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POSITIVE SOLUTIONS FOR A CLASS OF INFINITE SEMIPOSITONE PROBLEMS

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Abstract. We analyze the positive solutions to the singular boundary value problem

$$-\Delta u = \lambda[f(u) - 1/u^{\alpha}]; x \in \Omega$$
$$u = 0;\ x \in \partial\Omega,$$

where f is a C^2 function in $(0, \infty)$, $f(0) \geq 0, f^{'} > 0, \lim_{s\to\infty} \frac{f(s)}{s} = 0, \lambda$ is a positive parameter, $\alpha \in (0,1)$ and Ω is a bounded region in $R^n, n \geq 1$ with $C^{2+\gamma}$ boundary for some $\gamma \in (0,1)$. In the case $n = 1$ we use the quadrature method and for $n > 1$ we use the method of sub-super solution to establish our results.

1. Introduction

We consider

$$\begin{aligned} -\Delta u &= \lambda[f(u) - 1/u^{\alpha}]; x \in \Omega \\ u &= 0;\ x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\alpha \in (0,1), \lambda$ is a positive parameter, and Ω is a bounded region in $R^n, n \geq 1$ with $C^{2+\gamma}$ boundary for some $\gamma \in (0,1)$. Throughout this paper we assume:

(H) $f \in C^2(0,\infty), f^{'}(s) > 0; s > 0$ and $\lim_{s\to\infty} \frac{f(s)}{s} = 0$.

Let $g(u) := f(u) - 1/u^{\alpha}$. Then $\lim_{u\to 0} g(u) = -\infty$, and hence (1.1) is a singular boundary value problem. In recent years, there is a rich history of research for the case when $g(0) < 0$ but finite (see [1-12, 21]). Such non-singular problems are referred to as semipositone problems. It is well

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known in the literature that the study of positive solutions to such semipositone problems are mathematically very challenging (see [5, 19]). In this paper, we consider the even more challenging semipositone problem when $\lim_{u\to 0} g(u) = -\infty$, which we refer to as an infinite semipositone problem. We will seek positive solutions in $C^2(\Omega) \cap C(\overline{\Omega})$. Our aim in this paper is to establish non existence of positive solution for λ near zero and existence of positive solution for λ large. In the case when $n = 1$ we will also discuss multiplicity and uniqueness results. In particular, in Section 2 we study

$$\begin{aligned} -u'' &= \lambda[f(u) - 1/u^\alpha]; x \in (0,1) \\ u(0) &= 0 = u(1), \end{aligned} \tag{1.2}$$

with $f(0) > 0$ and establish the following result:

Theorem 1.1. *Assume hypothesis (H) holds, $f(0) > 0, f''(s) < 0$, and $f(\theta) \leq \theta f'(\theta) + \frac{1+\alpha}{\theta^\alpha}$, where θ is the positive real zero of $F(s) := \int_0^s [f(z) - 1/z^\alpha]dz$. Then there exists positive constants $\mu_i, i = 1, 2, 3$ such that $\mu_1 < \mu_2 < \mu_3$ and the BVP (1.2) has no positive solution for $\lambda < \mu_1$, has at least one positive solution for $\lambda \geq \mu_1$, has at least two positive solutions for $\mu_1 < \lambda \leq \mu_2$ and a unique positive solution for $\lambda > \mu_3$.*

Next, in Section 3, we study (1.1) in the case when $n \geq 1$ and prove our main Theorems. Namely, we establish the following results:

Theorem 1.2. *Assume hypothesis (H) holds and $f(0) > 0$. Then there exists positive constants μ_1, μ_2 such that $\mu_1 < \mu_2$ and the BVP (1.1) has no positive solution for $\lambda < \mu_1$ and has at least one positive solution for $\lambda > \mu_2$.*

Theorem 1.3. *Assume hypothesis (H) holds, $f(0) = 0$ and $\lim_{s\to 0} \frac{f(s)}{s^\beta} = k > 0\, (0 < \beta \leq 1)$. Then there exists positive constants μ_1, μ_2 such that $\mu_1 < \mu_2$ and the BVP (1.1) has no positive solution for $\lambda < \mu_1$ and has at least one positive solution for $\lambda > \mu_2$.*

Finally, in Section 4, we discuss some examples satisfying our hypothesis.

In [22], the author considers the boundary value problem

$$-\Delta u = \lambda u^p - 1/u^\alpha; x \in \Omega$$

$$u = 0;\ x \in \partial\Omega.$$

Note that in their equation the parameter λ multiplies the term u^p and not the singular term $1/u^\alpha$. See also [16] for some extensions of the results in [22]. Here we extend this study to the more challenging problem (1.1) where we deal with a class of non-linear $f(u)$ rather than the specific u^p non-linear, and the parameter λ also multiplying the singular term $1/u^{\alpha}$. Note that this more difficult case was not treated in [22] and [16].

We prove Theorem 1.1 via the quadrature method developed in [7, 12, 17]. See also [13] where exact multiplicity results for such problems in the case $n = 1$ was recently discussed under more restrictive conditions. We prove our main results (Theorem 1.2 and Theorem 1.3) by the method of sub-super solution discussed in [14]. Here a subsolution is a function $\underline{u} : \overline{\Omega} \to R$ such that $\underline{u} \in C^2(\Omega) \bigcap C(\overline{\Omega})$ and

$$\begin{aligned} -\Delta\underline{u} &\leq \lambda[f(\underline{u}) - 1/\underline{u}^{\alpha}]; x \in \Omega \\ \underline{u} &> 0; x \in \Omega \\ \underline{u} &= 0; x \in \partial\Omega, \end{aligned}$$

and a supersolution is a function $\overline{u} : \overline{\Omega} \to R$ such that $\overline{u} \in C^2(\Omega) \bigcap C(\overline{\Omega})$ and

$$\begin{aligned} -\Delta\overline{u} &\geq \lambda[f(\overline{u}) - 1/\overline{u}^{\alpha}]; x \in \Omega \\ \overline{u} &> 0; x \in \Omega \\ \overline{u} &= 0; x \in \partial\Omega. \end{aligned}$$

Then the following results hold:

Lemma 1.1. (See Lemma 3 in [14]) *For $\lambda > 0$, if there exist a subsolution $\underline{u}_\lambda$ and a supersolution $\overline{u}_\lambda$ of (1.1) such that $\underline{u}_\lambda \leq \overline{u}_\lambda$ on $\overline{\Omega}$, then* (1.1) *has at least one solution $u_\lambda \in C^{2+\gamma}(\Omega) \bigcap C(\overline{\Omega})$ satisfying $\underline{u}_\lambda \leq u_\lambda \leq \overline{u}_\lambda$ on $\overline{\Omega}$.*

Lemma 1.2. (See [18]) *Let ϕ denote an eigenfunction corresponding to the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions, then $\int_{\Omega}(1/\phi)^s dx < +\infty$ if and only if $s < 1$.*

Lemma 1.3. (See [15]) *Let $g \in L^1(\Omega)$, $g \geq 0$ and $f \in L^{\infty}(\Omega)$. Let $u \geq 0$ be a solution of*

$$\begin{aligned} -\Delta u + g = f;\ x \in \Omega \\ u = 0;\ x \in \partial\Omega, \end{aligned}$$

then $u \in H_0^1(\Omega)$.

We prove Theorems 1.2-1.3 by establishing a crucial subsolution of the form $\lambda^r \phi^{\frac{2}{1+\alpha}}$ and applying Lemma 1.1. Hence, our solution u will be in the space $C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$ and satisfies $u \geq \lambda^r \phi^{\frac{2}{1+\alpha}}$. Here $\phi > 0; \Omega$ is an eigenfunction corresponding to the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions. By Lemma 1.2, $1/\phi^{\frac{2\alpha}{1+\alpha}} \in L^1(\Omega)$ since $\frac{2\alpha}{1+\alpha} <$

1 when $\alpha < 1$. Thus, $\frac{\lambda}{u^\alpha} \in L^1(\Omega)$ and by Lemma 1.3 we also have that $u \in H_0^1(\Omega)$. Hence, our classical solution is indeed in a much better space, namely $C^{2+\gamma}(\Omega) \cap C(\overline{\Omega}) \cap H_0^1(\Omega)$. The non existence result when λ small follows easily by using a linear upper bound for $g(u)$.

We note that our results for these infinite semipositone problems (with nonlinearities f that are sublinear at ∞) resemble results for the corresponding non-singular semipositone problems (See [7, 9]).

2. Proof Of Theorem 1.1

We first recall some of the results of the Quadrature method introduced by Laetsch in [17] and extended to semipositone problems in [7]. See also [20] where this quadrature method was used to study classes of singular positone problems.

If (1.2) has a positive solution u, then u increases in $[0, 1/2)$ and is symmetric about $x = 1/2$. Multiplying (1.2) by $u'(x)$ and integrating we obtain

$$\frac{[u'(x)]^2}{2} = \lambda[F(\rho) - F(u(x))], \tag{2.1}$$

where $F(s) = \int_0^s [f(z) - 1/z^\alpha]dz$ and $\rho = u(1/2) = \| u \|_\infty$. Further, integrating (2.1), $u(x)$ is determined by the equation

$$\int_0^{u(x)} \frac{1}{\sqrt{F(\rho) - F(s)}} ds = \sqrt{2\lambda} x; \quad 0 \le x \le 1/2. \tag{2.2}$$

Setting $x = 1/2, \lambda$ and ρ must satisfy

$$\sqrt{\lambda} = \sqrt{2} \int_0^\rho \frac{1}{\sqrt{F(\rho) - F(s)}} ds := G(\rho). \tag{2.3}$$

Note that $G(\rho)$ is well defined for $\rho \in [\theta, \infty)$ where θ is the positive zero of F. This follows from the fact that $F'(\rho) > 0$ for $\rho \in [\theta, \infty)$. Also clearly $G(\rho) > 0$. Further, since $F(\rho) < 0$ for $0 < \rho < \theta$, by (2.1) there are no positive solutions with $\| u \|_\infty = \rho < \theta$. On the other hand if $\sqrt{\lambda} = G(\rho)$ for some $\rho \in [\theta, \infty)$, then (2.2) defines the positive solution of (1.2) with $u(1/2) = \rho$. Therefore, (2.3) describes the bifurcation diagram for positive solutions of (1.2). Further, rewriting $G(\rho)$ as

$$G(\rho) = \sqrt{2}\rho \int_0^1 \frac{1}{\sqrt{F(\rho) - F(\rho v)}} dv,$$

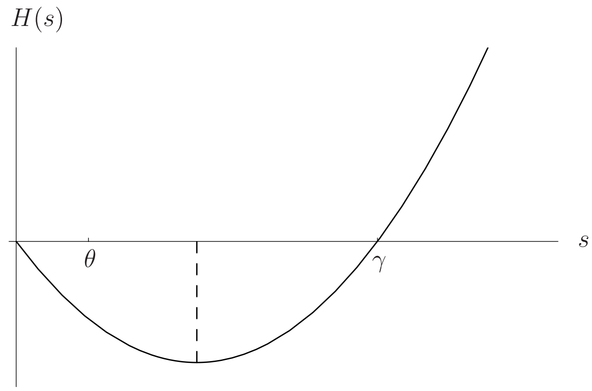

Figure 1

it follows that

$$G^{'}(\rho) = \sqrt{2}\int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{3/2}} dv,$$

where $H(s) = F(s) - \frac{1}{2}s[f(s) - \frac{1}{s^\alpha}]$ (see [7]). Thus, $G^{'}(\rho) > 0$ if $H(\rho) > H(s); 0 \leq s < \rho$ and $G^{'}(\rho) < 0$ if $H(\rho) < H(s); 0 \leq s < \rho$.

In order to prove Theorem 1.1, we first analyze the properties of the function H. In fact, $H'(s) = \frac{1}{2}[f(s) - sf^{'}(s) - \frac{1+\alpha}{s^\alpha}]$ and $H''(s) = -\frac{1}{2}f^{''}(s) + \frac{1}{2}\frac{\alpha(1+\alpha)}{s^{\alpha+1}}$. Hence, $H^{''}(s) > 0; s > 0, \lim_{s\to 0^+} H^{'}(s) = -\infty$ and

$$\begin{aligned}\lim_{s\to\infty} H^{'}(s) &= \lim_{s\to\infty} \frac{s}{2}[\frac{f(s) - f(0)}{s} - f^{'}(s)] + \frac{1}{2}f(0)\\ &\geq \frac{1}{2}f(0) > 0 \text{ (since } f'' < 0).\end{aligned}$$

Also $H^{'}(\theta) = \frac{1}{2}[f(\theta) - \theta f^{'}(\theta) - \frac{1+\alpha}{\theta^\alpha}] \leq 0$. Thus, H has the described shape in Figure 1. Here γ is the positive zero of H. Now Theorem 1.1 follows easily by proving the following (see Figure 2):

$$(a)\ G^{'}(\theta) < 0, \quad (b)\ \lim_{\rho\to\infty} G(\rho) = \infty \text{ and } \ (c)\ G^{'}(\rho) > 0 \text{ if } \rho \text{ is large.}$$

In fact, if (a), (b) and (c) are proven then μ_1 in Theorem 1.1 turn out to be $\mu_1 = (\min\{G(\rho) : \rho \in [\theta, \gamma]\})^2$. Now from the properties of H and the expression for $G^{'}(\rho)$ it is easy to see that $G^{'}(\theta) < 0$ and $G^{'}(\rho) > 0; \rho \geq \gamma$. Hence (a) and (c) are proven. Next note that

$$F(\rho v) = \int_0^{\rho v} [f(s) - \frac{1}{s^\alpha}]ds \geq [f(0) - \frac{1}{1-\alpha}\frac{1}{(\rho v)^\alpha}]\rho v$$

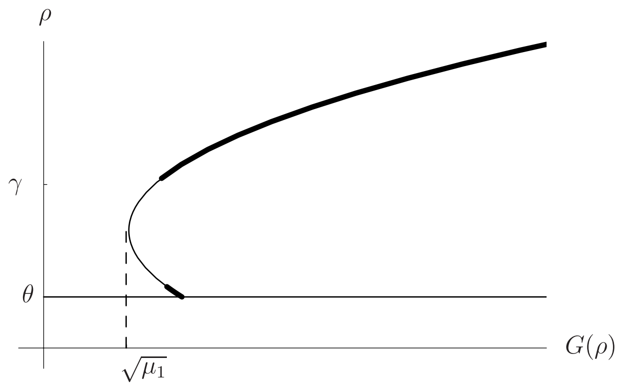

Figure 2

$$\geq \frac{1}{2}f(0)(\rho v) \text{ if } v > [\frac{2}{(1-\alpha)f(0)}]^{1/\alpha}\frac{1}{\rho} = \beta \text{ (say)}.$$

Thus, for ρ large

$$G(\rho) \geq \sqrt{2}\frac{\rho}{\sqrt{F(\rho)}}\int_\beta^1 \frac{dv}{\sqrt{1-\frac{f(0)}{2F(\rho)}\rho v}} \geq 2\sqrt{2}\frac{\sqrt{F(\rho)}}{f(0)}\int_{\frac{f(0)\rho\beta}{2F(\rho)}}^{\frac{f(0)\rho}{2F(\rho)}} \frac{dw}{\sqrt{1-w}}$$

$$= 4\sqrt{2}[\frac{\sqrt{F(\rho)}}{f(0)}][\sqrt{1-\frac{f(0)\rho\beta}{2F(\rho)}} - \sqrt{1-\frac{f(0)\rho}{2F(\rho)}}]$$

$$= 2\sqrt{2}\frac{\rho}{\sqrt{F(\rho)}}(1-\beta)\frac{1}{\sqrt{1-\frac{f(0)\rho\beta}{2F(\rho)}}+\sqrt{1-\frac{f(0)\rho}{2F(\rho)}}}.$$

But $\lim_{\rho\to\infty}\frac{\rho^2}{F(\rho)} = \lim_{\rho\to\infty}\frac{2\rho}{f(\rho)-\frac{1}{\rho^\alpha}} = \infty$. Hence, $\lim_{\rho\to\infty} G(\rho) = \infty$ and (b) and Theorem 1.1 is proven.

3. PROOF OF THEOREM 1.2 AND THEOREM 1.3

3.1. **Proof of Theorem 1.2.** Since $\lim_{s\to\infty}\frac{f(s)}{s} = 0$, there exists constants $a > 0$, $b > 0$ such that $f(s) - \frac{1}{s^\alpha} < as - b$. Let $\lambda_1 > 0$ be the principle eigenvalue and $\phi > 0; \Omega$ be a corresponding eigenfunction of $-\Delta$ with Dirichlet boundary conditions. Suppose $u > 0; \Omega$ is a positive solution of (1.1). Then

$$\int_\Omega (-\Delta u)\phi dx \leq \lambda \int_\Omega (au - b)\phi dx.$$

But

$$\int_\Omega (-\Delta u)\phi dx = \int_\Omega u(-\Delta\phi)dx = \int_\Omega \lambda_1 u\phi dx.$$

Thus

$$\int_\Omega (\lambda_1 - \lambda a)u\phi dx \le \int_\Omega (-\lambda b)\phi dx.$$

This is impossible if $\lambda < \lambda_1/a$ and the first part of Theorem 1.2 is proven.

Next choose the eigenfunction $\phi > 0$ such that $\| \phi \|_\infty = 1$ and let $\psi := \lambda^r \phi^{\frac{2}{1+\alpha}}$ where the parameter $r \in (\frac{1}{1+\alpha}, 1)$. Then

$$\nabla\psi = \lambda^r(\frac{2}{1+\alpha})\phi^{\frac{1-\alpha}{1+\alpha}}\nabla\phi$$

and

$$\Delta\psi = \lambda^r(\frac{2}{1+\alpha})\{\phi^{\frac{1-\alpha}{1+\alpha}}\Delta\phi + \frac{1-\alpha}{1+\alpha}\phi^{-\frac{2\alpha}{1+\alpha}} \mid \nabla\phi \mid^2\}$$
$$= \lambda^r(\frac{2}{1+\alpha})\{-\lambda_1\phi^{\frac{2}{1+\alpha}} + \frac{1-\alpha}{1+\alpha}\frac{\mid \nabla\phi \mid^2}{\phi^{\frac{2\alpha}{1+\alpha}}}\}.$$

Thus,

$$-\Delta\psi = \lambda^r(\frac{2}{1+\alpha})\{\lambda_1\phi^{\frac{2}{1+\alpha}} - \frac{1-\alpha}{1+\alpha}\frac{\mid \nabla\phi \mid^2}{\phi^{\frac{2\alpha}{1+\alpha}}}\}.$$

Let $\delta > 0, \mu > 0, m > 0$ be such that $\mid \nabla\phi \mid^2 \ge m$, in $\overline{\Omega_\delta}$, and $\phi^{\frac{2}{1+\alpha}} \in [\mu, 1]$ in $\Omega - \overline{\Omega_\delta}$ where $\overline{\Omega_\delta} := \{x \in \Omega \mid d(x, \partial\Omega) \le \delta\}$. This is possible since $\mid \nabla\phi \mid \ne 0; \partial\Omega$. Then in $\overline{\Omega_\delta}$ if $\lambda \gg 1$

$$-\lambda^r(\frac{2}{1+\alpha})\frac{1-\alpha}{1+\alpha}\frac{\mid \nabla\phi \mid^2}{\phi^{\frac{2\alpha}{1+\alpha}}} \le \lambda[-\frac{1}{(\lambda^r\phi^{\frac{2}{1+\alpha}})^\alpha}]$$

since $1 - r - r\alpha < 0$. Also in $\overline{\Omega_\delta}$ (in fact in Ω),

$$\lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}} \le \lambda f(0) \le \lambda f(\lambda^r\phi^{\frac{2}{1+\alpha}}) \text{ if } \lambda \gg 1.$$

Hence, in $\overline{\Omega_\delta}$,

$$-\Delta\psi \le \lambda[f(\lambda^r\phi^{\frac{2}{1+\alpha}}) - \frac{1}{(\lambda^r\phi^{\frac{2}{1+\alpha}})^\alpha}] = \lambda[f(\psi) - \frac{1}{\psi^\alpha}]. \tag{3.1}$$

Next in $\Omega - \overline{\Omega_\delta}$, since $\phi^{\frac{2}{1+\alpha}} \ge \mu, \lambda[f(\psi) - \frac{1}{\psi^\alpha}] \ge \lambda[f(\lambda^r\mu) - \frac{1}{(\lambda^r\mu)^\alpha}]$. But if $\lambda \gg 1, -\Delta\psi \le \lambda^r\frac{2}{(1+\alpha)}\lambda_1 \le \lambda[f(\lambda^r\mu) - \frac{1}{(\lambda^r\mu)^\alpha}]$, since $r < 1$. Hence, if $\lambda \gg 1$, in $\Omega - \overline{\Omega_\delta}$, we have

$$-\Delta\psi \leq \lambda[f(\psi) - \frac{1}{\psi^\alpha}]. \tag{3.2}$$

Combining (3.1) and (3.2) we see that $\psi = \lambda^r \phi^{\frac{2}{1+\alpha}}$ is a positive subsolution of (1.1).

Now we construct a supersolution $Z \geq \psi$. Since $\lim_{s\to\infty} \frac{f(s)}{s} = 0, \forall\lambda > 0, \exists m(\lambda) > 0$ such that $m(\lambda) \geq \lambda f(m(\lambda)\|e\|_\infty)$ where e is the unique positive solution of $-\Delta e = 1; x \in \Omega,\ \ e = 0; x \in \partial\Omega$.

Let $Z := m(\lambda)e$. Then

$$-\Delta Z = m(\lambda) \geq \lambda f(m(\lambda) \parallel e \parallel_\infty) \geq \lambda f(m(\lambda)e) = \lambda f(Z)$$

Thus Z is a supersolution. Further, $m(\lambda)$ can be chosen large enough so that $Z = m(\lambda)e \geq \psi$ in $\overline{\Omega}$. Hence for $\lambda \gg 1$, (1.1) has a positive solution $u \in [\psi, Z]$ and the second part of Theorem 1.2 is proven.

3.2. **Proof of Theorem 1.3.** Let $\psi = \lambda^r \phi^{\frac{2}{1+\alpha}}$ as earlier. In the proof of Theorem 1.2, when proving ψ is a subsolution of (1.1) for $\lambda \gg 1$, we used the fact that $f(0) > 0$ to show $\lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}} \leq \lambda f(\lambda^r\phi^{\frac{2}{1+\alpha}})$ in $\overline{\Omega_\delta}$. Here with $f(0) = 0$ we establish the above inequality by using $\lim_{s\to 0} \frac{f(s)}{s^\beta} = k > 0$. (The rest of the proof of Theorem 1.3 is exactly as the proof of Theorem 1.2.) Let $A > 0$ be such that $f(x) \geq \frac{k}{2}x^\beta$ for $x \in [0, A]$. Choose $\lambda^* \gg 1$ such that

$$\lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}} \leq \lambda\{\frac{k}{2}\lambda^{r\beta}\phi^{\frac{2\beta}{1+\alpha}}\} \text{ holds } \forall\lambda \geq \lambda^*.$$

Hence, if $(\lambda^*)^r\phi^{\frac{2}{1+\alpha}} \leq A$, then

$$f((\lambda^*)^r\phi^{\frac{2}{1+\alpha}}) \geq (\lambda^*)^{r-1}(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}}.$$

Further, for $\lambda \geq \lambda^*$ we have

$$\begin{aligned} f(\lambda^r\phi^{\frac{2}{1+\alpha}}) \geq f((\lambda^*)^r\phi^{\frac{2}{1+\alpha}}) &\geq (\lambda^*)^{r-1}(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}} \\ &\geq \lambda^{r-1}(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}}. \end{aligned}$$

Multiplying by λ we get

$$\lambda f(\psi) \geq \lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}}.$$

Next if $(\lambda^*)^r\phi^{\frac{2}{1+\alpha}} > A$, then $\forall\lambda(\geq\lambda^*)$ and sufficiently large we have

$$\lambda f(\psi) \geq \lambda f(A) \geq \lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}}.$$

Hence, here for $\lambda \gg 1$

$$\lambda f(\psi) \geq \lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}}$$

must hold in $\overline{\Omega_\delta}$ (in fact throughout Ω). Thus Theorem 1.3 is proven.

4. Some examples

In this section we discuss the following examples:

(A) $f(u) = (u+1)^{\frac{1}{2}}$,
(B) $f(u) = e^{\frac{u}{u+1}}$,
(C) $f(u) = u^{\frac{1}{2}}$.

4.1. **Example (A):** $f(u) = (u+1)^{\frac{1}{2}}$**.** Here $f(0) > 0, f^{'}(u) = \frac{1}{2\sqrt{u+1}} > 0$ and $\lim_{s\to\infty}\frac{f(s)}{s} = 0$. Hence, Theorem 1.2 holds $\forall\alpha\in(0,1)$. Next for $\alpha = \frac{1}{8}$ we estimate the root θ for $F(\theta) = 0$. Here,

$$F(s) = \int_0^s(\sqrt{u+1} - \frac{1}{u^{0.125}})du = \frac{2}{3}(s+1)^{\frac{3}{2}} - \frac{2}{3} - \frac{8}{7}s^{\frac{7}{8}},$$

and $F(1) = \frac{4}{3}\sqrt{2} - \frac{2}{3} - \frac{8}{7} > 0$. But $F^{''} = f^{'} + \frac{\alpha}{s^{\alpha+1}} > 0$. Hence, $\theta < 1$. Let $T(s) = f(s) - sf^{'}(s) - \frac{1+0.125}{s^{0.125}}$. Then $T^{'}(s) = -sf^{''}(s) + \frac{0.140625}{s^{1.125}} > 0$, and $T(1) = \sqrt{2} - \frac{1}{2\sqrt{2}} - 1.125 < 0$. So $T(\theta) < 0$. Thus, $f(\theta) < \theta f^{'}(\theta) + \frac{1+\alpha}{\theta^\alpha}$. Also $f^{''}(u) = -\frac{1}{4(u+1)^{3/2}} < 0$. Therefore, this example satisfies all the hypotheses in Theorem 1.1 when $\alpha = \frac{1}{8}$.

4.2. **Example (B):**$f(u) = e^{\frac{u}{u+1}}$**.** Here, $f(0) > 0, f^{'}(u) = e^{\frac{u}{u+1}}\frac{1}{(u+1)^2} > 0$ and $\lim_{s\to\infty}\frac{f(s)}{s} = 0$. Hence, Theorem 1.2 holds $\forall\alpha\in(0,1)$. Next for $\alpha = 0.3$ we estimate the root θ for $F(\theta) = 0$. Here,

$$F(s) = \int_0^s(e^{\frac{u}{u+1}} - \frac{1}{u^{0.3}})du$$
$$\geq \int_0^s[1 + \frac{u}{u+1} + \frac{1}{2}\frac{u^2}{(u+1)^2} + \frac{1}{6}\frac{u^3}{(u+1)^3}]du - \frac{10}{7}s^{0.7}$$
$$= \int_0^s[2 - \frac{1}{u+1} + \frac{1}{2} - \frac{1}{2}\frac{2u+2}{u^2+2u+1} + \frac{1}{2}\frac{1}{(u+1)^2} + \frac{1}{6}\frac{u^3}{(u+1)^3}]du - \frac{10}{7}s^{0.7}$$

$$= 2.5s - 2ln(s+1) + \frac{1}{2} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\int_0^s \frac{u^3}{(u+1)^3}du - \frac{10}{7}s^{0.7}.$$

Hence,

$$F(1.09) > 2.725 - 1.4744 + 0.5 - 0.2393 + \frac{1}{6}\frac{1}{2.09^3}\frac{1}{4}1.09^4 - \frac{10}{7}1.09^{0.7}$$
$$> 1.5113 + 0.0064 - 1.5175 = 0.0002 > 0.$$

So $\theta < 1.09$. Let $T(s) = f(s) - sf^{'}(s) - \frac{1+0.3}{s^{0.3}}$, then $T^{'}(s) = -sf^{''}(s) + \frac{0.39}{s^{1.3}} > 0$, and $T(1.09) = e^{\frac{1.09}{2.09}} - 1.09e^{\frac{1.09}{2.09}}(\frac{1}{2.09})^2 - \frac{1.3}{1.09^{0.3}} < 1.6864 - 0.4204 - 1.2669 = -0.0027 < 0$. So $T(\theta) < 0$. Thus $f(\theta) < \theta f^{'}(\theta) + \frac{1+\alpha}{\theta^\alpha}$. Also $f^{''}(u) = e^{\frac{u}{u+1}}(\frac{1}{(u+1)^4} - \frac{2}{(u+1)^3}) = e^{\frac{u}{u+1}}\frac{1}{(u+1)^4}(-2u-1) < 0$, for $u > 0$. Therefore this example satisfies all hypotheses in Theorem 1.1 when $\alpha = 0.3$.

4.3. **Example (C):** $f(u) = u^{\frac{1}{2}}$. Here, $f(0) = 0, f^{'}(u) = \frac{1}{2\sqrt{u}} > 0$ and $\lim_{s\to\infty}\frac{f(s)}{s} = 0$. Also since $\frac{f(s)}{s^{1/2}} = 1$, choose $\beta = 1/2$ all the hypotheses of Theorem 1.3 are satisfied $\forall\alpha \in (0,1)$.

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