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A REMARK ON GLOBAL WELL-POSEDNESS BELOW L^2 FOR THE GKDV-3 EQUATION

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Abstract. The I-method in its first version as developed by Colliander et al. in [2] is applied to prove that the Cauchy-problem for the generalized Korteweg-de Vries equation of order three (gKdV-3) is globally well-posed for large real-valued data in the Sobolev space $H^s(\mathbb{R} \to \mathbb{R})$, provided $s > -\frac{1}{42}$.

1. Introduction

In a recently published paper of Tao [12] concerning the Cauchy problem for the generalized Korteweg-de Vries equation of order three (for short, gKdV-3), i.e.,

$$u_t + u_{xxx} \pm (u^4)_x = 0, \qquad u(0,x) = u_0(x), \;\; x \in \mathbb{R}, \tag{1.1}$$

it was shown that this problem is locally well-posed for data u_0 in the critical Sobolev space $\dot{H}^{-\frac{1}{6}}(\mathbb{R} \to \mathbb{C})$ and globally well-posed for data with sufficiently small $\dot{H}^{-\frac{1}{6}}$-norm. Moreover, scattering results in $H^1 \cap \dot{H}^{-\frac{1}{6}}(\mathbb{R} \to \mathbb{R})$ for the radiation component of a perturbed soliton were obtained. Tao's local

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result improves earlier work of Kenig, Ponce, and Vega ($s \geq \frac{1}{12}$; see [9, Theorem 2.6]) and of the first author ($s > -\frac{1}{6}$; see [7]), while the global *small* data theory seems to be completely new in Sobolev spaces of negative index. For *large real-valued* data $u_0 \in H^s(\mathbb{R} \to \mathbb{R})$, $s \geq 0$, global well-posedness of (1.1) was obtained in [7] by combining the conservation of the L^2-norm with the local L^2-result; for $s \geq 1$ this was already in [9, Corollary 2.7], where the energy conservation was used.

Starting with Bourgain's splitting argument [1], which was followed by the "I-method" or "method of almost conservation laws" introduced and further refined by Colliander, Keel, Staffilani, Takaoka, and Tao in a series of papers—see e.g. [2], [3], [4], [5], and [6]—effective techniques have been developed, which are capable of showing large data global well-posedness *below* certain conserved quantities such as the energy or the L^2-norm. The question of whether and to what extent these methods apply to the Cauchy problem for gKdV-3 was raised by Linares and Ponce [10, p. 177 and p. 183] as well as by Tao; see Remark 5.3 in [12]. In this note we establish global well-posedness of (1.1) for large data $u_0 \in H^s(\mathbb{R} \to \mathbb{R})$, provided $s > -\frac{1}{42}$, thus giving a partial answer to this question. Our proof combines the first version of the I-method as in [2] with a sharp four-linear $X_{s,b}$-estimate exhibiting an extra[3] gain of half a derivative.

Before we turn to the details, let us point out that substantial difficulties appear if we try to push the analysis further to lower values of s; by following the construction of a sequence of "modified energies" in [5] we are led already in the second step to a Fourier multiplier, say μ_8, corresponding to M_4 in [5], with a quadratic singularity, and the argument breaks down.[4] Our fruitless effort in this direction seems to confirm Tao's remark, that "it is unlikely that these methods would get arbitrarily close to the scaling regularity $s = -\frac{1}{6}$" [12, Remark 5.3].

2. A variant of local well-posedness, the decay estimate, and the main result

Here we follow the lines of [2]: The operator I_N is defined via the Fourier transform by

$$\widehat{I_N u}(\xi) := m(\frac{|\xi|}{N})\widehat{u}(\xi),$$

[3] I.e., beyond the cancellation of the derivative in the nonlinearity

[4] A similar problem was observed by Tzirakis for the quintic semilinear Schrödinger equation in one dimension; see the concluding remark in [13].

where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth monotonic function with $m(x) = 1$ for $x \leq 1$ and $m(x) = x^s$, $x \geq 2$. Here $s < 0$, so that $0 < m(x) \leq 1$. $I_N : H^s \to L^2$ is isomorphic and $\|I_N \cdot\|_{L^2}$ defines an equivalent norm on H^s, with implicit constants depending on N.

The crucial nonlinear estimate in the proof of local well-posedness for (1.1) with H^s-data, $s > -\frac{1}{6}$, is

$$\|\partial_x \prod_{i=1}^{4} u_i\|_{X_{s,b'}} \lesssim \prod_{i=1}^{4} \|u_i\|_{X_{s,b}}, \tag{2.1}$$

which holds true whenever $0 \geq s > -\frac{1}{6}$, $-\frac{1}{2} < b' < s - \frac{1}{3}$ and $b > \frac{1}{2}$; see [7, Theorem 1]. The $X_{s,b}$-norms used here are given by

$$\|u\|_{X_{s,b}} = \left(\int d\xi d\tau \langle \tau - \xi^3 \rangle^{2b} \langle \xi \rangle^{2s} |\mathcal{F}u(\xi, \tau)|^2 \right)^{\frac{1}{2}},$$

where $\mathcal{F}$ denotes the Fourier transform in both variables. Later on we shall also use the restriction norms $\|v\|_{X_{s,b}(\delta)} = \inf \{\|u\|_{X_{s,b}} : u|_{[0,\delta]\times\mathbb{R}} = v\}$. Applying the interpolation lemma [6, Lemma 12.1] to (2.1) we obtain, under the same assumptions on the parameters s, b' and b,

$$\|I_N \partial_x \prod_{i=1}^{4} u_i\|_{X_{0,b'}} \lesssim \prod_{i=1}^{4} \|I_N u_i\|_{X_{0,b}}, \tag{2.2}$$

where the implicit constant is *independent* of N. Now familiar arguments invoking the contraction-mapping principle give the following variant of local well-posedness.

Lemma 1. *For $s > -\frac{1}{6}$ the Cauchy problem* (1.1) *is locally well-posed for data $u_0 \in (H^s, \|I_N \cdot\|_{L^2})$. The lifespan δ of the local solution u satisfies*

$$\delta \gtrsim \|I_N u_0\|_{L^2}^{-\frac{18}{6s+1}-}, \tag{2.3}$$

and moreover we have for $b = \frac{1}{2}+$

$$\|I_N u\|_{X_{0,b}(\delta)} \lesssim \|I_N u_0\|_{L^2}. \tag{2.4}$$

Replacing u^2 by u^4 in the calculation on page 2 of [2], we obtain for a solution u of (1.1)

$$\|I_N u(\delta)\|_{L^2}^2 - \|I_N u(0)\|_{L^2}^2 \lesssim \|\partial_x(I_N u^4 - (I_N u)^4)\|_{X_{0,-b}(\delta)} \|I_N u\|_{X_{0,b}(\delta)}. \tag{2.5}$$

The next section will be devoted to the proof that for $b > \frac{1}{2}$ and $0 \geq s \geq -\frac{1}{8}$

$$\|\partial_x(I_N u^4 - (I_N u)^4)\|_{X_{0,-b}(\delta)} \lesssim N^{-\frac{1}{2}} \|I_N u(0)\|_{L^2}^4 \tag{2.6}$$

(see Corollary 1 below), which together with (2.5) and (2.4) gives

$$\|I_N u(\delta)\|_{L^2} - \|I_N u(0)\|_{L^2} \lesssim N^{-\frac{1}{2}} \|I_N u(0)\|^4_{L^2}. \tag{2.7}$$

Now the decay estimate (2.7) allows us to prove our main result:

Theorem 1. *Let* $s > -\frac{1}{42}$ *and* $u_0 \in H^s(\mathbb{R} \to \mathbb{R})$. *Then the solution* u *of* (1.1) *according to Lemma* 1 *extends uniquely to any time interval* $[0, T]$ *and satisfies*

$$\sup_{0 \le t \le T} \|u(t)\|_{H^s} \lesssim \langle T \rangle^{\frac{-2s}{1+42s}} \|u_0\|_{H^s}. \tag{2.8}$$

Proof. We choose ε_0 so that Lemma 1 gives the lifespan $\delta = 1$ for all data $\phi \in H^s$ with $\|I_N \phi\|_{L^2} \le 2\varepsilon_0$. Moreover, we demand $16C\varepsilon_0^3 \le 1$, where C is the implicit constant in the decay estimate (2.7). Assuming without loss that $T \gg 1$, we fix parameters C_1, N and λ with

$$2C_1^{-\frac{1}{6}-s} \|u_0\|_{H^s} = \varepsilon_0, \qquad N^{\frac{1+42s}{2(1+6s)}} = C_1^3 T, \qquad \text{and} \qquad \lambda = C_1 N^{\frac{-6s}{1+6s}}.$$

Then $N^{\frac{1}{2}} = \lambda^3 T$ and for $u_0^\lambda(x) = \lambda^{-\frac{2}{3}} u_0(\frac{x}{\lambda})$ it is easily checked that $\|I_N u_0^\lambda\|_{L^2} \le \varepsilon_0$. For any $k \in \mathbb{N}$, repeated applications of Lemma 1 give a solution u^λ of gKdV-3 with $u^\lambda(0) = u_0^\lambda$ on $[0, k]$, as long as

$$\|I_N u^\lambda(j)\|_{L^2} \le 2\varepsilon_0, \qquad 1 \le j < k. \tag{2.9}$$

Since by (2.7) and the second assumption on ε_0

$$\|I_N u^\lambda(j)\|_{L^2} \le \varepsilon_0 + jCN^{-\frac{1}{2}}(2\varepsilon_0)^4 \le \varepsilon_0(1 + jN^{-\frac{1}{2}}),$$

condition (2.9) is fulfilled for $j \le N^{\frac{1}{2}} = \lambda^3 T$. Thus u^λ is defined on $[0, \lambda^3 T]$, and with $u(x,t) = \lambda^{\frac{2}{3}} u^\lambda(\lambda x, \lambda^3 t)$ we obtain a solution of (1.1) on $[0, T]$. Finally we have for $0 \le t \le T$

$$\|u(t)\|_{H^s} \lesssim \|I_{\lambda N} u(t)\|_{L^2} \lesssim \lambda^{\frac{1}{6}} \|I_N u^\lambda(\lambda^3 t)\|_{L^2}$$

with $\lambda^{\frac{1}{6}} \sim T^{\frac{-2s}{1+42s}}$ and $\|I_N u^\lambda(\lambda^3 t)\|_{L^2}$ being bounded during the whole iteration process by $2\varepsilon_0 \lesssim \|u_0\|_{H^s}$. This gives the growth bound (2.8). □

3. The decisive four-linear estimate

Let us first recall several linear and bilinear Airy estimates (in their $X_{s,b}$-versions), which shall be used below; by interpolation between the sharp version of Kato's smoothing effect (see [8, Theorem 4.1]) and the maximal function estimate from [11, Theorem 3] we have

$$\|J^s u\|_{L^p_x(L^q_t)} \lesssim \|u\|_{X_{0,b}}, \tag{3.1}$$

whenever $b > \frac{1}{2}$, $-\frac{1}{4} \le s \le 1$ and $(\frac{1}{p}, \frac{1}{q}) = (\frac{1-s}{5}, \frac{1+4s}{10})$. We will use (3.1) with $s = 0$, i.e.,

$$\|u\|_{L^5_x(L^{10}_t)} \lesssim \|u\|_{X_{0,b}}, \tag{3.2}$$

and the dual version of (3.1) with $s = \frac{1}{2}$, which is

$$\|u\|_{X_{\frac{1}{2},-b}} \lesssim \|u\|_{L_x^{\frac{10}{9}}(L_t^{\frac{10}{7}})}. \tag{3.3}$$

Moreover, we shall rely on the Strichartz-type estimate

$$\|u\|_{L^8_{xt}} \lesssim \|u\|_{X_{0,b}} \quad (b > \tfrac{1}{2}) \tag{3.4}$$

(cf. [8, Theorem 2.4]) and the bilinear estimate

$$\|I^{\frac{1}{2}} I_-^{\frac{1}{2}}(u, v)\|_{L^2_{xt}} \lesssim \|u\|_{X_{0,b}} \|v\|_{X_{0,b}} \quad (b > \tfrac{1}{2}) \tag{3.5}$$

from [7, Corollary 1]. Here I^s (J^s) denotes the Riesz (Bessel) potential operator of order $-s$ and I^s_- is defined via the Fourier transform by

$$\widehat{I^s_-(f,g)}(\xi) := \int_{\xi_1+\xi_2=\xi} d\xi_1 |\xi_1 - \xi_2|^s \widehat{f}(\xi_1)\widehat{g}(\xi_2).$$

Now we turn to the crucial four-linear $X_{s,b}$-estimate:

Lemma 2. *Let $b > \frac{1}{2}$ and $s_i \le 0$, $1 \le i \le 4$, with $\sum_{i=1}^4 s_i = -\frac{1}{2}$. Then*

$$\|\partial_x \prod_{i=1}^4 v_i\|_{X_{0,-b}} \lesssim \prod_{i=1}^4 \|v_i\|_{X_{s_i,b}}. \tag{3.6}$$

Proof. We write

$$\|\partial_x \prod_{i=1}^4 v_i\|_{X_{0,-b}} = c\|\xi\langle\tau - \xi^3\rangle^{-b} \int d\nu \prod_{i=1}^4 \mathcal{F} v_i(\xi_i, \tau_i)\|_{L^2_{\xi,\tau}},$$

where $d\nu = d\xi_1 \cdots d\xi_3 d\tau_1 \cdots d\tau_3$ and $\sum_{i=1}^4 (\xi_i, \tau_i) = (\xi, \tau)$, and divide the domain of integration into three regions A, B and $C = (A \cup B)^c$. In region A we assume that[5] $|\xi_{max}| \le 1$ and hence $|\xi| \le 4$, so for this region we get the upper bound

$$\|\prod_{i=1}^4 J^{s_i} v_i\|_{L^2_{xt}} \le \prod_{i=1}^4 \|J^{s_i} v_i\|_{L^8_{xt}} \lesssim \prod_{i=1}^4 \|v_i\|_{X_{s_i,b}},$$

where in the last step we have used the L^8_{xt}-Strichartz-type estimate (3.4). Concerning the region B we shall assume—besides $|\xi_{\max}| \ge 1$, implying $\langle\xi_{\max}\rangle \lesssim |\xi_{\max}|$—that

[5]Here ξ_{max} is defined by $|\xi_{max}| = \max_{i=1}^4 |\xi_i|$, similarly ξ_{min}.

i) $|\xi_{\min}| \le 0.99|\xi_{\max}|$ or
ii) $|\xi_{\min}| > 0.99|\xi_{\max}|$, and there are exactly two indices $i \in \{1,2,3,4\}$ with $\xi_i > 0$.

Then the region B can be split further into a finite number of subregions, so that for any of these subregions there exists a permutation π of $\{1,2,3,4\}$ with

$$|\xi| \lesssim |\xi|^{\frac{1}{2}}|\xi_{\pi(1)} + \xi_{\pi(2)}|^{\frac{1}{2}}|\xi_{\pi(1)} - \xi_{\pi(2)}|^{\frac{1}{2}} \prod_{i=1}^{4} \langle \xi_i \rangle^{s_i}.$$

Assume $\pi = id$ for the sake of simplicity now. Then we get the upper bound

$$\begin{aligned} &\|(I^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^{s_1}v_1, J^{s_2}v_2))(J^{s_3}v_3)(J^{s_4}v_4)\|_{X_{\frac{1}{2},-b}} \\ \lesssim\ &\|(I^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^{s_1}v_1, J^{s_2}v_2))(J^{s_3}v_3)(J^{s_4}v_4)\|_{L_x^{\frac{10}{9}}(L_t^{\frac{10}{7}})} \\ \lesssim\ &\|I^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^{s_1}v_1, J^{s_2}v_2)\|_{L^2_{xt}} \|J^{s_3}v_3\|_{L^5_x(L^{10}_t)} \|J^{s_4}v_4\|_{L^5_x(L^{10}_t)} \lesssim \prod_{i=1}^{4} \|v_i\|_{X_{s_i,b}}. \end{aligned}$$

Here we have applied the estimates (3.3), Hölder, (3.5) and (3.2). Finally we consider the remaining region C: Here the $|\xi_i|$, $1 \le i \le 4$, are all very close together and $\gtrsim \langle \xi_i \rangle$. Moreover, at least three of the variables ξ_i have the same sign. Thus for the quantity *c.q.* controlled by the expressions $\langle \tau - \xi^3 \rangle$ and $\langle \tau_i - \xi_i^3 \rangle$, $1 \le i \le 4$, we have in this region

$$c.q. := |\xi^3 - \sum_{i=1}^{4} \xi_i^3| \gtrsim \sum_{i=1}^{4} \langle \xi_i \rangle^3 \gtrsim \langle \xi \rangle^3.$$

So the contribution of the subregion, where $\langle \tau - \xi^3 \rangle \ge \max_{i=1}^4 \langle \tau_i - \xi_i^3 \rangle$, is bounded by

$$\|\prod_{i=1}^{4} J^{s_i}v_i\|_{L^2_{xt}} \le \prod_{i=1}^{4} \|J^{s_i}v_i\|_{L^8_{xt}} \lesssim \prod_{i=1}^{4} \|v_i\|_{X_{s_i,b}},$$

where (3.4) was used again. On the other hand, if $\langle \tau_1 - \xi_1^3 \rangle$ is dominant, we write $\Lambda^{\frac{1}{2}} = \mathcal{F}^{-1}\langle \tau - \xi^3 \rangle^{\frac{1}{2}}\mathcal{F}$ and obtain the upper bound

$$\begin{aligned} \|(\Lambda^{\frac{1}{2}} J^{s_1}v_1) \prod_{i=2}^{4} J^{s_i}v_i\|_{X_{0,-b}} &\lesssim \|(\Lambda^{\frac{1}{2}} J^{s_1}v_1) \prod_{i=2}^{4} J^{s_i}v_i\|_{L_{xt}^{\frac{8}{7}}} \\ \le \|\Lambda^{\frac{1}{2}} J^{s_1}v_1\|_{L^2_{xt}} \prod_{i=2}^{4} \|J^{s_i}v_i\|_{L^8_{xt}} &\lesssim \prod_{i=1}^{4} \|v_i\|_{X_{s_i,b}}. \end{aligned}$$

Here the dual version $X_{0,-b} \supset L_{xt}^{\frac{8}{7}}$ of the L_{xt}^8 estimate was used first, followed by Hölder's inequality and the estimate itself. The remaining subregions, where $\langle \tau_k - \xi_k^3 \rangle$, $2 \leq k \leq 4$, are maximal, can be treated in precisely the same manner. □

Corollary 1. *Let $b > \frac{1}{2}$ and $0 \geq s \geq -\frac{1}{8}$. Then*

$$\|\partial_x(I_N(\prod_{i=1}^4 u_i) - \prod_{i=1}^4 I_N u_i)\|_{X_{0,-b}(\delta)} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^4 \|I_N u_i\|_{X_{0,b}(\delta)}. \tag{3.7}$$

In particular, if u is a solution of gKdV-3 according to Lemma 1 with $u(0) = u_0$, then

$$\|\partial_x(I_N u^4 - (I_N u)^4)\|_{X_{0,-b}(\delta)} \lesssim N^{-\frac{1}{2}} \|I_N u_0\|_{L^2}^4. \tag{3.8}$$

Proof. By (2.4) the estimate (3.7) implies (3.8). Thus it suffices to show

$$\|\partial_x(I_N(\prod_{i=1}^4 u_i) - \prod_{i=1}^4 I_N u_i)\|_{X_{0,-b}} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^4 \|I_N u_i\|_{X_{0,b}}. \tag{3.9}$$

Now let ξ_i denote the frequencies of the u_i, $1 \leq i \leq 4$. If all the $|\xi_i| \leq N$, then either $|\xi| \leq N$, such that there is no contribution at all, or we have $|\xi| \geq N$, so that at least, say, $|\xi_1| \geq \frac{N}{4}$. In this case, by Lemma 2, the norm on the left of (3.9) is bounded by

$$\|\partial_x \prod_{i=1}^4 u_i\|_{X_{0,-b}} \lesssim \|u_1\|_{X_{-\frac{1}{2},b}} \prod_{i=2}^4 \|u_i\|_{X_{0,b}} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^4 \|I_N u_i\|_{X_{0,b}}.$$

Otherwise there are k large frequencies for some $1 \leq k \leq 4$. By symmetry we may assume that $|\xi_1|, \dots, |\xi_k| \geq N$ and $|\xi_{k+1}|, \dots, |\xi_4| \leq N$. Then we have, again by Lemma 2,

$$\|\partial_x \prod_{i=1}^4 I_N u_i\|_{X_{0,-b}} \lesssim \prod_{i=1}^k \|I_N u_i\|_{X_{-\frac{1}{2k},b}} \prod_{i=k+1}^4 \|I_N u_i\|_{X_{0,b}} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^4 \|I_N u_i\|_{X_{0,b}}$$

as well as

$$\|\partial_x I_N \prod_{i=1}^4 u_i\|_{X_{0,-b}} \lesssim \prod_{i=1}^k \|u_i\|_{X_{-\frac{1}{2k},b}} \prod_{i=k+1}^4 \|u_i\|_{X_{0,b}}.$$

Since for any $s_1 \leq s$ and any v with frequency $|\xi| \geq N$ it holds that

$$\|v\|_{X_{s_1,b}} \lesssim N^{s_1 - s} \|v\|_{X_{s,b}} \sim N^{s_1} \|I_N v\|_{X_{0,b}},$$

the latter is again bounded by the right-hand side of (3.9). □

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