then

$$\mathcal{B}w(x) = c_0(x)w(x) + C\sum_{j=1}^N c_j(x)\nu_j(x)w(x) \ge 0, \qquad x \in \partial\Omega.$$

Set $v := uw^{-1}$ and choose $\lambda > \lambda_1 := \max\{\lambda_0, \|w^{-1}\mathcal{A}w\|_{C_b(\mathbb{R}^N)}\}$. Then, v satisfies

$$\begin{cases} \left(\lambda - \dfrac{\mathcal{A}w}{w}\right)v - \mathcal{A}'v &= 0, \quad \text{in } \Omega,\\ \mathcal{B}'v &= 0, \quad \text{on } \partial\Omega,\end{cases}$$

where

$$\mathcal{A}'v = \mathcal{A} + 2w^{-1}\sum_{i,j=1}^N q_{ij}D_iwD_jv, \qquad \mathcal{B}'v = \sum_{j=1}^N c_jD_jv + \frac{\mathcal{B}w}{w}v. \tag{A.6}$$

Since $c_0' = \frac{\mathcal{B}w}{w}$ is nonnegative on $\partial\Omega$ and φ satisfies Hypothesis $(HCb)_\Omega$-(ii) with $\mathcal{A}$ being replaced by the operator $\mathcal{A}'' = \mathcal{A}' + \frac{\mathcal{A}w}{w}I$, we may use the arguments given in Step 1 to conclude that v, and hence also u, are nonpositive in Ω.

Let us now prove property (ii) in the cases when $c_0 \ge 0$. To this end, for $n \in \mathbb{N}$ and $\lambda \ge \lambda_0$ define the function $u_n : [0,\infty) \times \Omega \to \mathbb{R}$ by

$$u_n(t,x) = e^{-\lambda t}u(t,x) - \frac{1}{n}\widehat{\varphi}(x), \qquad t > 0,\ x \in \Omega,$$

where $\widehat{\varphi}$ is as in Step 1. Then,

$$\begin{cases} D_tu_n(t,x) - (\mathcal{A} - \lambda)u_n(t,x) &\le 0, \quad t > 0, \quad x \in \Omega,\\ \mathcal{B}u_n(t,x) &\le 0, \quad t > 0, \quad x \in \partial\Omega,\\ u_n(0,x) &\le 0, \qquad\qquad x \in \overline{\Omega},\end{cases} \tag{A.7}$$

for $\lambda \ge \lambda_0$. Arguing as in the elliptic case and taking into account [11, Theorem 2.14], it follows that $u_n \le 0$ in $[0,\infty) \times \Omega$.

In the case when c_0 assumes also negative values, we apply the above arguments to the function $v = w^{-1}u$, where w is the same function as in the elliptic case, taking $\lambda > \lambda_1$, λ_1 being as in Step 2. Indeed, the function v satisfies the problem (A.7), with the operators $\mathcal{A}$ and $\mathcal{B}$ being replaced by the operators $\mathcal{A}''$ and $\mathcal{B}'$ in (A.6).

To conclude the proof, let us check estimate (A.4). We first assume that c_0 is everywhere nonnegative in $\partial\Omega$. Then, both the functions $u - \|f\|_{C_b(\overline{\Omega})}$ as well as $-u - \|f\|_{C_b(\overline{\Omega})}$ satisfy (A.3). Hence, (A.4) follows.

Let us assume that c_0 is not everywhere nonnegative on $\partial\Omega$ and fix $\lambda > \|w^{-1}\mathcal{A}w\|_\infty$, where w is as above. Then, the functions $(t,x) \mapsto e^{-\lambda t}u(t,x)w^{-1}(x) - \|w^{-1}f\|_\infty$ and $(t,x) \mapsto -e^{-\lambda t}u(t,x)w^{-1}(x) - \|w^{-1}f\|_\infty$ turn out to satisfy problem (A.7) with $(\mathcal{A},\mathcal{B})$ being replaced by $(\mathcal{A}'',\mathcal{B}'')$. Hence, they are both nonpositive in Ω. It follows that

$$|\exp(-\lambda t + C\psi(x))u(t,x)| \le \|w^{-1}f\|_\infty \le e^{2C\delta}\|f\|_\infty, \qquad (t,x) \in \Omega_T,$$

which yields (A.4) with $K_T = e^{2C\delta+\lambda T}$. □

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