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MULTIPLE SOLUTIONS FOR SEMILINEAR ELLIPTIC SYSTEMS INVOLVING CRITICAL SOBOLEV EXPONENT

Michinori Ishiwata
Mathematical Institute, Tohoku university, Sendai 980-8578, Japan

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Abstract. The effect of the domain topology on the multiplicity of solutions for a semilinear elliptic system with critical Sobolev exponent is discussed. We show that if the coupling term is sufficiently small, then there exist at least $\operatorname{cat}\Omega$ positive solutions, where cat denotes the Ljusternik-Schnirelman category.

1. Problem and results

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 4)$ be a bounded domain and let $a_{11}, a_{12}, a_{22} \geq 0$. In this work, we consider the multiplicity of solutions of the following elliptic system involving critical Sobolev exponent:

$$\text{(ES)} \quad \begin{cases} -\Delta u = a_{11}u - a_{12}\dfrac{\alpha_1}{\alpha}|u|^{\alpha_1-2}u|v|^{\alpha_2} + u|u|^{p-2} & \text{in } \Omega, \\ -\Delta v = a_{22}v - a_{12}\dfrac{\alpha_2}{\alpha}|u|^{\alpha_1}|v|^{\alpha_2-2}v + v|v|^{p-2} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases}$$

where p is the critical Sobolev exponent defined by $p = 2^* = 2N/(N-2)$ and α_1 and α_2 are numbers satisfying $\alpha_1, \alpha_2 > 1$ and $\alpha := \alpha_1 + \alpha_2 < 2^*$.

It is well known that if $p \leq 2^*$, then the Sobolev embedding $H_0^1 \hookrightarrow L^p$ is a continuous map. Moreover, the above embedding is compact for $p < 2^*$, and this is not true for $p = 2^*$. Because of this lack of compactness, a nonlinear equation with critical Sobolev exponent exhibits a several interesting features.

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For example, the existence and the multiplicity of solutions for

$$\text{(P)} \quad \begin{cases} -\Delta u = \gamma u + u|u|^{p-2} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u \not\equiv 0 & \text{in} \quad \Omega \end{cases}$$

are highly nontrivial from the variational point of view. In fact, different from the subcritical case $p < 2^*$, (P) with $p = 2^*$ has no solution [18] if Ω is star-shaped and $\gamma \leq 0$. On the other hand, if $\gamma > 0$, then there exists a solution [6], [9]. Moreover, Coron [8] showed that if Ω is an "annular-shaped" domain in an appropriate sense, then (P) with $\gamma = 0$ has at least one solution. Thus there exists a delicate dependence of the (non)existence of a solution on the sign of the lower-order perturbation term γu and the geometry of the domain. The multiplicity of solutions is also affected by the geometry and the topology of the domain. Indeed, Rey [19] and Lazzo [15] proved that (P) has at least $\operatorname{cat}\Omega$ solutions if $\gamma > 0$ is sufficiently small, where $\operatorname{cat}\Omega$ is the Ljusternik-Schnirelman category of Ω (see Definition 1.1). These solutions cannot be obtained by the simple minimization argument or the mountain-pass-type argument. The effect of the domain topology on the multiplicity of solutions is also observed in the "nearly critical" problem and in the singularly perturbed problem; see e.g. [4] and [7] for the pioneering works. In particular, one standard machinery for the verification of the effect of the domain topology, which is called the method of photography, was first introduced in [4] by Benci and Cerami. The scheme of this method relies on the notion of a local Palais-Smale condition, barycenter map, comparison between the category of sublevel sets of the energy functional and the category of Ω. The present paper relies on this scheme.

Recently, the semilinear elliptic system with critical Sobolev exponent has been treated by many authors. For example, Alves et al. [2] showed that

$$\text{(PES)} \quad \begin{cases} -\Delta u = a_1 u + a_2 v + \frac{2q_1}{q}|u|^{q_1-2}u|v|^{q_2} & \text{in} \quad \Omega, \\ -\Delta v = a_2 u + a_3 v + \frac{2q_2}{q}|u|^{q_1}|v|^{q_2-2}v & \text{in} \quad \Omega, \\ u, v = 0 & \text{on} \quad \partial\Omega \end{cases}$$

with $q_1 + q_2 = 2^*$ has at least one nontrivial solution if $a_2 \geq 0$ and $0 < \mu_1 \leq \mu_2 < \lambda_1$, where μ_1 and μ_2 denote eigenvalues of $A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$ and λ_1 the first eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition. A similar existence result for

$$\begin{cases} -\Delta u_i = |u_i|^{2^*-2}u_i + \sum_{i=1}^n a_{ij}u_j & \text{in} \quad \Omega, \\ u_i = 0 & \text{on} \quad \partial\Omega \end{cases}$$

is treated in [3]. For another type of result, see e.g. [1, 12, 13, 14] and references therein.

For elliptic systems, most of the known works treat the existence of a solution, and there does not seem to exist much work concerning the relationship between the multiplicity of solutions and the topology of the domain. By using the method of photography, Han proved in [14] that (PES) has at least $\operatorname{cat}\Omega$ positive solutions if the coupling term is sufficiently small. These solutions have *low energy,* and the variational argument on the standard Nehari manifold together with the method of photography is used in [14]. On the other hand, we prove the existence of multiple positive solutions for (ES) with *high energy* in this paper. Hence we have to work with a constraint manifold different from the standard Nehari manifold.

Now we introduce the Ljusternik-Schnirelman category.

Let M be a topological space. A subset K of M is said to be contractible in M if there exist $x_0 \in M$ and $\eta \in C([0,1] \times K; M)$ such that $\eta(0,x) = x$ and $\eta(1,x) = x_0$ for $x \in K$.

Definition 1.1 (category)**.** *Let M be a topological space and let K be a closed subset of M. The natural number n is called the "category of K in M," denoted by $n = \operatorname{cat}_M K$, if n is the smallest number among m such that there exists a covering $(K_j)_{j=1}^m$ of K which consists of closed and contractible sets in M.*

Here and henceforth, we simply denote $\operatorname{cat}_K K$ by $\operatorname{cat} K$ and $\operatorname{cat}\overline{\Omega}$ by $\operatorname{cat}\Omega$. Throughout this paper, we assume that p, α_1, and α_2 satisfy the following:

(A) $p = 2^* = 2N/(N-2)$, $\alpha_1, \alpha_2 > 1$ and $\alpha := \alpha_1 + \alpha_2 < 2^*$.

Our main result reads as follows.

Theorem 1.1. *Assume* (A)*. Then there exists $\mu_1 > 0$ which satisfies the following: for a_{11} and a_{22} with $a_{11}, a_{22} \in (0, \mu_1)$, there exists $\mu_2 > 0$ such that* (ES) *has at least* $\operatorname{cat}\Omega$ *positive solutions for a_{12} satisfying $a_{12} \in (0, \mu_2)$.*

Notation. $p = 2^*$ denotes the critical Sobolev exponent,

$$S := \inf_{u \in H_0^1 \setminus \{0\}} \|\nabla u\|_2^2 / \|u\|_p^2$$

the best Sobolev constant (which is independent of Ω; see [6]),

$$S_\gamma(\Omega) := \inf_{u \in H_0^1 \setminus \{0\}} (\|\nabla u\|_2^2 - \gamma \|u\|_2^2)/\|u\|_p^2,$$

$S_{a_{11}} := S_{a_{11}}(\Omega)$, $S_{a_{22}} := S_{a_{22}}(\Omega)$ and $S_* := \min(S_{a_{11}}, S_{a_{22}})$.

$E = H_0^1(\Omega) \times H_0^1(\Omega)$ and $z = (u, v)$ represents an arbitrary element of E. The positive part of a function w is denoted by w^+. For $\omega \subset \mathbb{R}^N$, $\|\cdot\|_{r,\omega}$ stands for the $L^r(\omega)$ norm. We occasionally omit the subscript ω when $\omega = \Omega$. Let

$$\begin{aligned}
\varphi_{\gamma,\omega}(u) &= \frac{1}{2}\|\nabla u\|_{2,\omega}^2 - \frac{\gamma}{2}\|u\|_{2,\omega}^2 - \frac{1}{p}\|u\|_{p,\omega}^p, \\
K(u,v) &= \int_\Omega |u^+(x)|^{\alpha_1}|v^+(x)|^{\alpha_2}\,dx, \\
I(u,v) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 - \frac{a_{11}}{2}\|u^+\|_2^2 + \frac{a_{12}}{\alpha}K(u,v) \\
&\quad - \frac{a_{22}}{2}\|v^+\|_2^2 - \frac{1}{p}\|u^+\|_p^p - \frac{1}{p}\|v^+\|_p^p.
\end{aligned}$$

Also let

$$\begin{aligned}
J_1(u,v) &= \|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2 + a_{12}\frac{\alpha_1}{\alpha}K(u,v) - \|u^+\|_p^p \\
J_2(u,v) &= \|\nabla v\|_2^2 - a_{22}\|v^+\|_2^2 + a_{12}\frac{\alpha_2}{\alpha}K(u,v) - \|v^+\|_p^p.
\end{aligned}$$

It is clear that $I, J_1, J_2 \in C^1(E;\mathbb{R})$ under the assumption (A).

The constraint manifold M considered in this paper is defined by $M = \{z \in E : u, v \neq 0,\ J_1(z) = J_2(z) = 0\}$. It is easy to see that M is a complete Banach-Finsler manifold of class C^1 modeled on E. The infimum of I on M is denoted by d_M. For $a \in \mathbb{R}$, $[I \leq a]_M := \{z \in M : I(z) \leq a\}$.

I, M, J_1 and J_2 for the sequence $(a_{ij}^n)_{n=1}^\infty$ $(i,\, j = 1,\, 2)$ are denoted by I_n, M_n, $J_{1,n}$ and $J_{2,n}$, respectively.

The barycenter maps β_1 and β_2 are defined by

$$\beta_1(z) = \int_\Omega x|u(x)|^p dx/\|u\|_p^p \quad \text{and} \quad \beta_2(z) = \int_\Omega x|v(x)|^p dx/\|v\|_p^p.$$

2. PRELIMINARIES

In the proof of Theorem 1.1, a comparison between the category of Ω and that of a certain subset of M is needed. The following lemma is used for this purpose. This lemma can be proved by an argument similar to that in [21, Proof of Lemma 5.25].

Lemma 2.1. *Let A (respectively B) be a topological space and let $a \subset A$ (respectively $b \subset B$) be a closed subset. Also let $\Phi \in C(a; b)$ and $\beta \in C(B; A)$ be maps such that $\beta \circ \Phi$ is homotopically equivalent to the injection from a to A. Then we have $\operatorname{cat}_B b \geq \operatorname{cat}_A a$.*

We next recall the Ljusternik-Schnirelman theorem (category version).

Definition 2.1 ((PS) condition)**.** *Let M be a C^1 Banach-Finsler manifold and $I \in C^1(M;\mathbb{R})$. Then*

(a) *$(z_n) \subset M$ is called a Palais-Smale sequence at level d ($(\mathrm{PS})_d$ sequence) if $I(z_n) \to d$ and $(dI)_{z_n} \to 0$ as $n \to \infty$, where $(dI)_{z_n}$ denotes the first derivative of I on the manifold M at z_n.*

(b) *I is said to satisfy the Palais-Smale condition at level d ($(\mathrm{PS})_d$ condition) if every $(\mathrm{PS})_d$ sequence has a convergent subsequence.*

Our main tool is the following abstract critical-point theorem. We can obtain this theorem by an argument similar to that in [21, p. 90, Theorem 5.20], replacing the usual deformation lemma for Banach-Finsler manifolds of class $C^{1,1}$ by the deformation lemma for manifolds of class C^1 (see e.g. [20, p. 87, Theorem 3.11] or [10, p. 55, Lemma 3.7]).

Proposition 2.1. (Ljusternik-Schnirelman theorem) *Let M be a complete C^1 Banach-Finsler manifold, $a < a'$ and let I be a C^1 functional on M which is bounded from below. Suppose that I satisfies $(\mathrm{PS})_d$ condition for all $d \le a'$ and $\{z \in M : I(z) = a,\ (dI)_z = 0\} = \emptyset$. Then $[I \le a]_M$ contains at least $\mathrm{cat}[I \le a]_M$ critical points.*

The following result is well known (see [6]).

Lemma 2.2.

(a) *$S_\gamma \to S$ as $\gamma \to 0$.*

(b) *If $0 < a < b$, then $0 < S_b < S_a < S$.*

The following representation of I for $z \in M$ follows from $S_{a_{ii}}\|w\|_p^2 \le \|\nabla w\|_2^2 - a_{ii}\|w\|_2^2$ and $J_i(z) = 0$, where $i = 1$ or 2.

Lemma 2.3. *Suppose that $a_{11}, a_{22} < \lambda_1$. Then for $z \in M$,*

$$
\begin{aligned}
I(z) &= a_{12}\frac{1}{\alpha}\Big(1 - \frac{\alpha}{p}\Big)K(z) + \Big(\frac{1}{2} - \frac{1}{p}\Big)(\|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2) \\
&\quad + \Big(\frac{1}{2} - \frac{1}{p}\Big)(\|\nabla v\|_2^2 - a_{22}\|v^+\|_2^2) && (2.1) \\
&\ge a_{12}\frac{1}{\alpha}\Big(1 - \frac{\alpha}{p}\Big)K(z) + \Big(\frac{1}{2} - \frac{1}{p}\Big)S_{a_{11}}\|u\|_p^2 + \Big(\frac{1}{2} - \frac{1}{p}\Big)S_{a_{22}}\|v\|_p^2. && (2.2)
\end{aligned}
$$

Lemma 2.4. *Assume that $a_{11}, a_{22} < \lambda_1$ and $a_{12} > 0$. Then*

$$S_{a_{11}}^{N/2} \le \|\nabla u\|_2^2 - a_{11}\|u\|_2^2, \qquad S_{a_{11}}^{N/2} \le \|u^+\|_p^p,$$

$$S_{a_{22}}^{N/2} \le \|\nabla v\|_2^2 - a_{22}\|v\|_2^2, \qquad S_{a_{22}}^{N/2} \le \|v^+\|_p^p$$

hold for $z = (u, v) \in M$.

Proof of Lemma 2.4. By the definition of $S_{a_{11}}$, it holds that

$$S_{a_{11}}\|u^+\|_p^2 \le S_{a_{11}}\|u\|_p^2 \le \|\nabla u\|_2^2 - a_{11}\|u\|_2^2 \le \|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2. \tag{2.3}$$

Moreover,

$$\|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2 = -a_{12}\frac{\alpha_1}{\alpha}K(u, v) + \|u^+\|_p^p \le \|u^+\|_p^p$$

follows from $J_1(z) = 0$. This relation and (2.3) imply that $S_{a_{11}}^{N/2} \le \|u^+\|_p^p$. Then again (2.3) yields $S_{a_{11}}^{N/2} \le \|\nabla u\|_2^2 - a_{11}\|u\|_2^2$. The argument for v is similar. □

Proposition 2.2. *Assume that* $a_{11}, a_{22} \in (0, \lambda_1)$ *and* $a_{12} > 0$. *Then*

$$\frac{1}{N}(S_{a_{11}}(\Omega)^{N/2} + S_{a_{22}}(\Omega)^{N/2}) \le d_M < \frac{2}{N}S^{N/2}$$

holds, where d_M *denotes the infimum of* I *on* M.

Proof of Proposition 2.2. For $z \in M$, $S_{a_{11}}^{N/2} \le \|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2$ and $S_{a_{22}}^{N/2} \le \|\nabla v\|_2^2 - a_{22}\|v^+\|_2^2$ hold in view of Lemma 2.4. Hence (2.1) together with $a_{12} > 0$ yield the first inequality. Let $B_r(y) = \{x \in \mathbb{R}^N : |x - y| < r\}$. Take $x_1, x_2 \in \Omega$ and $r > 0$ such that $B_r(x_1) \cap B_r(x_2) = \emptyset$, $B_r(x_1)$ and $B_r(x_2) \subset \Omega$. Let w_i be the least-energy solution of $-\Delta w = a_{ii}w + w|w|^{p-2}$ in $B_r(0)$ with the homogeneous Dirichlet boundary condition ($i = 1, 2$). It is well known that w_i satisfies $\varphi_{a_{ii}}(w_i) = S_{a_{ii}}(B_r(0))^{N/2}/N$; see [6]. Then for $z_0 := (w_1(\cdot - x_1), w_2(\cdot - x_2))$, we see that $z_0 \in M$ and

$$d_M \le I(z_0) = \varphi_{a_{11}}(w_1) + \varphi_{a_{22}}(w_2) = \frac{1}{N}(S_{a_{11}}(B_r(0))^{N/2} + S_{a_{22}}(B_r(0))^{N/2}).$$

This relation and Lemma 2.2 yield the second inequality. □

Lemma 2.5. *Let* $z \in M$. *Then for any* $K \in (0, \lambda_1)$ *and* $L \ge 2S^{N/2}/N$, *there exists* $C(K, L) > 0$ *such that*

$$\|\nabla u\|_2^2, \|\nabla v\|_2^2 < C$$

holds for $z = (u, v) \in [I \le L]_M$ *if* $a_{11}, a_{22} < K$ *and* $a_{12} > 0$.

Proof of Lemma 2.5. Proposition 2.2 yields $[I \le L]_M \ne \emptyset$ for $L \ge 2S^{N/2}/N$. Then (2.1) implies that

$$L \ge I(z) \ge \Big(\frac{1}{2} - \frac{1}{p}\Big)(\|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2) + \Big(\frac{1}{2} - \frac{1}{p}\Big)(\|\nabla v\|_2^2 - a_{22}\|v^+\|_2^2)$$

by virtue of $a_{12} > 0$, $p = 2^*$ and $\alpha < 2^*$. This relation together with the Poincaré inequality and $a_{11}, a_{22} < K$ yields

$$L \geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \frac{K}{\lambda_1}\right)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2).$$

Hence for $C(K, L) := L/(1/2 - 1/p)(1 - K/\lambda_1)$, the conclusion of the lemma holds. □

Proposition 2.3. *For $L \geq 2S^{N/2}/N$ and $K \in (0, \lambda_1)$, there exists $\overline{a} > 0$ such that every $(\mathrm{PS})_d$ sequence of I in M is a $(\mathrm{PS})_d$ sequence of I in E if $a_{12} \in (0, \overline{a})$, $a_{11}, a_{22} \in (0, K)$ and $d < L$. In particular, for the a_{ij} above, every critical point z of I in M with $I(z) \leq d$ is a critical point of I in E.*

Proof of Proposition 2.3. Let $d < L$ and let (z_n) be a $(\mathrm{PS})_d$ sequence in M. It is clear that $(dJ_1)_{z_n}, (dJ_2)_{z_n} \neq 0$ in E^*. Hence by the Lagrange multiplier rule, there exist $\rho_{1,n}, \rho_{2,n} \in \mathbb{R}$ such that

$$(dI)_{z_n} = \rho_{1,n}(dJ_1)_{z_n} + \rho_{2,n}(dJ_2)_{z_n} + o(1) \text{ in } E^*. \tag{2.4}$$

Lemma 2.5 implies that (z_n) is bounded in E. Hence, testing (2.4) by $(u_n, 0)$ and $(0, v_n)$, we obtain

$$\begin{aligned}(dI)_{z_n}(u_n, 0) &= \rho_{1,n}(dJ_1)_{z_n}(u_n, 0) + \rho_{2,n}(dJ_2)_{z_n}(u_n, 0) + o(1),\\ (dI)_{z_n}(0, v_n) &= \rho_{1,n}(dJ_1)_{z_n}(0, v_n) + \rho_{2,n}(dJ_2)_{z_n}(0, v_n) + o(1),\end{aligned}$$

respectively. By using $(dI)_{z_n}(u_n, 0) = J_1(z_n) = 0$ and $(dI)_{z_n}(0, v_n) = J_2(z_n) = 0$, we have

$$\begin{aligned}0 &= \rho_{1,n}\left(a_{12}\frac{\alpha_1}{\alpha}(\alpha_1 - 2)K(u_n, v_n) + (2 - p)\|u_n^+\|_p^p\right)\\ &\quad + \rho_{2,n}a_{12}\frac{\alpha_1\alpha_2}{\alpha}K(u_n, v_n) + o(1),\\ 0 &= \rho_{1,n}a_{12}\frac{\alpha_1\alpha_2}{\alpha}K(u_n, v_n)\\ &\quad + \rho_{2,n}\left(a_{12}\frac{\alpha_2}{\alpha}(\alpha_2 - 2)K(u_n, v_n) + (2 - p)\|v_n^+\|_p^p\right) + o(1).\end{aligned}$$

These relations are written in the form

$$o(1) = A_n \begin{pmatrix} \rho_{1,n} \\ \rho_{2,n} \end{pmatrix}, \tag{2.5}$$

where $A_n = \begin{pmatrix} a_n & b_n \\ b_n & c_n \end{pmatrix}$ and

$$a_n = a_{12}\frac{\alpha_1}{\alpha}(\alpha_1 - 2)K(u_n, v_n) + (2 - p)\|u_n^+\|_p^p,$$

$$\begin{aligned} b_n &= a_{12}\frac{\alpha_1\alpha_2}{\alpha}K(u_n,v_n),\\ c_n &= a_{12}\frac{\alpha_2}{\alpha}(\alpha_2-2)K(u_n,v_n)+(2-p)\|v_n^+\|_p^p. \end{aligned}$$

Then

$$\begin{aligned} \det A_n &= a_{12}^2\frac{\alpha_1\alpha_2}{\alpha^2}(2-\alpha_1)(2-\alpha_2)K(u_n,v_n)^2\\ &\quad +a_{12}\frac{\alpha_1(2-\alpha_1)(p-2)}{\alpha}\|v_n^+\|_p^pK(u_n,v_n)\\ &\quad +a_{12}\frac{\alpha_2(2-\alpha_2)(p-2)}{\alpha}\|u_n^+\|_p^pK(u_n,v_n)\\ &\quad +(p-2)^2\|u_n^+\|_p^p\|v_n^+\|_p^p - a_{12}^2\frac{\alpha_1^2\alpha_2^2}{\alpha^2}K(u_n,v_n)^2. \end{aligned} \tag{2.6}$$

By virtue of Lemma 2.5, we have that there exists $C_1(K,L)>0$ such that $\|\nabla u_n\|_2, \|\nabla v_n\|_2 < C_1$. This relation together with Lemma 2.4 yields that, for sufficiently small $a_{12}>0$ and for some $C_2(K,L)>0$,

$$\begin{aligned} \text{(RHS) of (2.6)} &\geq (p-2)^2 S_{a_{11}}^{N/2} S_{a_{22}}^{N/2} - a_{12}C_2(K,L)\\ &\geq (p-2)^2 (S_K^{N/2})^2 - a_{12}C_2(K,L) > 0. \end{aligned}$$

Note that the bound for a_{12} does not depend on n. Hence A_n^{-1} exists and all entries of A_n^{-1} are bounded; thus, $\rho_{1,n}, \rho_{2,n} = o(1)$ by virtue of (2.5). Relation (2.4) and the boundedness of $\|(dJ_1)_{z_n}\|_{E^*}$ and $\|(dJ_2)_{z_n}\|_{E^*}$ yield the conclusion. □

Hereafter we set $S_* = \min(S_{a_{11}}, S_{a_{22}})$.

Proposition 2.4. *For $K \in (0,\lambda_1)$, there exists $\overline{a}>0$ such that I with $a_{12} \in (0,\overline{a})$ and $a_{11}, a_{22} \in (0,K)$ satisfies the $(\mathrm{PS})_d$ condition in M for $d < (S^{N/2} + S_*^{N/2})/N$.*

Proof of Proposition 2.4. Proposition 2.3 says that for $L = 2S^{N/2}/N$ and $K \in (0,\lambda_1)$, there exists $\overline{a}>0$ such that every $(\mathrm{PS})_d$ sequence of I in M is that of I in E if $a_{11}, a_{22} \in (0,K)$, $a_{12} \in (0,\overline{a})$ and if $d<L$. Take any a_{ij} which satisfy these conditions.

Let $(z_n) \subset M$ be a nonconvergent $(\mathrm{PS})_d$ sequence. By (2.2), $J_1(z_n)=0$ and by $J_2(z_n)=0$, we see that (z_n) is a bounded sequence in E. Hence there exist $u, v \in H_0^1$ such that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in H_0^1 as $n\to\infty$. Let $w_n := u_n - u$ and let $w_n' := v_n - v$. Then the Brezis-Lieb lemma ([5]; see also [21, p. 21, Lemma 1.32]) implies that

$$\|\nabla u_n\|_2^2 - a_{11}\|u_n^+\|_2^2 = \|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2 + \|\nabla w_n\|_2^2 + o(1), \tag{2.7}$$

$$\|u_n^+\|_p^p = \|u^+\|_p^p + \|w_n^+\|_p^p + o(1), \tag{2.8}$$

$$\|\nabla v_n\|_2^2 - a_{22}\|v_n^+\|_2^2 = \|\nabla v\|_2^2 - a_{22}\|v^+\|_2^2 + \|\nabla w_n'\|_2^2 + o(1), \tag{2.9}$$

$$\|v_n^+\|_p^p = \|v^+\|_p^p + \|w_n'^+\|_p^p + o(1). \tag{2.10}$$

From these relations, we obtain

$$\begin{aligned} 0 = J_1(z_n) &= \|\nabla u_n\|_2^2 - a_{11}\|u_n^+\|_2^2 + a_{12}\frac{\alpha_1}{\alpha}K(z_n) - \|u_n^+\|_p^p \qquad (2.11)\\ &= \|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2 + a_{12}\frac{\alpha_1}{\alpha}K(z) - \|u^+\|_p^p + \|\nabla w_n\|_2^2 - \|w_n^+\|_p^p + o(1) \end{aligned}$$

in view of $(z_n) \subset M$, (2.7) and (2.8). Moreover,

$$\|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2 + a_{12}\frac{\alpha_1}{\alpha}K(z) - \|u^+\|_p^p = 0 \tag{2.12}$$

holds since (u, v) is a critical point of I in E. From (2.11) and (2.12), we can find $\gamma \in \mathbb{R}$ such that $\gamma := \|\nabla w_n\|_2^2 + o(1) = \|w_n^+\|_p^p + o(1)$.

Assume that $u_n \not\to u$ strongly in H_0^1 as $n \to \infty$. Then $w_n \not\to 0$ in H_0^1 as $n \to \infty$ and $\gamma > 0$. Combining this fact with the Sobolev inequality, we get

$$\gamma = \|\nabla w_n\|_2^2 + o(1) \geq S^{N/2}. \tag{2.13}$$

Also by Lemma 2.4, we have

$$\|\nabla u_n\|_2^2 - a_{11}\|u_n^+\|_2^2 \geq S_{a_{11}}^{N/2}, \tag{2.14}$$

$$\|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2 \geq 0. \tag{2.15}$$

If v_n does not converge to v, then

$$\|\nabla w_n'\|_2^2 + o(1) \geq S^{N/2}, \tag{2.16}$$

$$\|\nabla v\|_2^2 - a_{22}\|v^+\|_2^2 \geq 0 \tag{2.17}$$

follow from an argument similar to that above, replacing (2.7) and (2.8) by (2.9) and (2.10), respectively.

Since z_n is a nonconvergent sequence in E, without loss of generality, we can assume that there holds (i) $u_n \to u$ and $v_n \not\to v$ or, (ii) $u_n \not\to u$ and $v_n \not\to v$, as $n \to \infty$.

In the first case, we have

$$\begin{aligned} d + o(1) = I(z_n) \geq \Big(\frac{1}{2} - \frac{1}{p}\Big)&(\|\nabla u_n\|_2^2 - a_{11}\|u_n^+\|_2^2)\\ &+ \Big(\frac{1}{2} - \frac{1}{p}\Big)(\|\nabla v\|_2^2 - a_{22}\|v^+\|_2^2) + \Big(\frac{1}{2} - \frac{1}{p}\Big)\|\nabla w_n'\|_2^2 \end{aligned}$$

by $a_{12} > 0$, (2.1) and by (2.9). Combining this relation with (2.14), (2.16) and (2.17), we obtain

$$d + o(1) \geq \Big(\frac{1}{2} - \frac{1}{p}\Big) S_*^{N/2} + \Big(\frac{1}{2} - \frac{1}{p}\Big) S^{N/2}. \tag{2.18}$$

Similarly, in case of (ii), we have

$$\begin{aligned} d + o(1) &\geq \Big(\frac{1}{2} - \frac{1}{p}\Big)(\|\nabla u\|_2^2 - a_{11}\|u^+\|_2^2) + \Big(\frac{1}{2} - \frac{1}{p}\Big)\|\nabla w_n\|_2^2 \\ &\quad + \Big(\frac{1}{2} - \frac{1}{p}\Big)(\|\nabla v\|_2^2 - a_{22}\|v^+\|_2^2) + \Big(\frac{1}{2} - \frac{1}{p}\Big)\|\nabla w_n'\|_2^2 \\ &\geq 0 + \Big(\frac{1}{2} - \frac{1}{p}\Big) S^{N/2} + 0 + \Big(\frac{1}{2} - \frac{1}{p}\Big) S^{N/2} \\ &> \Big(\frac{1}{2} - \frac{1}{p}\Big) S_*^{N/2} + \Big(\frac{1}{2} - \frac{1}{p}\Big) S^{N/2} \end{aligned}$$

in view of (2.13), (2.15), (2.16), (2.17) and Lemma 2.2. The conclusion follows from this relation and (2.18). □

The proof of Theorem 1.1 relies on Lemma 2.1. Thus we need mappings β and Φ. Hereafter we are devoted to this subject.

Lemma 2.6. *Let $a_{ij}^n \downarrow 0$ and $\varepsilon_n \downarrow 0$ as $n \to \infty$, where $i, j = 1, 2$. Then*

$$\begin{aligned} \|u_n^+\|_p^p &= \|\nabla u_n\|_2^2 + o(1) = S^{N/2} + o(1), \\ \|v_n^+\|_p^p &= \|\nabla v_n\|_2^2 + o(1) = S^{N/2} + o(1) \end{aligned}$$

holds for $z_n \in [I_n < 2S^{N/2}/N + \varepsilon_n]_{M_n}$.

Proof of Lemma 2.6. We show only the first relation. The second relation is verified in the same way.

By (2.1) and $a_{ii}^n \downarrow 0$ $(i = 1, 2)$ as $n \to \infty$, we have

$$\frac{2}{N} S^{N/2} + o(1) \geq I(z_n) \geq \frac{1}{N}\Big[\Big(1 - \frac{a_{11}^n}{\lambda_1}\Big)\|\nabla u_n\|_2^2 + \Big(1 - \frac{a_{22}^n}{\lambda_1}\Big)\|\nabla v_n\|_2^2\Big].$$

Hence, (z_n) is a bounded sequence in E. This relation and Lemma 2.4 yield

$$\begin{aligned} \frac{1}{N}\left(\|\nabla u_n\|_2^2 - a_{11}^n\|u_n^+\|_2^2\right) &\leq I(z_n) - \frac{1}{N}\left(\|\nabla v_n\|_2^2 - a_{22}^n\|v_n^+\|_2^2\right) \\ &\leq \frac{2}{N} S^{N/2} - \frac{1}{N} S_{a_{22}^n}^{N/2} + o(1). \end{aligned}$$

This relation together with the boundedness of (z_n), the fact that $a_{11}^n, a_{22}^n \to 0$ as $n \to \infty$ and Lemma 2.2 assure that

$$\frac{1}{N}\|\nabla u_n\|_2^2 \le \frac{a_{11}^n}{N}\|u_n^+\|_2^2 + \frac{2}{N}S^{N/2} - \frac{1}{N}S_{a_{22}^n}^{N/2} + o(1) = \frac{1}{N}S^{N/2} + o(1). \quad (2.19)$$

On the other hand, by Lemma 2.2 and Lemma 2.4 and by the boundedness of (z_n),

$$S^{N/2} + o(1) = S_{a_{11}^n}^{N/2} \le \|\nabla u_n\|_2^2 - a_{11}^n\|u_n^+\|_2^2 = \|\nabla u_n\|_2^2 + o(1)$$

holds. This relation and (2.19) yield

$$\|\nabla u_n\|_2^2 = S^{N/2} + o(1). \quad (2.20)$$

The facts that $J_{1,n}(z_n) = 0$ and $a_{ij}^n \downarrow 0$ as $n \to \infty$ yield

$$\|u_n^+\|_p^p = \|\nabla u_n\|_2^2 - a_{12}^n\frac{\alpha_1}{\alpha}K(u_n, v_n) + a_{11}^n\|u_n^+\|_2^2 = \|\nabla u_n\|_2^2 + o(1).$$

The conclusion follows from (2.20). □

Let $d_{\mathbb{R}^N}(\cdot,\cdot)$ be a standard Euclidean distance.

Since Ω is a smooth bounded domain, there exists $r > 0$ such that $(\Omega)_r^+$, $(\Omega)_r^-$ and Ω are all homotopically equivalent, where $(\Omega)_r^- = \{x \in \Omega : d_{\mathbb{R}^N}(x, \partial\Omega) \ge r\}$ and $(\Omega)_r^+ = \{x \in \mathbb{R}^N : d_{\mathbb{R}^N}(x, \Omega) \le r\}$.

Proposition 2.5. *For $r > 0$ above, there exist $\varepsilon > 0$ and $\overline{a} > 0$ such that for a_{ij} with $0 < a_{ij} < \overline{a}$ $(i, j = 1, 2)$ and for $l = 1, 2$,*

$$\beta_l\left([I \le 2S^{N/2}/N + \varepsilon]_M\right) \subset (\Omega)_r^+$$

holds.

Proof of Proposition 2.5. Assume that the conclusion is not true for $l = 1$. Then there exist $\varepsilon_n \downarrow 0$, $a_{ij}^n \downarrow 0$ and $z_n \in [I_n \le 2S^{N/2}/N + \varepsilon_n]_{M_n}$ such that

$$\beta_1(z_n) \notin (\Omega)_r^+. \quad (2.21)$$

By Lemma 2.6,

$$\|\nabla u_n\|_2^2 + o(1) = \|u_n^+\|_p^p + o(1) = S^{N/2}.$$

This relation together with the Sobolev inequality implies that

$$S^{N/2} + o(1) = S\|u_n^+\|_p^2 \le S\|u_n\|_p^2 \le \|\nabla u_n\|_2^2 = S^{N/2},$$

thus $\|u_n\|_p^p = S^{N/2} + o(1)$. Assume that there exists a subsequence of (u_n) and u such that $u_n \to u$ strongly in H_0^1 as $n \to \infty$. Then $w := S^{-(N-2)/4}u$ achieves $S = \inf_{\|u\|_p^2=1}\|\nabla u\|_2^2$, which is absurd since S is never achieved in

$\Omega \neq \mathbb{R}^N$ (see e.g. [20, p. 171, Theorem 1.2]). Hence u_n is a nonconvergent sequence in H_0^1 and the concentration compactness principle yields $|u_n|^p \rightharpoonup S^{N/2}\delta_{x_0}$ weakly in $\mathcal{M}(\Omega)$ as $n \to \infty$ for some $x_0 \in \overline{\Omega}$, where $\mathcal{M}(\Omega)$ denotes the set consisting of Radon measures in Ω (see [16, 17]; see also [21, Lemma 1.40]). Therefore $\beta_1(z_n) = \int x|u_n|^p/\|u_n\|_p^p + o(1) = x_0 + o(1) \in (\Omega)_r^+$ as $n \to \infty$, which is absurd in view of (2.21). The case $l = 2$ can be handled by a similar argument with u_n and β_1 replaced by v_n and β_2, respectively. □

Now we introduce the mapping Φ. Without loss of generality, we assume that $a_{11} \geq a_{22}$; thus, $S_* = S_{a_{11}}(\Omega)$ by Lemma 2.2. Let $B_r(0) = \{x \in \mathbb{R}^N : |x| < r\}$, where $r > 0$ is the number which satisfies the condition that $(\Omega)_r^-$, Ω and $(\Omega)_r^+$ are homotopically equivalent. Let $u_{a_{11}}$ (respectively $w_{a_{22}}$) be a function which attains the infimum of $\varphi_{a_{11}}$ (respectively $\varphi_{a_{22},B_r(0)}$) on $\{u \in H_0^1(\Omega) \setminus \{0\} : \|\nabla u\|_2^2 = a_{11}\|u\|_2^2 + \|u\|_p^p\}$ (respectively $\{u \in H_0^1(B_r(0)) \setminus \{0\} : \|\nabla u\|_2^2 = a_{22}\|u\|_2^2 + \|u\|_p^p\}$), where $\varphi_{a_{11}}(u) = \|\nabla u\|_{2,\Omega}^2/2 - a_{11}\|u\|_{2,\Omega}^2/2 - \|u\|_{p,\Omega}^p/p$ and $\varphi_{a_{22},B_r(0)}(w) = \|\nabla u\|_{2,B_r(0)}^2/2 - a_{22}\|w\|_{2,B_r(0)}^2/2 - \|w\|_{p,B_r(0)}^p/p$. The existence and the positivity of $u_{a_{11}}$ and $w_{a_{22}}$ and the radial symmetry of $w_{a_{22}}$ are well known (see [6, 11]). It is also well known that $\varphi_{a_{11}}(u_{a_{11}}) = S_{a_{11}}(\Omega)^{N/2}/N$ and $\varphi_{a_{22}}(w_{a_{22}}) = S_{a_{22}}(B_r(0))^{N/2}/N$ (see [6]). In the rest of this section, $u_{a_{11}}$, $w_{a_{22}}$ and $w_{a_{22}}(\cdot - y)$ for $y \in (\Omega)_r^-$ are simply denoted by u, w and w_y, respectively.

At first we claim that, for each $y \in (\Omega)_r^-$, there exist $s = s(y)$, $t = t(y) > 0$ such that $(tu, sw_y) \in M$.

Lemma 2.7. *Let $u(\cdot) := u_{a_{11}}(\cdot)$ and $w_y(\cdot) := w_{a_{22}}(\cdot - y)$ for $y \in (\Omega)_r^-$. Then for sufficiently small $a_{12} > 0$ and $a_{11}, a_{22} \in (0, \lambda_1)$, there exist mappings $t(\cdot), s(\cdot) \in C((\Omega)_r^-; \mathbb{R})$ such that $(tu, sw_y) \in M$ for $y \in (\Omega)_r^-$. Moreover, we have*

$$t, s > 1 \quad \text{and} \quad t, s \to 1 \text{ as } a_{12} \to 0 \text{ uniformly in } y. \tag{2.22}$$

The proof of this lemma is rather technical, and hence is given in Section 4.

Let us define the mapping Φ from $(\Omega)_r^-$ to M by

$$\Phi(y) = (t(y)u_{a_{11}}, s(y)w_{a_{22}}(\cdot - y)).$$

From Lemma 2.7, it is obvious that Φ is a well-defined and continuous mapping from $(\Omega)_r^-$ to M if a_{12} is sufficiently small.

Proposition 2.6. *For $a_{11}, a_{22} \in (0, \lambda_1)$, there exist $\overline{a}, \sigma > 0$ such that*

$$\Phi((\Omega)_r^-) \subset [I \leq (S^{N/2} + S_*^{N/2})/N - \sigma]_M$$

holds for $a_{12} \in (0, \overline{a})$.

Proof of Proposition 2.6. Recall our assumption that $a_{11} \geq a_{22}$. This fact and Lemma 2.2 yield $S_{a_{11}} \leq S_{a_{22}}$; thus, $S_* = S_{a_{11}}$. The fact that $u_{a_{11}}$ satisfies $-\Delta u_{a_{11}} = a_{11}u_{a_{11}} + u_{a_{11}}|u_{a_{11}}|^{p-2}$ implies that $\|\nabla u_{a_{11}}\|_2^2 - a_{11}\|u_{a_{11}}\|_2^2 = \|u_{a_{11}}\|_p^p$. Moreover, since $u_{a_{11}}$ is the least-energy solution of the above elliptic equation, we have $S_{a_{11}}^{N/2}/N = \varphi_{a_{11}}(u_{a_{11}}) = \|\nabla u_{a_{11}}\|_2^2/2 - a_{11}\|u_{a_{11}}\|_2^2/2 - \|u_{a_{11}}\|_p^p$ (see [6]). Thus we obtain $\|u_{a_{11}}\|_p^p = S_{a_{11}}^{N/2} = S_*^{N/2}$. Similarly, $\|w_{a_{22}}\|_p^p = S_{a_{22}}(B_r(0))^{N/2}$ holds. Hence by (2.22) and by Lemma 2.2,

$$
\begin{aligned}
I \circ \Phi(y) &= \left(t^2/2 - t^p/p\right)\|u_{a_{11}}\|_p^p + \left(s^2/2 - s^p/p\right)\|w_{a_{22}}(\cdot - y)\|_p^p \\
&\quad + \frac{a_{12}}{\alpha} t^{\alpha_1} s^{\alpha_2} K(u_{a_{11}}, w_{a_{22}}(\cdot - y)) \\
&= S_*^{N/2}/N + S_{a_{22}}^{N/2}(B_r(0))/N + o(1) < S_*^{N/2}/N + S^{N/2}/N
\end{aligned}
$$

uniformly in y as $a_{12} \to 0$, where $t = t(y)$ and $s = s(y)$. □

3. Proof of Theorem 1.1

It is easy to see that critical points of I correspond to positive solutions of (ES). Hence, in order to prove Theorem 1.1, it is enough to show the existence of multiple critical points of I in E. Let $L := 2S^{N/2}/N$ and $K := \lambda_1/2$. Then Proposition 2.3 gives $\overline{a}_0 > 0$ such that all critical points of I in $[I \leq d]_M$ for $d < L$ are that of I in E if $a_{12} \in (0, \overline{a}_0)$ and $a_{11}, a_{22} \in (0, K)$. Proposition 2.1 and Proposition 2.4 say that the number of critical points in $[I \leq d]_M$ is equal to or greater than $\operatorname{cat}[I \leq d]_M$. Hence, it is enough to show the following:

Claim. *There exists $\mu_1 > 0$ which satisfies the following: for $a_{11}, a_{22} \in (0, \mu_1)$, there exist $d \in [d_M, S_*^{N/2}/N + S^{N/2}/N)$ and $\mu_2 > 0$ satisfying*

$$\operatorname{cat}[I \leq d]_M \geq \operatorname{cat}\Omega$$

for $a_{12} \in (0, \mu_2)$, where d_M is the infimum of I in M.

Proof of Claim. Take $r > 0$ such that $(\Omega)_r^-$, Ω and $(\Omega)_r^+$ are all homotopically equivalent. Then by Proposition 2.5, there exist $\varepsilon > 0$ and $\overline{a}_1 > 0$ which satisfy the conclusion of Proposition 2.5. Let $\mu_1 = \min(\overline{a}_0, \overline{a}_1)$. Take any $a_{11}, a_{22} \in (0, \mu_1)$. Without loss of generality, we assume that $a_{11} \geq a_{22}$. For this a_{11} and a_{22}, Proposition 2.6 yields the existence of $\overline{a}_2$ and $\sigma > 0$ which satisfy the conclusion of Proposition 2.6. Take any a_{12} with $0 < a_{12} < \mu_2 := \min(\overline{a}_0, \overline{a}_1, \overline{a}_2)$.

Without loss of generality, we can assume that the existence of a regular value $d \in (S_*^{N/2}/N + S^{N/2}/N - \sigma, S_*^{N/2}/N + S^{N/2}/N)$, since otherwise we already have infinitely many critical points. Then by Proposition 2.5 and Proposition 2.6, $(\Omega)_r^- \xrightarrow{\Phi} [I \le d]_M \xrightarrow{\beta_2} (\Omega)_r^+$ and $\beta_2 \circ \Phi$ is a canonical injection in view of the radial symmetry of $w_{a_{22}}(\cdot - y)$ with respect to y. Hence by Lemma 2.1 with $a = (\Omega)_r^-$, $A = (\Omega)_r^+$ and $b = B = [I \le d]_M$, and by the homotopy equivalence of $(\Omega)_r^-$, $(\Omega)_r^+$ and Ω, $\operatorname{cat}[I \le d]_M \ge \operatorname{cat}\Omega$ holds. □

4. Appendix

In this appendix, we give the proof of Lemma 2.7. We denote $u_{a_{11}}$, $w_{a_{22}}$ and $w_{a_{22}}(\cdot - y)$ by u, w and w_y, respectively.

At first we show that there exist $s = s(y)$ and $t = t(y) > 0$ such that $(tu_{a_{11}}, sw_{a_{22}}(\cdot - y)) \in M$ for each $y \in (\Omega)_r^-$. In order to find such a (t, s), it is enough to solve

$$J_1(tu, sw_y) = 0, \quad J_2(tu, sw_y) = 0 \tag{4.1}$$

with respect to t and s. Let us introduce $f_y : \mathbb{R}^2 \to \mathbb{R}$ by $f_y(t, s) := I(tu, sw_y)$ for each $y \in (\Omega)_r^-$. It is easy to see that (t, s) satisfies (4.1) if and only if $(\nabla f_y)_{(t,s)} = 0$. Since f_y is bounded from above, there exists a maximum point $(t(y), s(y))$ of f_y. We can assume that $t(y), s(y) > 0$.

Let $(t_*, s_*) \in (\mathbb{R}^+)^2$ be a critical point of f_y with $t_*, s_* > 0$. Then by using $\|\nabla u\|_2^2 = a_{11}\|u^+\|_2^2 + \|u^+\|_p^p$, $\|\nabla w_y\|_2^2 = a_{22}\|w_y^+\|_2^2 + \|w_y^+\|_p^p$ and (4.1), we have

$$\begin{aligned}
({t_*}^2 - {t_*}^p)\|u^+\|_p^p &= -a_{12}{t_*}^{\alpha_1}{s_*}^{\alpha_2}\frac{\alpha_1 K(u, w_y)}{\alpha} < 0, \\
({s_*}^2 - {s_*}^p)\|w_y^+\|_p^p &= -a_{12}{t_*}^{\alpha_1}{s_*}^{\alpha_2}\frac{\alpha_2 K(u, w_y)}{\alpha} < 0,
\end{aligned}$$

thus (2.22).

Next we show that (4.1) has a unique solution if a_{12} is sufficiently small. Again let $(t_*, s_*) \in (\mathbb{R}^+)^2$ be a critical point of f_y such that t_*, $s_* > 0$. Since $(\nabla^2 f_y)_{(t,s)} = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$, where

$$p = (1 - (p-1)t^{p-2})\|u^+\|_p^p + a_{12}\frac{\alpha_1(\alpha_1 - 1)}{\alpha}t^{\alpha_1 - 2}s^{\alpha_2}K(u, w_y), \tag{4.2}$$

$$q = a_{12}\frac{\alpha_1\alpha_2}{\alpha}t^{\alpha_1 - 1}s^{\alpha_2 - 1}K(u, w_y), \tag{4.3}$$

$$r = (1 - (p-1)s^{p-2})\|w_y^+\|_p^p + a_{12}\frac{\alpha_2(\alpha_2 - 1)}{\alpha}t^{\alpha_1}s^{\alpha_2 - 2}K(u, w_y), \tag{4.4}$$

we have

$$\begin{aligned}&\det(\nabla^2 f_y)_{(t,s)=(t_*,s_*)}\\&> ((p-1)t_*^{p-2}-1)\|u^+\|_p^p((p-1)s_*^{p-2}-1)\|w_y^+\|_p^p - a_{12}C\\&\geq (p-2)^2\|u^+\|_p^p(p-2)\|w^+\|_p^p/2 \qquad (4.5)\end{aligned}$$

uniformly in y for $a_{12} > 0$ sufficiently small, where we use (2.22). Therefore

$$f_y \text{ has a unique critical point } (t,s) = (t(y), s(y)) \qquad (4.6)$$

for each $y \in (\Omega)_r^-$ if $a_{12} > 0$ is sufficiently small.

Now we prove the continuity of $(t(\cdot), s(\cdot))$: $(\Omega)_r^- \to \mathbb{R}^2$. For $y \in (\Omega)_r^-$ and $(t,s) \in \mathbb{R}^2$, let $H(y,(t,s)) := (\nabla f_y)_{(t,s)}$. Then $H(y,(t,s)) = 0$ if and only if (t,s) is a solution of (4.1).

Let $y_0 \in (\Omega)_r^-$, $t_0 := t(y_0)$ and $s_0 = s(y_0)$; thus, $H(y_0,(t_0,s_0)) = 0$. Since $(\nabla_{(t,s)}H)_{(y,(t,s))} = (\nabla^2 f_y)_{(t,s)}$, (4.5) with $(t_*,s_*) = (t_0,s_0)$ yields

$$\ker(\nabla_{(t,s)}H)_{(y,(t,s))=(y_0,(t_0,s_0))} = \{0\}. \qquad (4.7)$$

Then the implicit function theorem implies that there exist a neighborhood U_0 of y_0 in $(\Omega)_r^-$ and $\tilde{t}_{y_0}(\cdot), \tilde{s}_{y_0}(\cdot) \in C(U_0;\mathbb{R})$ such that $H(y,(\tilde{t}_{y_0}(y), \tilde{s}_{y_0}(y))) = 0$. The uniqueness result (4.6) yields $\tilde{t}_{y_0}(y) = t(y)$ and $\tilde{s}_{y_0}(y) = s(y)$; i.e., $\tilde{t}$ and $\tilde{s}$ are independent of y_0. The continuity of $t(\cdot)$ and $s(\cdot)$ follows from that of $\tilde{t}_{y_0}(\cdot)$ and $\tilde{s}_{y_0}(\cdot)$. □

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