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A FORMULA FOR PRINCIPAL EIGENVALUES OF DIRICHLET PERIODIC PARABOLIC PROBLEMS WITH INDEFINITE WEIGHT

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and let m be a T-periodic function such that $m_{|\Omega\times(0,T)} \in L^r(\Omega \times (0,T))$ for some $r > N+2$ and $\int_0^T \operatorname{esssup}_{x\in\Omega} m(x,t)\,dt > 0$. Let $\lambda_1(m)$ be the (unique) positive principal eigenvalue of the Dirichlet periodic parabolic problem $Lu = \lambda mu$ in $\Omega \times \mathbb{R}$, $u = 0$ on $\partial\Omega \times \mathbb{R}$, $u > 0$ in $\Omega \times \mathbb{R}$. We prove a formula for $\lambda_1(m)$ which is an analogous of the well known variational expression for principal eigenvalues of self-adjoint elliptic problems. As a direct consequence we obtain monotonicity results for $\lambda_1(m)$ with respect to the domain Ω and with respect to the zero order coefficient of the differential operator L.

1. Introduction

Let Ω be a $C^{2+\theta}$ bounded domain in $\mathbb{R}^N$, $\theta \in (0,1)$, $N \geq 2$. For $T > 0$ and $1 \leq p \leq \infty$, let L_T^p be the Banach space of T-periodic functions f on $\Omega \times \mathbb{R}$ (i.e., satisfying $f(x,t) = f(x,t+T)$ a.e. $(x,t) \in \Omega\times\mathbb{R}$) such that $f_{|\Omega\times(0,T)} \in L^p(\Omega \times (0,T))$, equipped with the norm $\|f\|_{L_T^p} := \left\|f_{|\Omega\times(0,T)}\right\|_{L^p(\Omega\times(0,T))}$.

Let $\{a_{ij}\}$, $\{b_j\}$, $1 \leq i,j \leq N$, be two families of T-periodic and Lipschitz continuous functions on $\overline{\Omega} \times \mathbb{R}$ such that $a_{ij} = a_{ji}$ and

$$\sum a_{ij}(x,t)\,\xi_i\xi_j \geq \alpha\,|\xi|^2,$$

for some $\alpha > 0$ and all $(x,t) \in \Omega \times \mathbb{R}$, $\xi \in \mathbb{R}^N$. Let A be the $N \times N$ matrix whose i,j entry is a_{ij}, let $\overline{b} = (b_1, ..., b_N)$, let $0 \leq c_0 \in L_T^r$ with $r > N+2$ and let L be the parabolic operator given by

$$Lu = u_t - div\,(A\nabla u) + \left\langle \overline{b}, \nabla u\right\rangle + c_0 u.$$

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For $1 \leq s \leq \infty$, let $W_s^{2,1}(\Omega \times (t_0, t_1))$ be the Sobolev space of the functions $u \in L^s(\Omega \times (t_0, t_1))$, $u = u(x,t)$, $x = (x_1, ..., x_N)$ such that u_t, u_{x_j} and $u_{x_i x_j}$ belong to $L^s(\Omega \times (t_0, t_1))$ for $1 \leq i, j \leq N$, and let $W_{s,T}^{2,1}$ be the space of T-periodic functions such that $u_{|\Omega \times (0,T)} \in W_s^{2,1}(\Omega \times (0,T))$. For $h : \Omega \times \mathbb{R} \to \mathbb{R}$ and $s > 1$ we will say that $u \in W_{s,T}^{2,1}$ is a (strong) solution of the periodic problem

$$\begin{cases} Lu = h & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \ T\text{-periodic} & , \end{cases} \tag{1.1}$$

if the equation holds a.e. in the pointwise sense. It is known that for $h \in L_T^p$ with $1 < p < \infty$ there exists a unique strong solution $u \in W_{2,T}^{p,1}$ of (1.1) and the associated solution operator $L^{-1} : L_T^p \to W_{p,T}^{2,1}$ is continuous (see e.g. [17], Section 4). Moreover, if $h \in L_T^2$ and u is a strong solution in $W_{2,T}^{2,1}$ then

$$\int_{\Omega \times (0,T)} \left[u_t g + \langle A \nabla u, \nabla g \rangle + \left\langle \overline{b}, \nabla u \right\rangle g + c_0 u g \right] = \int_{\Omega \times (0,T)} hg, \tag{1.2}$$

for all $g \in L_T^2$ such that $\nabla g \in L_T^2$.

The Dirichlet periodic parabolic principal eigenvalue problem with a T-periodic weight function $m : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} Lu = \lambda m u & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \ T\text{-periodic} & , \end{cases} \tag{1.3}$$

has been widely studied. For applications we refer to [5], [13]. If m belongs to $C_T^{\theta,\theta/2}\left(\overline{\Omega} \times \mathbb{R}\right)$, Beltramo and Hess found in [1] (for a $C^{2+\theta}$ bounded domain and a parabolic operator with regular coefficients and where the elliptic part is not in divergence form) that $\int_0^T \max_{x \in \overline{\Omega}} m(x,t)\, dt > 0$ is a necessary and sufficient condition on m for the existence of a positive principal eigenvalue $\lambda_1(L, m, \Omega)$ (or $\lambda_1(L, m)$ or simply $\lambda_1(m)$ if no confusion arises) for (1.3) which is unique and simple. Later on there were many extensions of their results, weakening the regularity assumptions in the weight and the domain and also allowing more general boundary conditions (see e.g. [12], [11], [3], [9], [4] [10], [15] and the references therein).

On the other hand, for self-adjoint elliptic operators with Dirichlet boundary condition it is well known that $\lambda_1(m)$ can be expressed as the minimum on $H_0^1(\Omega)$ of the Rayleigh´s quotient, and similar results exist for Robin or natural Neumann boundary conditions. For non-selfadjoijnt elliptic problems, min-max formulas for principal eigenvalues were first established in [6] for $m \equiv 1$ and in [14] for a definite weight (i.e. $m(x) > c > 0$ in Ω). More recently, for sign changing weights, a similar formula was derived in [7] in the case of a Neumann-Robin boundary condition, and in [8] for the Dirichlet problem. To our knowledge, no formula for $\lambda_1(m)$ is known (even if $m \equiv 1$) in the periodic parabolic case.

Our aim in this paper is to prove, adapting the general approach introduced in [14] (and further developed in [7] and [8] for Neumann-Robin and Dirichlet boundary conditions respectively), a similar expression for the Dirichlet periodic parabolic problem. As in [8], the main difficulty in dealing with the Dirichlet condition is that several equations degenerate on $\partial\Omega$. Therefore, a large part of the paper is devoted to the study of these equations, in the context of weighted Sobolev spaces. Let us mention also that the method used to derive the min-max formula in [8] gives an estimate for $\lambda_1(m)$ as proved in Proposition 3.1 in [12]. In this paper we show that a similar estimate to that in [12] is sharp, and the key for this is the solution of a minimization problem. In fact, we shall prove that

$$\lambda_1(m) = \min_{u \in \mathcal{U}} \frac{\int_{\Omega\times(0,T)} \left[\langle A\nabla u, \nabla u\rangle + \left\langle \overline{b}, u\nabla u\right\rangle + u^2 \left(c_0 + \langle A\nabla W_u, \nabla W_u\rangle\right)\right]}{\int_{\Omega\times(0,T)} mu^2},$$

where $\mathcal{U}$ is a suitable set of functions and W_u satisfies a specific degenerate equation (for more precisions see Theorem 3.5 and Lemma 2.5 below).

As a direct consequence we shall obtain monotonicity results for $\lambda_1(m)$ with respect to the domain Ω and with respect to the zero order coefficient of the differential operator L (see Corollary 3.6), i.e., we shall see that $\lambda_1(\Omega_0, m) > \lambda_1(\Omega, m)$ for any smooth bounded domain $\Omega_0 \subset \Omega$, and that $\lambda_1(L + a, m) > \lambda_1(L, m)$ if $a \geq 0$.

2. Notation and preliminaries

Along the whole paper V will denote an open subset of Ω with C^2 boundary. For $x \in V$ let $d_V(x) = dist(x, \partial V)$. We will say that a function $u : V \times \mathbb{R}$ is comparable with d_V if there exist $c_1, c_2 > 0$ such that

$$c_1 d_V(x) \leq u(x,t) \leq c_2 d_V(x), \tag{2.1}$$

for all $(x,t) \in \Omega \times \mathbb{R}$.

C_T will denote the space of continuous and T-periodic functions on $\overline{\Omega} \times \mathbb{R}$ provided with the L^∞ norm and, for $0 < \theta < 1$, we will write $C_T^{1+\theta,(1+\theta)/2}$

for the space of T-periodic functions belonging to $C^{1+\theta,(1+\theta)/2}\left(\overline{\Omega}\times\mathbb{R}\right)$ and $C_{T,\mathcal{B}}^{1+\theta,(1+\theta)/2}$ will denote the space of functions $u \in C_{T}^{1+\theta,(1+\theta)/2}$ satisfying the boundary condition $u = 0$ on $\partial\Omega \times \mathbb{R}$.

For a normed space X we will write $C_T(\mathbb{R}, X)$ for the space of continuous and T-periodic functions $u : \mathbb{R} \to X$ and $L_T^p(\mathbb{R}, X)$ for the space of T-periodic and measurable (in the Bochner's sense) functions $u : \mathbb{R} \to X$ such that $\int_0^T \|u(t)\|_X^p \, dt < \infty$. If X is a space of functions defined on V we will identify a function belonging to $C_T(\mathbb{R}, V)$ or $L_T^p(\mathbb{R}, V)$ with the function, defined on $V \times \mathbb{R}$ and still denoted by u, given by $u(x,t) = u(t)(x)$.

Finally, we will write $L_T^2(V \times \mathbb{R})$ for the space of T periodic functions $u : V \times \mathbb{R} \to \mathbb{R}$ that restricted to $V \times (0,T)$ belong to $L^2(V \times (0,T))$.

We start collecting some necessary facts concerning linear problems with weight and weighted Sobolev spaces.

Remark 2.1. Let $m \in L_T^r$ with $r > N+2$, and consider problem (1.3). Let

$$P(m) := \int_0^T \operatorname{esssup}_{x\in\Omega} m(x,t)\, dt, \tag{2.2}$$

let P be the positive cone in $C_{T,\mathcal{B}}^{1+\theta,(1+\theta)/2}$ and let P° be its interior. It is known (cf. [10]) that $P(m) > 0$ is necessary and sufficient for the existence of a positive principal eigenvalue $\lambda_1(m)$ for (1.3). We note that the case $P(m) = +\infty$ is allowed (cf. [10], p. 218). Moreover, $\lambda_1(m)$ (if exists) is unique and algebraically simple (see [10], Theorem 3.6). Let u^* be a positive eigenfunction of (1.3). Since $L^{-1}\left(L_T^p\right) \subset W_{p,T}^{2,1}$ for $1 < p < \infty$, a bootstrap argument gives that $u^* \in W_{r,T}^{2,1}$, and from the Sobolev imbedding theorems (e.g. [16], Lemma 3.3, p. 80) and the strong maximum principle (e.g. [5], Theorem 13.5) it follows that $u^* \in P^{\circ}$. In particular, $\partial u^*/\partial\nu < 0$ on $\partial\Omega \times \mathbb{R}$ and u^* is comparable with d_Ω. We fix from now on such a u^*, normalized by $\|u^*\|_\infty = 1$. □

Remark 2.2. For k large enough and $(N+2)/2 < s < \infty$, let $T_k : L_T^s \to L_T^s$ be defined by $T_k = (L + \lambda_1(m)(k-m))^{-1}$. Then T_k is a compact, positive and irreducible operator on L_T^s (cf. [10] Proposition 2.4 and Lemma 2.6), where is s' be defined by $\frac{1}{s} + \frac{1}{s'} = 1$. Let T_k^* be the adjoint of T_k, let $\rho(T_k)$ and $\rho(T_k^*)$ be the spectral radius of T_k and T_k^* respectively, and let u^* be as in Remark 2.1. Since $T_k u^* = \frac{1}{k\lambda_1(m)} u^*$, by the Krein-Rutman theorem (as stated e.g. in [5], Theorem 12.3) we have that $\rho(T_k) = \rho(T_k^*) = \frac{1}{k\lambda_1(m)}$ and that there exists a positive (and unique up to a multiplicative constant)

$z^* \in L_T^{s'}$ satisfying $T_k^* z^* = \frac{1}{k\lambda_1(m)} z^*$. Denote by $(L^* + \lambda_1(m)(k-m))^{-1}$ the solution operator of the problem $L^* u + \lambda_1(m)(k-m) u = h$ in $\Omega \times \mathbb{R}$, $u = 0$ on $\partial\Omega \times \mathbb{R}$, u T-periodic, where L^* is the operator given by $L^* u = -u_t - div(A\nabla u) - \langle \overline{b}, \nabla u \rangle + \left(c_0 - div\left(\overline{b}\right)\right) u$ (we note that this problem can be reduced to the form (1.1) by considering $v(x,t) := u(x, T-t)$). A computation shows that $T_k^* = (L^* + \lambda_1(m)(k-m))^{-1}$ and so z^* satisfies

$$\begin{cases} L^* z^* = \lambda_1(m) m z^* & \text{in } \Omega \times \mathbb{R} \\ z^* = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ z^* \ T\text{-periodic} & , \end{cases} \tag{2.3}$$

and a bootstrap argument gives that $z^* \in W_{r,T}^{2,1}$. Moreover, as in the case of u^*, we have that $z^* \in P^\circ$ and so $\partial z^* / \partial \nu$ is negative everywhere on $\partial\Omega \times \mathbb{R}$ and z^* is comparable with d_Ω. From now on z^* will denote such an eigenfunction, normalized by $\|z^*\|_\infty = 1$. □

Remark 2.3. Let $H^1\left(V, d_V^2\right)$ be the space of functions $\varphi \in H_{loc}^1(V)$ such that

$$\|\varphi\|_{H^1\left(V,d_V^2\right)} := \left(\int_V \varphi^2 d_V^2 + \int_V |\nabla\varphi|^2 d_V^2\right)^{1/2} < \infty.$$

We fix an open ball D such that $\overline{D} \subset V$ and we also set

$$H_D^1\left(V, d_V^2\right) := \left\{\varphi \in H^1\left(V, d_V^2\right) : \int_D \varphi = 0\right\}.$$

Then $H^1\left(V, d_V^2\right)$ and its closed subspace $H_D^1\left(V, d_V^2\right)$, equipped with the norm $\|.\|_{H^1\left(V,d_V^2\right)}$ and the induced norm respectively, are Hilbert spaces and there exists some $c_V > 0$ such that

$$\|\varphi\|_{L^2\left(V.d_V^2\right)} \le c_V \|\nabla\varphi\|_{L^2\left(V,d_V^2\right)}, \tag{2.4}$$

for all $\varphi \in H_D^1\left(V, d_V^2\right)$ (cf. [8], Lemma 4.2). Moreover, $H^1\left(V, d_V^2\right) \subset L^2(V)$ with continuous inclusion (cf. [8], Lemma 4.1) and so, modifying c_V if necessary, we also have that

$$\|\varphi\|_{L^2(V)} \le c_V \|\nabla\varphi\|_{L^2\left(V,d_V^2\right)}, \tag{2.5}$$

for all $\varphi \in H_D^1\left(V, d_V^2\right)$. For $\varphi \in H_D^1\left(V, d_V^2\right)$ we set

$$\|\varphi\|_{H_D^1\left(V,d_V^2\right)} := \|\nabla\varphi\|_{L^2\left(V,d_V^2\right)}.$$

From (2.4) it follows that $\|.\|_{H_D^1\left(V,d_V^2\right)}$ and $\|.\|_{H^1\left(V,d_V^2\right)}$ are equivalent norms on $H_D^1\left(V, d_V^2\right)$.

Lemma 2.4. *Let* $u \in L^2_T(V \times \mathbb{R})$ *satisfying*

(H1) u *is comparable with* d_V

(H2) $u/d_V \in C_T(\mathbb{R}, L^\infty(V))$

(H3) $(u^2)_t \in C_T(\mathbb{R}, L^2(V))$

(H4) $\int_V (u^2)_t(x,t)\,dx = 0$ *for all* t.

Then there exists a unique $\widetilde{W}_u \in C_T(\mathbb{R}, H^1_D(V, d_V^2))$ *satisfying*

$$\int_V \left[(u^2)_t(.,t)\,\varphi + u^2(.,t)\left\langle 2A(.,t)\,\nabla\widetilde{W}_u(.,t) - b(.,t), \nabla\varphi\right\rangle\right] = 0. \qquad (2.6)$$

for all $\varphi \in H^1(V, d_V^2)$ *and all* $t \in \mathbb{R}$.

Proof. For each t let $Q^t_u : H^1_D(V, d_V^2) \to \mathbb{R}$ be defined by

$$Q^t_u(\varphi) = \int_V \left[(u^2)_t(.,t)\,\varphi + u^2(.,t)\langle A(.,t)\,\nabla\varphi - b(.,t), \nabla\varphi\rangle\right].$$

Let $c_1, c_2 > 0$ such that $c_1 d_V(x) \le u(x,t) \le c_2 d_V(x)$ for all $(x,t) \in V \times \mathbb{R}$. From (2.5) we have

$$\begin{aligned} Q^t_u(\varphi) \ge{}& c_1^2\alpha\,\|\nabla\varphi\|^2_{L^2(V,d_V^2)} - c_2\,\||b(.,t)|\|_\infty\,\|u(.,t)\|_{L^2(V)}\,\|\nabla\varphi\|_{L^2(V,d_V^2)} \\ &- \left\|(u^2)_t(.,t)\right\|_{L^2(V)}\|\varphi\|_{L^2(V)} \ge c_1^2\alpha\,\|\nabla\varphi\|^2_{L^2(V,d_V^2)} \\ &- \left(c_2\,\||b(.,t)|\|_\infty\,\|u(.,t)\|_{L^2(V)} + c_V\left\|(u^2)_t(.,t)\right\|_{L^2(V)}\right)\|\nabla\varphi\|_{L^2(V,d_V^2)}, \end{aligned}$$

where α is the ellipticity constant of A and so Q^t_u is coercive on $H^1_D(V, d_V^2)$. It is also easy to see that Q^t_u is strictly convex and lower semi-continuous on $H^1_D(V, d_V^2)$. Thus, Q^t_u has a unique minimum $\widetilde{W}_u(.,t) \in H^1_D(V, d_V^2)$ which is T-periodic in t and satisfies (2.6) for all $\varphi \in H^1_D(V, d_V^2)$ and $t \in \mathbb{R}$. The case where $\varphi \in H^1(V, d_V^2)$ is arbitrary reduces to the previous one by considering

$$\widetilde{\varphi} = \varphi - \frac{1}{|D|}\int_D \varphi,$$

and using H4.

Now, we take $\varphi = \widetilde{W}_u(,.t)$ in (2.6) to obtain

$$\begin{aligned} &2\int_V u^2(.,t)\left\langle A(.,t)\,\nabla\widetilde{W}_u(.,t), \nabla\widetilde{W}_u(.,t)\right\rangle \qquad (2.7) \\ &= \int_V \left[-(u^2)_t(.,t)\,\widetilde{W}_u(.,t) + u^2(.,t)\left\langle b(.,t), \nabla\widetilde{W}_u(.,t)\right\rangle\right], \end{aligned}$$

Then

$$2\alpha c_1^2 \left\|\nabla \widetilde{W}_u(.,t)\right\|^2_{L^2\left(V,d_V^2\right)} \leq c_V \left\|\left(u^2\right)_t(,.t)\right\|_{L^2(V)} \left\|\nabla \widetilde{W}_u(.,t)\right\|_{L^2\left(V,d_V^2\right)} + c_2^2 \left\|b(.,t)\right\|_\infty \left\|\nabla \widetilde{W}_u(.,t)\right\|_{L^2\left(V,d_V^2\right)},$$

and so

$$\left\|\nabla \widetilde{W}_u(,.t)\right\|_{L^2\left(V,d_V^2\right)} \leq \gamma(t) := \frac{c_V \left\|\left(u^2\right)_t(.,t)\right\|_{L^2(V)} + c_2^2 \left\|b(.,t)\right\|_\infty}{2\alpha c_1^2}, \quad (2.8)$$

and thus, recalling (2.4) we get

$$\left\|\widetilde{W}_u(.,t)\right\|_{L^2\left(V,d_V^2\right)} \leq c_V \gamma(t) := \eta(t). \quad (2.9)$$

Let $t_1, t_2 \in \mathbb{R}$ and let $\psi := \widetilde{W}_u(.,t_2) - \widetilde{W}_u(.,t_1)$. From (2.6) we have

$$2\int_V u^2(.,t_2)\left\langle A(.,t_2)\nabla \widetilde{W}_u(.,t_2), \nabla\psi\right\rangle = \int_V \left[-\left(u^2\right)_t(.,t_2)\psi + u^2(.,t_2)\, b(.,t_2), \nabla\psi\right],$$

and we also have the analogous equation with t_2 replaced by t_1. Subtracting the second of these equations from the first one we find that

$$2\int_V \left\langle \left(u^2 A\right)(.,t_2)\nabla \widetilde{W}_u(.,t_2) - \left(u^2 A\right)(.,t_1)\nabla \widetilde{W}_u(.,t_1), \nabla\psi\right\rangle = -\int_V \left(\left(u^2\right)_t(.,t_2) - \left(u^2\right)_t(.,t_1)\right)\psi - \left\langle \left(u^2 b\right)(.,t_2) + \left(u^2 b\right)(.,t_1), \nabla\psi\right\rangle,$$

and so

$$\begin{aligned} &2\int_V u^2(.,t)\left\langle A(.,t_2)\nabla\psi, \nabla\psi\right\rangle \\ &= -2\int_V \left\langle \left(\left(u^2 A\right)(.,t_2) - \left(u^2 A\right)(.,t_1)\right)\nabla \widetilde{W}_u(.,t_1), \nabla\psi\right\rangle \\ &- \int_V \left(\left(u^2\right)_t(.,t_2) - \left(u^2\right)_t(.,t_1)\right)\psi - \left\langle \left(u^2 b\right)(.,t_2) - \left(u^2 b\right)(.,t_1), \nabla\psi\right\rangle, \end{aligned}$$

which, taking into account (2.5) and (2.8), gives

$$\begin{aligned} &2\alpha c_1^2 \left\|\nabla\psi\right\|^2_{L^2\left(V,d_V^2\right)} \\ &\leq \left[2\gamma(t_1)\,\varepsilon_1(t_1,t_2) + \varepsilon_2(t_1,t_2)\, c_V + \varepsilon_3(t_1,t_2)\right] \left\|\nabla\psi\right\|_{L^2\left(V,d_V^2\right)}, \end{aligned} \quad (2.10)$$

with $\gamma(t)$ given by (2.8) and

$$\varepsilon_1(t_1,t_2) := \left\| u^2(.,t_2)\, A(.,t_2)/d_V^2 - u^2(.,t_1)\, A(.,t_1)/d_V^2 \right\|_{L^\infty(V,M_n(\mathbb{R}))},$$

$$\varepsilon_2(t_1,t_2) := \left\| \left(u^2\right)_t(.,t_2) - \left(u^2\right)_t(.,t_1) \right\|_{L^2(V)},$$

$$\varepsilon_3(t_1,t_2) := \left\| u^2(.,t_2)\, b(.,t_2)/d_V^2 - u^2(.,t_1)\, b(.,t_1)/d_V^2 \right\|_{L^2(V)}.$$

Since $\lim_{t_2\to t_1} \varepsilon_i(t_1,t_2) = 0$ for $i = 1,2,3$ and $\psi = \widetilde{W}_u(.,t_2) - \widetilde{W}_u(.,t_1)$, (2.10) gives that $\lim_{t_2\to t_1} \nabla\widetilde{W}_u(.,t_2) = \nabla\widetilde{W}_u(.,t_1)$ in $L^2\left(V, d_V^2\right)$ and so, by (2.4), we also have that $\lim_{t_2\to t_1} \widetilde{W}_u(.,t_2) = \widetilde{W}_u(.,t_1)$ in the same space. Thus, $\widetilde{W}_u \in C_T\left(\mathbb{R}, H_D^1\left(V, d_V^2\right)\right)$ and the lemma follows. □

Lemma 2.5. *Let $u \in L_T^2(V\times\mathbb{R})$ satisfying*

(H1') *u is comparable with d_V*

(H2') $u/d_V \in C_T\left(\overline{V}\times\mathbb{R}\right)$

(H3') $u_t \in L_T^2\left(\overline{V}\times\mathbb{R}\right)$

(H4') $\int_V \left(u^2\right)_t(x,t)\,dx = 0$ *for all t.*

Then there exists a unique $W_u \in L_T^2\left(\mathbb{R}, H_D^1\left(V, d_V^2\right)\right)$ satisfying

$$\int_{V\times\mathbb{R}} \left[\left(u^2\right)_t g + u^2\left\langle 2A\nabla W_u - \overline{b}, \nabla g\right\rangle\right] = 0, \tag{2.11}$$

for all $g \in L_T^2\left(\mathbb{R}, H^1\left(V, d_V^2\right)\right)$.

Proof. We pick an even and non-negative function $h \in C^\infty(\mathbb{R})$ satisfying that $\operatorname{supp}(h) \subset (-T/2, T/2)$, $\int_{-T/2}^{T/2} h = 1$ and $h'(s) \le 0$ for $s \in [0,\infty)$. For $j \in \mathbb{N}$ and $s \in \mathbb{R}$ let $h_j(s) := jh(sj)$ and for $(x,t) \in V\times\mathbb{R}$ let

$$u_j(x,t) := \left(\int_{-T/2}^{T/2} u^2(x,t-s)\, h_j(s)\,ds\right)^{1/2}.$$

Clearly, u_j satisfies the assumptions H1-H4 of Lemma 2.4. We observe that

$$\sup_{j\in\mathbb{N}} \left\| \left(u_j^2\right)_t \right\|_{L^2(V\times(0,T))} < \infty. \tag{2.12}$$

Indeed, let c_1, c_2 satisfying (2.1). We have

$$\left\| \left(u_j^2\right)_t \right\|^2_{L^2(V\times(0,T))} = \int_{V\times(0,T)} \left(\int_{-T/2}^{T/2} \left(u^2\right)_t(x,t-s)\, h_j(s)\,ds\right)^2 dxdt$$

$$\le 4c_2^2 d(V)^2 \int_{V\times(0,T)} \left(\int_{-T/2}^{T/2} u_t(x,t-s)\, h_j(s)\,ds\right)^2 dxdt,$$

where $d(V)$ denotes the diameter of V. Since

$$\int_{-T/2}^{T/2} h_j = 1,$$

Jensen's inequality gives that

$$\left(\int_{-T/2}^{T/2} u_t(x,t-s)\,h_j(s)\,ds\right)^2 \leq \int_{-T/2}^{T/2} u_t^2(x,t-s)\,h_j(s)\,ds,$$

and so

$$\begin{aligned}\left\|\left(u_j^2\right)_t\right\|^2_{L^2(V\times(0,T))} &\leq 4c_2^2 d(V)^2 \int_{V\times(0,T)}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} u_t(x,t-s)^2\,h_j(s)\,ds\right)dxdt\\ &= 4c_2^2 d(V)^2\,\|u_t\|^2_{L^2(V\times(0,T))}.\end{aligned}$$

Now, let $\widetilde{W}_{u_j}$ be given by Lemma 2.4. From (2.8), (2.9) and (2.12), $\widetilde{W}_{u_j}$ and $A\nabla\widetilde{W}_{u_j}$ are bounded in $L_T^2\left(V\times\mathbb{R}, d_V^2\right)$. Thus, passing to a subsequence if necessary, we can assume that $\widetilde{W}_{u_j}$ converges weakly in $L_T^2\left(V\times\mathbb{R}, d_V^2\right)$ to some W_u and that each coordinate of $A\nabla\widetilde{W}_{u_j}$ converges weakly to the corresponding coordinate of $A\nabla W_u$ in the same space. Clearly, $W_u \in L_T^2\left(\mathbb{R}, H^1\left(V, d_V^2\right)\right)$, and since $\int_{D\times(0,T)}\widetilde{W}_{u_j} = 0$, we have that

$$W_u \in L_T^2\left(\mathbb{R}, H_D^1\left(V, d_V^2\right)\right).$$

We note that if $X = (X_1, ..., X_n)$ with $X_i \in L_T^2\left(V\times\mathbb{R}, d_V^2\right)$ for $i = 1, ..., n$, then

$$\lim_{j\to\infty}\int_{V\times(0,T)} u_j^2\left\langle A\nabla\widetilde{W}_{u_j}, X\right\rangle = \int_{V\times(0,T)} u^2\left\langle A\nabla W_u, X\right\rangle. \tag{2.13}$$

Indeed,

$$\int_{V\times(0,T)}\left\langle u_j^2 A\nabla\widetilde{W}_{u_j} - u^2 A\nabla W_u, X\right\rangle \tag{2.14}$$

$$= \int_{V\times(0,T)}\left\langle \frac{u_j^2-u^2}{d_V^2}A\nabla\widetilde{W}_{u_j}, X\right\rangle d_V^2 + \left\langle\left(A\nabla\widetilde{W}_{u_j} - A\nabla W_u\right), \frac{u^2}{d_V^2}X\right\rangle d_V^2.$$

Taking into account (2.12), and since

$$\lim_{j\to\infty}\left\|\left(u_j^2-u^2\right)/d_V^2\right\|_{L^\infty(V\times\mathbb{R})} = 0,$$

the first summand in the right member of (2.14) converges to 0 as j tends to ∞ and, since $X_i\left(u^2/d_V^2\right) \in L_T^2\left(V \times \mathbb{R}, d_V^2\right)$ for each i, the same is true for the second one. Thus (2.13) holds.

Now, for $g \in L_T^2\left(\mathbb{R}, H^1\left(V, d_V^2\right)\right)$,

$$\int_V \left[\left(u_j^2\right)_t(.,t)\, g(.,t) + u_j^2(.,t)\left\langle 2A(.,t)\nabla\widetilde{W}_{u_j}(.,t) - b(.,t), \nabla g(.,t)\right\rangle\right] = 0$$

$a.e.$ $t \in \mathbb{R}$ and so

$$\int_{V\times(0,T)} \left[\left(u_j^2\right)_t g + u_j^2\left\langle 2A\nabla W_{u_j} - \bar{b}, \nabla g\right\rangle\right] = 0,$$

which implies (2.11). Finally, if $Z_u \in L_T^2\left(\mathbb{R}, H_D^1\left(V, d_V^2\right)\right)$ is another solution of (2.11), then

$$\int_{V\times(0,T)} u^2\left\langle 2A\nabla\left(Z_u - W_u\right) - \bar{b}, \nabla g\right\rangle = 0,$$

for all $g \in L_T^2\left(\mathbb{R}, H^1\left(V, d_V^2\right)\right)$. Taking $g = Z_u - W_u$ we get $Z_u = W_u$ and this ends the proof. □

Remark 2.6. If $u \in L_T^2\left(\mathbb{R}, H^1(V)\right)$ is comparable with d_V then $\log u \in L_T^2\left(\mathbb{R}, H^1\left(V, d_V^2\right)\right)$. Indeed, we have $c_1 d_V \le u \le c_2 d_V$ for some $c_1, c_2 > 0$. Thus,

$$\left|\log u(.,t)\right| \le \left|\log c_2\right| + \left|\log d_V\right|,$$

in the set where $u(.,t) > 1$ and

$$\left|\log u(.,t)\right| \le \left|\log c_1\right| + \left|\log d_V\right|,$$

if $u \le 1$. So, $\log u \in L_T^2\left(V \times \mathbb{R}, d_V^2\right)$ and, since

$$d_V\left|\nabla \log u\right| = d_V\left|\nabla u\right| / \left|u\right| \le c_2\left|\nabla u\right| \in L_T^2\left(V \times \mathbb{R}\right),$$

we have also that $\nabla \log u \in L_T^2\left(V \times \mathbb{R}, d_V^2\right)$. □

3. A formula for the principal eigenvalue

We assume for the rest of the paper that $m \in L_T^r$ for some $r > N + 2$ and that $P(m) > 0$, where $P(m)$ is given by (2.2). For $u \in L_T^2\left(\mathbb{R}, H^1(V)\right)$ we set

$$\Lambda(u) := \int_{V\times(0,T)} \left[\left\langle A\nabla u, \nabla u\right\rangle + \left\langle \bar{b}, u\nabla u\right\rangle + c_0 u^2\right]. \tag{3.1}$$

Proposition 3.1. *Let $u \in L^2_T\left(\mathbb{R}, H^1(V)\right)$ satisfying the conditions H1'-H4' of Lemma* 2.5. *Then*

$$\lambda_1(m)\int_{V\times(0,T)} mu^2 \leq \Lambda(u) + \int_{V\times(0,T)} u^2 \langle A\nabla W_u, \nabla W_u\rangle .$$

Proof. Let $v^* := -\log u^*$. Thus v^* is T -periodic and $v^* \in W^{2,1}_{r,loc}(\Omega\times\mathbb{R})$. Moreover, from (1.3) we have that

$$-v^*_t + div\left(A\nabla v^*\right) - \left\langle \overline{b}, \nabla v^*\right\rangle - \langle A\nabla v^*, \nabla v^*\rangle + c_0 = \lambda_1(m)\, m \quad \text{in } \Omega\times\mathbb{R}, \quad (3.2)$$

and hence, multiplying this equation by u^2 we obtain

$$\lambda_1(m)\int_{V\times(0,T)} mu^2$$
$$= \int_{V\times(0,T)} \left[-v^*_t + div\left(A\nabla v^*\right) - \left\langle \overline{b}, \nabla v^*\right\rangle - \langle A\nabla v^*, \nabla v^*\rangle + c_0\right] u^2.$$

Let us note that since $v^*_t = -u^*_t/u^*$ and $\nabla v^* = -\left(\nabla u^*\right)/u^*$, the assumptions on u and the properties of u^* stated in Remark 2.1 imply the existence of the above integrals.

Next, we note that

$$\int_{V\times(0,T)} u^2 div\left(A\nabla v^*\right) = -\int_{V\times(0,T)} \left\langle A\nabla v^*, \nabla\left(u^2\right)\right\rangle . \quad (3.3)$$

Indeed, let $X = (X_1, ..., X_n) : \overline{V}\times\mathbb{R} \to \mathbb{R}^n$ be the vector field defined by $X(x,t) = \left(u^2 A\nabla v^*\right)(x,t)$ for $(x,t)\in V\times\mathbb{R}$, and by $X(x,t) = 0$ for $x \in \partial V\times\mathbb{R}$ and observe that for each t, $X(.,t) \in C^1\left(\overline{V}\right)$ and its distributional derivatives $\partial X_i/\partial x_j$ belong to $L^1(V)$. Thus the divergence theorem (as stated in [2]) gives

$$\int_V div\left(u^2 A\nabla v^*\right)(.,t) = 0,$$

and so, taking into account that

$$u^2 div\left(A\nabla v^*\right) = div\left(u^2 A\nabla v^*\right) - \left\langle A\nabla v^*, \nabla\left(u^2\right)\right\rangle ,$$

we get (3.3). Observe also that, since u and v^* are T-periodic,

$$\int_{V\times(0,T)} \left(v^*\right)_t u^2 = -\int_{V\times(0,T)} v^*\left(u^2\right)_t . \quad (3.4)$$

Define now $\omega_u := -b + 2A\left(\nabla u/u + \nabla W_u\right)$, where W_u is the solution of (2.11) given by Lemma 2.5. For $\xi \in \mathbb{R}^N$ we have

$$-\left\langle A\xi, \xi\right\rangle - \left\langle \overline{b}, \xi\right\rangle = \min_{\eta\in\mathbb{R}^N}\Big(\left\langle \xi, \eta\right\rangle + \frac{1}{4}\left\langle \overline{b}+\eta, A^{-1}\left(\overline{b}+\eta\right)\right\rangle\Big). \tag{3.5}$$

Thus,

$$-u^2\left\langle A\nabla v^*, \nabla v^*\right\rangle - u^2\left\langle \overline{b}, \nabla v^*\right\rangle \leq u^2\left\langle \nabla v^*, \omega_u\right\rangle + \frac{u^2}{4}\left\langle \overline{b}+\omega_u, A^{-1}\left(\overline{b}+\omega_u\right)\right\rangle, \tag{3.6}$$

in $V\times\mathbb{R}$. From this inequality and using (3.2), (3.3), (3.4) and (3.6), we obtain

$$\begin{aligned}\lambda_1(m)\int_{V\times(0,T)} mu^2 \leq \int_{V\times(0,T)}\Big[&\left(u^2\right)_t v^* + \left\langle u^2\omega_u - A\nabla\left(u^2\right), \nabla v^*\right\rangle \\ &+\frac{u^2}{4}\left\langle \overline{b}+\omega_u, A^{-1}\left(\overline{b}+\omega_u\right)\right\rangle + c_0 u^2\Big].\end{aligned} \tag{3.7}$$

Now, $d_V\left|v^*\right|$ and $d_V\left|\nabla v^*\right|$ are bounded and so $v^* \in L^2_T\left(\mathbb{R}, H^1\left(V, d_V^2\right)\right)$. Therefore, from the definition of ω_u and Lemma 2.5,

$$\begin{aligned}\int_{\Omega\times(0,T)}\left\langle u^2\omega_u - A\nabla\left(u^2\right), \nabla v^*\right\rangle &= \int_{\Omega\times(0,T)}\left\langle u^2\left(-\overline{b}+2A\nabla W_u\right), \nabla v^*\right\rangle \\ &= -\int_{\Omega\times(0,T)}\left(u^2\right)_t v^*,\end{aligned}$$

which combined with (3.7) gives

$$\lambda_1(m)\int_{V\times(0,T)} mu^2 \leq \int_{V\times(0,T)}\frac{u^2}{4}\left\langle \overline{b}+\omega_u, A^{-1}\left(\overline{b}+\omega_u\right)\right\rangle + c_0u^2. \tag{3.8}$$

Also,

$$\begin{aligned}&\frac{1}{4}\int_{V\times(0,T)} u^2\left\langle \overline{b}+\omega_u, A^{-1}\left(\overline{b}+\omega_u\right)\right\rangle \\ &= \int_{V\times(0,T)} u^2\left\langle A\left(\nabla u/u + \nabla W_u\right), \nabla u/u + \nabla W_u\right\rangle \\ &= \int_{V\times(0,T)}\left[\left\langle A\nabla u, \nabla u\right\rangle + u^2\left\langle 2A\nabla W_u, \nabla \log u\right\rangle + u^2\left\langle A\nabla W_u, \nabla W_u\right\rangle\right],\end{aligned}$$

and since $\log u \in L_T^2\left(\mathbb{R}, H^1\left(V, d_V^2\right)\right)$ by Remark 2.6, recalling Lemma 2.5 we get

$$\int_{V\times(0,T)} u^2 \langle 2A\nabla W_u, \nabla \log u\rangle = \int_{V\times(0,T)} \left[u^2 \langle b, \nabla \log u\rangle - \left(u^2\right)_t \log u\right]$$
$$= \int_{V\times(0,T)} \left[u \left\langle \overline{b}, \nabla u\right\rangle + uu_t\right] = \int_{V\times(0,T)} u \left\langle \overline{b}, \nabla u\right\rangle,$$

the last equality because u is T-periodic. Thus

$$\frac{1}{4}\int_{\Omega\times(0,T)} u^2 \left\langle \overline{b} + \omega_u, A^{-1}\left(\overline{b} + \omega_u\right)\right\rangle$$
$$= \int_{\Omega\times(0,T)} u^2 \left\langle A\left(\nabla u/u + \nabla W_u\right), \nabla u/u + \nabla W_u\right\rangle$$
$$= \int_{\Omega\times(0,T)} \langle A\nabla u, \nabla u\rangle + u\left\langle \overline{b}, \nabla u\right\rangle + u^2 \langle A\nabla W_u, \nabla W_u\rangle,$$

which combined with (3.8) gives the proposition. □

Lemma 3.2. *Let $\widetilde{u} = (u^* z^*)^{1/2}$. Then $\widetilde{u}$ satisfies the hypothesis H1'-H4' of Lemma* 2.5 *and $\nabla\widetilde{u} \in C_T$.*

Proof. u^* and z^* belong to $C_T^{1+\gamma,\gamma/2}$ for some $\gamma \in (0,1)$ and so $\widetilde{u} \in C_T\left(\mathbb{R}, L^2(\Omega)\right)$. Also, $u^*, z^* > 0$ in $\Omega\times\mathbb{R}$, $u^* = z^* = 0$ on $\partial\Omega\times\mathbb{R}$ and $\partial u^*/\partial\nu, \partial z^*/\partial\nu < 0$ on $\partial\Omega\times\mathbb{R}$. Moreover,

$$\lim_{(x,t)\to\left(\overline{x},\overline{t}\right)} u^*(x,t)/d_\Omega(x) = \partial u^*/\partial\nu\left(\overline{x},\overline{t}\right),$$

for $\left(\overline{x},\overline{t}\right) \in \partial\Omega\times\mathbb{R}$ and the analogous assertion holds also for z^*. Then, u^*/d_Ω and z^*/d_Ω can be extended to functions belonging to C_T which are positive everywhere in $\overline{\Omega}\times\mathbb{R}$. This implies H1' and H2' and it implies also that u^*/z^* and z^*/u^* have extensions belonging to C_T and positive everywhere in $\overline{\Omega}\times\mathbb{R}$. Now, taking into account these facts and since

$$\nabla\widetilde{u} = \left(\left(u^*/z^*\right)\nabla z^* + \left(z^*/u^*\right)\nabla u^*\right)/2,$$

and

$$\widetilde{u}_t = \left(\left(u^*/z^*\right) z_t^* + \left(z^*/u^*\right) u_t^*\right)/2,$$

we get H3'.

To see H4' we note that from the equations satisfied by u^* and z^* and the divergence theorem we have

$$\int_\Omega \left(u^* z^*\right)_t(.,t) = \int_\Omega \left[u_t^* z^* + u^* z_t^*\right](.,t)$$

$$= \int_{\Omega} \left[\lambda_1(m) m u^* + div(A \nabla u^*) - \langle \overline{b}, \nabla u^* \rangle - c_0 u^* \right](.,t) z^*(.,t)$$

$$+ \int_{\Omega} \left[-\lambda_1(m) m z^* - div(A\nabla z^*) - \langle \overline{b}, \nabla z^* \rangle + \left(c_0 - div\left(\overline{b}\right)\right) z^* \right](.,t) u^*(.,t)$$

$$= \int_{\Omega} - \langle \overline{b}, \nabla u^* \rangle (.,t) z^*(.,t) + \left[- \langle \overline{b}, \nabla z^* \rangle - div\left(\overline{b}\right) z^* \right](.,t) u^*(.,t)$$

$$= - \int_{\Omega} div \langle u^* z^* \overline{b} \rangle (.,t) = 0.$$

Lemma 3.3. *Let* $\varphi \in L_T^2(\mathbb{R}, H^1(\Omega, d_\Omega^2))$*. Then* $\varphi u^*, \varphi z^* \in L_T^2\left(\mathbb{R}, H_0^1(\Omega)\right)$*.*

Proof. For $j \in \mathbb{N}$ we set

$$\Omega_j := \left\{ x \in \Omega : d_\Omega(x) > j^{-1} \right\}, \qquad A_j = \left\{ x \in \Omega : (2j)^{-1} < d_\Omega(x) < j^{-1} \right\}$$

and $\psi_j \in C_T^\infty(\Omega \times \mathbb{R})$ such that $\psi_j \geq 0$ in $\Omega \times \mathbb{R}$, supp $(\psi_j) \subset \Omega_{2j} \times \mathbb{R}$, $\psi_j = 1$ in $\Omega_j \times \mathbb{R}$ and $|\nabla \psi_j| \leq 4j$ in $A_j \times \mathbb{R}$. Clearly, $\psi_j \varphi u^* \in L_T^2\left(\mathbb{R}, H_0^1(\Omega)\right)$. Moreover, by Lebesgue´s convergence theorem $\lim_{j\to\infty}(\psi_j \varphi u^*) = \varphi u^*$ in $L_T^2\left(\mathbb{R}, H_0^1(\Omega)\right)$. Also,

$$\nabla(\psi_j \varphi u^*) = \varphi u^* \nabla \psi_j + \psi_j u^* \nabla \varphi + \psi_j \varphi \nabla u^*,$$

and, again by Lebesgue's theorem,

$$\lim_{j\to\infty} \psi_j u^* \nabla \varphi + \psi_j \varphi \nabla u^* = u^* \nabla \varphi + \varphi \nabla u^*,$$

in $L_T^2\left(\mathbb{R}, H_0^1(\Omega)\right)$. On the other hand $0 \leq u^* \leq c d_\Omega$ for some $c > 0$ and so

$$\|\varphi u^* \nabla \psi_j\|^2_{L_T^2(\Omega \times (0,T))} = \int_{A_j \times (0,T)} \left(\varphi^2 u^{*2} \left|\nabla(\psi_j)\right|^2 \right) \leq 16 c^2 \int_{A_j \times (0,T)} \varphi^2.$$

Now, since $L_T^2\left(\mathbb{R}, H^1\left(\Omega, d_\Omega^2\right)\right) \subset L_T^2$ and $\lim_{j\to\infty}|A_j| = 0$ it follows that

$$\lim_{j\to\infty} \int_{A_j \times (0,T)} \varphi^2 = 0,$$

and so $\lim_{j\to\infty} \varphi u^* \nabla \psi_j = 0$ in L_T^2. Thus $\varphi u^* \in L_T^2\left(\mathbb{R}, H_0^1(\Omega)\right)$. Similarly, $\varphi z^* \in L_T^2\left(\mathbb{R}, H_0^1(\Omega)\right)$ and the lemma is proved. □

We see from Lemma 3.2 that $\widetilde{u}$ satisfies the assumptions of Lemma 2.5, and so we can consider the function $W_{\widetilde{u}}$ given there. The following lemma gives an explicit description of it.

Lemma 3.4. *Let* $\widetilde{u}$ *be as in Lemma* 3.2*. Then*

$$W_{\widetilde{u}} = \frac{1}{2}(\log u^* - \log z^*) - \frac{1}{2|D|T} \int_{D \times (0,T)} (\log u^* - \log z^*). \tag{3.9}$$

Proof. Let Z be the function defined by the right member of (3.9). The properties of u^* and z^* imply that $Z \in L^2_T\left(\mathbb{R}, H^1\left(\Omega, d^2_\Omega\right)\right)$, and clearly $\int_{D\times(0,T)} Z = 0$. Since the solution of (2.11) is unique, it is enough to show that for all $g \in L^2_T\left(\mathbb{R}, H^1\left(\Omega, d^2_\Omega\right)\right)$

$$\int_{\Omega\times(0,T)} \left[\left(\widetilde{u}^2\right)_t g + \widetilde{u}^2 \left\langle 2A\nabla Z - \overline{b}, \nabla g\right\rangle\right] = 0. \tag{3.10}$$

Taking into account that $\nabla Z = \left(\nabla u^*/u^* - \nabla z^*/z^*\right)/2$ and that $\left(\widetilde{u}^2\right)_t = z^* u^*_t + u^* z^*_t$, a computation using (1.3) and (2.3) shows that

$$\begin{aligned}
&\int_{\Omega\times(0,T)} \left(\widetilde{u}^2\right)_t g + \widetilde{u}^2 \left\langle 2A\nabla Z - \overline{b}, \nabla g\right\rangle \\
&= \int_{\Omega\times(0,T)} \left(u^*_t z^* + z^*_t u^*\right) g + z^* \left\langle A\nabla u^*, \nabla g\right\rangle - u^* \left\langle A\nabla z^*, \nabla g\right\rangle - u^* z^* \left\langle \overline{b}, \nabla g\right\rangle \\
&= \int_{\Omega\times(0,T)} \left[\lambda_1(m)\, m u^* z^* g - \left\langle A\nabla u^*, \nabla\left(z^* g\right)\right\rangle - \left(\left\langle \overline{b}, \nabla u^*\right\rangle z^* - c_0 u^* z^*\right) g \right.\\
&\quad -\lambda_1(m)\, m z^* u^* g + \left\langle A\nabla z^*, \nabla\left(u^* g\right)\right\rangle - \left(\left\langle \overline{b}, \nabla z^*\right\rangle + \left(c_0 - div\left(\overline{b}\right)\right) z^*\right) u^* g \\
&\quad \left. + z^* \left\langle A\nabla u^*, \nabla g\right\rangle - u^* \left\langle A\nabla z^*, \nabla g\right\rangle - u^* z^* \left\langle \overline{b}, \nabla g\right\rangle\right] \\
&= -\int_{\Omega\times(0,T)} div\left(g u^* z^* \overline{b}\right) = 0,
\end{aligned}$$

the last equality from the divergence theorem and the fact that for each t, $\left(g u^* z^* b\right)(.,t)$ is a continuous vector field on $\overline{\Omega}$ with divergence in $L^1(\Omega)$. □

Theorem 3.5. *Let $\Lambda(u)$ be given by* (3.1), *and let $\mathcal{U}$ be the set of the functions u satisfying the conditions of Lemma* 2.5 *for $V = \Omega$ and such that $\int_{\Omega\times(0,T)} mu^2 > 0$. Let W_u be given by Lemma* 2.5. *We have*

$$\lambda_1(m) = \min_{u\in\mathcal{U}} \frac{\Lambda(u) + \int_{\Omega\times(0,T)} u^2 \left\langle A\nabla W_u, \nabla W_u\right\rangle}{\int_{\Omega\times(0,T)} mu^2}.$$

Proof. Let $\widetilde{u}$ be as in the above lemmas. Taking into account Proposition 3.1, it suffices to show the following facts

$$\int_{\Omega\times(0,T)} m\widetilde{u}^2 > 0, \tag{3.11}$$

$$\lambda_1(m) \int_{\Omega\times(0,T)} m\widetilde{u}^2 = \Lambda(\widetilde{u}) + \int_{\Omega\times(0,T)} \widetilde{u}^2 \left\langle A\nabla W_{\widetilde{u}}, \nabla W_{\widetilde{u}}\right\rangle. \tag{3.12}$$

Let us prove (3.11) We first multiply (3.2) by $\widetilde{u}^2 = u^*z^*$ and use that $\nabla v^* = -\nabla u^*/u^*$ and $v_t^* = -u_t^*/u^*$ to obtain

$$\int_{\Omega\times(0,T)} \lambda_1(m)\, m\widetilde{u}^2 - z^*u_t^* - \widetilde{u}^2 div\,(A\nabla v^*) \tag{3.13}$$
$$= \int_{\Omega\times(0,T)} \left[-\left\langle \overline{b}, \nabla v^*\right\rangle - \langle A\nabla v^*, \nabla v^*\rangle + c_0\right]\widetilde{u}^2.$$

Let $\eta := -b - 2A\nabla v^*$, where $v^* := -\log u^*$. Since

$$-\left\langle \overline{b}, \nabla v^*\right\rangle - \langle A\nabla v^*, \nabla v^*\rangle = \langle \nabla v^*, \eta\rangle + \left\langle a+\eta, A^{-1}(a+\eta)\right\rangle/4,$$

we can rewrite (3.13) as

$$\int_{\Omega\times(0,T)} \lambda_1(m)\, m\widetilde{u}^2 - z^*u_t^* - \widetilde{u}^2 div\,(A\nabla v^*) \tag{3.14}$$
$$= \int_{\Omega\times(0,T)} \widetilde{u}^2\langle \eta, \nabla v^*\rangle + \frac{1}{4}\widetilde{u}^2\left\langle \overline{b}+\eta, A^{-1}\left(\overline{b}+\eta\right)\right\rangle + c_0\widetilde{u}^2.$$

Now, we have $\nabla\widetilde{u}/\widetilde{u} = ((\nabla u^*/u^*) + (\nabla z^*/z^*))/2$ and, from Lemma 3.4, $\nabla W_{\widetilde{u}} = ((\nabla u^*/u^*) - (\nabla z^*/z^*))/2$. Thus, a computation gives

$$\eta = -\overline{b} - 2A\nabla v^* = -\overline{b} + 2A\left(\nabla\widetilde{u}/\widetilde{u} + \nabla W_{\widetilde{u}}\right).$$

Then, from (3.14),

$$\int_{\Omega\times(0,T)} \lambda_1(m)\, m\widetilde{u}^2 - z^*u_t^* + \left\langle A\nabla\left(\widetilde{u}^2\right), \nabla v^*\right\rangle$$
$$= \int_{\Omega\times(0,T)}\left[\left\langle -\overline{b} + 2A(\nabla\widetilde{u}/\widetilde{u} + \nabla W_{\widetilde{u}}), \nabla v^*\right\rangle + \left\langle \overline{b}+\eta, A^{-1}(\overline{b}+\eta)\right\rangle/4 + c_0\right]\widetilde{u}^2.$$

As observed in the proof of Proposition 3.1, $v^* \in L_T^2\left(\mathbb{R}, H^1\left(\Omega, d_\Omega^2\right)\right)$. Hence, from the last equation and (2.6) applied to $u = \widetilde{u}$ and $\varphi = v^*$ we get

$$\int_{\Omega\times(0,T)} \lambda_1(m)\, m\widetilde{u}^2 - z^*u_t^*$$
$$= \int_{\Omega\times(0,T)} -\left(\widetilde{u}^2\right)_t v^* + \left(\left\langle \overline{b}+\eta, A^{-1}\left(\overline{b}+\eta\right)\right\rangle/4 + c_0\right)\widetilde{u}^2.$$

Now, $\int_{\Omega\times(0,T)}\left(\widetilde{u}^2\right)_t v^* = \int_{\Omega\times(0,T)} z^*u_t^*$ and so

$$\lambda_1(m)\int_{\Omega\times(0,T)} m\widetilde{u}^2 = \int_{\Omega\times(0,T)}\left(\left\langle \overline{b}+\eta, A^{-1}\left(\overline{b}+\eta\right)\right\rangle/4 + c_0\right)\widetilde{u}^2 \tag{3.15}$$

$$= \int_{\Omega\times(0,T)} \langle A\nabla v^*, \nabla v^*\rangle + c_0\widetilde{u}^2,$$

which (since $c_0 \geq 0$ and since v^* is not a constant because $\lambda_1(m) > 0$) gives (3.11). On the other hand, a computation shows that

$$\frac{1}{4}\int_{\Omega\times(0,T)} \widetilde{u}^2 \left\langle b+\eta, A^{-1}(b+\eta)\right\rangle \tag{3.16}$$

$$= \int_{\Omega\times(0,T)} \widetilde{u}^2 \left\langle A\left(\nabla\widetilde{u}/\widetilde{u} + \nabla W_{\widetilde{u}}\right), \nabla\widetilde{u}/\widetilde{u} + \nabla W_{\widetilde{u}}\right\rangle$$

$$= \int_{\Omega\times(0,T)} \langle A\nabla\widetilde{u}, \nabla\widetilde{u}\rangle + \left(\langle A\nabla W_{\widetilde{u}}, \nabla W_{\widetilde{u}}\rangle + \langle 2A\nabla W_{\widetilde{u}}, \nabla \log\widetilde{u}\rangle\right)\widetilde{u}^2.$$

Note that $\widetilde{u}$ satisfies the conditions of Lemma 2.5 for $V = \Omega$ and that, by Remark 2.6, $\log\widetilde{u} \in L^2_T\left(\mathbb{R}, H^1\left(\Omega, d^2_\Omega\right)\right)$. Thus, using (2.6),

$$\int_{\Omega\times(0,T)} \widetilde{u}^2 \langle 2A\nabla W_{\widetilde{u}}, \nabla\log\widetilde{u}\rangle = \int_{\Omega\times(0,T)} \widetilde{u}^2 \left\langle \overline{b}, \nabla\log\widetilde{u}\right\rangle - \left(\widetilde{u}^2\right)_t \log\widetilde{u}, \tag{3.17}$$

but, since $\widetilde{u}$ is T-periodic, $\int_{\Omega\times(0,T)} \left(\widetilde{u}^2\right)_t \log\widetilde{u} = -\int_{\Omega\times(0,T)} \widetilde{u}\widetilde{u}_t = 0$. From this fact, combining the first equality in (3.15) with (3.16) and (3.17), we obtain (3.12) and this concludes the proof. □

Let us write $P(\Omega, m)$ instead of $P(m)$. As an immediate consequence of the above theorem we have:

Corollary 3.6. (i) *Let $\Omega_0 \subset \Omega$ be a smooth bounded domain, and assume $P(\Omega_0, m) > 0$. Then $\lambda_1(\Omega_0, m) > \lambda_1(\Omega, m)$.*

(ii) *Let $0 \leq a \in L^r_T$ for some $r > N + 2$, and assume $P(\Omega, m) > 0$. Then $\lambda_1(L + a, m) > \lambda_1(L, m)$.*

Proof. Clearly (ii) is a direct consequence of Theorem 3.5. To see (i), consider the eigenfunctions u_0^* and z_0^* given in Remarks 2.1 and 2.2 corresponding to the domain Ω_0 instead of Ω. Let $\widetilde{u}_0 = (u_0^* z_0^*)^{1/2}$. From Theorem 3.5 applied on $\Omega_0 \times \mathbb{R}$ and Proposition 3.1 used with $V = \Omega_0$ we have

$$\begin{aligned} &\lambda_1(\Omega_0, m) \\ &= \int_{\Omega_0\times(0,T)} \left[\langle A\nabla\widetilde{u}_0, \nabla\widetilde{u}_0\rangle + \left\langle \overline{b}, \nabla\widetilde{u}_0\right\rangle \widetilde{u}_0 + c_0\widetilde{u}_0^2 + \widetilde{u}_0^2 \langle A\nabla W_{\widetilde{u}_0}, \nabla W_{\widetilde{u}_0}\rangle\right] \\ &> \lambda_1(\Omega, m). \end{aligned}$$

We note that both (i) and (ii) are well-known in the elliptic case. For the periodic parabolic problem, (ii) can be proved in another way using the monotonicity of $\lambda_1(m)$ with respect to the weight (see e.g. [10], Remark 3.7), while we have found no alternative proof of (i) in the literature.

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