Differential and Integral Equations Volume 20, Number 11 (2007), 1201–1209

INFINITE-DIMENSIONAL ATTRACTORS FOR EVOLUTION EQUATIONS WITH p-LAPLACIAN AND THEIR KOLMOGOROV ENTROPY

Messoud A. Efendiev
Center of Mathematical Sciences, Technical University of München
Boltzman Str.3, 85747 Garching/München, Germany

Mitsuharu Ôtani[1]
Department of Applied Physics, School of Science and Engineering
Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo, Japan 169-8555

(Submitted by: L.A. Peletier)

Abstract. In this paper we give a detailed study on the attractors for the parabolic equations in bounded domains involving p-Laplacian as the principal term. Not only the existence of attractors but also their new properties are presented, which cannot be observed for the non-degenerate parabolic equations. In particular, we construct infinite-dimensional attractors whose ε-Kolmogorov entropy behave as the polynomial of $1/\varepsilon$ as ε tends to zero.

1. Introduction

It is well known that the long-time behaviour of many dissipative systems generated by the evolution of PDE's in mathematical physics can be described in terms of the so-called global attractor, which is a compact invariant set in the phase space X which attracts the images of all bounded subsets under the temporal evolution. Hence, if the global attractor exists, its property guarantees that the dynamical system (DS) reduced to the attractor $\mathcal{A}$ contains all of the nontrivial dynamics of the original system and the reduced phase space $\mathcal{A}$ is really "thinner" than the initial phase space X (we recall that in infinite-dimensional spaces, a compact set cannot contain, for instance, balls and as a result should be nowhere dense). One of

Accepted for publication: July 2007.
AMS Subject Classifications: 37L30, 35K65 .
[1]Partly supported by Waseda University Grant for Special Research Projects #2003B-027 and the Grant-in-Aid for Scientific Research, #15654024 and #16340043, the Ministry of Education, Culture, Sports, Science and Technology, Japan.

the important questions in this theory is the following: *In what sense is the dynamics reduced to the attractor finite or infinite dimensional?*

Usually for regular (non-degenerate) dissipative autonomous PDE's in a bounded domain spatial Ω, the Kolmogorov ε-entropy of their attractors has asymptotics such as

$$C_1|\Omega|\log_2\frac{1}{\varepsilon} \leq H_\varepsilon(\mathcal{A}, X) \leq C_2|\Omega|\log_2\frac{1}{\varepsilon},$$

where $|\Omega|$ denotes the volume of Ω and C_i $(i = 1, 2)$ are some constants which are independent of $|\Omega|$.

In contrast to the non-degenerate parabolic PDE's, not so much is known about the long-time behaviour for degenerate parabolic PDE's (see [1, 6, 10, 8] and [5]). In this paper we are mainly interested in the long-time behaviour of the solutions of parabolic equations involving the p-Laplacian, in a bounded domain. More precisely, we present some new features, to the best of our knowledge, related to the attractor of such equations which one cannot observe in non-degenerate cases, namely

a) Infinite dimensionality of attractor,

b) Polynomial asymptotics of its ε-Kolmogorov entropy,

c) Difference of the asymptotics of the ε-Kolmogorov entropy depending on the choice of the underlying phase spaces.

Remark 1. It is worth noting that, although global attractors of non-degenerate parabolic equations in unbounded domains are also infinite-dimensional, the asymptotics of their Kolmogorov entropy are always logarithmic (see [4]).

Remark 2. We emphasize that the usual method for obtaining lower bounds of the Kolmogorov entropy (or dimension) of attractors is based on the instability index of hyperbolic equilibria (see [11]), which in turn requires differentiability of the associated semigroup with respect to the initial data. However, this method is not applicable for the degenerate parabolic equations, since the associated semigroups are usually not differentiable. To obtain such a lower bound, we shall here directly use the concrete properties of solutions of (E1)–(E3) given below, such as the self-similarity and the finite-speed-of-propagation property.

The main goal of the present paper is to give a detailed study of the attractors for the following class of parabolic equations with p-Laplacian, that is,

$$(\mathrm{E})_p \begin{cases} \frac{\partial u}{\partial t} \in \Delta_p u(x,t) - g(u(x,t)) + h(x), & (x,t) \in \Omega \times [0,\infty), \quad \text{(E1)} \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \quad \text{(E2)} \\ u(x,0) = u_0(x), & x \in \Omega, \quad \text{(E3)} \end{cases}$$

where

$$\Delta_p u = \mathrm{div}\,(|\nabla u|^{p-2} \nabla u), \quad p \in [\tfrac{2N}{N+2}, \infty)$$

and Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$.

This paper is organized as follows: in Section 1, we fix some basic results on the existence, uniqueness and regularity of solutions. In Section 2, we prove the existence of the global attractor for the semigroup generated by (E1)–(E3). Infinite dimensionality of the attractor and asymptotics of its Kolmogorov entropy is given in Section 3. In particular, we show that its Kolmogorov entropy admits polynomial asymptotics, which sheds light on completely new phenomena (see Remark 1).

2. Existence of global solutions and a priori estimates

Consider the initial-/boundary-value problem $(\mathrm{E})_p$. We assume that $g(\xi) = g_0(\xi) + g_1(\xi)$, where

a) $g_0(\cdot)$ is a maximal monotone graph in $\mathbf{R}^2$ such that $0 \in g_0(0), D(g_0) = \{\xi \in \mathbf{R}^1 : |g_0(\xi)| < +\infty\} = \mathbf{R}^1$.

b) $g_1(\cdot)$ is a globally Lipschitz function from $\mathbf{R}^1$ into $\mathbf{R}^1$ and $g_1(0) = 0$.

Note that condition a) implies that there exists a lower-semicontinous convex function $j(\cdot)$ from $\mathbf{R}^1$ into $[0, +\infty]$ such that $g_0(u) = \partial j(u)$. Here ∂j denotes the subdifferential of j.

Put $D(\varphi) = \{u : u \in W_0^{1,p}(\Omega) \text{ and } j(u) \in L^1(\Omega)\}$, and define φ by

$$\varphi(u) = \begin{cases} \dfrac{1}{p} \displaystyle\int_\Omega |\nabla u|^p \, dx + \int_\Omega j(u(x)) \, dx & \text{if } u \in D(\varphi), \\ +\infty & \text{otherwise.} \end{cases}$$

Then φ becomes a lower-semicontinous convex functional from $L^2(\Omega)$ into $[0, \infty]$.

Furthermore, since for the Yosida approximation $(g_0)_\lambda$ of g_0, we obtain

$$(-\Delta_p u, (g_0)_\lambda(u))_{L^2(\Omega)} = \int_\Omega |\nabla u|^p (g_0)'_\lambda(u) \, dx \geq 0 \quad \forall \lambda > 0,$$

by virtue of Proposition 2.17 and parts (i) and (ii) of Theorem 4.4 in [2], we find that $u \mapsto -\Delta_p u + g_0(u)$ is maximal monotone in $L^2(\Omega)$. Hence, we obtain

$$\partial\varphi(u) = -\Delta_p u + g_0(u) \qquad \forall u \in D(\partial\varphi), \tag{2.1}$$

$$D(\partial\varphi) = \Big\{ u \in D(\varphi) : \Delta_p u \in L^2(\Omega),\ \exists b \in L^2(\Omega) \text{ s.t. } b(x) \in g_0(u(x)) \text{ a.e. } x \in \Omega \Big\}.$$

Then, by putting $B(u(x)) = g_1(u(x)) - h(x)$, we see that $(\mathrm{E})_p$ can be reduced to the following abstract Cauchy problem in $L^2(\Omega)$:

$$\frac{du(t)}{dt} + \partial\varphi(u(t)) + B(u(t)) \ni 0, \quad u(0) = u_0.$$

By virtue of b), $B(u)$ becomes a Lipschitz function from $L^2(\Omega)$ into itself, if $h(\cdot) \in L^2(\Omega)$. Thus, Proposition 3.12 in [2] ensures the existence of unique global solutions of $(\mathrm{E})_p$.

Theorem 1. *Assume that* a) *and* b) *are satisfied. Then, for any* $u_0(\cdot)$, $h(\cdot) \in L^2(\Omega)$, *there exists a unique solution* $u \in C([0,\infty); L^2(\Omega)) \cap W^{1,2}_{loc}((0,\infty); L^2(\Omega)) \cap L^p_{loc}([0,\infty); W^{1,p}_0(\Omega))$ *of* $(\mathrm{E})_p$ *which depends continuously on the initial data in the strong topology of* $L^2(\Omega)$.

Remark 3. If $g \in C^1(\mathbf{R}^1)$ satisfies $g(0) = 0$ and $g'(\tau) \geq -K$ for all $\tau \in \mathbf{R}^1$, then by putting $g_0(\tau) = g(\tau) + K\tau$ and $g_1(\tau) = -K\tau$, we find that $g(\cdot)$ falls within our class. In particular, $g(\tau) = C_1|\tau|^{q_1-2}\tau - C_2|\tau|^{q_2-2}\tau$, where $2 \leq q_2 < q_1 < +\infty$ with a suitable decomposition satisfies a) and b).

As to the a priori bounds for solutions of (E1)–(E3), we obtain the following result.

Theorem 2. *Assume that* a) *and* b) *are satisfied,* $\frac{2N}{N+2} < p < \infty$ *and* $h \in L^\infty(\Omega)$. *For the case* $\frac{2N}{N+2} < p \leq 2$, *we further assume*

$$|\overset{\circ}{g}_0(s)| \geq k_0|s|^{1+\theta} - k_1 \quad (\theta, k_0, k_1 > 0), \tag{2.2}$$

where $\overset{\circ}{g}_0(s)$ *denotes the minimal section of* $g_0(s)$, *i.e., the unique nearest point of* $g_0(s)$ *from the origin. Then there exist constants* C_1, C_2 *and* C_3, *independent of the initial data* $u_0 \in L^2(\Omega)$, *such that every solution of* (E1)–(E3) *satisfies*

$$\|u(t)\|_{L^2} \leq C_1 \qquad \text{for all } t \in [1, +\infty), \tag{2.3}$$

$$\|u(t)\|_{L^\infty} \leq C_2 \qquad \text{for all } t \in [2, \infty), \tag{2.4}$$

$$\|u(t)\|_{C^{1,\alpha}(\bar{\Omega})} \leq C_3 \qquad \text{for all } t \in [3, \infty). \tag{2.5}$$

Proof. Multiply (E1) by u. Then integration by parts and using a)–b) gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2} + \|\nabla u\|_{L^p}^p \leq C_4\|u(t)\|_{L^2}^2 + \|h\|_{L^2} \cdot \|u\|_{L^2}. \tag{2.6}$$

Then by Poincaré's inequality, we get for the case $p > 2$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \gamma_1\|u\|_{L^2}^p \leq \gamma_2, \tag{2.7}$$

for some $\gamma_1, \gamma_2 > 0$. Hence, a Ghidaglia-type estimate (see [11]) assures (3).

As for the case $\frac{2N}{N+2} \leq p \leq 2$, the lower bound (2) yields the same estimate as (7) with $p = 2 + \theta$, from which (3) follows.

In order to derive the L^∞-estimate, we need the following lemma. □

Lemma 3. *Let $v^\pm$ be the unique solution of*

$$v_t^\pm = \Delta_p v^\pm - \widetilde{k}_0|v^\pm|^\theta v^\pm - g_1(v^\pm) \pm (|h| + \widetilde{k}_1), \quad v^\pm|_{\partial\Omega} = 0 \tag{2.8}$$

with the initial condition $v^\pm(x,0) = \pm|u_0(x)|$ respectively, where $\widetilde{k}_i = 0$ for $p > 2$ and $\widetilde{k}_i = k_i$ for $p \leq 2$ $(i = 0,1)$. Then the solution u of $(\mathrm{E})_p$ *satisfies*

$$v^-(x,t) \leq u(x,t) \leq v^+(x,t) \qquad \text{for a.e. } (x,t) \in \Omega \times [0,\infty). \tag{2.9}$$

Proof of Lemma 3. Let $u^+(x,t)$ be the unique solution of $(\mathrm{E})_p$ with h and u_0 replaced by $|h|$ and $|u_0|$ respectively. Then it is easy to see that $u^+ \geq 0$ and $(u^+ - v^+)$ satisfies

$$(u^+ - v^+)_t \in \Delta_p u^+ - \Delta_p v^+ - g_0(u^+) + \widetilde{k}_0|v^+|^\theta v^+ - \widetilde{k}_1 - g_1(u^+) + g_1(v^+). \tag{2.10}$$

Multiply (10) by $[u^+ - v^+]^+ = \max(u^+(x,t) - v^+(x,t), 0)$. Then, noting that

$$\begin{aligned}
&(\Delta_p u^+ - \Delta_p v^+, [u^+ - v^+]^+) = -|\nabla([u^+ - v^+]^+)|_{L^p}^p \leq 0,\\
&(-g_0(u^+) + \widetilde{k}_0|v^+|^\theta v^+ - \widetilde{k}_1, [u^+ - v^+]^+)\\
&\qquad \leq -\widetilde{k}_0\left(|u^+|^\theta u^+ - |v^+|^\theta v^+, [u^+ - v^+]^+\right) \leq 0,
\end{aligned}$$

and

$$(-g_1(u^+) + g_1(v^+), [u^+ - v^+]^+) \leq L|[u^+ - v^+]^+|_{L^2}^2,$$

we get

$$|[u^+ - v^+]^+(t)|_{L^2}^2 \leq |[u^+ - v^+]^+(0)|_{L^2}^2 e^{2tL},$$

whence it follows that $0 \leq u^+ \leq v^+$. By much the same argument as above, we also find $u \leq u^+$. Thus repeating this procedure for u^- (the solution of $(\mathrm{E})_p$ with h and u_0 replaced by $-|h|$ and $-|u_0|$) and v^-, we obtain (9). □

Proof of Theorem 2 (continued). Since equation (8) has a simple form, L^∞-estimates for v^+ and v^- have been fully investigated by many authors. For instance, Theorem 3.2 of Chapter 5 of [3] assures us that the following estimate holds:

$$\sup_{x\in\Omega}|v^\pm(x,t)| \le \max\left(1, C\left(t^{\frac{p}{N}}+\frac{1}{t}\right)^{\frac{1}{q-\delta}}\left(\int_0^t\int_\Omega |v^\pm|^\delta\,dx\,d\tau\right)^{\frac{p}{N(q-\delta)}}\right), \quad (2.11)$$

where $\delta = 2+\theta$ for $p \le 2$, $\delta = p$ for $p > 2$ and $q = p(N+2)/N$.

Here [3] requires the condition $p \le \delta < p\frac{N+2}{N}$, which is obviously satisfied for $p > 2$. As for the case where $\frac{2N}{N+2} < p \le 2$, since $\frac{2N}{N+2} < p$ implies $2 < p\frac{N+2}{N}$, we can choose a sufficiently small θ_0 such that $p \le 2+\theta_0 < p\frac{N+2}{N}$. If $\theta_0 < \theta$, it clear that (2) is satisfied with θ and k_1 replaced by θ_0 and k_0+k_1 respectively. Thus the estimate (11) is derived.

On the other hand, by the same argument as used in the first half of the proof of Theorem 2, we easily see that an estimate similar to (7) ensures that $\sup_{t\ge1}|v^\pm(t)|_{L^\delta(t,t+1;L^\delta(\Omega))} \le C_1$. Thus (4) can be derived from (9) and (11).

Now we can rewrite (E1) as $u_t(x,t) = \Delta_p u(x,t) + \tilde{h}(x,t)$, where $\tilde{h}(x,t) = -h_0(x,t) - g_1(u(x,t)) + h(x)$ and $h_0(x,t)$ is the section of $g_0(u(x,t))$ satisfying $u_t(x,t) = \Delta_p u(x,t) - h_0(x,t) - g_1(u(x,t)) + h(x)$. Note that (4) ensures that $\tilde{h} \in L^\infty(\Omega\times[2,+\infty))$. Consequently, by virtue of Theorem 1.2 in Chapter 10 of [3], we can derive the $C^{1,\alpha}(\bar{\Omega})$-bound for u on $[3,+\infty)$. □

3. Global attractor and its Kolmogorov's ε-entropy

Let Φ be a Banach space. The set $\mathcal{A} \subset \Phi$ is called a global attractor of the semigroup $S(t)$ if the following conditions are satisfied:

1) The set $\mathcal{A}$ is a compact subset of the phase space Φ.

2) It is strictly invariant; i.e., $S(t)\mathcal{A} = \mathcal{A}$, for all $t \ge 0$.

3) For every bounded subset $B \subset \Phi$, $\lim_{t\to\infty} \operatorname{dist}(S(t)B, \mathcal{A}) = 0$, where $\operatorname{dist}(X,Y) = \sup_{x\in X}\inf_{y\in Y}\|x-y\|_\Phi$.

The next result states the existence of a global attractor in $L^2(\Omega)$.

Theorem 4. *Let all the assumptions in Section* 2 *be satisfied. Then a semigroup* $S(t)$ *associated with equation* (E1)–(E3) *possesses a global attractor* $\mathcal{A}$ *in the phase space* $L^2(\Omega)$ *which is globally bounded in* $C^{1,\alpha}(\bar{\Omega})$ *(for some* α*) and has the following structure:* $\mathcal{A} = \mathcal{K}\big|_{t=0}$*, where* $\mathcal{K}$ *is a set of all bounded solutions of* (E1) *defined for all* $t \in \mathbb{R}$*. Moreover, this set satisfies* $\|\mathcal{K}\|_{C^{1,\alpha}(\mathbb{R}\times\Omega)} \le Q(\|h\|_{L^\infty(\Omega)})$ *for some monotone-increasing function* Q*.*

A proof of the existence of a global attractor $\mathcal{A}$ is an immediate consequence of Theorem 2. The characterization of $\mathcal{A}$ in terms of $\mathcal{K}$ is derived from standard arguments. Next we present lower bounds for the Kolmogorov's ε-entropy of an attractor $\mathcal{A}$ (denoted by $\mathcal{H}_\varepsilon(\mathcal{A},\Phi)$), which is by definition the binary logarithm of $N_\varepsilon(\mathcal{A},\Phi)$, that is $\mathcal{H}_\varepsilon(\mathcal{A},\Phi) := \log_2 N_\varepsilon(\mathcal{A},\Phi)$. Here we denote by $N_\varepsilon(\mathcal{A},\Phi)$ the minimal number of ε-balls in Φ that covers $\mathcal{A}$ (recall that $\mathcal{A}$ is a compact in Φ).

From now on we assume that $p > 2$, $h \equiv 0$ and $g(u) = \tilde{g}(u) - u$, where $\tilde{g} \equiv 0$ in some small neighborhood of $u = 0$, say in $\{u : |u| \le \rho_0\}$. The special form of $g(u)$ is made for simplicity of presentation (see Remark 4). Then we can show that $\dim_F(\mathcal{A}, L^2(\Omega))$, the fractal dimension of $\mathcal{A}$ in $L^2(\Omega)$, is infinite by the following arguments. Let $\mathcal{M}^+_{\rho_0} = \{u_0 \in L^\infty(\Omega) : \exists u(t,u_0) \in \mathcal{K}^+ \text{ s.t. } u(0,u_0) = u_0,\ \|u(t,u_0)\|_{L^\infty} \le \rho_0 \forall t \in (-\infty,0]\}$, where $u \in \mathcal{K}^+$ implies that u is a bounded solution of (E1)–(E3) on $(-\infty,0]$. Since $\mathcal{M}^+_{\rho_0} \subset \mathcal{A}$, it is enough to prove $\dim_F \mathcal{M}^+_{\rho_0} = +\infty$.

We consider the following equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + u, & (x,t) \in \Omega \times (-\infty,0], \\ u|_{\partial\Omega} = 0 \quad t \in (-\infty,0], \quad u|_{t=0} = u_0 \in L^\infty(\Omega) & x \in \Omega. \end{cases} \tag{3.1}$$

Note that any solution $u(t,x)$ with small L^∞-norm (less than ρ_0) of (3.1) is also a solution of (E1)–(E2) on $(-\infty,0]$. Let u be a solution of (3.1) and put $u(t,x) = e^t v(t,x)$, then $v(t,x)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} = e^{(p-2)t}\operatorname{div}\left(|\nabla_x v|^{p-2}\nabla_x v\right) & (x,t) \in \Omega \times (-\infty,0], \\ v|_{\partial\Omega} = 0 \quad t \in (-\infty,0], \quad v|_{t=0} = u_0 \in L^\infty(\Omega) & x \in \Omega. \end{cases} \tag{3.2}$$

Put $s = e^{(p-2)t}$ and $\tilde{v}(s,x) = v(t,x)$, then we get

$$\begin{cases} \dfrac{\partial \tilde{v}}{\partial s} = \dfrac{1}{p-2}\operatorname{div}\left(|\nabla \tilde{v}|^{p-2}\nabla \tilde{v}\right) & (x,s) \in \Omega \times (0,1], \\ \tilde{v}|_{\partial\Omega} = 0 \quad s \in (0,1], \quad \tilde{v}(1,x) = u_0(x) \quad x \in \Omega. \end{cases} \tag{3.3}$$

We now consider the following initial-/boundary-value problem.

$$\begin{cases} \partial_t w = \dfrac{1}{p-2}\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right), \\ w|_{\partial\Omega} = 0 \quad t \in [0,1], \quad w|_{t=0} = w_0 \in L^\infty(\Omega) \quad x \in \Omega. \end{cases} \tag{3.4}$$

For any (sufficiently small) compact set $K \subset \Omega$, we fix the initial data w_0 such that the corresponding solution $w(x,t)$ of (3.4) satisfies $\|w(\cdot,1)\|_{L^\infty} = 1$, $\sup_{0\le t\le 1}\|w(\cdot,t)\|_{L^\infty} < \rho_1$ and $\operatorname{supp} w(x,t) \subset K$ $\forall t \in [0,1]$. This is always possible since the L^∞-norm is a Lyapunov function for (3.4) (cf. [9]) which has also the finite-speed propagation property. (Such a solution can be given explicitly.) Furthermore, it is easy to see that $w_\varepsilon(x,t) = \varepsilon w(\varepsilon^{\frac{2-p}{p}}x,t)$ becomes a solution of (3.4) and $\sup_{0\le t\le 1}\|w_\varepsilon\|_{L^\infty} < \varepsilon\rho_1$, $\operatorname{supp} w_\varepsilon \subset \varepsilon^{\frac{p-2}{p}}K$ for all $t \in [0,1]$. Hence, in view of the transformations given above, we find that $w_\varepsilon(x,1) \in \mathcal{M}^+_{\rho_0}$ for sufficiently small ε. Moreover, $w_\varepsilon(x-x_i,t)$ gives also a solution of (3.4), as long as $x_i + K \subset \Omega$. Therefore, for sufficiently small ε, there exists a finite set $R_\varepsilon := \{x_i\} \subset \Omega$ such that

(1) $(x_i + K_\varepsilon) \cap (x_j + K_\varepsilon) = \emptyset$, $\forall x_i, x_j \in R_\varepsilon$, $i \ne j$, $K_\varepsilon := \varepsilon^{\frac{p-2}{p}}K$
(2) $\# R_\varepsilon \ge C\left(\frac{1}{\varepsilon}\right)^{\frac{N(p-2)}{p}}$,
(3) $x_i + K_\varepsilon \subset\subset \Omega$, $\forall x_i \in R_\varepsilon$.

Consequently, for every $\vec{m} \in \{0,1\}^{R_\varepsilon}$ the function

$$w_{\vec{m},\varepsilon}(x,t) = \sum_{i=1}^{\# R_\varepsilon} m_i w_\varepsilon(x-x_i,t) \tag{3.5}$$

solves (3.4) in Ω. On the other hand, obviously for $\vec{m}^1 \ne \vec{m}^2$

$$\|w_{\vec{m}^1,\varepsilon}(x,1) - w_{\vec{m}^2,\varepsilon}(x,1)\|_{L^\infty} = \varepsilon\|w(x,1)\|_{L^\infty} = \varepsilon. \tag{3.6}$$

Thus we find $2^{\# R_\varepsilon}$ different functions of the form (3.5) in $\mathcal{M}^+_{\rho_0}$. Hence

$$H_\varepsilon(\mathcal{A}, L^\infty) \ge H_\varepsilon(\mathcal{M}^+_{\rho_0}, L^\infty) \ge \# R_\varepsilon \ge C\left(\frac{1}{\varepsilon}\right)^{\frac{N(p-2)}{p}}. \tag{3.7}$$

As for the measurement in the topology of L^r $(1 \le r < \infty)$, instead of (3.6), we get

$$\|w_{\vec{m}^1,\varepsilon}(x,1) - w_{\vec{m}^2,\varepsilon}(x,1)\|_{L^r} \ge \varepsilon^{\frac{pr+(p-2)N}{pr}}\|w(x,1)\|_{L^r}. \tag{3.8}$$

Hence it is easy to derive $H_\varepsilon(\mathcal{A}, L^r) \ge \left(\frac{1}{\varepsilon}\right)^{\frac{Nr(p-2)}{pr+N(p-2)}}$. Moreover, since Kolmogorov's ε-entropy of a bounded set of $C^1(\bar\Omega)$ in the topology of L^∞ is estimated above by $C\left(\frac{1}{\varepsilon}\right)^N$ (see [7]), we obtain

$$\left(\frac{1}{\varepsilon}\right)^{\frac{Nr(p-2)}{pr+N(p-2)}} \le H_\varepsilon(\mathcal{A}, L^r) \le C\left(\frac{1}{\varepsilon}\right)^N \qquad \forall r \in [1,\infty]. \tag{3.9}$$

It remains to recall that $\dim_F(\mathcal{A}, L^r)$, the fractal dimension of $\mathcal{A}$, can be expressed in terms of entropy via $\dim_F(\mathcal{A}, L^r) := \limsup_{\varepsilon\to 0} \frac{\mathcal{H}_\varepsilon(\mathcal{A}, L^r)}{\log_2 \frac{1}{\varepsilon}}$. Letting $\varepsilon \to 0$ in (20), we find

$$\dim_F(\mathcal{A}, L^r) = \infty \qquad \forall r \in [1, \infty].$$

Remark 4. By using a comparison theorem similar to Lemma 3, we can show the infinite dimensionality of the attractor for more general quasilinear parabolic equations $\frac{\partial u}{\partial t} = \mathrm{div}(a(x, u, \nabla u)) - g(x, u(x, t)) + h(x)$ with $\partial g(x, 0)/\partial u < 0$, which will be discussed in our forthcoming papers.

References

[1] D.G. Aronson, M.G. Crandall, and L.A. Peletier, *Stabilization of solutions of a degenerate nonlinear diffusion problem,* Nonlinear Anal., 6 (1982), 1001–1022

[2] H. Brézis, “Opérateurs Maximaux Monotone et Semi-Groupes de Contractions dans les Espaces Hilbert,” North-Holland Math. Studies 5, 1973.

[3] E. DiBenedetto, “Degenerate Parabolic Equations,” Universitext, Springer-Verlag, New York, 1993.

[4] M.A. Efendiev and S. Zelik, *Upper and lower bounds for the Kolmogorov entropy of the attractor for reaction-diffusion equation in an unbounded domain,* J. Dynamics and Diff. Eqns., 14 (2002), 369–403.

[5] M.A. Efendiev and S. Zelik, *Finite and infinite dimensional attractors for porous media equations,* submitted.

[6] E. Feireisl, Ph. Laurencot, and F. Simondon, *Global attractors for degenerate parabolic equations on unbounded domains,* J. Diff. Equ., 129 (1996), 239–261.

[7] A.N. Kolmogorov and V.M. Tikhomirov, *$\mathcal{E}$-entropy and ε-capacity of sets in functional space,* Amer. Math. Soc. Transl., 17 (1961), 277–364.

[8] M. Nakao and N. Aris, *On global attractor for nonlinear parabolic equations of m-Laplacian type,* submitted.

[9] M. Ôtani, *L^∞-energy method and its applications,* in “Nonlinear Partial Differential Equations and Their Applications,” 505–516, GAKUTO Internat. Ser. Math. Sci. Appl., 20, Gakkotosho, Tokyo, 2004.

[10] S. Takeuchi and T. Yokota, *Global attractors for a class of degenerate diffusion equations,* Electronic Journal of Differential Equations, Vol. 2003 (2003), 1–13.

[11] R. Temam, “Infinite-dimensional Dynamical Systems in Mechanics and Physics,” Springer-Verlag, New York, 1988.